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<th>Title</th>
<th>Quasiconformal metric and its application to quasiregular mappings</th>
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Osaka University
The quasi conformal metric introduced by Kuusalo [5] seems to me useful for studying the $n$-dimensional quasiregular mappings but has not ever been fully utilized in these connections except what are found in V.M. Gol'dstein-S.K. Vodop'yanov [2] and H. Tanaka [14].

In this paper we shed light on some features of quasiconformal metrics on subdomains of $\mathbb{R}^n$ and apply those to quasiregular mappings to obtain several important properties of them, among others, a characterization for quasiregularity which comes to a generalization of the result in O. Martio, S. Rickman and J. Väisälä [6, Theorem 7.1]. Most of the statements in the sequel remain to hold in $\mathbb{R}^n$, but we often confine ourselves to $\mathbb{R}^n$ in order to avoid inessential complexities in technique.

1. Notations and terminologies

$\mathbb{R}^n\ (n \geq 2)$: the $n$-dimensional euclidean space.
$\overline{\mathbb{R}}^n$: the one point compactification of $\mathbb{R}^n$.
$m_\alpha$: the $\alpha$-dimensional Hausdorff measure.
$m=m_n$: the $n$-dimensional Lebesgue measure.
$q$: the spherical metric.

For a point $x \in \mathbb{R}^n$, the coordinates of $x$ are denoted by $x^1$, $\cdots$, $x^n$ and $|x|$ is the euclidean norm.

Let $E$ be a subset of $\overline{\mathbb{R}}^n$, then $\overline{E}$, $\partial E$, $E^c$ denote the closure, the boundary, the complement of $E$ respectively, all taken with respect to $\overline{\mathbb{R}}^n$.

Given two sets $E, F \subset \mathbb{R}^n$, $d(E, F)$ is the euclidean distance between $E$ and $F$, $d(E)$ is the euclidean diameter of $E$ and $E \setminus F$ is the set-theoretical difference.

Suppose given a non-empty compact proper subset $E$ of $\overline{\mathbb{R}}^n$ and an open set $G \subset \overline{\mathbb{R}}^n$, including $E$, then we call the pair $(E, G)$ a condenser and we may define the (conformal) capacity $\text{cap}(E, G)$ as the (conformal) modulus of the family of all paths connecting $E$ and $\partial G$ in $G$ (cf. [3]). If $E=\phi$ or $\partial G=\phi$, then we set $\text{cap}(E, G)=0$. 
A compact proper subset $E$ of $\bar{\mathbb{R}}^n$ is said of capacity zero if $\text{cap}(E, G) = 0$ for some open set $G \subset \bar{\mathbb{R}}^n$ such that $E \subset G$ and $G \neq \bar{\mathbb{R}}^n$, otherwise of positive capacity. A subset $E$ of $\bar{\mathbb{R}}^n$ is of capacity zero if and only if all compact subsets of $E$ are of capacity zero, or else $E$ is of positive capacity. We refer to [6], [10] for the properties of the capacities.

2. Quasiconformal metrics

Let $G$ be a domain in $\bar{\mathbb{R}}^n$. Given two points $x, y \in G$, the quasiconformal distance $c_G(x, y)$ between $x$ and $y$, relative to $G$, is defined by

$$c_G(x, y) = \inf \text{cap}(E, G),$$

where the infimum is taken over all continua $E$ in $G$, which contain both $x$ and $y$. It is easy to see that $c_G$ is a pseudometric and a conformal invariant. According to [5] we call $c_G$ a quasiconformal metric.

From the definition of quasiconformal metrics and the properties of condenser capacities follows immediately the following

**Proposition 1.** Let $G, G' \subset \bar{\mathbb{R}}^n$ such that $G \subset G'$. Then

$$c_G(x, y) \geq c_{G'}(x, y)$$

for any two points $x, y \in G$.

**Proposition 2.** Let $G$ be a domain in $\bar{\mathbb{R}}^n$ and let $F$ be a set closed relative to $G$, which is of capacity zero. Then

$$c_{G \setminus F}(x, y) = c_G(x, y)$$

for any two points $x, y \in G \setminus F$.

**Remark 1.** Note that $G \setminus F$ is also a domain since $F$ is of $(n-1)$-dimensional Hausdorff measure zero ([10, Corollary 1 of Theorem 8], [4, Corollary 1 of Theorem IV 4 and Theorem VII 3]).

Proof of Proposition 2. Let $x, y \in G \setminus F$ and let $E$ be an arbitrary continuum in $G$, which contains both $x$ and $y$. Select a non-increasing sequence \{\(D_j\)\} of subdomains of $G$ such that each $D_j$ is relatively compact in $G$ and $\bigcap_j D_j = E$. Then for each $j$, we can find a path $\gamma_j$ joining $x$ with $y$ in $D_j \setminus F$ since $D_j \setminus F$ is a domain (Remark 1) and $x, y \in D_j \setminus F$.

From the properties of condenser capacities we obtain

$$c_{G \setminus F}(x, y) \leq \text{cap}(|\gamma_j|, G \setminus F)$$

$$= \text{cap}(|\gamma_j|, G)$$

$$\leq \text{cap}(\bar{D}_j, G),$$
where $|\gamma_j|$ is the locus of $\gamma_j$.

Letting $j \to \infty$, since $\lim_{j \to \infty} \text{cap}(D_j, G) = \text{cap}(E, G)$ ([6, Lemma 5.7]), we have

$$c_{G, \rho}(x, y) \leq \text{cap}(E, G),$$

from which it follows that

$$c_{G, \rho}(x, y) \leq c_G(x, y).$$

The reverse inequality is derived from Proposition 1. q.e.d.

**Theorem 1** (cf. [5, Theorem 2]). Let $G$ be a domain in $\mathbb{R}^n$. Then either $c_g$ is a metric or $c_g$ equals identically to zero according as $G^e$ is of positive capacity or not. Furthermore whenever $c_g$ is a metric, the topology induced by $c_g$ is equivalent to the one induced by $q$ and the identity mapping of $G$ is the uniformly continuous mapping of the metric space $(G, c_g)$ onto the metric space $(G, q)$.

**Proof.** If $G^e$ is of capacity zero, then $\text{cap}(E, G) = 0$ for all continua $E$ in $G$, hence $c_g(x, y) = 0$ for all $x, y \in G$.

If $G^e$ is of positive capacity, then [7, Lemma 3.11] proves that $c_g$ is a metric and the identity mapping of $G$ is the uniformly continuous mapping of $(G, c_g)$ onto $(G, q)$. Now for every $x \in G$ with $x \neq \infty$ and all $y \in \{y \in \mathbb{R}^n : |x - y| < d(x, \partial G)\}$, we have

$$c_g(x, y) \leq \text{cap}(\tilde{B}(x, |y - x|), B^*(x, d(x, \partial G)))$$

$$= \omega_{n-1} \left( \log \frac{d(x, \partial G)}{|y - x|} \right)^{1-n},$$

where $B^*(x, r) = \{z \in \mathbb{R}^n : |x - z| < r\}$ and $\omega_{n-1}$ is the area of the unit $(n-1)$-sphere. Suppose $\infty \in G$. If we set $\phi(x) = \frac{x}{|x|}$, then since $\phi$ is conformal, we have

$$c_g(\infty, y) = c_{\phi(G)}(0, \phi(y)) \leq \omega_{n-1} \left( \log \frac{|\phi(y)|}{r} \right)^{1-n}$$

for all $y \in B^*(r)^c$, where $B^*(r)$ is a ball with the center 0 such that $B^*(r)^c \subset G$. These inequalities imply that the topology induced by $c_g$ is weaker than the one induced by $q$, which completes the proof.

Here we refer to two estimates of quasiconformal metrics from below. From [6, Lemma 5.9] we have the following

**Proposition 3.** Let $G$ be a domain in $\mathbb{R}^n$ with $m(G) < \infty$. Then

$$c_g(x, y) \leq b_n \frac{|x - y|^n}{m(G)}$$
for all \( x, y \in G \), where \( b_n \) is the constant in [6, Lemma 5.9].

**Proposition 4.** If \( G \) is a domain in \( \mathbb{R}^n \) with a continuum \( C \subset \partial G \), then

\[
c^G(x, y) \geq 2^{-1}c_n \log \left[ 1 + \frac{\min \{|x-y|^2, d(C)^2\}}{2 \min \{d(x, C)^2, d(y, C)^2\}} \right]
\]

for all \( x, y \in G \), where \( c_n \) is the constant in [16, Theorem 10.12].

Proof. Let \( E \) be an arbitrary continuum in \( G \), containing \( x, y \). Select two points \( x_1 \in E, x_2 \in C \) with \(|x_1-x_2| = d(E, C)\) and let \( x_0 \) be the midpoint of the line segment joining \( x_1 \) with \( x_2 \). Then we see easily that both \( E \) and \( C \) meet \( S^{-1}(x_0, r) = \partial B^*(x_0, r) \) for each \( r, r_1 < r < r_2 \), where \( r_1 = 2^{-1}d(E, C) \), \( r_2 = 2^{-1} \sqrt{d(E, C)^2 + 2\delta^2} \) and \( \delta = 2^{-1} \min \{d(E), d(C)\} \). Hence if we let \( \Gamma \) be the family of all paths connecting \( E \) and \( C \) in \( B^*(x_0, r_2) \setminus \overline{B}^*(x_0, r_1) \), then using [16, Theorem 10.12], we obtain the following estimate of the modulus \( M(\Gamma) \).

\[
M(\Gamma) \geq c_n \log \frac{r_2}{r_1} = 2^{-1}c_n \log \left[ 1 + \frac{2\delta^2}{d(E, C)^2} \right] \geq 2^{-1}c_n \log \left[ 1 + \frac{\min \{|x-y|^2, d(C)^2\}}{2 \min \{d(x, C)^2, d(y, C)^2\}} \right].
\]

Since \( \Gamma \) is minorized by the family \( \Gamma \) of all paths connecting \( E \) and \( \partial G \) in \( G \), we have

\[
\text{cap}(E, G) = M(\Gamma) \geq M(\Gamma) \geq 2^{-1}c_n \log \left[ 1 + \frac{\min \{|x-y|^2, d(C)^2\}}{2 \min \{d(x, C)^2, d(y, C)^2\}} \right],
\]

from which the required inequality follows. q.e.d.

**Corollary 1.** Suppose that \( G \) is a domain in \( \mathbb{R}^n \), all of whose boundary components contain at least two points. Then \( c_G \) is a metric and the set \( \{y \in G : c_G(x, y) \leq r\} \) is compact for any \( x \in G \) and any \( r > 0 \). Therefore \( (G, c_G) \) is a complete metric space.

**Example 1.** \( c_{\mathbb{R}^n} = 0 \) for all \( x, y \in \mathbb{R}^n \).

**Example 2.** If \( G \) is a bounded domain in \( \mathbb{R}^n \), then \( c_G \) is a metric since \( G^c \) is of positive capacity. Moreover \( (G, c_G) \) is a complete metric space whenever \( \partial G \) is a continuum.

**Example 3.** It is known by Gehring that
for all \( x \in \mathbb{B}^n \), where \( \mathbb{B}^n \) is the unit ball and \( J(|x|) = \{ y \in \mathbb{B}^n : 0 \leq y^1 \leq |x|, y^2 = \cdots = y^n = 0 \} \). From this relation we have

\[
\max \left\{ c_n \log \frac{1 + |x|}{1 - |x|}, \omega_{n-1} \left( \log \frac{\lambda_n}{|x|} \right)^{1-n} \right\} \leq c_{\mathbb{B}^n}(0, x) \leq \omega_{n-1} \left( \log \frac{1}{|x|} \right)^{1-n},
\]

where \( \lambda_n \) is a constant depending only on \( n \) and \( \omega_{n-1} \) is the area of the unit sphere.

### 3. Quasiregular mappings

In the following the notation \( f: G \to \mathbb{R}^n \) always implies that \( G \) is a domain in \( \mathbb{R}^n \) and \( f \) is a continuous mapping of \( G \) into \( \mathbb{R}^n \), unless otherwise stated.

Given \( f: G \to \mathbb{R}^n \), we employ the following notations:

- \( L(x, f, r) = \sup \{|f(y) - f(x)| : |y - x| = r\} \) for \( x \in \mathbb{R}^n \) and \( r > 0 \);
- \( L(x, f) = \lim_{r \to 0} \frac{L(x, f, r)}{r} \);
- \( J(x, f) = \sup \lim_{j \to \infty} \frac{m(f(A_j))}{m(A_j)} \),

where the supremum is taken over all regular sequences of closed sets tending to \( x \) in the sense explained in [13];

- \( N(y, f, A) \) is the cardinal number of \( \{ x \in A : f(x) = y \} \) for any \( y \in \mathbb{R}^n \) and any \( A \subset G \);
- \( N(f, A) = \sup \{ N(y, f, A) : y \in \mathbb{R}^n \} \) for any \( A \subset G \);

Given an arbitrary relatively compact subdomain \( D \) in \( G \) and any \( y \in f(\partial D) \), \( \mu(y, f, D) \) denotes the topological index in the sense stated in [9] (cf. [6], [10]);

- \( f'(x) \) denotes the Jacobian matrix whenever all partial derivatives exist at \( x \);
- \( |f'(x)| = \sup \{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\} \).

According to [6] we say that \( f \) is \textit{quasiregular} if \( f \) is ACL* and \( |f'(x)|^* \leq K \det f'(x) \) a.e. in \( G \) for some constant \( K \geq 1 \). We refer to [6], [10] for the basic properties of quasiregular mappings. Here we quote only the following fundamental facts.

If \( f: G \to \mathbb{R}^n \) is a non-constant quasiregular mapping, then \( f \) is sense-preserving, discrete and open, and hence \( f(G) \) is a domain. "\( f \) is sense-preserving" means that \( \mu(y, f, D) > 0 \) for every relatively compact subdomain \( D \) in \( G \) and for all \( y \in f(D) \setminus f(\partial D) \). Let \( (E, D) \) be an arbitrary condenser in \( G \), i.e. \( D \subset G \), then the inequality

\[
\text{cap}(f(E), f(D)) \leq K_c(f) \text{cap}(E, D)
\]

holds and further

\[
\text{cap}(E, D) \leq K_c(f) N(f, D) \text{cap}(f(E), f(D))
\]
also holds if $D$ is a normal domain for $f$, that is, $D$ is a relatively compact subdomain of $G$ and $f(\partial D) = \partial f(D)$, where $K_i(f), K_o(f)$ are the inner, the outer dilatation of $f$ respectively. From the above capacity inequalities we obtain easily the following

**Theorem 2.** Let $f: G \rightarrow \mathbb{R}^n$ be a quasiregular mapping. Then

$$c_{\nu}(f(x), f(y)) \leq K_i(f) c_{D}(x, y)$$

for any two domains $D \subset G, D' \ni f(D)$ and for all $x, y \in D$. Further if $f$ is not constant and $D$ is a normal domain for $f$, then

$$\inf \{c_{D}(x, y) : y \in f^{-1}(f(y))\} \leq K_0(f) c_{f(D)}(f(x), f(y))$$

for any $x, y \in D$.

**REMARK 2.** Let $f$ be a mapping of a domain $D$ into a domain $D'$. Suppose that there exists a constant $K > 0$ with the property:

$$(*) \quad c_{\nu}(f(x), f(y)) \leq K c_{D}(x, y) \quad \text{for all } x, y \in D.$$  

If $c_{\nu}$ is a metric, then $f$ is continuous. Furthermore if $c_{D}, c_{\nu}$ are metrics, then $f$ is a uniformly continuous mapping of $(D, c_{D})$ into $(D', c_{\nu})$ and hence $f$ is also a uniformly continuous mapping of $(D, c_{D})$ into $(\mathbb{R}^n, q)$ (Theorem 1).

The condition $(*)$ assures the quasiregularity for mappings under some assumptions. To see this, we need some preliminaries.

Given $f: G \rightarrow \mathbb{R}^n$, we say, according to [9], that $f$ is locally of bounded variation in the Banach sense (briefly, locally BV in $G$) if $\int_{\mathbb{R}^n} N(y, f, D) dm(y) < \infty$ for every relatively compact subdomain $D$ of $G$.

Suppose that $f: G \rightarrow \mathbb{R}^n$ is locally BV and that $D$ is a relatively compact subdomain of $G$. Set

$$\Phi_i(E, D) = \int_{\mathbb{R}^n} N(y, f, D \cap \Pi_i^{-1}(E)) dm(y)$$

for each $i, 1 \leq i \leq n$, and for Borel sets $E \in P_i(D)$, where $\Pi_i$ is the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}_i = \{x \in \mathbb{R}^n : x^i = 0\}$. Then $\Phi_i(E, D)$ is a countably additive set function of Borel sets in $P_i(D)$. The (symmetrical) derivative $\Phi'_i(z, D)$ of $\Phi_i(E, D)$, i.e.

$$\Phi'_i(z, D) = \lim_{r \rightarrow 0} \Phi_i(B^{n-1}_i(z, r), D) / m_{n-1}(B^{n-1}_i(z, r))$$

exists and is finite $m_{n-1}$-a.e. in $P_i(D)$. 

}\]
Lemma 1 (cf. [6, Lemma 2.17]). Let $f: G \rightarrow \mathbb{R}^n$ be locally BVB. If there exists a constant $c > 0$ such that

\[(\#) \quad [\sum_i d(f(\Delta_i))]^n \leq c \Phi_i(z, Q) [\sum_i m_i(\Delta_i)]^{n-1}\]

for each relatively compact open n-interval $Q$ in $G$, each $i$, $1 \leq i \leq n$, a.e. $z \in P_i(Q)$ and any disjoint finite sequence $\{\Delta_1, \ldots, \Delta_k\}$ of closed subintervals of $Q \cap P_i^{-1}(z)$, then $f$ is ACL$^n$.

Proof. The proof is much the same as that of [6, Lemma 2.17].

It is easy to see that $f$ is ACL. To prove that $f$ is ACL$^n$, since the situation is the same in any case, it is sufficient to show that $\left| \frac{\partial f}{\partial x^i} \right|^n$ is integrable on each relatively compact open n-interval $Q$ in $G$.

Suppose $Q = Q_0 \times J$, where $Q_0$ is an open $(n-1)$-interval in $\mathbb{R}^{n-1}$ and $J$ is an open 1-interval in $\mathbb{R}$. Set

\[g(x, u) = \left| \frac{\partial f}{\partial x^i}(x, u) \right|, \quad g_j(x, u) = \frac{j}{2} \int_{-1/j}^{1/j} g(x, u+t) \, dt\]

for each positive integer $j$ with $0 < \frac{1}{j} < d(Q, \partial G)$, whenever these make sense. Then we see, as in [6], that $g, g_j$ are all measurable in $Q$ and

\[(1) \quad g_j(z, u) \rightarrow g(z, u) \quad \text{a.e. in } Q_0\]

for a.e. $u \in J$.

Now given each $u \in J$ and each $j$, we set

\[F_{x, j}(E) = \Phi_n(E, Q_0 \times \left( u - \frac{1}{j}, u + \frac{1}{j} \right))\]

for Borel sets $E$ in $Q_0$. Since $F_{x, j}(z) < \infty$ a.e. in $Q_0$ the condition ($\#$) implies that $f(x, t)$ is absolutely continuous on $\left[ u - \frac{1}{j}, u + \frac{1}{j} \right]$ as the function of $t$ and the $n$th power of its total variation is not greater than $c F_{x, j}(z) \left( \frac{2}{j} \right)^{n-1}$ for a.e. $z \in Q_0$. Hence we obtain

\[g_j(z, u)^n \leq c \frac{j}{2} F_{x, j}(z)\]

a.e. in $Q_0$. Integrating over $Q_0$

\[(2) \quad \int_{Q_0} g_j(z, u)^n \, dm_{n-1}(z) \leq c \frac{j}{2} \int_{Q_0} F_{x, j}(z) \, dm_{n-1}(z)\]
\[ \leq c \frac{j}{2} P_{x,j}(Q_0) \]
\[ = c \frac{j}{2} \int_{\mathbb{R}^n} N(y, f, Q_0 \times (u - \frac{1}{j}, u + \frac{1}{j})) \, dm(y) \]

for each \( u \in J \).

If we let

\[ \Psi(E) = \int_{\mathbb{R}^n} N(y, f, Q_0 \times E) \, dm(y) \]

for Borel sets \( E \subseteq J \), then \( \Psi \) is countably additive for Borel sets in \( J \) and hence the derivative \( \Psi'(u) \) of \( \Psi \) exists and is finite a.e. in \( J \). For \( u \in J \) such that (1) holds and \( \Psi'(u) \) exists, Fatou's lemma and (2) yield

\[
\int_{Q_0} g(x, u)^* \, dm_{n-1}(x) \leq \liminf_{j \to \infty} \int_{Q_0} g_j(x, u)^* \, dm_{n-1}(x) \\
\leq c \lim_{j \to \infty} \left[ \frac{j}{2} \Psi \left( \left( u - \frac{1}{j}, u + \frac{1}{j} \right) \right) \right] \\
= c \Psi'(u). 
\]

Integrating over \( J \), we have

\[
\int_{Q} g(x)^* \, dm(x) \leq c \int_{J} \Psi'(u) \, dm_J(u) \\
\leq c \Psi(J) \\
= c \int_{\mathbb{R}^n} N(y, f, Q) \, dm(y) < \infty,
\]

which completes the proof.

**Lemma 2.** Given \( f: G \to \mathbb{R}^n \), if there exists a constant \( K > 0 \) such that the property \((\ast)\) is satisfied for any two domains \( D \subseteq G, D' \supseteq f(D) \), then

\[ L(x, f)^* \leq \bar{K} J(x, f) \]

for all \( x \in G \), where \( \bar{K} \) is a constant depending only on \( n, K \).

Proof. Given \( x \in G \) and \( r, 0 < r < \frac{1}{2} d(x, \partial G) \), choose \( y \in G \) such that \( |x - y| = r \) and \( |f(x) - f(y)| = L(x, f, r) \). Let \( J_r \) be the line segment joining \( x \) with \( y \) and set \( D_r = \{ z \in \mathbb{R}^n : d(z, J_r) < r \} \).

If \( D' \) is an arbitrary domain containing \( f(D_r) \), then the condition \((\ast)\) and Proposition 3 yield
It is easy to see that both $c_{D_r}(x, y)$ and $\frac{m(D_r)}{r^n}$ are constant for all $x, r$ and $y$ which are taken as above. Set $K = \frac{K^*-1}{b_n} c_{D_r}(x, y) m(D_r)$ and if we bring $D'$ arbitrarily close to $f(D_r)$, then we have

$$L(x, f, r)^n \leq K \frac{m(f(D_r))}{m(D_r)} \leq K \frac{m(D_r)}{m(D_r)}.$$

Obviously, $K$ depends only on $n, K$.

Letting $r \to 0$, we obtain

$$L(x, f)^n \leq K J(x, f).$$

q.e.d.

Theorem 3. Suppose that $f: G \to \mathbb{R}^n$ is as in Lemma 2. If $f$ is sense-preserving and locally BVB, then $f$ is quasiregular.

Proof. First of all we assert that $f$ is ACL*. To do so, we have only to show that there exists a constant $c>0$ with the property in Lemma 1. Let $Q$ be an arbitrary open $n$-interval with $\bar{Q} \subset G$. Fix $i, 1 \leq i \leq n$, and let $z \in P_i(Q)$ with $\Phi([z, Q]) < \infty$. Given any disjoint finite sequence $\{\Delta_1, \cdots, \Delta_k\}$ of closed subintervals of $P_i^{-1}(z) \cap Q$, set $D_j, r = \{x \in \mathbb{R}^n: d(x, \Delta_j) < r\}$ for each $j, 1 \leq j \leq k$, and for $r > 0$. Let $D_j, r$ be an arbitrary domain containing $f(D_j, r)$ whenever $D_j, r \subset G$.

Suppose that $r$ is so small as the following properties hold: $D_j, r \subset Q$ for each $j, 1 \leq j \leq k$; $D_{1, r}, \cdots, D_{k, r}$ are disjoint; $r \leq n m_i(\Delta_j)$ for all $j, 1 \leq j \leq k$. Then owing to the manner in which $r$ was chosen we have

$$c_{D_j, r}(x, y) \leq \frac{m(D_j, r)}{r^n} \leq \frac{2 \omega_{n-1}}{r} m(\Delta_j)$$

for each $j (j = 1, \cdots, k)$ and all $x, y \in \Delta_j$.

On the other hand Proposition 3 yields

$$c_{D_j, r}(f(x), f(y))^{n-1} \geq \frac{b_n |f(x) - f(y)|^n}{m(D_j, r)}.$$
By these two inequalities and the condition (*) we obtain
\[ |f(x) - f(y)| \leq c_1 r^{(1-n)/s} m(D_{j,r})^{1/s} m_i(\Delta_j)^{(n-1)/s} \]
for all \( x, y \in \Delta_j \) \((j = 1, \ldots, k)\), where \( c_1 \) is a constant depending only on \( n, K \).
It follows from this inequality that
\[ d(f(\Delta_j)) \leq c_1 r^{(1-n)/s} m(f(D_{j,r}))^{1/s} m_i(\Delta_j)^{(n-1)/s} \]
for each \( j \) \((j = 1, \ldots, k)\).
Summing over \( 1 \leq j \leq k \) and using Hölder's inequality, we have
\[ \left\{ \sum_{i=1}^{k} m(f(D_{j,r})) \right\}^{n} \leq \frac{c}{m_{n-1}(B^{n-1}(z, r))} \left\{ \sum_{i=1}^{k} m_i(\Delta_j) \right\}^{n-1}, \]
where \( c \) depends only on \( n, K \). Now
\[ \sum_{i=1}^{k} m(f(D_{j,r})) = \sum_{i=1}^{k} \int_{f(D_{j,r})} 1 \, dm \]
\[ \leq \sum_{i=1}^{k} \int_{f(D_{j,r})} N(y, f, D_{j,r}) \, dm(y) \]
\[ = \int_{g^n} N(y, f, \bigcup_{i=1}^{k} D_{j,r}) \, dm(y) \]
\[ \leq \int_{g^n} N(y, f, Q \cap P^{-1}(B^{n-1}(z, r))) \, dm(y) \]
\[ = \Phi_i(B^{n-1}(z, r), Q). \]
Hence
\[ \left\{ \sum_{i=1}^{k} d(f(\Delta_j)) \right\}^{n} \leq c \frac{\Phi_i(B^{n-1}(z, r), Q)}{m_{n-1}(B^{n-1}(z, r))} \left\{ \sum_{i=1}^{k} m_i(\Delta_j) \right\}^{n-1}. \]
Thus letting \( r \to 0 \), we obtain
\[ \left\{ \sum_{i=1}^{k} d(f(\Delta_j)) \right\} \leq c \Phi_i(z, Q) \left\{ \sum_{i=1}^{k} m_i(\Delta_j) \right\}^{n-1}, \]
from which it follows that \( f \) is ACL* (Lemma 1).
Since \( f \) is continuous and sense-preserving, \( f \) is monotone in the sense that if \( D \) is an arbitrary relatively compact subdomain of \( G \), then the unbounded connected component of \( f(\partial D)^c \) contains no point of \( f(D) \). Hence all components of \( f \) are monotone functions in the sense of Lebesgue. It is known that a monotone continuous ACL*-function is differentiable almost everywhere in the domain of the function ([11]). So \( f \) is differentiable a.e. in \( G \), from which it follows that \( L(x, f) = |f'(x)| \) and \( J(x, f) = \det f'(x) \) (as \( f \) is sense-preserving), a.e. in \( G \). Consequently Lemma 2 implies that
|f'(x)|^n \leq \bar{K} \det f'(x)
a.e. in \( G \), where \( \bar{K} \) depends only on \( n, K \), which concludes the proof.

**Remark 3.** If \( f: G \to \mathbb{R}^n \) is sense-preserving, discrete and open, then \( f \) is locally \( BVB \) in \( G \), since \( N(f, A) < \infty \) for every relatively compact subset \( A \) of \( G \) ([6, Lemma 2.12]). Hence the above Theorem 3 generalizes a part of the Theorem 7.1 in [6].

As applications of the preceding results we prove alternatively the several known properties of quasiregular mappings.

**Theorem 4.** Let \( f: G \to \mathbb{R}^n \) be a non-constant quasiregular mapping. If \( G' \) is of capacity zero, then \( f(G') \) is also of capacity zero.

**Proof.** On account of Theorem 2,
\[ c_{f(G)}(f(x), f(y)) \leq K_0(f)c_G(x, y) \]
holds for any \( x, y \in G \). The right-hand side of this inequality is always zero since \( c_G \) is identically equal to zero (Theorem 1). Hence \( c_{f(G)}(f(x), f(y)) = 0 \) for all \( x, y \in G \). Therefore if \( c_{f(G)} \) is a metric, that is, \( f(G)' \) is of positive capacity, then \( f \) is constant, which comes to a contradiction. Thus \( f(G)' \) is of capacity zero.

**Theorem 5** ([7, Theorem 3.17]). Let \( G, G' \) be domains in \( \mathbb{R}^n \) and let \( K \geq 1 \) be a constant. Suppose that \( G' \) is of positive capacity. Then a family of quasiregular mappings \( f \) of \( G \) into \( G' \) such that \( K_0(f) \leq K \) is equicontinuous if we consider \( G' \) as a metric space with the metric \( q \).

**Proof.** If \( G \) is of capacity zero, then all mappings belonging to the family in the theorem are constant and hence the theorem is trivial. Suppose that \( G' \) is of positive capacity. Given \( x \in G \) and \( \varepsilon > 0 \), choose \( \eta > 0 \) such that \( c_G(x, y) < \eta \) implies \( q(\bar{x}, \bar{y}) < \varepsilon \). If \( U \) is a neighbourhood of \( x \) such that \( c_G(x, y) < \frac{\eta}{K} \) for all \( y \in U \), then \( q(f(x), f(y)) < \varepsilon \) for any \( f \) belonging to the family under consideration and for all \( y \in U \).

**Theorem 6** ([7, Theorem 4.1]). Let \( G \) be a domain in \( \mathbb{R}^n \) and let \( F \) be a relatively closed subset of \( G \), which is of capacity zero. Suppose that \( f: G \setminus F \to \mathbb{R}^n \) is a quasiregular mapping for which \( f(G \setminus F)' \) is of positive capacity. Then \( f \) is uniquely extended to a continuous mapping \( \bar{f}: G \to \mathbb{R}^n \) such that the restriction \( f^* \) of \( \bar{f} \) to \( G \setminus \bar{f}^{-1}(\infty) \) is quasiregular. Furthermore \( K_0(f^*) = K_0(f) \) and \( K_1(f^*) = K_1(f) \).

**Proof.** If \( G' \) is of capacity zero, then \( (G \setminus F)' = G' \cup F \) is also of capacity zero. Hence \( f \) is constant or else a contradiction arises (Theorem 4), from
which the theorem is obvious. Hereafter we suppose that $G'$ is of positive capacity. Further we may assume that $f$ is not constant. Then $f$ is a uniformly continuous mapping of $(G' \setminus F, c_G)$ into $(\mathbb{R}^n, q)$ (Remark 2) as $c_{G \setminus F} = c_G$ on $G' \setminus F$. Since $G' \setminus F$ is dense everywhere in $G$ and $(\mathbb{R}^n, q)$ is a complete metric space, $f$ is uniquely extended to a continuous mapping $\bar{f}: G \to \mathbb{R}^n$. $\bar{f}(F)$ contains no non-empty open set, because owing to the way of path lifting ([12]) and a modulus inequality under quasiregular mappings ([8]), we can show that $\bar{f}(F)$ is of capacity zero. Therefore since $F$ is $0$-dimensional, it follows from [15, Theorem 9 and Corollary to Theorem 4] that $\bar{f}$ is locally sense-preserving discrete, open, and hence $f^*$ is sense-preserving, locally $BVB$ (Remark 3) as the local sense-preservingsness implies obviously the sense-preservingness.

To see that $f^*$ is quasiregular, it remains to be proved that the condition (*) holds for a constant $K > 0$. Let $D \subseteq G \setminus \bar{f}^{-1}(\infty), D' \supset f^*(D)$ be any domains. Then we have

$$c_D(f(x), f(y)) \leq c_{f(D'), D}(f(x), f(y)) \leq K_1(f)c_{D', D}(x, y) = K_1(f)c_D(x, y)$$

for all $x, y \in D \setminus F$, and hence

$$c_{D'}(f^*(x), f^*(y)) \leq K_1(f)c_D(x, y)$$

for all $x, y \in D$ since $F \cap D$ is nowhere dense in $D$. It is obvious that $K_0(f^*) = K_0(f), K_1(f^*) = K_1(f)$, since $F \cap \bar{f}^{-1}(\infty)$ is of Lebesgue measure zero. q.e.d.

**Remark 4.** The $\bar{f}$ in Theorem 6 is, in fact, quasimeromorphic in the sense stated in [7].

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**References**


