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PROJECTIONS OF SINGULAR VARIETIES AND CASTELNUOVO-MUMFORD REGULARITY

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Abstract

We find upper bounds for the regularity of a singular projective variety over an algebraically closed field of characteristic zero. In particular we study the cases when the variety has finitely many isolated singularities and when it has a 1-dimensional singular locus.

1. Introduction

Let $X \subseteq \mathbb{P}^r$ be an integral locally Cohen-Macaulay non-degenerate variety of dimension $n \leq r-2$, over an algebraically closed field K of characteristic zero. It is known that the canonical restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(j)) \to H^0(\mathcal{O}_X(j))$ is surjective for large j, and the knowledge of the exact point where it becomes surjective is relevant for understanding the geometry of X. More generally it is interesting to bound the Castelnuovo-Mumford regularity of X, defined by:

$$\operatorname{reg}(X) := \min\{t \in \mathbb{Z} \mid H^i(\mathcal{I}_X(t-i)) = 0 \text{ for } i > 0\}$$

There is the following well-known conjecture on the Castelnuovo-Mumford regularity:

CONJECTURE. *If X is smooth of degree d, then* $\operatorname{reg}(X) \leq d - r + n + 1$.

This conjecture is true for integral curves (see [7]) and for smooth surfaces (see [11]). In higher dimension the problem is still open; lot of work has been done recently, see e.g. [10] and [5] and the bibliography of these papers. See also [4] for interesting variants.

On the contrary, for singular varieties only the case of curves has been dealt with, in the already quoted [7], where the conjecture is proved.

For higher dimensional singular varieties, the previous bound is easily seen to be false; hence it is meaningful to look for a bound in terms of the singularities.

This is the aim of the present paper. Our approach is the one due to Lazarsfeld [11], as generalized in [10], and with the point of view of [5]. The main idea is to use

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a general projection $f\colon X \to \mathbb{P}^{n+1}$ and to find a surjective morphism of $\mathcal{O}_{\mathbb{P}^{n+1}}$-modules $w\colon \mathcal{F} \to f_*(\mathcal{O}_X)$, where $\mathcal{F}$ is a direct sum of line bundles: the sheaf $\mathcal{F}$ has to be found "as small as possible," as its shape will provide a bound (see e.g. [5], Theorem 1.4).

In order to find a suitable pair $(w, \mathcal{F})$ we need a good understanding of the fibers of f, and of their stratification by length or, better, by regularity. For the fibers coming from the regular locus of X we can rely on a result of Mather, generalized in [2]. So our main problem is to study the fibers corresponding to the singular points: this will be the object of this paper.

Before listing our main results let's fix some notation and definitions.

DEFINITION 1.1. If $P \in X$ is a singular point we can write $\mathrm{emdim}_P(X) = n + k$, where $1 \leq k \leq r - n$ and we have, since X is locally Cohen-Macaulay,

$$m(P) \geq k + 1$$

(see Remark 2.4).

If $k = 1$, P will be called *hypersuperficial*.

If $m(P) = k+1$, P is said to be *of minimal multiplicity* (for details see Section 2).

DEFINITION 1.2. If $P \in X$ is a singular point, we define the *general regularity* $\rho(P)$ of P as follows.

Let L be a general linear space of dimension $r-n-1$ passing through P and let W be the largest subscheme of $L \cap X$ supported on P. Then we put $\rho(P) := \mathrm{reg}(W)-1$, namely:

$$\rho(P) = \min\{j \in \mathbb{Z} \mid H^1(\mathcal{I}_{W,L}(j)) = 0\}.$$

We also put $\rho(\mathrm{Sing}X) := \max\{\rho(P) \mid P \in \mathrm{Sing}(X)\}$.

Now we are ready to describe how the paper is organized.

After some preparation, done in Section 2, in Section 3 we evaluate the length of the fibers of a general projection of X over the images of singular points and we bound their regularity.

The results we obtain are applied in the next sections.

In Section 4 we study the case when X has a finite number of isolated singularities.

Here we introduce the notion of "good projection with respect to X_{reg}" (see Definition 4.1) and we prove:

Theorem 1.3. *Assume that* $\mathrm{Sing}(X)$ *is finite and set* $p := \max\{n, \rho(\mathrm{Sing}\,X)\}$. *Assume further that there is a good projection of* X *with respect to* X_{reg}, *e.g. assume*

$\dim X \leq 14$. *Then*

$$\operatorname{reg}(X) \leq d - r + n + 1 + \frac{(p-1)(p-2)}{2}.$$

Moreover, using the bounds given in Section 3, we deduce a bound for the regularity in terms of the multiplicities and the embedding dimensions of the singular points, as shown in Corollary 4.3. As an immediate consequence we get:

Corollary 1.4. *Let X be as in* Theorem 1.3, *and assume that each singular point of X is either hypersuperficial or of minimal multiplicity. Then*

$$\operatorname{reg}(X) \leq d - r + n + 1 + \frac{(n-1)(n-2)}{2}.$$

The previous bound is exactly the bound for the regularity of a smooth variety given in [5], Theorem 2.5 (i). This shows that the singularities which are either hypersuperficial or of minimal multiplicity are irrelevant with regard to this type of bound.

When the singular locus of X has higher dimension, the problem becomes much more difficult.

In Section 5 we study the case when X has a 1-dimensional singular locus and almost all singularities are hypersuperficial. The main result is Theorem 5.6.

2. Multiplicity and embedding dimension

We collect here some known facts on the notion of multiplicity of a scheme at a point, and its relations with the embedding dimension.

DEFINITION 2.1. If A is an Artinian ring, we denote by $\lambda(A)$ the length of A.

Likewise, if Z is a zero-dimensional scheme, we denote by $\lambda(Z)$ the length (or degree) of Z.

Moreover if $P \in Z$, we denote by $\lambda_P(Z)$ the length of Z at P, namely the length of the largest subscheme of Z supported on P. In other words $\lambda_P(Z)$ is the length of the Artinian local ring $\mathcal{O}_{Z,P}$.

DEFINITION 2.2. If A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ one can define the integer $e(\mathfrak{q})$ for every $\mathfrak{m}$-primary ideal $\mathfrak{q}$; $m(A) := e(\mathfrak{m})$ is called the *multiplicity* of A (see e.g, [12], [6], [9]).

If X is a locally Noetherian scheme and $P \in X$, the multiplicity of P is $m_P(X) := m(\mathcal{O}_{X,P})$. We simply write $m(P)$ instead of $m_P(X)$ if X is understood.

REMARK 2.3. Let $Y \subseteq \mathbb{P}^r$ be a closed subscheme and let $P \in Y$ be a closed point. Set $\dim_P(Y) := \dim(\mathcal{O}_{Y,P})$. We have:

(i) if $\dim_P(Y) = 0$, then $m_P(Y) = \lambda_P(Y)$ (this is immediate from the quoted definition);
(ii) Y is locally Cohen-Macaulay at P (i.e. $\mathcal{O}_{Y,P}$ is Cohen-Macaulay) if and only if $m(P) = \lambda_P(Y \cap L)$ for any general linear space L of dimension $r - \dim_P(Y)$ passing through P (this follows, e.g., from [12], Theorems 14.14, 14.13 and 17.11);
(iii) if Y is locally Cohen-Macaulay at P, then for any general hyperplane H through P we have $m_P(Y) = m_P(Y \cap L)$ (this is immediate from (ii)).

Let's now recall that the embedding dimension of a Noetherian local ring A with maximal ideal $\mathfrak{m}$ and residue field k is $\operatorname{emdim}(A) := \dim_k \mathfrak{m}/\mathfrak{m}^2$; this is also the cardinality of a minimal set of generators of $\mathfrak{m}$.

If Y is a scheme and $P \in Y$ we set $\operatorname{emdim}_P(Y) = \operatorname{emdim}(\mathcal{O}_{Y,P})$. Note that if $Y \subseteq \mathbb{P}^r$ and $P \in Y$ is a closed point, then $\operatorname{emdim}_P(Y) = \dim(T_P(Y))$, where $T_P(Y)$ is the embedded tangent space to Y at P.

REMARK 2.4. The embedding dimension and the multiplicity of a local Cohen-Macaulay ring A are related by an inequality due to Abhyankar (see [1]), which implies:

$$m(P) \geq \operatorname{emdim}_P(Y) - \dim_P(Y) + 1$$

if Y is locally Cohen-Macaulay at P (this inequality follows also from Remark 2.3).

Note that this inequality can fail for non Cohen-Macaulay points: take for example an improper double point of a surface in $\mathbb{P}^4$, e.g. a singular point obtained by projecting generically in $\mathbb{P}^4$ a smooth surface spanning $\mathbb{P}^5$ (except the Veronese surface).

REMARK 2.5. We will use freely the fact that embedding dimension and multiplicity are upper semicontinuous functions on the space of closed points P of any scheme $Y \subseteq \mathbb{P}^r$.

The first assertion follows from the isomorphism $\mathfrak{m}_P/(\mathfrak{m}_P)^2 \cong \Omega_{Y/K} \otimes \kappa(P)$, where $\Omega_{Y/K}$ is the (coherent) sheaf of Kähler differentials (see e.g. [8], p.187, Example 8.1 (a)). For the second one see e.g. [9], Appendix, Chapter II, Theorem 5.2.

3. The fiber of a general projection at a singular point

In order to produce our bounds for the regularity of an integral locally Cohen-Macaulay non-degenerate singular variety $X \subseteq \mathbb{P}^r$ we have to evaluate the length and the regularity of the fibers of a general projection over the images of singular points. In this section we make some steps in this direction. Applications will be given in the next sections.

We denote by $G(t, N)$ the Grassmannian of the linear subspaces of dimension t in $\mathbb{P}^N$. Recall that $\dim(G(t, N)) = (N - t)(t + 1)$.

Let $X \subseteq \mathbb{P}^r$ be an integral locally Cohen-Macaulay non-degenerate variety of dimension $n \leq r-2$, and let's denote by h its codimension.

DEFINITION 3.1. Let $P \in X$. We set

$$\alpha_P := \min\{\lambda_P(L \cap X) \mid L \text{ linear } (h-1)\text{-subspace through } P\}$$

This makes sense because a general linear subspace of dimension $h-1$ through P intersects X in a zero-dimensional scheme.

Obviously $\alpha_P = 1$ if and only if P is non-singular. Moreover since X is locally Cohen-Macaulay it follows from Remark 2.3 that $\alpha_P \leq m(P)$. This inequality will be improved later.

DEFINITION 3.2. Let $T \in G(h-2, r)$ be a linear space not meeting X, let $f_T\colon X \to \mathbb{P}^{n+1}$ be the projection with center T and let $P \in X$. We say that f_T is a *good projection at* P if $\lambda\big(f_T^{-1}(f_T(P))\big) = \alpha_P$. This is equivalent to the following two conditions:
(a) $\lambda_P(X \cap \langle T, P\rangle) = \alpha_P$;
(b) the fiber of f_T at $f_T(P)$ is supported on P.

Lemma 3.3. *If $P \in X$ and $T \in G(h-2, r)$ is general, then f_T is a good projection at P.*

Proof. We show that conditions (a) and (b) of Definition 3.2 hold.

Let W be the family of the linear spaces of dimension $h-1$ passing through P. It is easy to see that W is isomorphic to $\mathrm{G}(h-2, r-1)$, whence it is irreducible and $\dim W = (r-h+1)(h-1)$.

Set now $\alpha := \alpha_P$ and consider the set

$$\overline{W} := \{L \in W \mid \lambda_P(L \cap X) > \alpha\}.$$

By semicontinuity we have that $\overline{W}$ is a proper closed subscheme of W, whence

$$\dim \overline{W} < \dim W = (r-h+1)(h-1)$$

since W is irreducible.

Set now

$$V := \{T \in \mathrm{G}(h-2, r) \mid T \cap X = \emptyset\}$$

and

$$Z := \{\Lambda \in V \mid \lambda_P(\langle \Lambda, P\rangle \cap X) > \alpha\} = \{\Lambda \in V \mid \langle \Lambda, P\rangle \in \overline{W}\}.$$

The map defined by $\Lambda \mapsto \langle \Lambda, P\rangle$ is a morphism $Z \to W$ whose fibers are isomorphic to non-empty open subschemes of $G(h-2, h-1)$. It follows:

$$\begin{aligned} \dim Z &= \dim \overline{W} + \dim(G(h-2, h-1)) \\ &< (r-h+1)(h-1) + h - 1 \\ &= \dim(G(h-2, r)). \end{aligned}$$

This shows that a general $T \in V$ does not belong to Z, whence condition (a) of Definition 3.2 holds.

To prove condition (b) it is sufficient to show that a general $L \in W$ meets X only at P, and then proceed as in the previous case. For this consider the incidence relation

$$\mathcal{W} := \{(Q, L) \in (X \setminus \{P\}) \times W \mid Q \in L\}$$

and let

$$\mathrm{pr}_1 \colon \mathcal{W} \to X \setminus \{P\} \quad \text{and} \quad \mathrm{pr}_2 \colon \mathcal{W} \to W$$

be the projections.

It is easy to see that the fibers of pr_1 are isomorphic to the Grassmannian $G(h-3, r-2)$, whence

$$\dim \mathcal{W} = n + (r-h+1)(h-2).$$

Since $n = r - h$ it follows that $\dim \mathcal{W} < \dim W$, whence $\dim(\mathrm{pr}_2(\mathcal{W})) < \dim W$. The conclusion follows easily. $\sqcap$

Lemma 3.4. *Let $Z \subseteq \mathbb{P}^r$ be a zero-dimensional scheme with $Z_{\mathrm{red}} = \{P\}$. Let $m := \lambda(Z)$ and $k := \mathrm{emdim}_P(Z)$ and assume $k \geq 1$.*

Let $H \subseteq \mathbb{P}^r$ be a general hyperplane through P. Then:

(a) $\lambda(H \cap Z) \leq (1/k)((k-1)m + 1)$;

(b) $H^1(\mathcal{I}_{H\cap Z}(j)) = 0$ *for* $j \geq [((k-1)(m-k)+1)/k]$.

Proof. (a) If $k = 1$ it is easy to see that $\lambda(H \cap Z) = 1$, whence the claim. Hence we assume $k \geq 2$. In order to prove (a) we need first some preparation. For $0 \leq i \leq k$ set

$$a_{k-i} := \min\{\lambda(L \cap Z) \mid L \text{ linear subspace of codimension } i \text{ through } P\}.$$

Let $X_1, \ldots, X_r$ be a system of affine coordinates with origin P, whence $A := \mathcal{O}_{Z,P} = k[X_1, \ldots, X_r]_{(X_1,\ldots,X_r)}/I_Z = k[x_1, \ldots, x_r]_{(x_1,\ldots,x_r)}$. Let $\mathfrak{m} = (x_1, \ldots, x_r)A$ be the maximal ideal of A. Let H_i be the hyperplane $X_i = 0$. We can choose the coordinates general enough so that $\lambda(H_1 \cap Z) = \lambda(H_2 \cap Z) = a_{k-1}$ and $\lambda(Z \cap H_1 \cap H_2) = a_{k-2}$, that is

$$a_{k-1} = \lambda(A/(x_1)) = \lambda(A/(x_2))$$

and

$$a_{k-2} = \lambda(A/(x_1, x_2)).$$

By assumption $\mathfrak{m}$ is minimally generated by k elements and by Nakayama's Lemma and by a general choice of coordinates we may also assume that $\mathfrak{m} = (x_1, \dots, x_k)A$.

Then from the Mayer-Vietoris sequence

$$0 \to A/(x_1) \cap (x_2) \to A/(x_1) \oplus A/(x_2) \to A/(x_1, x_2) \to 0$$

we get

$$2a_{k-1} \le a_{k-2} + m. \tag{1}$$

Now we can prove (a) by induction on k. If $k = 2$, then $a_{k-2} = 1$ and the conclusion follows from (1).

Assume now that $k > 2$ and that the statement is true for $k - 1$.

Let $Z_1 := Z \cap H_1$. Then $\operatorname{emdim}_P(Z_1) = k - 1$ and $\lambda_P(Z_1) = a_{k-1}$. Then by the induction hypothesis we have:

$$(k-1)a_{k-2} \le (k-2)a_{k-1} + 1. \tag{2}$$

Multiplying both sides of (1) by $k - 1$ and using (2) we get

$$\begin{aligned} 2(k-1)a_{k-1} &\le (k-1)a_{k-2} + (k-1)m \\ &\le (k-2)a_{k-1} + 1 + (k-1)m \end{aligned}$$

whence

$$ka_{k-1} \le (k-1)m + 1$$

and (a) follows.

(b) Clearly $h^0(\mathcal{I}_{H\cap Z}(1)) \le r - \operatorname{emdim}_P(H \cap Z) = r - (k-1)$. The conclusion follows from (a) and the exact sequence $0 \to H^0(\mathcal{I}_{H\cap Z}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{O}_{H\cap Z}(1)) \to H^1(\mathcal{I}_{H\cap Z}(1)) \to 0$ and from the fact that $h^1(\mathcal{I}_{H\cap Z}(j))$ is strictly decreasing for $j \ge 0$ until it vanishes. $\square$

Lemma 3.5. *Let $P \in X$ be a singular point with* $\operatorname{emdim}_P(X) = n+k$ *and $m(P) = m$. Then we have*:

$$\rho(P) \le \left[\frac{(k-1)(m-k)+1}{k}\right].$$

In particular if P is either hypersuperficial or of minimal multiplicity (Definition 1.1) *then* $\rho(P) = 1$.

Proof. Let M be a general linear space of dimension h passing through P. Then $\dim(L \cap X) = 0$ and we denote by Z the largest subscheme of $L \cap X$ supported by P. Then we have $\lambda(Z) = m$ and $\operatorname{emdim}_P(Z) = k$. Let $L \subseteq M$ be a general linear space of dimension $h - 1$ passing through P and let $W := L \cap Z$. It is clear that L is general in the sense of Definition 1.2.

The conclusion follows easily by Lemma 3.4. □

4. The case of isolated singularities

We give some upper bounds for the regularity of an integral variety $X \subseteq \mathbb{P}^r$ with only finitely many isolated singularities. Recall that $n := \dim(X)$.

DEFINITION 4.1. We say that a finite birational projection $\sigma : X \to \mathbb{P}^{n+1}$ is *good with respect to* X_{reg} if the locally closed sets

$$S_j := \{Q \in \sigma(X_{\mathrm{reg}}) \mid \deg \sigma^{-1}(Q) \cap X_{\mathrm{reg}} = j\}$$

have the expected dimension, namely

$$\dim S_j \leq \min\{-1, n - j + 1\}$$

(see [5], def. 1.2).

REMARK 4.2. (i) By [2] a general projection is good with respect to X_{reg} if $\dim X \leq 14$.
(ii) It is not difficult to see that if X admits a good projection with respect to X_{reg}, then a general projection is a good projection with respect to X_{reg}.

Proof of Theorem 1.3. The proof follows exactly the pattern of [5] (proofs of Proposition 2.3 and Corollary 2.4). Here we highlight the role of the singular points.

Let $f := f_T : X \to \mathbb{P}^{n+1}$ be a general projection (from a general linear space T of dimension $h - 2$).

CLAIM. There is a surjective morphism of sheaves

$$w_p : \mathcal{G} \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(-3) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(-p) \to f_*(\mathcal{O}_X)$$

where

$$\mathcal{G} := \mathcal{O}_{\mathbb{P}^{n+1}} \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{r-n+1} \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(-2)^{\binom{r+2}{2}}.$$

Moreover $\ker w_p$ is a locally free sheaf.

Proof of Claim. For each $P \in \mathrm{Sing}(X)$ the fiber $f^{-1}(f(P))$ is supported on P and $\rho(f^{-1}(f(P)) \leq \rho(\mathrm{Sing}(X))$ (see Lemma 3.3). It follows also that if $Q \in f(X_{\mathrm{reg}})$ we have $f^{-1}(Q) \subseteq X_{\mathrm{reg}}$. Hence (with the notation of Definition 4.1) we have $S_j := \{Q \in f(X_{\mathrm{reg}}) \mid \deg f^{-1}(Q) = j\}$.

Now by assumption we have $\dim S_j \leq \max\{-1, n-j+1\}$ and since $\mathrm{Sing}\, X$ is finite, the existence of w_p follows as in [5], with obvious changes.

Now, since X is locally Cohen-Macaulay we have that $f_*(\mathcal{O}_X)$ is a coherent $\mathcal{O}_{\mathbb{P}^{n+1}}$-module, with local depth n at each closed point, and this implies that $\ker w_p$ is locally free by a theorem of Auslander-Buchsbaum (see, e.g., [6], Theorem 19.9). This proves the Claim. $\square$

The conclusion follows by the Claim and by [5], Theorem 1.4. $\square$

Corollary 4.3. *Let the notation and the assumptions be as in* Theorem 1.3, *Let* $\mathrm{Sing}(X) = \{P_1, \ldots, P_s\}$ *and for each* $i = 1, \ldots, s$ *put* $m_i := m(P_i)$, $k_i := \mathrm{emdim}_{P_i} X - n$ *and* $\sigma_i := [((k_i - 1)(m_i - k_i) + 1)/k_i]$. *Let* $q := \max\{n, \sigma_1, \ldots, \sigma_s\}$. *Then*

$$\mathrm{reg}(X) \leq d - r + n + 1 + \frac{(q-1)(q-2)}{2}.$$

Proof. It is an immediate consequence of Theorem 1.3 and Lemma 3.5. $\square$

5. The case of one-dimensional singular locus

If $\dim(\mathrm{Sing}(X)) > 0$ and $f\colon X \to \mathbb{P}^{n+1}$ is a general projection the fiber $f^{-1}(f(P))$, ($P \in \mathrm{Sing}(X)$), may not be supported on P. So we need to introduce a new invariant, which is motivated also by Lemma 5.3.

DEFINITION 5.1. Let $P \in \mathrm{Sing}(X)$. We define the non-negative integer γ_P to be the integer such that the general line connecting P to a point of X_{reg} intersects X_{reg} in a scheme of length $\gamma_P + 1$.

If T is an irreducible one-dimensional component of $\mathrm{Sing}(X)$, set $\gamma(X, T) := \gamma_P$, for a general $P \in T$.

Moreover, set $\tilde{\gamma}(X, T) := \gamma(X, T)$, if the union of all secant lines connecting P to a point of X_{reg} has dimension $n+2$, and $\tilde{\gamma}(X, T) := 0$ otherwise.

EXAMPLE 5.2. Fix an integer $\gamma \geq 2$. Let $Y \subseteq \mathbb{P}^{r-2}$ be a smooth non-degenerate subvariety. See $\mathbb{P}^{r-2}$ as a hyperplane, A, of $\mathbb{P}^{r-1}$ and $\mathbb{P}^{r-1}$ as a hyperplane, B, of $\mathbb{P}^r$. Fix $Q_1 \in B \setminus A$ and $Q_2 \in \mathbb{P}^r \setminus A$.

Let $C := [Q_1 : Y]$ be the cone with vertex Q_1 and Y as a basis.

Let Z be the intersection of C with a general degree $\gamma + 1$ hypersurface of A. Thus $\deg(Z) = (\gamma + 1)\deg(Y)$ and $Q_1 = \mathrm{Sing}(Z)$.

Let X be the cone with vertex Q_2 and Z as a basis. Thus $\mathrm{Sing}(X)$ is the line $T := \langle Q_1, Q_2\rangle$.

Fix any $P \in T \setminus Q_2$. For every hyperplane H with $P \in H, Q_2 \notin H$, the pair $(X \cap H, P)$ is projectively equivalent to the pair (Z, Q_1).

Thus every line through P intersecting $X \cap H$ at another point, intersects $X \cap H$, outside P, in exactly γ points.

Therefore the general secant line of X intersecting T, intersects X in $\gamma+1$ points, exactly one of them being in T. Hence $\gamma(X, T) = \gamma$.

Notice that in this example we have $\tilde{\gamma} = 0$.

Lemma 5.3. *Let $T \subseteq \mathrm{Sing}(X)$ be a curve. Then for a general projection $f_\Lambda\colon X \to \mathbb{P}^{n+1}$ the following holds: for any $P \in T$, $f_\Lambda^{-1}(f_\Lambda(P)) \cap X_{\mathrm{reg}}$ cannot contain any length 2 subscheme not collinear with P.*

Proof. For each subscheme $Z \subseteq X_{\mathrm{reg}}$ of length 2 not collinear with P, let $\pi_Z := \langle P, Z\rangle$. The planes π_Z form a family $\mathcal{F}$ of dimension $\leq 2n+1$. Our claim is equivalent to say that for a general $\Lambda \in G(h-2, r)$ and for any $\pi_Z \in \mathcal{F}$ we have $\dim(\Lambda \cap \pi_Z) \leq 0$.

Let now $\mathcal{G}$ be the family consisting of all linear subspaces of dimension $h - 2$ such that $\Lambda \cap \pi$ contains a line of π, for some $\pi \in \mathcal{F}$.

Now, if L is a line, the family $\{\Lambda \in G(h - 2, r) \mid L \subseteq \Lambda\}$ is isomorphic to $G(h-4, r-2)$, whence $\dim \mathcal{G} \leq 2n+1+2+\dim G(h-4, r-2) = (r-h+2)(h-1)-1 < \dim G(h - 2, r)$ and the conclusion follows. $\square$

Lemma 5.4. *Let $T \subseteq \mathrm{Sing}(X)$ be a 1-dimensional irreducible component. Consider the family $\mathcal{L}'$ of secant lines $\langle P, A\rangle$, with $P \in T$ and $A \in X_{\mathrm{reg}}$, with $\lambda(\langle P, A\rangle \cap X_{\mathrm{reg}}) > \tilde{\gamma}(X, T)$. Then none of these secants meets the general $\Lambda \in G(h - 2, r)$.*

Proof. Let $\mathcal{L}$ be the family of secant lines of the form $\langle P, A\rangle$, with $P \in T$ and $A \in X_{\mathrm{reg}}$. $\mathcal{L}$ is irreducible, of dimension $n + 1$ and $\mathcal{L}'$ is a proper subfamily of $\mathcal{L}$, whence $\dim \mathcal{L}' \leq n$. For every line L the family of centers of projection Λ meeting L has dimension $1 + \dim G(h - 3, r - 1)$. Therefore the family of those $\Lambda \in G(h - 2, r)$ meeting at least one line of $\mathcal{L}'$ has dimension $\leq n+1+\dim G(h-3, r-1) = \dim G(h-2, r) - 1$ and this proves the claim. $\square$

Lemma 5.5. *Assume $\dim(\mathrm{Sing}(X)) = 1$ and let T be a 1-dimensional irreducible locally closed subscheme of $\mathrm{Sing}(X)$ such that every $P \in T$ is hypersuperficial and satisfies the equalities $m(P) = m(T)$ and $\tilde{\gamma}_P = \tilde{\gamma}(X, T)$. Then for a general $f := f_\Lambda$ and for every $P \in T$ we have $\deg(f^{-1}(f(P)) \leq m(T) + \tilde{\gamma}(X, T)$, whence $\mathrm{reg}(f^{-1}(f(P)) \leq m(T) + \tilde{\gamma}(X, T) - 1$.*

Proof. Since every $P \in T$ is hypersuperficial it is easy to see that for every $P \in T$ the tangent cone $C_P(X)$ is a hypersurface in $T_P(X)$, and hence is arithmetically Cohen-Macaulay. This means that the graded ring $\mathrm{gr}(\mathcal{O}_{X,P})$ is Cohen-Macaulay.

Set now $W := \bigcup_{P \in T} T_P(X)$, which is a locally closed subscheme of $\mathbb{P}^r$. We have $\dim W \leq n+2$, since every $P \in T$ is hypersuperficial. Then a general $\Lambda \in G(n-2, r)$ meets W in finitely many points $A_1 \in T_{P_1}(X), \dots, A_l \in T_{P_l}(X)$, where $P_1, \dots, P_l \in T$ are distinct points. Moreover $\Lambda \cap C_P(X) = \emptyset$ for every $P \in T$.

Fix any Λ satisfying the above conditions. Fix any i with $1 \leq i \leq l$ and put $P := P_i$ and $A := A_i$. Let $L = H_1 \cap \cdots \cap H_n$ be a general linear space of dimension h containing Λ and P, where $H_1, \dots, H_n$ are hyperplanes. Then $\pi := L \cap T_P(X)$ is a general plane of $T_P(X)$ through the line $\langle A, P \rangle$, and since $A \notin C_P(X)$ we have that $\dim(L \cap C_P(X)) = 1$. For $j = 1, \dots, n$ let $l_j \in \mathcal{O}_{X,P}$ be a generator of the ideal of H_j and let $l_j^* \in \mathrm{gr}(\mathcal{O}_{X,P})$ be the leading form of l_j. Then $(l_1^*, \dots, l_n^*)\,\mathrm{gr}(\mathcal{O}_{X,P})$ is the homogeneous ideal of $L \cap C_P(X)$ with respect to $C_P(X)$. Since $\mathrm{gr}(\mathcal{O}_{X,P})$ is Cohen-Macaulay of dimension $n+1$ it follows that $(l_1^*, \dots, l_n^*)$ is a regular sequence. Then $\lambda_P(L \cap X) = m(P)$ (see [3], Ch. VIII, §7, Prop. 7), whence $\lambda_P(\langle \Lambda, P \rangle \cap X) \leq m(P)$.

Now by Lemma 5.3 $f_\Lambda^{-1}(f_\Lambda(P))$ is the disjoint union of a scheme supported on P and of a scheme of length $\tilde{\gamma}_P$ contained in X_{reg} collinear with P. A similar (and shorter) argument shows that $\lambda\big(f_\Lambda^{-1}(f_\Lambda(P))\big) = 1$ for every $P \in T \setminus \{P_1, \dots, P_l\}$ and the conclusion follows. □

Theorem 5.6. *Let $X \subseteq \mathbb{P}^r$ be an integral locally Cohen-Macaulay non-degenerate singular variety of dimension $n \leq r-2$ and assume that it admits a good projection with respect to X_{red} (e.g. $\dim(X) \leq 14$). Assume further that $\dim(\mathrm{Sing}(X)) = 1$ and that all singular points, except finitely many, are hypersuperficial. Let $T_1, \dots, T_l$ be the 1-dimensional irreducible components of $\mathrm{Sing}(X)$, let $m_i := m(X, T_i)$ and $\tilde{\gamma}_i := \tilde{\gamma}(X, T_i)$, for $i = 1, \dots, l$. Let $p := \sup\{n, \rho(\mathrm{Sing}(X)), m_i + \tilde{\gamma}_i - 1\}$. Then*

$$\mathrm{reg}(X) \leq d - r + n + 1 + \frac{(p-1)(p-2)}{2}.$$

Proof. Let E be the finite subset of $\mathrm{Sing}(X)$ consisting of the following points:

- isolated points;
- points contained in $T_i \cap T_j$, $i \neq j$;
- points $P \in T_i$ such that $m(P) > m_i$;
- points $P \in T_i$ such that $\tilde{\gamma}_P > \gamma_i$;
- non-hypersuperficial points.

Then for a general projection $f = f_\Lambda$ the regularity of each fiber $f^{-1}(f(P))$, $P \in E$ is $\leq \rho(\mathrm{Sing}(X))$ (see Lemma 3.3).

Set now $T_i' := T_i \setminus E$ for $i = 1, \dots, s$. Then T_i satisfies the assumptions of Lemma 5.5, whence $\mathrm{reg}\big(f^{-1}(f(P))\big) \leq m(T) + \tilde{\gamma}(X, T) - 1$

The conclusion follows as in the proof of Theorem 1.3. □

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