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BRANCHED COVER HOMOLOGY AND Q EVALUATIONS

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1. Introduction

The Alexander polynomial [1] is a classical invariant of knots and links in S^3, which has been known since its discovery 70 years ago to be closely related to the topology of the knot (or link) complement, and which has been playing a central role in knot theory over decades (see [16]). Thus, after the appearance of the relatives of the Alexander polynomial, the Jones polynomial V [8] and its immediate successors [6, 2, 10], it has been hoped to find a topological understanding of these invariants, too. This succeeded only for special evaluations of the above polynomials [13, 9]. (A summary on this matter can be found in [12].) In all cases these evaluations have been related to the homology modules of branched coverings of S^3 over the link with coefficients in some finite field. Of particular interest is the Q polynomial of Brandt-Lickorish-Millett-Ho [2, 5], a polynomial invariant with values in $\mathbb{Z}[z, z^{-1}]$, from which the rank of the homology of the double branched cover with coefficients in $\mathbb{Z}_3$ can be obtained from the evaluation at $z = -1$ (see [2, p. 570] and Theorem 8.4.8 (2) of [11]) or the one with coefficients in $\mathbb{Z}_5$, recoverable from the (Galois equivalent) evaluations at $z = (\pm\sqrt{5} - 1)/2$ [9]. See also [17].

To prove that such an evaluation gives lower bounds for the unknotting number was initiated by Traczyk [19] for V and the continued by myself [18] for Q by considering the skein/Kauffman relation of the polynomial. When the evaluation is *entirely* determined by the rank of the homology of the double branched cover with values in some finite field, these bounds, in view of the homological interpretation of the value, are only weaker versions of the inequality already written down by Wendt [20, theorem p. 690]. However, the diagrammatic view on this inequality has the advantage that it can make use of the additional information carried by the *sign* of the other evaluations (this sign is understood in [14] for the Lickorish-Millett value $V(e^{\pi i/3})$ and in [17] for the one of Jones). This enabled, in [19, 18], a decision to be made about the unknotting number of 9 open cases in Kawauchi's tables [11].

An important point in the argumentation of [19, 18] was that the relevant evaluation at $z \in \mathbb{C}$ is *discrete*, i.e., the set

$$S(z) := \{ P_L(z) : L \text{ link} \} \subset \mathbb{C}$$

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(with $P = V$ resp. Q) is a discrete subset ($\forall x \in S(z) \exists \varepsilon > 0 : S(z) \cap B(x, \varepsilon) = \{x\}$, with $B(x, \varepsilon)$ being the ball in $\mathbb{C}$ around x of radius ε). This suggests that every discrete polynomial evaluation may give some information on unknotting numbers, and thus the reasonable question comes about: are the known ones all discrete evaluations of V and Q?

As a partial (and disappointing) result towards this problem, in this note we begin by showing that $z = -1$ and $z = (\pm\sqrt{5} - 1)/2$ are indeed the only evaluations of Q (beside the other special values $z = 1$ and $z = \pm 2$, where the picture was clarified already in [2]), which are, not only up to sign, but even up to multiplication with unit norm complex numbers, determined by the homology of the double branched cover with values in some finite field.

For the proofs we consider certain rational functions $f(x, z)$, which are generating series associated to polynomials of twist sequences. They are closely related to Przytycki's k-moves [15]. He showed, as a special case of his result on the Kauffman polynomial [15, Corollary 1.17], that $Q(z)$ for $z = 2\cos 2\pi n/k$ (except for a couple of special values of this type) is invariant under a k-move. This result will follow more elegantly from our approach by considering the (periodicity of the) Taylor development of f. Moreover, our arguments will show the converse.

Theorem 1.1. *If for some $z \in \mathbb{C} \setminus \{0\}$ the evaluation $Q(z)$, or even just its norm $|Q(z)|$, is invariant under a k-move, then z must be of the form $2\cos 2\pi n/k$.*

Also, by writing out the Taylor coefficients of f in terms of (negative powers of) the zeros of its denominator polynomial, one could show that, except possibly for values of the above form, any $z \in [-2, 2]$ is not a discrete evaluation of Q. The problem with the other (including complex) values of z, however, appears more complicated.

The reason we chose to consider Q rather than V is because the additional term in the relation causes the denominator of the generating functions we obtain to be cubic (rather than quadratic), which makes the discussion of its residues more interesting. Nevertheless, a similar reasoning can be applied for the Jones polynomial as well, and we leave it to an interested reader to do so.

Finally we should remark that there is a more elementary approach to show our results on the unrelatedness of Q evaluations and branched cover homology by examining explicitly the polynomials of some low crossing number knots/links. This method, however, did not seem less awkward than the one we will choose here, although it is certainly less elegant, and it also does not reveal the relation to the k-moves.

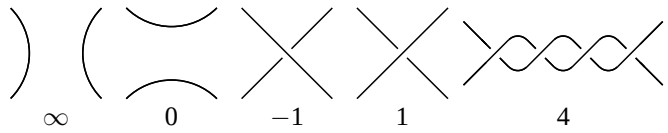


Fig. 1. The Conway tangles.

2. Special Q evaluations determined completely by $\dim H_1(D_L;\mathbb{Z}_n)$

Recall, that the Q polynomial is a Laurent polynomial in one variable z for links without orientation, defined by being 1 on the unknot and the relation

$$A_1 + A_{-1} = z(A_0 + A_\infty)\,, \tag{1}$$

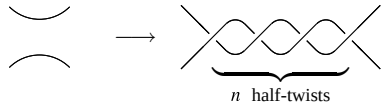
where A_i are the Q polynomials of links K_i and K_i ($i \in \mathbb{Z} \cup \{\infty\}$) possess the same diagrams except in one room, where an i-tangle (in the Conway sense) is inserted; see Fig. 1.

First, for simplicity we let $Q(z)$ depend entirely on the homology of the double branched cover of L. We start with the following

Proposition 2.1. *Let $z \in \mathbb{C} \setminus \{0\}$ be so that $Q_L(z)$ is determined by the dimension over $\mathbb{Z}_n$ of the homology $H_1(D_L;\mathbb{Z}_n)$ of the double branched cover D_L of L with values in some finite field $\mathbb{Z}_n$, n prime. Then $z = -2, 1$, or $z = -1$ and $n = 3$.*

REMARK 2.1. The Jones values $z = (\pm\sqrt{5} - 1)/2$ do not occur here, because for them $Q(z)$ is determined *just up to a sign* by $\dim H_1(D_L;\mathbb{Z}_5)$. We will later prove a stronger version of Proposition 2.1, where $Q_L(z)$ is replaced by $|Q_L(z)|$ (but, unfortunately, with much more effort).

Proof of Proposition 2.1 and Theorem 1.1 for $Q(z)$. We use the observation of Przytycki [15] using the Goeritz matrix [4], that $H_1(D_L;\mathbb{Z}_n)$ is unchanged by an n-move. An n-move is a move on a knot or link diagram, replacing a 0 tangle (in the Conway [3] sense) by an n or $-n$ tangle (where a $-n$ tangle is the obverse of an n tangle):



Two links are called n-equivalent if there is a sequence of Reidemeister and n-moves transforming a diagram of one link into one of the other link.

Now, assume that $z \neq -2, 1$ and consider the series

$$f(x,z) := \sum_{i=0}^{\infty} A_i(z)x^i\,.$$

As the coefficients of Q are exponentially bounded in the crossing number, if all K_i, except K_∞, are knots, the series has a positive convergence radius around $(0,0)$ and converges absolutely within this radius and defines an analytic function. More precisely, the series converges absolutely for $|x| < 1$ and $|z| < 1$, and for $|z| > 1$ and $|xz| < 1$. Therefore, for any given z we can choose x small enough so as to perform the following calculations. If some K_i are links, their number of components is bounded and the arguments that follow will apply by rescaling f by a power of z.

From (1) we obtain

$$f(x,z) = \frac{-f(x,z) + A_0(z) + A_1(z)x}{x^2} + z\,\frac{f(x,z) - A_0(z)}{x} + \frac{z}{1-x}A_\infty(z)\,,$$

whence

$$f = \frac{(1-zx)(1-x)A_0 + x(1-x)A_1 + zx^2A_\infty}{(1-zx+x^2)(1-x)}\,. \tag{2}$$

The dependency of f on $A_{0,1,\infty}$ we will not mark explicitly, but should implicitly keep in mind.

Assume now that for some concrete value of z, $Q(z)$ depends just on $H_1(D_L;\mathbb{Z}_n)$. Then $f(x,z)$ would have a n-periodic Taylor expansion in x around 0 (for any z_0, $f(x,z_0)$ converges absolutely for $|x| < \varepsilon_{z_0}$). So

$$\big((1-zx)(1-x)A_0 + x(1-x)A_1 + zx^2A_\infty\big)(1-x^n) = (1-x)(x^2-zx+1)P(x,z)\,, \tag{3}$$

for some polynomial $P \in \mathbb{Z}[z,x]$ of degree at most $n-1$ in x.

Now, first we show that $z = 2\cos 2\pi k/n$ for some natural number k, $0 \leq k \leq n-1$. Assume that it is not the case. Then x^2-zx+1 has zeros, which are not zeros of $1-x^n$. Then $x^2 - zx + 1$ must divide the first factor

$$L = \big((1-zx)(1-x)A_0 + x(1-x)A_1 + zx^2A_\infty\big)$$

on the left side of (3). But, already making the simplest non-trivial choice $A_\infty = A_1 = 1$, $A_0 = (2/z) - 1$, we find

$$z \cdot \big(\,L \bmod (x^2 - zx + 1)\,\big) = x\big(2z^2 - 2\big) + (2 - 2z) \neq 0\,,$$

unless $z = -1$, which is a contradiction.

Therefore, $z = 2\cos 2\pi k/n$. Now, connected sum shows that the map $\operatorname{rank} H_1 \mapsto Q(z)$ sends addition to multiplication, and hence must be an exponential. The only

candidate for a base is found by examining the 2 component unlink. So we must have

$$Q\left(2\cos 2\pi\frac{k}{n}\right) = \left(\frac{1}{\cos 2\pi k/n} - 1\right)^{\dim H_1(D_L;\mathbb{Z}_n)}.$$

Consider the trefoil with polynomial $Q = -3 + 2z + 2z^2$ and $n > 3$ prime. The above equality specializes to

$$-3 + 4c + 8c^2 = 1,$$

with $c := \cos 2\pi k/n$, whence $c = 1/2$ or $c = -1$, which implies $z = 1$ or $z = -2$. So we are done checking the cases $n \leq 3$ directly. □

REMARK 2.2. Proposition 2.1 is also, if not implied, at least strongly suggested by the complexity results of Vertigan, Jaeger and Welsh [7, §6] (at least if the determination is supposed to be of polynomial complexity), and it is also a special case of [17, Proposition 1] (which I noticed after the preparation of the initial version of this paper). Moreover, the proof given above does not necessarily need the use of the generating functions. They are, however, relevant for the proof of Theorem 1.1 and will later be importantly used in the proof of Theorem 3.1.

3. The evaluations determined by norm

More effort is necessary, when considering signed evaluations (which are the really significant ones from the point of view of unknotting numbers). We need some integration procedure. But this procedure turns out to work equally well not only when considering $Q(z)$ up to sign, but up to norm (that is, up to multiplication with unit complex numbers). Therefore, we now replace $Q(z)$ by $|Q(z)|$. This makes life somewhat more complicated. The generating series is now

$$\tilde{f}(x,z) := \sum_{j=0}^{\infty} |A_j(z)|^2 x^j.$$

This series converges absolutely for $|x| < 1$ and $|z| < 1$, and for $|z| > 1$ and $|xz| < 1$ (again possibly up to multiplying by a power of z). Therefore, for any given z we can choose x small enough so as to perform the following calculations.

$\tilde{f}$ still can be expressed in terms of f by Fourier calculus, but rather complicatedly. One way is to use the substitution $z \mapsto e^{2\pi ik}$ and the formula

$$\int_0^1 e^{2\pi imk} e^{-2\pi ink}\, dk = \delta_{m,n},$$

where $\delta_{m,n}$ is Krenecker's delta. Then one has

$$\begin{aligned}\tilde{f}(x,z) &= \int_0^1 f(\sqrt{x}e^{2\pi ik},z)\,\overline{f(\overline{\sqrt{x}}e^{2\pi ik},z)}\,dk \\ &= \int_0^1 f(\sqrt{x}e^{2\pi ik},z)\,f(\sqrt{x}e^{-2\pi ik},\overline{z})\,dk\end{aligned} \tag{4}$$

(we take the same branch of the square root for both '$\sqrt{x}$'; which one of the two branches we choose is of no importance after the integration).

Now we need to examine for which z there is an n such that

$$\frac{\partial^n}{\partial x^n}\Big|_{|x|<\varepsilon} \int_0^1 f(\sqrt{x}e^{2\pi ik},z)\,\overline{f(\overline{\sqrt{x}}e^{2\pi ik},z)}\,(1-x^n)\,dk \equiv 0$$

for *any* choice of $A_{0,1,\infty}$. This leads to the result we alluded to in the introduction.

Theorem 3.1. *Let $z\in\mathbb{C}\backslash\{0\}$ be so that $|Q_L(z)|$ is determined by* $\dim H_1(D_L;\mathbb{Z}_n)$, *$n$ prime. Then $z=-2,1$, or $z=-1$ and $n=3$, or $z=(-1\pm\sqrt{5})/2$ and $n=5$.*

Proof of Theorem 3.1 and Theorem 1.1 for $|Q(z)|$. Again we are interested in a periodic Taylor expansion with regard to x around $x=0$, this time of the integral in (4). This integral can be expressed as a curve integral

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{|k|=\sqrt{x}} f(k,z)\,f\left(\frac{x}{k},\overline{z}\right)\frac{1}{k}\,dk \\ &= \frac{1}{2\pi i}\oint_{|k|=\sqrt{x}} \frac{\left[(1-zk)(1-k)A_0(z)+k(1-k)A_1(z)+zk^2A_\infty(z)\right]}{(1-kz+k^2)(1-k)} \\ &\qquad\times\left\{\left[\left(1-\frac{x}{k}\overline{z}+\frac{x^2}{k^2}\right)\left(1-\frac{x}{k}\right)k\right]^{-1}\right. \\ &\qquad\qquad\left.\times\left[\left(1-\overline{z}\frac{x}{k}\right)\left(1-\frac{x}{k}\right)A_0(\overline{z})+\frac{x}{k}\left(1-\frac{x}{k}\right)A_1(\overline{z})+\overline{z}\frac{x^2}{k^2}A_\infty(\overline{z})\right]\right\}dk \\ &= \frac{1}{2\pi i}\oint_{|k|=\sqrt{x}} \left\{(1-kz+k^2)(1-k)(k^2-xk\overline{z}+x^2)(k-x)\right\}^{-1} \\ &\qquad\times\left\{\left[(1-zk)(1-k)A_0(z)+k(1-k)A_1(z)+zk^2A_\infty(z)\right]\right. \\ &\qquad\qquad\left.\times\left[\left(1-\overline{z}\frac{x}{k}\right)\left(1-\frac{x}{k}\right)A_0(\overline{z})+\frac{x}{k}\left(1-\frac{x}{k}\right)A_1(\overline{z})+\overline{z}\frac{x^2}{k^2}A_\infty(\overline{z})\right]k^2\right\}dk\end{aligned}$$

where '$|k|=\sqrt{x}$' means the circle in $\mathbb{C}$ with origin zero and radius $\sqrt{x}$, positively

oriented. As $x \to 0$, the relevant zeros of the denominator are x, $x\overline{k_0}$ and $x/\overline{k_0}$, where $k = k_0$ is one zero of $k^2 - zk + 1$, so that

$$z = k_0 + \frac{1}{k_0}\,. \tag{5}$$

To express ourselves more briefly, set

$$\begin{aligned}\Phi(k,x,z) := &\left[(1-zk)(1-k)A_0(z) + k(1-k)A_1(z) + zk^2A_\infty(z)\right]\\ &\times\left[(k-\overline{z}x)(k-x)A_0(\overline{z}) + x(k-x)A_1(\overline{z}) + \overline{z}x^2A_\infty(\overline{z})\right].\end{aligned}$$

The denominators of the residues are (in the order of appearance)

$$\begin{gathered}\left(1 - xz + x^2\right)(1-x)x^2(2-\overline{z})\,,\\ \left(1 - x\overline{k_0}z + x^2\overline{k_0}^2\right)\left(1 - x\overline{k_0}\right)\left(-x + x\overline{k_0}\right)\left(x\overline{k_0} - \frac{x}{\overline{k_0}}\right)\,,\\ \left(1 - \frac{x}{\overline{k_0}}z + \frac{x^2}{\overline{k_0}^2}\right)\left(1 - \frac{x}{\overline{k_0}}\right)\left(-x + \frac{x}{\overline{k_0}}\right)\left(\frac{x}{\overline{k_0}} - x\overline{k_0}\right)\,.\end{gathered}$$

We may assume that $k_0 \neq \pm 1$ (else $z = \pm 2$, which is clearly not of the kind we want). Then, regarded as functions in x, they have the following zeros (with multiplicities)

$$\begin{array}{cccccc}\overline{k_0}, & 1/\overline{k_0}, & 1, & 0, & 0\,;\\ 1/\overline{k_0}^2, & 1, & 1/\overline{k_0}, & 0, & 0\,;\\ \overline{k_0}^2, & 1, & \overline{k_0}, & 0, & 0\,.\end{array}$$

If k_0 is an n-th root of unity we would have $z = 2\cos 2\pi l/n$. Now assume, that k_0 is not an n-th unity root. Then, because $\overline{k_0}^{\pm 2}$ appear as zeros in only one of the denominators above, we must have that they divide the corresponding numerators. Therefore, $x - \overline{k_0}^{-2}$ divides the numerator of the second residue, and $x - \overline{k_0}^2$ divides the numerator of the third residue for *any* choice of $A_{0,1,\infty}$. These numerators are $\Phi(x\overline{k_0}, x, z)$ and $\Phi(x/\overline{k_0}, x, z)$, respectively.

Therefore, $\Phi(\overline{k_0}, \overline{k_0}^2, z) = \Phi(1/\overline{k_0}, 1/\overline{k_0}^2, z) = 0$, for the given choice of z. Set again $A_\infty = A_1 = 1$ and $A_0 = 2/z - 1$. The equality $\Phi(\overline{k_0}, \overline{k_0}^2, z) = 0$ yields

$$\hat{z}^2k_0 + \hat{z}(k_0^2 - 1) + (2 - 2k_0) = 0\,, \tag{6}$$

where '$\hat{z}$' means that one of both identities with $\hat{z}$ replaced by z or $\overline{z}$ is to be satisfied. But because of (5), the substitution $k_0 \to 1/k_0$ does not change z (and $\overline{z}$), and hence

from (6) we obtain under this substitution

$$\hat{z}^2 k_0 + \hat{z}(1 - k_0^2) + (2k_0^2 - 2k_0) = 0\,,$$

with '$\hat{z}$' meaning the *same* choice between z and $\overline{z}$ as in (6).

Taking difference with (6), we find $k_0 = \pm 1$, hence $z = \pm 2$, or $z = -1$, which are evaluations already completely understood in [2].

Hence, assume that $z = 2\cos 2\pi k/n$. As in the previous proof, we get

$$\left| Q\left(2\cos 2\pi \frac{k}{n}\right)\right| = \left|\frac{1}{\cos 2\pi k/n} - 1\right|^{\dim H_1(D_L;\mathbb{Z}_n)}\,.$$

Again with the trefoil we find for $n > 3$ this implies

$$|-3 + 4c + 8c^2| = 1\,,$$

with c as before. But $-3 + 4c + 8c^2$ is real, so $-3 + 4c + 8c^2 = \pm 1$. In former case we get where we did in the previous proof, and in latter case we obtain $c = (-1 \pm \sqrt{5})/4$, giving the case $n = 5$ of Jones. □

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