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QUASICONFORMAL METRIC AND ITS APPLICATION TO QUASIREGULAR MAPPINGS

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The quasiconformal metric introduced by Kuusalo [5] seems to me useful for studying the n -dimensional quasiregular mappings but has not ever been fully utilized in these connections except what are found in V.M. Gol'dstein-S.K. Vodop'yanov [2] and H. Tanaka [14].

In this paper we shed light on some features of quasiconformal metrics on subdomains of \bar{R}^n and apply those to quasiregular mappings to obtain several important properties of them, among others, a characterization for quasiregularity which comes to a generalization of the result in O. Martio, S. Rickman and J. Väisälä [6, Theorem 7.1]. Most of the statements in the sequel remain to hold in \bar{R}^n , but we often confine ourselves to R^n in order to avoid inessential complexities in technique.

1. Notations and terminologies

R^n ($n \geq 2$): the n -dimensional euclidean space.

\bar{R}^n : the one point compactification of R^n .

m_α : the α -dimensional Hausdorff measure.

$m = m_n$: the n -dimensional Lebesgue measure.

q : the spherical metric.

For a point $x \in R^n$, the coordinates of x are denoted by x^1, \dots, x^n and $|x|$ is the euclidean norm.

Let E be a subset of \bar{R}^n , then \bar{E} , ∂E , E^c denote the closure, the boundary, the complement of E respectively, all taken with respect to \bar{R}^n .

Given two sets $E, F \subset R^n$, $d(E, F)$ is the euclidean distance between E and F , $d(E)$ is the euclidean diameter of E and $E \setminus F$ is the set-theoretical difference.

Suppose given a non-empty compact proper subset E of \bar{R}^n and an open set $G \subset \bar{R}^n$, including E , then we call the pair (E, G) a condenser and we may define the (conformal) capacity $\text{cap}(E, G)$ as the (conformal) modulus of the family of all paths connecting E and ∂G in G (cf. [3]). If $E = \emptyset$ or $\partial G = \emptyset$, then we set $\text{cap}(E, G) = 0$.

A compact proper subset E of \bar{R}^n is said of capacity zero if $\text{cap}(E, G)=0$ for some open set $G \subset \bar{R}^n$ such that $E \subset G$ and $\bar{G} \neq \bar{R}^n$, otherwise of positive capacity. A subset E of \bar{R}^n is of capacity zero if and only if all compact subsets of E are of capacity zero, or else E is of positive capacity. We refer to [6], [10] for the properties of the capacities.

2. Quasiconformal metrics

Let G be a domain in \bar{R}^n . Given two points $x, y \in G$, the *quasiconformal distance* $c_G(x, y)$ between x and y , relative to G , is defined by

$$c_G(x, y) = \inf \text{cap}(E, G),$$

where the infimum is taken over all continua E in G , which contain both x and y . It is easy to see that c_G is a pseudometric and a conformal invariant. According to [5] we call c_G a *quasiconformal metric*.

From the definition of quasiconformal metrics and the properties of condenser capacities follows immediately the following

Proposition 1. *Let G, G' be domains in \bar{R}^n such that $G \subset G'$. Then*

$$c_G(x, y) \geq c_{G'}(x, y)$$

for any two points $x, y \in G$.

Proposition 2. *Let G be a domain in \bar{R}^n and let F be a set closed relative to G , which is of capacity zero. Then*

$$c_{G \setminus F}(x, y) = c_G(x, y)$$

for any two points $x, y \in G \setminus F$.

REMARK 1. Note that $G \setminus F$ is also a domain since F is of $(n-1)$ -dimensional Hausdorff measure zero ([10, Corollary 1 of Theorem 8], [4, Corollary 1 of Theorem IV 4 and Theorem VII 3]).

Proof of Proposition 2. Let $x, y \in G \setminus F$ and let E be an arbitrary continuum in G , which contains both x and y . Select a non-increasing sequence $\{D_j\}_1^\infty$ of subdomains of G such that each D_j is relatively compact in G and $\bigcap_1^\infty \bar{D}_j = E$. Then for each j , we can find a path γ_j joining x with y in $D_j \setminus F$ since $D_j \setminus F$ is a domain (Remark 1) and $x, y \in D_j \setminus F$.

From the properties of condenser capacities we obtain

$$\begin{aligned} c_{G \setminus F}(x, y) &\leq \text{cap}(|\gamma_j|, G \setminus F) \\ &= \text{cap}(|\gamma_j|, G) \\ &\leq \text{cap}(\bar{D}_j, G), \end{aligned}$$

where $|\gamma_j|$ is the locus of γ_j .

Letting $j \rightarrow \infty$, since $\lim_{j \rightarrow \infty} \text{cap}(\bar{D}_j, G) = \text{cap}(E, G)$ ([6, Lemma 5.7]), we have

$$c_{G \setminus E}(x, y) \leq \text{cap}(E, G),$$

from which it follows that

$$c_{G \setminus E}(x, y) \leq c_G(x, y).$$

The reverse inequality is derived from Proposition 1. q.e.d.

Theorem 1 (cf. [5, Theorem 2]). *Let G be a domain in \bar{R}^n . Then either c_G is a metric or c_G equals identically to zero according as G^c is of positive capacity or not. Furthermore whenever c_G is a metric, the topology induced by c_G is equivalent to the one induced by q and the identity mapping of G is the uniformly continuous mapping of the metric space (G, c_G) onto the metric space (G, q) .*

Proof. If G^c is of capacity zero, then $\text{cap}(E, G) = 0$ for all continua E in G , hence $c_G(x, y) = 0$ for all $x, y \in G$.

If G^c is of positive capacity, then [7, Lemma 3.11] proves that c_G is a metric and the identity mapping of G is the uniformly continuous mapping of (G, c_G) onto (G, q) . Now for every $x \in G$ with $x \neq \infty$ and all $y \in \{y \in R^n: |x - y| < d(x, \partial G)\}$, we have

$$\begin{aligned} c_G(x, y) &\leq \text{cap}(\bar{B}^n(x, |y - x|), B^n(x, d(x, \partial G))) \\ &= \omega_{n-1} \left(\log \frac{d(x, \partial G)}{|y - x|} \right)^{1-n}, \end{aligned}$$

where $B^n(x, r) = \{\bar{x} \in R^n: |\bar{x} - x| < r\}$ and ω_{n-1} is the area of the unit $(n-1)$ -sphere. Suppose $\infty \in G$. If we set $\phi(x) = \frac{x}{|x|}$, then since ϕ is conformal, we have

$$c_G(\infty, y) = c_{\phi(G)}(0, \phi(y)) \leq \omega_{n-1} \left(\log \frac{|\phi(y)|}{r} \right)^{1-n}$$

for all $y \in B^n(r)^c$, where $B^n(r)$ is a ball with the center 0 such that $B^n(r)^c \subset G$. These inequalities imply that the topology induced by c_G is weaker than the one induced by q , which completes the proof.

Here we refer to two estimates of quasiconformal metrics from below. From [6, Lemma 5.9] we have the following

Proposition 3. *Let G be a domain in R^n with $m(G) < \infty$. Then*

$$c_G(x, y)^{n-1} \geq b_n \frac{|x - y|^n}{m(G)}$$

for all $x, y \in G$, where b_n is the constant in [6, Lemma 5.9].

Proposition 4. *If G is a domain in R^n with a continuum $C \subset \partial G$, then*

$$c_G(x, y) \geq 2^{-1}c_n \log \left[1 + \frac{\min\{|x-y|^2, d(C)^2\}}{2 \min\{d(x, C)^2, d(y, C)^2\}} \right]$$

for all $x, y \in G$, where c_n is the constant in [16, Theorem 10.12].

Proof. Let E be an arbitrary continuum in G , containing x, y . Select two points $x_1 \in E, x_2 \in C$ with $|x_1 - x_2| = d(E, C)$ and let x_0 be the midpoint of the line segment joining x_1 with x_2 . Then we see easily that both E and C meet $S^{n-1}(x_0, r) = \partial B^n(x_0, r)$ for each $r, r_1 < r < r_2$, where $r_1 = 2^{-1}d(E, C), r_2 = 2^{-1}\sqrt{d(E, C)^2 + 2\delta^2}$ and $\delta = 2^{-1} \min\{d(E), d(C)\}$. Hence if we let Γ be the family of all paths connecting E and C in $B^n(x_0, r_2) \setminus \bar{B}^n(x_0, r_1)$, then using [16, Theorem 10.12], we obtain the following estimate of the modulus $M(\Gamma)$.

$$\begin{aligned} M(\Gamma) &\geq c_n \log \frac{r_2}{r_1} \\ &= 2^{-1}c_n \log \left[1 + \frac{2\delta^2}{d(E, C)^2} \right] \\ &\geq 2^{-1}c_n \log \left[1 + \frac{\min\{|x-y|^2, d(C)^2\}}{2 \min\{d(x, C)^2, d(y, C)^2\}} \right]. \end{aligned}$$

Since Γ is minorized by the family $\tilde{\Gamma}$ of all paths connecting E and ∂G in G , we have

$$\begin{aligned} \text{cap}(E, G) = M(\tilde{\Gamma}) &\geq M(\Gamma) \\ &\geq 2^{-1}c_n \log \left[1 + \frac{\min\{|x-y|^2, d(C)^2\}}{2 \min\{d(x, C)^2, d(y, C)^2\}} \right], \end{aligned}$$

from which the required inequality follows.

q.e.d.

Corollary 1. *Suppose that G is a domain in \bar{R}^n , all of whose boundary components contain at least two points. Then c_G is a metric and the set $\{y \in G: c_G(x, y) \leq r\}$ is compact for any $x \in G$ and any $r > 0$. Therefore (G, c_G) is a complete metric space.*

EXAMPLE 1. $c_{R^n} = 0$ for all $x, y \in R^n$.

EXAMPLE 2. If G is a bounded domain in R^n , then c_G is a metric since G^c is of positive capacity. Moreover (G, c_G) is a complete metric space whenever ∂G is a continuum.

EXAMPLE 3. It is known by Gehring that

$$c_{B^n}(0, x) = \text{cap}(J(|x|), B^n)$$

for all $x \in B^n$, where B^n is the unit ball and $J(|x|) = \{y \in B^n: 0 \leq y^1 \leq |x|, y^2 = \dots = y^n = 0\}$. From this relation we have

$$\max \left\{ c_n \log \frac{1+|x|}{1-|x|}, \omega_{n-1} \left(\log \frac{\lambda_n}{|x|} \right)^{1-n} \right\} \leq c_{B^n}(0, x) \leq \omega_{n-1} \left(\log \frac{1}{|x|} \right)^{1-n},$$

where λ_n is a constant depending only on n and ω_{n-1} is the area of the unit sphere.

3. Quasiregular mappings

In the following the notation $f: G \rightarrow R^n$ always implies that G is a domain in R^n and f is a continuous mapping of G into R^n , unless otherwise stated.

Given $f: G \rightarrow R^n$, we employ the following notations:

$$L(x, f, r) = \sup \{ |f(y) - f(x)| : |y - x| = r \} \text{ for } x \in R^n \text{ and } r > 0;$$

$$L(x, f) = \limsup_{r \rightarrow 0} \frac{L(x, f, r)}{r};$$

$$J(x, f) = \sup \limsup_{j \rightarrow \infty} \frac{m(f(A_j))}{m(A_j)},$$

where the supremum is taken over all regular sequences of closed sets tending to x in the sense explained in [13];

$N(y, f, A)$ is the cardinal number of $\{x \in A: f(x) = y\}$ for any $y \in R^n$ and any $A \subset G$;

$$N(f, A) = \sup \{N(y, f, A): y \in R^n\} \text{ for any } A \subset G;$$

Given an arbitrary relatively compact subdomain D in G and any $y \notin f(\partial D)$, $\mu(y, f, D)$ denotes the topological index in the sense stated in [9] (cf. [6], [10]);

$f'(x)$ denotes the Jacobian matrix whenever all partial derivatives exist at x ;

$$|f'(x)| = \sup \{ |f'(x)h| : h \in R^n, |h| = 1 \}.$$

According to [6] we say that f is *quasiregular* if f is *ACL*ⁿ and $|f'(x)|^n \leq K \det f'(x)$ a.e. in G for some constant $K \geq 1$. We refer to [6], [10] for the basic properties of quasiregular mappings. Here we quote only the following fundamental facts.

If $f: G \rightarrow R^n$ is a non-constant quasiregular mapping, then f is sense-preserving, discrete and open, and hence $f(G)$ is a domain. “ f is *sense-preserving*” means that $\mu(y, f, D) > 0$ for every relatively compact subdomain D in G and for all $y \in f(D) \setminus f(\partial D)$. Let (E, D) be an arbitrary condenser in G , i.e. $D \subset G$, then the inequality

$$\text{cap}(f(E), f(D)) \leq K_r(f) \text{cap}(E, D)$$

holds and further

$$\text{cap}(E, D) \leq K_o(f) N(f, D) \text{cap}(f(E), f(D))$$

also holds if D is a *normal domain* for f , that is, D is a relatively compact subdomain of G and $f(\partial D) = \partial f(D)$, where $K_I(f)$, $K_O(f)$ are the inner, the outer dilatation of f respectively. From the above capacity inequalities we obtain easily the following

Theorem 2. *Let $f: G \rightarrow R^n$ be a quasiregular mapping. Then*

$$c_{D'}(f(x), f(y)) \leq K_I(f) c_D(x, y)$$

for any two domains $D \subset G$, $D' \supset f(D)$ and for all $x, y \in D$. Further if f is not constant and D is a normal domain for f , then

$$\inf \{c_D(x, \tilde{y}) : \tilde{y} \in f^{-1}(f(y))\} \leq K_O(f) N(f, D) c_{f(D)}(f(x), f(y))$$

for any $x, y \in D$.

REMARK 2. Let f be a mapping of a domain D into a domain D' . Suppose that there exists a constant $K > 0$ with the property:

$$(*) \quad c_{D'}(f(x), f(y)) \leq K c_D(x, y) \quad \text{for all } x, y \in D.$$

If $c_{D'}$ is a metric, then f is continuous. Furthermore if $c_D, c_{D'}$ are metrics, then f is a uniformly continuous mapping of (D, c_D) into $(D', c_{D'})$ and hence f is also a uniformly continuous mapping of (D, c_D) into (\bar{R}^n, q) (Theorem 1).

The condition (*) assures the quasiregularity for mappings under some assumptions. To see this, we need some preliminaries.

Given $f: G \rightarrow R^n$, we say, according to [9], that f is *locally of bounded variation in the Banach sense* (briefly, locally *BVB* in G) if $\int_{R^n} N(y, f, D) dm(y) < \infty$ for every relatively compact subdomain D of G .

Suppose that $f: G \rightarrow R^n$ is locally *BVB* and that D is a relatively compact subdomain of G . Set

$$\Phi_i(E, D) = \int_{R^n} N(y, f, D \cap P_i^{-1}(E)) dm(y)$$

for each i , $1 \leq i \leq n$, and for Borel sets E in $P_i(D)$, where P_i is the orthogonal projection of R^n onto $R_i^{n-1} = \{x \in R^n : x^i = 0\}$. Then $\Phi_i(E, D)$ is a countably additive set function of Borel sets in $P_i(D)$. The (symmetrical) derivative $\Phi'_i(z, D)$ of $\Phi_i(E, D)$, i.e.

$$\Phi'_i(z, D) = \lim_{r \rightarrow 0} \frac{\Phi_i(B^{n-1}(z, r), D)}{m_{n-1}(B^{n-1}(z, r))}$$

exists and is finite m_{n-1} -a.e. in $P_i(D)$.

Lemma 1 (cf. [6, Lemma 2.17]). *Let $f: G \rightarrow R^n$ be locally BVB. If there exists a constant $c > 0$ such that*

$$(\#) \quad \left[\sum_1^k d(f(\Delta_j)) \right]^n \leq c \Phi'_i(z, Q) \left[\sum_1^k m_1(\Delta_j) \right]^{n-1}$$

for each relatively compact open n -interval Q in G , each $i, 1 \leq i \leq n$, a.e. $z \in P_i(Q)$ and any disjoint finite sequence $\{\Delta_1, \dots, \Delta_k\}$ of closed subintervals of $Q \cap P_i^{-1}(z)$, then f is ACL ^{n} .

Proof. The proof is much the same as that of [6, Lemma 2.17].

It is easy to see that f is ACL. To prove that f is ACL ^{n} , since the situation is the same in any case, it is sufficient to show that $\left| \frac{\partial f}{\partial x^n} \right|^n$ is integrable on each relatively compact open n -interval Q in G .

Suppose $Q = Q_0 \times J$, where Q_0 is an open $(n-1)$ -interval in R^{n-1} and J is an open 1-interval in R^1 . Set

$$g(z, u) = \left| \frac{\partial f}{\partial x^n}(z, u) \right|, \quad g_j(z, u) = \frac{j}{2} \int_{-1/j}^{1/j} |g(z, u+t)| dt$$

for each positive integer j with $0 < \frac{1}{j} < d(Q, \partial G)$, whenever these make sense.

Then we see, as in [6], that g, g_j are all measurable in Q and

$$(1) \quad g_j(z, u) \rightarrow g(z, u) \quad \text{a.e. in } Q_0$$

for a.e. $u \in J$.

Now given each $u \in J$ and each j , we set

$$F_{u,j}(E) = \Phi_n \left(E, Q_0 \times \left(u - \frac{1}{j}, u + \frac{1}{j} \right) \right)$$

for Borel sets E in Q_0 . Since $F'_{u,j}(z) < \infty$ a.e. in Q_0 the condition (#) implies that $f(z, t)$ is absolutely continuous on $\left[u - \frac{1}{j}, u + \frac{1}{j} \right]$ as the function of t

and the n th power of its total variation is not greater than $c F'_{u,j}(z) \left(\frac{2}{j} \right)^{n-1}$

for a.e. $z \in Q_0$. Hence we obtain

$$g_j(z, u)^n \leq c \frac{j}{2} F'_{u,j}(z)$$

a.e. in Q_0 . Integrating over Q_0

$$(2) \quad \int_{Q_0} g_j(z, u)^n dm_{n-1}(z) \leq c \frac{j}{2} \int_{Q_0} F'_{u,j}(z) dm_{n-1}(z)$$

$$\begin{aligned} &\leq c \frac{j}{2} F_{u,j}(Q_0) \\ &= c \frac{j}{2} \int_{R^n} N\left(y, f, Q_0 \times \left(u - \frac{1}{j}, u + \frac{1}{j}\right)\right) dm(y) \end{aligned}$$

for each $u \in J$.

If we let

$$\Psi(E) = \int_{R^n} N(y, f, Q_0 \times E) dm(y)$$

for Borel sets $E \subset J$, then Ψ is countably additive for Borel sets in J and hence the derivative $\Psi'(u)$ of Ψ exists and is finite a.e. in J . For $u \in J$ such that (1) holds and $\Psi'(u)$ exists, Fatou's lemma and (2) yield

$$\begin{aligned} \int_{Q_0} g(z, u)^n dm_{n-1}(z) &\leq \liminf_{j \rightarrow \infty} \int_{Q_0} g_j(z, u)^n dm_{n-1}(z) \\ &\leq c \lim_{j \rightarrow \infty} \left[\frac{j}{2} \Psi\left(\left(u - \frac{1}{j}, u + \frac{1}{j}\right)\right) \right] \\ &= c \Psi'(u). \end{aligned}$$

Integrating over J , we have

$$\begin{aligned} \int_Q g(x)^n dm(x) &\leq c \int_J \Psi'(u) dm_1(u) \\ &\leq c \Psi(J) \\ &= c \int_{R^n} N(y, f, Q) dm(y) < \infty, \end{aligned}$$

which completes the proof.

Lemma 2. *Given $f: G \rightarrow R^n$, if there exists a constant $K > 0$ such that the property (*) is satisfied for any two domains $D \subset G$, $D' \supset f(D)$, then*

$$L(x, f)^n \leq \tilde{K} J(x, f)$$

for all $x \in G$, where \tilde{K} is a constant depending only on n, K .

Proof. Given $x \in G$ and r , $0 < r < \frac{1}{2} d(x, \partial G)$, choose $y \in G$ such that $|x - y| = r$ and $|f(x) - f(y)| = L(x, f, r)$. Let J_r be the line segment joining x with y and set $D_r = \{z \in R^n : d(z, J_r) < r\}$.

If D' is an arbitrary domain containing $f(D_r)$, then the condition (*) and Proposition 3 yield

$$\begin{aligned} \frac{L(x, f, r)^n}{r^n} &= \frac{|f(x) - f(y)|^n}{r^n} \\ &\leq \frac{K^{n-1}}{b_n} \frac{m(D')}{r^n} c_{D_r}(x, y)^{n-1} \\ &= \frac{K^{n-1}}{b_n} c_{D_r}(x, y)^{n-1} \frac{m(D_r)}{r^n} \frac{m(D')}{m(D_r)}. \end{aligned}$$

It is easy to see that both $c_{D_r}(x, y)$ and $\frac{m(D_r)}{r^n}$ are constant for all x, r and y which are taken as above. Set $\tilde{K} = \frac{K^{n-1}}{b_n} c_{D_r}(x, y) \frac{m(D_r)}{r^n}$ and if we bring D' arbitrarily close to $f(D_r)$, then we have

$$\frac{L(x, f, r)^n}{r^n} \leq \tilde{K} \frac{m(f(D_r))}{m(D_r)} \leq \tilde{K} \frac{m(f(\bar{D}_r))}{m(\bar{D}_r)}.$$

Obviously, \tilde{K} depends only on n, K .

Letting $r \rightarrow 0$, we obtain

$$L(x, f)^n \leq \tilde{K} J(x, f).$$

q.e.d.

Theorem 3. *Suppose that $f: G \rightarrow R^n$ is as in Lemma 2. If f is sense-preserving and locally BVB, then f is quasiregular.*

Proof. First of all we assert that f is ACLⁿ. To do so, we have only to show that there exists a constant $c > 0$ with the property in Lemma 1. Let Q be an arbitrary open n -interval with $\bar{Q} \subset G$. Fix $i, 1 \leq i \leq n$, and let $z \in P_i(Q)$ with $\Phi'_i(z, Q) < \infty$. Given any disjoint finite sequence $\{\Delta_1, \dots, \Delta_k\}$ of closed subintervals of $P_i^{-1}(z) \cap Q$, set $D_{j,r} = \{x \in R^n: d(x, \Delta_j) < r\}$ for each $j, 1 \leq j \leq k$, and for $r > 0$. Let $D'_{j,r}$ be an arbitrary domain containing $f(D_{j,r})$ whenever $D_{j,r} \subset G$.

Suppose that r is so small as the following properties hold: $D_{j,r} \subset Q$ for each $j, 1 \leq j \leq k$; $D_{1,r}, \dots, D_{k,r}$ are disjoint; $r \leq nm_1(\Delta_j)$ for all $j, 1 \leq j \leq k$. Then owing to the manner in which r was chosen we have

$$c_{D_{j,r}}(x, y) \leq \frac{m(D_{j,r})}{r^n} \leq \frac{2\omega_{n-1}m_1(\Delta_j)}{r}$$

for each j ($j = 1, \dots, k$) and all $x, y \in \Delta_j$.

On the other hand Proposition 3 yields

$$c_{D'_{j,r}}(f(x), f(y))^{n-1} \geq b_n \frac{|f(x) - f(y)|^n}{m(D'_{j,r})}.$$

By these two inequalities and the condition (*) we obtain

$$|f(x) - f(y)| \leq c_1 r^{(1-n)/n} m(D'_{j,r})^{1/n} m_1(\Delta_j)^{(n-1)/n}$$

for all $x, y \in \Delta_j$ ($j=1, \dots, k$), where c_1 is a constant depending only on n, K . It follows from this inequality that

$$d(f(\Delta_j)) \leq c_1 r^{(1-n)/n} m(f(D_{j,r}))^{1/n} m_1(\Delta_j)^{(n-1)/n}$$

for each j ($j=1, \dots, k$).

Summing over $1 \leq j \leq k$ and using Hölder's inequality, we have

$$\left\{ \sum_1^k d(f(\Delta_j)) \right\}^n \leq c \frac{\sum_1^k m(f(D_{j,r}))}{m_{n-1}(B^{n-1}(z, r))} \left\{ \sum_1^k m_1(\Delta_j) \right\}^{n-1},$$

where c depends only on n, K . Now

$$\begin{aligned} \sum_1^k m(f(D_{j,r})) &= \sum_1^k \int_{f(D_{j,r})} 1 dm \\ &\leq \sum_1^k \int_{f(D_{j,r})} N(y, f, D_{j,r}) dm(y) \\ &= \int_{R^n} N(y, f, \bigcup_1^k D_{j,r}) dm(y) \\ &\leq \int_{R^n} N(y, f, Q \cap P_i^{-1}(B^{n-1}(z, r))) dm(y) \\ &= \Phi_i(B^{n-1}(z, r), Q). \end{aligned}$$

Hence

$$\left\{ \sum_1^k d(f(\Delta_j)) \right\}^n \leq c \frac{\Phi_i(B^{n-1}(z, r), Q)}{m_{n-1}(B^{n-1}(z, r))} \left\{ \sum_1^k m_1(\Delta_j) \right\}^{n-1}.$$

Thus letting $r \rightarrow 0$, we obtain

$$\left\{ \sum_1^k d(f(\Delta_j)) \right\} \leq c \Phi'_i(z, Q) \left\{ \sum_1^k m_1(\Delta_j) \right\}^{n-1},$$

from which it follows that f is ACLⁿ (Lemma 1).

Since f is continuous and sense-preserving, f is monotone in the sense that if D is an arbitrary relatively compact subdomain of G , then the unbounded connected component of $f(\partial D)^c$ contains no point of $f(D)$. Hence all components of f are monotone functions in the sense of Lebesgue. It is known that a monotone continuous ACLⁿ-function is differentiable almost everywhere in the domain of the function ([11]). So f is differentiable a.e. in G , from which it follows that $L(x, f) = |f'(x)|$ and $J(x, f) = \det f'(x)$ (as f is sense-preserving), a.e. in G . Consequently Lemma 2 implies that

$$|f'(x)|^n \leq \tilde{K} \det f'(x)$$

a.e. in G , where \tilde{K} depends only on n, K , which concludes the proof.

REMARK 3. If $f: G \rightarrow R^n$ is sense-preserving, discrete and open, then f is locally *BVB* in G , since $N(f, A) < \infty$ for every relatively compact subset A of G ([6, Lemma 2.12]). Hence the above Theorem 3 generalizes a part of the Theorem 7.1 in [6].

As applications of the preceding results we prove alternatively the several known properties of quasiregular mappings.

Theorem 4. *Let $f: G \rightarrow R^n$ be a non-constant quasiregular mapping. If G^c is of capacity zero, then $f(G)^c$ is also of capacity zero.*

Proof. On account of Theorem 2,

$$c_{f(G)}(f(x), f(y)) \leq K_I(f) c_G(x, y)$$

holds for any $x, y \in G$. The right-hand side of this inequality is always zero since c_G is identically equal to zero (Theorem 1). Hence $c_{f(G)}(f(x), f(y)) = 0$ for all $x, y \in G$. Therefore if $c_{f(G)}$ is a metric, that is, $f(G)^c$ is of positive capacity, then f is constant, which comes to a contradiction. Thus $f(G)^c$ is of capacity zero. q.e.d.

Theorem 5 ([7, Theorem 3.17]). *Let G, G' be domains in \bar{R}^n and let $K \geq 1$ be a constant. Suppose that G'^c is of positive capacity. Then a family of quasiregular mappings f of G into G' such that $K_I(f) \leq K$ is equicontinuous if we consider G' as a metric space with the metric q .*

Proof. If G^c is of capacity zero, then all mappings belonging to the family in the theorem are constant and hence the theorem is trivial. Suppose that G^c is of positive capacity. Given $x \in G$ and $\varepsilon > 0$, choose $\eta > 0$ such that $c_G(\tilde{x}, \tilde{y}) < \eta$ implies $q(\tilde{x}, \tilde{y}) < \varepsilon$. If U is a neighbourhood of x such that $c_G(x, y) < \frac{\eta}{K}$ for all $y \in U$, then $q(f(x), f(y)) < \varepsilon$ for any f belonging to the family under consideration and for all $y \in U$. q.e.d.

Theorem 6 ([7, Theorem 4.1]). *Let G be a domain in R^n and let F be a relatively closed subset of G , which is of capacity zero. Suppose that $f: G \setminus F \rightarrow R^n$ is a quasiregular mapping for which $f(G \setminus F)^c$ is of positive capacity. Then f is uniquely extended to a continuous mapping $\tilde{f}: G \rightarrow \bar{R}^n$ such that the restriction f^* of \tilde{f} to $G \setminus \tilde{f}^{-1}(\infty)$ is quasiregular. Furthermore $K_o(f^*) = K_o(f)$ and $K_I(f^*) = K_I(f)$.*

Proof. If G^c is of capacity zero, then $(G \setminus F)^c = G^c \cup F$ is also of capacity zero. Hence f is constant or else a contradiction arises (Theorem 4), from

which the theorem is obvious. Hereafter we suppose that G^c is of positive capacity. Further we may assume that f is not constant. Then f is a uniformly continuous mapping of $(G \setminus F, c_G)$ into (\bar{R}^n, q) (Remark 2) as $c_{G \setminus F} = c_G$ on $G \setminus F$. Since $G \setminus F$ is dense everywhere in G and (\bar{R}^n, q) is a complete metric space, f is uniquely extended to a continuous mapping $\tilde{f}: G \rightarrow \bar{R}^n$. $\tilde{f}(F)$ contains no non-empty open set, because owing to the way of path lifting ([12]) and a modulus inequality under quasiregular mappings ([8]), we can show that $\tilde{f}(F)$ is of capacity zero. Therefore since F is 0-dimensional, it follows from [15, Theorem 9 and Corollary to Theorem 4] that \tilde{f} is locally sense-preserving discrete, open, and hence f^* is sense-preserving, locally *BVB* (Remark 3) as the local sense-preservingness implies obviously the sense-preservingness.

To see that f^* is quasiregular, it remains to be proved that the condition (*) holds for a constant $K > 0$. Let $D \subset G \setminus \tilde{f}^{-1}(\infty)$, $D' \supset f^*(D)$ be any domains. Then we have

$$\begin{aligned} c_{D'}(f(x), f(y)) &\leq c_{f(D \setminus F)}(f(x), f(y)) \\ &\leq K_I(f) c_{D \setminus F}(x, y) \\ &= K_I(f) c_D(x, y) \end{aligned}$$

for all $x, y \in D \setminus F$, and hence

$$c_{D'}(f^*(x), f^*(y)) \leq K_I(f) c_D(x, y)$$

for all $x, y \in D$ since $F \cap D$ is nowhere dense in D . It is obvious that $K_o(f^*) = K_o(f)$, $K_I(f^*) = K_I(f)$, since $F \cap \tilde{f}^{-1}(\infty)$ is of Lebesgue measure zero. q.e.d.

REMARK 4. The \tilde{f} in Theorem 6 is, in fact, quasimeromorphic in the sense stated in [7].

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