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<td>Ju, XianMeng</td>
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THE SMITH SET OF THE GROUP $S_5 \times C_2 \times \cdots \times C_2$

XIANMENG JU

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Abstract

In 1960, P.A. Smith raised an isomorphism problem. Is it true that the tangential $G$-modules at two fixed points of an arbitrary smooth $G$-action on a sphere with exactly two fixed points are isomorphic to each other? Given a finite group, the Smith set of the group means the subset of real representation ring consisting of all differences of Smith equivalent representations. Many researchers have studied the Smith equivalence for various finite groups. But the Smith sets for non-perfect groups were rarely determined. In particular, the Smith set for a non-gap group has not been determined unless it is trivial. We determine the Smith set for the non-gap group $G = S_5 \times C_2 \times \cdots \times C_2$.

1. Introduction

Throughout this paper, let $G$ be a finite group. In 1960, P.A. Smith [30] raised the next problem.

Smith isomorphism problem. Is it true that the tangential $G$-modules at two fixed points of an arbitrary smooth $G$-action on a sphere with exactly two fixed points are isomorphic to each other?

Following [25], two real $G$-modules $V$ and $W$ are called Smith equivalent if there exists a smooth action of $G$ on a homotopy sphere $S$ such that $S^G = \{x, y\}$ for two points $x$ and $y$ at which $T_x(S) \cong V$ and $T_y(S) \cong W$ as real $G$-modules.

Let $RO(G)$ denote the real representation ring of $G$. Define the Smith set $Sm(G)$ to be

$$Sm(G) := \{[V] - [W] \in RO(G) \mid V \text{ and } W \text{ are Smith equivalent}\}.$$ 

In general, we don’t know whether $Sm(G)$ is a subgroup of $RO(G)$. The Smith isomorphism problem can be restated as follows.

Smith isomorphism problem. Is it true that $Sm(G) = 0$?

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It is easy to show that the answer is affirmative if \( G \) is a group such that each element has the order 1, 2 or 4. Important breakthroughs on the problem came in the following.

1. M.F. Atiyah–R. Bott [1]: If \( G = C_p \), a cyclic group of order \( p \), where \( p \) an odd prime, then \( \text{Sm}(G) = 0 \).
2. J. Milnor [11]: If \( G \) is a compact group and the action semi-free, then \( T_0(S) \cong T_0(S) \).
3. C.U. Sanchez [28]: If \( G \) is a group with odd-prime-power order or \( G \) is a group with \( |G| = pq \), where \( p \) and \( q \) are odd primes, then \( \text{Sm}(G) = 0 \).
4. T. Petrie [24], [26]: If \( G \) is an odd order finite abelian group with at least four non-cyclic Sylow subgroups, then \( \text{Sm}(G) \neq 0 \).
5. S.E. Cappell–J.L. Shaneson [2]: If \( G \) is a cyclic group of order \( 4m \) such that \( m \geq 2 \) then \( \text{Sm}(G) \neq 0 \).

By the character theory, we have \( \text{Sm}(C_6) = 0 \) and \( \text{Sm}(D_6) = 0 \) where \( D_6 \) is a dihedral group of order 6. So, \( C_8 \) is the smallest group with \( \text{Sm}(G) \neq 0 \). T. Petrie and his collaborators found various pairs of non-isomorphic Smith equivalent real \( G \)-modules, e.g. K.H. Dovermann–T. Petrie [3], K.H. Dovermann–D.Y. Suh [5].

In 1996, in the case where \( G \) is an Oliver group, E. Laitinen [10, Appendix] lighted the problem again with the next conjecture.

**\( A_G \)-Conjecture.** If \( G \) is an Oliver group with \( a_G \geq 2 \), then \( \text{Sm}(G) \neq 0 \).

After E. Laitinen–M. Morimoto [8], a finite group \( G \) is called an *Oliver group* if and only if \( G \) never admits a normal series

\[
P \trianglelefteq H \trianglelefteq G
\]

such that \([P] \) and \([G : H] \) are prime powers and \( H/P \) is a cyclic group. For an element \( g \in G \), let \( (g) \) denote the conjugacy class of \( g \) in \( G \). The union \( (g)^\pm = (g) \cup (g^{-1}) \) is called the *real conjugacy class* of \( g \) in \( G \). Let \( a_G \) denote the number of the real conjugacy classes \( (g)^\pm \) in \( G \) such that the order of \( g \) is not a prime power.

We have affirmative answers for the \( A_G \)-Conjecture in the following cases.

- E. Laitinen–K. Pawalowski [10]: \( G \) is a finite perfect group.
- E. Laitinen–K. Pawalowski [10]: \( G \cong A_n, \text{SL}(2, p) \) or \( \text{PSL}(2, q) \) where \( n \) is a natural number, and \( p \) and \( q \) are primes.
- K. Pawalowski–R. Solomon [21]: \( G \) is a finite Oliver group of odd order.
- K. Pawalowski–R. Solomon [21]: \( G \) is a finite Oliver group with a cyclic quotient of order \( pq \) for two distinct odd primes \( p \) and \( q \).
- K. Pawalowski–R. Solomon [21]: \( G \) is a finite non-solvable gap group and \( G \not\cong \text{PSL}(2, 27) \), where \( \text{PSL}(2, 27) \) is the splitting extension of \( \text{PSL}(2, 27) \) by the group \( \text{Aut}(\mathbb{F}_{27}) \).
- M. Morimoto [13]: \( G \cong \text{PSL}(2, 27) \).

In 2006, M. Morimoto gave a counterexample to the \( A_G \)-Conjecture.
M. Morimoto [14]: If \( G = \text{Aut}(A_6) \), then \( a_G = 2 \) and \( \text{Sm}(G) = 0 \).

We refer to the articles [27], [4], [20], [6] for survey of related results. K. Pawalowski–T. Sumi claim \( \text{Sm}(G) \neq 0 \) for many Oliver groups \( G \) such that \( a_G \geq 2 \) and \( G \) is not a gap group. Recent information of this topic is found in [22], [32] and [23].

For a prime \( p \), let \( G^{[p]} \) denote the smallest normal subgroup \( H \) of \( G \) such that \( [G : H] \) is a power of \( p \) (possibly 1). Let \( \text{G}^{\text{nil}} \) denote the smallest normal subgroup \( H \) of \( G \) such that \( G/H \) is nilpotent. It is known that

\[
\text{G}^{\text{nil}} = \bigcap_p G^{[p]}.
\]

We introduce notation for several families consisting of subgroups of \( G \).

\[
\begin{align*}
\mathcal{S}(G) &:= \{ H \leq G \}, \\
\mathcal{P}(G) &:= \{ P \in \mathcal{S}(G) \mid P \text{ is a } p\text{-subgroup for some prime } p \text{ (possibly a trivial group)} \}, \\
\mathcal{L}(G) &:= \{ L \in \mathcal{S}(G) \mid G^{[p]} \subseteq L \text{ for some prime } p \}, \\
\mathcal{G}^{(1)}(G) &:= \{ H \in \mathcal{S}(G) \mid \exists P \leq H \text{ and } H/P \text{ is cyclic for some } P \in \mathcal{P}(G) \}.
\end{align*}
\]

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be families consisting of subgroups of \( G \). A real \( G \)-module \( V \) is said to be \( \mathcal{X}\text{-free} \) if \( V^H = 0 \) for any \( H \in \mathcal{X} \). If \( M \) is a subset of \( \text{RO}(G) \) then for the families \( \mathcal{X} \), \( \mathcal{Y} \), we define

\[
\begin{align*}
M_{\mathcal{X}} &:= \{ x = V - W \in M \mid \text{Res}_H^G V \cong \text{Res}_H^G W \text{ for all } H \in \mathcal{X} \}, \\
M_{\mathcal{Y}} &:= \{ x = V - W \in M \mid V \text{ and } W \text{ are } \mathcal{Y}\text{-free} \}, \\
M_{\mathcal{X}} \cap M_{\mathcal{Y}} &:= M_{\mathcal{X}} \cap M_{\mathcal{Y}}.
\end{align*}
\]

Let \( \mathcal{H}(G) \) denote the set of all pairs \( (H, P) \) consisting of \( H \in \mathcal{S}(G) \) and \( P \in \mathcal{P}(H) \) such that \( P \neq H \). A real \( G \)-module \( V \) is called a \textit{gap module} if it satisfies \( \dim V^P > 2 \dim V^H \) for all pairs \( (H, P) \in \mathcal{H}(G) \). A finite group \( G \) is called a \textit{gap group} if \( G \) admits a \( \mathcal{L}(G) \)-free gap module. Let \( V^{=H} \) denote the set consisting of all points \( x \in V \) with isotropy subgroup \( G_x = H \), and \( \dim V^{=H} \) as the maximum of the dimension of all connected components of \( V^{=H} \). A real \( G \)-module \( V \) is said to satisfy the \textit{weak gap condition} if it satisfies the following.

(WG1) \( \dim V^P \geq 2 \dim V^H \) for all pairs \( (H, P) \in \mathcal{H}(G) \).

(WG2) If \( \dim V^P = 2 \dim V^H \) for a pair \( (H, P) \in \mathcal{H}(G) \), then \( [H : P] = 2 \).

(WG3) If \( \dim V^P = 2 \dim V^H \) and \( \dim V^P = 2 \dim V^K \) for pairs \( (H, P), (K, P) \in \mathcal{H}(G) \) respectively, then \( (H, K) \) belongs to \( \mathcal{S}(G) \setminus \mathcal{L}(G) \).

(WG4) \( \dim V^P \geq 5 \) for all \( P \in \mathcal{P}(G) \).

(WG5) \( \dim V^{=H} \geq 2 \) for all \( H \in \mathcal{G}^{(1)}(G) \).

(WG6) If \( \dim V^P = 2 \dim V^H \) for a pair \( (H, P) \in \mathcal{H}(G) \), then for all \( g \in N_G(P) \cap N_G(H) \), the associated transformations \( g : V^H \to V^H \) are orientation preserving.
Throughout this paper let $X_2$ be a finite group isomorphic to a direct product of groups isomorphic to $C_2$, namely $X_2 \cong C_2 \times \cdots \times C_2$ (n-fold) where $C_2$ is the cyclic group of order 2. Let $S_5$ be the symmetric group on the five letters, and $A_5$ be the alternating group on the five letters.

Many authors have studied the Smith equivalence for various finite groups. But the Smith sets $\text{Sm}(G)$ were rarely determined. In particular, when $G$ is a non-solvable, non-perfect group, the Smith set $\text{Sm}(G)$ was not determined except the case $\text{Sm}(G) = 0$. Most finite Oliver groups are gap group, while neither $S_5$ nor $\text{Aut}(A_6)$ is a gap group. We have interested in the group $S_5$, because it is an Oliver group which is not a gap group, but its subgroup $A_5$ is an Oliver and gap group. In fact $\text{Sm}(S_5) = \text{Sm}(A_5) = 0$ ([21, Example E4, E5]). But what about the case $S_5 \times X_2$ and $A_5 \times X_2$?

**Theorem A.** If $K = A_5 \times X_2$ then $\text{Sm}(K) = \text{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)} \cong \mathbb{Z}^{2^{(2^n-1)}}$.

This theorem follows from the following 4 lemmas, and the rank of the Smith set follows from Lemma 6.1 and Proposition 6.2.

**Lemma 1.1** (K. Pawałowski–R. Solomon). If $G$ is an Oliver, gap group, then $\text{Sm}(G) \supseteq \text{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$.

This result was given as [21, p.850, Realization Theorem]. The next lemma is well known (see [10, Lemma 2.6]).

**Lemma 1.2.** If $G$ contains no elements of order 8, then $\text{Sm}(G) = \text{Sm}(G)_{\mathcal{P}(G)}$.

**Lemma 1.3.** If $G/G^{\text{nil}}$ is isomorphic to a direct product of groups isomorphic to $C_2$, then $\text{Sm}(G)_{\mathcal{P}(G)} \subseteq \text{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$.

This lemma immediately follows from [14, Proposition 2.2].

**Lemma 1.4.** If $K = A_5 \times X_2$ then the following hold.

1. $K$ is an Oliver, gap group.
2. $K$ does not contain an element of order 8.
3. $K^{\text{nil}} = A_5$ and $K/K^{\text{nil}} \cong X_2$.

The purpose of this paper is to show the next.

**Theorem B.** If $G = S_5 \times X_2$ then $\text{Sm}(G) = \text{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \cong \mathbb{Z}^{2^{2^n-1}}$.

For $G = S_5 \times X_2$, we can check the following.

1. $G$ is an Oliver, but not a gap group.
2. $G$ does not contain an element of order 8.
(3) \(G^\text{nil} = A_5 \subseteq S_5\) and \(G/G^\text{nil} \cong C_2 \times X_2\).

To prove Theorem B, we need to obtain an extended result of Lemma 1.1. Thus we will prove the next lemma.

**Lemma 1.5.** Let \(G\) be an Oliver group. For \(x = V_0 - W_0 \in \text{RO}(G)_{P(G)}^{L(G)}\) such that \(V_0\) and \(W_0\) are \(L(G)\)-free real \(G\)-modules, if there exists a real \(G\)-module \(U\) such that \(V_0 \oplus U\) and \(W_0 \oplus U\) are \(L(G)\)-free and satisfy the weak gap condition, then \(x \in \text{Sm}(G)\).

In addition, we will show

**Lemma 1.6.** Let \(G = S_5 \times X_2\). For each \(x \in \text{RO}(G)_{P(G)}^{L(G)}\), there exist real \(G\)-modules \(U, V\) and \(W\) such that \(x = V - W\), and \(V \oplus U\) and \(W \oplus U\) are \(L(G)\)-free and satisfy the weak gap condition.

Hence Theorem B follows from Lemmas 1.2, 1.3, 1.5 and 1.6, and the rank of the Smith set follows from Lemma 6.1 and Proposition 6.2. A key to proving Lemma 1.6 is the next.

**Lemma 1.7.** If \(K = A_5 \times X_2\) and \(G = S_5 \times X_2\) then

\[
\text{Ind}_K^G(\text{RO}(K)_{P(K)}^{L(K)}) = \text{RO}(G)_{P(G)}^{L(G)}.
\]

The organization of the paper is as follows. Section 2 is devoted to describing lemmas which are useful to construct smooth \(G\)-actions on spheres with non-isomorphic Smith equivalent tangential modules for a general Oliver group \(G\), and we give a proof of Lemma 1.5. In Section 3 we exhibit results on the groups \(K = A_5 \times C_2\) and \(G = S_5 \times C_2\) obtained by concrete computation and show that \(\text{Sm}(K)\) and \(\text{Sm}(G)\) are isomorphic to \(\mathbb{Z}^2\) and \(\mathbb{Z}\), respectively. In Section 4 we observe the induction homomorphism \(\text{Ind}_K^G : \text{RO}(K) \to \text{RO}(G)\) and the restriction homomorphism \(\text{Res}_K^G : \text{RO}(G) \to \text{RO}(K)\), and prove Lemma 1.7. In Section 5 we introduce the notion of orientation triviality. Section 6 completes proofs of Theorems A and B.

### 2. Construction of non-isomorphic Smith equivalent \(G\)-modules

If \(G\) is not of prime power order, define a real \(G\)-module \(V(G)\) by

\[
V(G) := (\mathbb{R}[G] - \mathbb{R}) \bigoplus_{p} (\mathbb{R}[G/G[p]] - \mathbb{R})
\]

where \(p\) runs over the set of primes dividing \(|G|\). Let \(kV(G) = V(G) \oplus \cdots \oplus V(G)\) (\(k\)-fold). We recall some properties of \(V(G)\).
Lemma 2.1 (E. Laitinen–M. Morimoto). For any finite group $G$, the module $V(G)$ satisfies the following properties.

(1) $\dim V(G)^P \geq 2 \dim V(G)^H$ for all $(H, P) \in \mathcal{H}(G)$.  
(2) Suppose $(H, P) \in \mathcal{H}(G)$ and $P \in S(G) \setminus \mathcal{L}(G)$. Then $\dim V(G)^P = 2 \dim V(G)^H$ holds if and only if $[H : P] = 2$, $\langle \{H, G^{[2]}\} : \{P, G^{[2]}\} \rangle = 2$ and $\langle P, G^{[p]} \rangle = G$ for all odd prime $p$.

This Lemma was given as [8, Theorem 2.3]. Reader can refer to [8] for fundamental properties of $V(G)$.

Lemma 2.2. Let $G$ be an Oliver group, $n$ an integer $\geq 1$, and $V$ and $W$ real $G$-modules. Suppose the following (1)–(3):

(1) There exists a smooth $G$-action on a homotopy sphere $\Sigma_1$ with exactly one $G$-fixed point, $x_1$ say, such that the tangential $G$-module $T_{x_1}(\Sigma_1)$ at $x_1$ of $\Sigma_1$ is isomorphic to $V \oplus nV(G)$.

(2) There exists a smooth $G$-action on a homotopy sphere $\Sigma_2$ with exactly one $G$-fixed point, $x_2$ say, such that $T_{x_2}(\Sigma_2)$ is isomorphic to $W \oplus nV(G)$.

(3) There exists a smooth $G$-action on a disk $\Delta$ with exactly two $G$-fixed points, $y_1$ and $y_2$ say, such that $T_{y_1}(\Delta)$ and $T_{y_2}(\Delta)$ are isomorphic to $V \oplus nV(G)$ and $W \oplus nV(G)$ respectively.

Then there exists a smooth $G$-action on a standard sphere $\Sigma$ with exactly two $G$-fixed points, $z_1$ and $z_2$ say, such that $T_{z_1}(\Sigma)$ and $T_{z_2}(\Sigma)$ are isomorphic to $V \oplus nV(G)$ and $W \oplus nV(G)$ respectively. Hence the element $V-W$ of $\text{RO}(G)$ belongs to $\text{Sm}(G)$.

Proof. Let $\Sigma_1$, $\Sigma_2$ and $\Delta$ be spheres and a disk appearing in (1)–(3) above. Let $\Sigma_3$ denote the sphere obtained as the double of $\Delta$, namely $\Sigma_3 = \Delta \cup \Delta'$, where $\Delta'$ is a copy of $\Delta$. Then $\Sigma_3^G$ consists of $y_1$, $y_2$, $y'_1$ and $y'_2$ such that $T_{y_1}(\Sigma_3) = T_{y'_1}(\Sigma_3) \cong V \oplus nV(G)$ and $T_{y_2}(\Sigma_3) = T_{y'_2}(\Sigma_3) \cong W \oplus nV(G)$. Let $\Sigma_4$ denote the $G$-connected sum of $\Sigma_3$ with $\Sigma_1$ and $\Sigma_2$ with respect to the pairs of points $(y'_1, x_1)$ and $(y'_2, x_2)$. Since $n \geq 1$, $\dim \Sigma_3^P \geq 2$ and $\Sigma_3$ contains (infinitely many) points of isotropy subgroup $P$ for each Sylow subgroup of $G$. By the [9, Proposition 1.3], we can obtain the standard sphere $\Sigma$ as the resulting manifold of iterated $G$-connected sum of $\Sigma_3$ with copies of $G \times_P \text{Res}_P^G \Sigma_3$, where $P$ runs over the set of all Sylow subgroups of $G$.  

Lemma 2.3 (M. Morimoto). Let $G$ be an Oliver group and $V$ an $\mathcal{L}(G)$-free real $G$-module satisfying the weak gap condition. Then there exists a smooth $G$-action on a sphere $\Sigma_1$ with exactly one $G$-fixed point, $x_1$ say, such that $T_{x_1}(\Sigma_1)$ is isomorphic to $V$.

Proof. By [18], Oliver group has a smooth fixed-point-free action on a disk. Thus we can construct a smooth action of $G$ on a disk $D = D(V)$ with exactly one $G$-fixed point $x_1$. Taking the double of $D$, we obtain a smooth action of $G$ on $\Sigma_1 = D \cup_{\partial D} D$.
with \( \Sigma^G_1 = \{x_1, x_2\} \). Clearly \( \Sigma_1 \cong G S(\mathbb{R} \oplus V) \). We can check that the action of \( G \) on \( \Sigma_1 \) satisfies Conditions (1)–(5) of [16, Theorem 36]. Therefore we can delete \( x_2 \) from \( \Sigma^G_1 \). Namely there exists a smooth action of \( G \) on a sphere \( \Sigma_2 \) with exactly one \( G \)-fixed point. \( \square \)

**Lemma 2.4** (B. Oliver, M. Morimoto–K. Pawalowski). Let \( G \) be an Oliver group and \( V_1 \) and \( W_1 \) \( L(G) \)-free real \( G \)-modules such that \( \text{Res}^G_P V_1 \) is isomorphic to \( \text{Res}^G_P W_1 \) for all Sylow subgroups \( P \). Then there exists an integer \( N \) such that for every \( n \geq N \), the \( m \)-dimensional disk \( \Delta \), where \( m = \dim V_1 + n \dim V(G) \), admits a smooth \( G \)-action with exactly two \( G \)-fixed points, \( y_1 \) and \( y_2 \), such that \( T_{y_1}(\Delta) \) and \( T_{y_2}(\Delta) \) are isomorphic to \( V_1 \oplus nV(G) \) and \( W_1 \oplus nV(G) \) respectively.

This lemma follows from [15, Theorem 0.3] but crucial part of the proof was due to [19].

Proof of Lemma 1.5. Set \( V_1 = V_0 \oplus U \) and \( W_1 = W_0 \oplus U \). Clearly, \( V_1 \) and \( W_1 \) are \( L(G) \)-free real \( G \)-modules such that \( \text{Res}^G_P V_1 \cong \text{Res}^G_P W_1 \) for all Sylow subgroups \( P \). Apply Lemma 2.4 to \( V_1 \) and \( W_1 \), for finding an integer \( N \) such that for each \( k \geq N \), putting \( n = 2k \), there exists a smooth \( G \)-action on a disk \( \Delta \) described in Lemma 2.4. Set \( V = V_1 \oplus 2kV(G) \), and \( W = W_1 \oplus 2kV(G) \), where \( k \geq N \). Apply Lemma 2.3 to \( V \) for obtaining a smooth \( G \)-action on a sphere \( \Sigma_1 \) described in Lemma 2.4. Then \( V_1 \oplus 2kV(G) \) and \( W_1 \oplus 2kV(G) \) satisfy the weak gap condition. Obtain \( \Sigma_2 \) for \( W \) similarly to \( \Sigma_1 \) replacing \( V \) by \( W \). Then by Lemma 2.2, we obtain a desired smooth \( G \)-action on the \( m \)-dimensional sphere \( \Sigma \) for arbitrary \( k \geq N \). \( \square \)

3. Computation of \( \text{Sm}(S_5 \times C_2) \)

**Proposition 3.1.** The following equalities hold for \( G = S_5 \times C_2 \) and \( K = A_5 \times C_2 \).

1. \( \text{Sm}(K) \cong \mathbb{Z}^2 \) and \( \text{Sm}(G) \cong \mathbb{Z} \).
2. \( \text{Ind}_K^G(\text{Sm}(K)) = \text{Sm}(G) \).

Here the map \( \text{Ind}_K^G : \text{RO}(K) \rightarrow \text{RO}(G) \) is the induction homomorphism:

\[
[V] \mapsto [\mathbb{R}[G] \otimes_{\mathbb{R}[K]} V].
\]

Proof. (1) By means of GAP [33], the irreducible complex characters of \( K = A_5 \times C_2 \) are as in Table 1. The notation in the table reads that, for example in the case “5b”, the first letter 5 of “5b” indicates the order of an element belonging to the corresponding conjugacy class and the second letter b of “5b” is used to distinguish conjugacy classes.

Since \( A_5 \) is a simple group, it follows that \( K^{[2]} = A_5 \) and \( K^{(p)} = K \) (\( p \neq 2 \)). Thus \( K^{\text{nil}} = A_5 \), and \( K/K^{\text{nil}} \cong C_2 \). Clearly \( K \) contains no elements of 8.

\( K \) is an Oliver group, because \( K \) is non-solvable. Let \( \{\delta_i, 1 \leq i \leq 8\} \) be the \( \mathbb{Z} \)-basis of \( \text{RO}(K) \) such that the complication of \( \delta_i \) is \( \delta_i C \). In fact, \( 4\delta_3 + 3\delta_5 + \delta_7 + 2\delta_8 + \)
1.2, and 1.3, we get $\text{Sm}(K)$ -free gap group was theoretically proved by T. Sumi [31, Proposition 3.3]. By Lemmas 1.1, are as in Table 2.

By a straightforward computation [33], a $Z$-basis of $\text{RO}(K)^{\mathcal{L}(K)}_{P(K)}$ is $\{x_1, x_2\}$, where $x_1 = \delta_3 - \delta_5 - 2\delta_7 + 2\delta_8 + \delta_9 - \delta_{10}$, $x_2 = \delta_4 - \delta_6 - 2\delta_7 + 2\delta_8 + \delta_9 - \delta_{10}$.

(2) By means of GAP [33], the irreducible complex characters of $G = S_5 \times C_2$ are as in Table 2.

Since $A_5$ is a simple group, it follows that $G^{(2)} = A_5$ and $G^{(p)} = G$ ($p \neq 2$). Thus $G^{\text{nil}} = A_5$, and $G/G^{\text{nil}} \cong C_2 \times C_2$. Clearly $G$ contains no elements of 8. $G$ is an Oliver group, because $G$ is non-solvable. By Lemmas 1.2 and 1.3, $\text{Sm}(G) \subseteq \text{RO}(G)^{\mathcal{L}(G)}_{P(G)}$.

Let $\{\xi_i, 1 \leq i \leq 14\}$ be the $Z$-basis of $\text{RO}(G)$ such that the complication of $\xi_i$ is $\xi_{iC}$. By a straightforward computation [33], $\text{RO}(G)^{\mathcal{L}(G)}_{P(G)} \cong Z$. We take the $Z$-basis element $y = V - W$ of $\text{RO}(G)^{\mathcal{L}(G)}_{P(G)}$ such that $V = 2\xi_5 + 2\xi_7 + \xi_{10} + \xi_{12} + \xi_{14}$ and $W = 2\xi_6 + 2\xi_8 + \xi_9 + \xi_{11} + \xi_{13}$. Let $U = \xi_5 + 2\xi_6 + 2\xi_8 + 3\xi_{10} + 3\xi_{12}$. We can check that $V \oplus 2U$ and $W \oplus 2U$ satisfy the weak gap condition. By Lemma 1.5, we obtain $ny \in \text{Sm}(G)$ for any $n \in Z$, thus $\{y\}$ is a $Z$-basis of $\text{Sm}(G)$.

Since the equalities

\[
\begin{align*}
\text{Ind}_K^G \delta_1 &= \xi_1 + \xi_4, \\
\text{Ind}_K^G \delta_2 &= \xi_2 + \xi_5, \\
\text{Ind}_K^G \delta_3 &= \xi_3, \\
\text{Ind}_K^G \delta_4 &= \xi_13, \\
\text{Ind}_K^G \delta_5 &= \xi_{14}, \\
\text{Ind}_K^G \delta_6 &= \xi_{14}, \\
\text{Ind}_K^G \delta_7 &= \xi_5 + \xi_7, \\
\text{Ind}_K^G \delta_8 &= \xi_6 + \xi_8, \\
\text{Ind}_K^G \delta_9 &= \xi_9 + \xi_{11}, \\
\text{Ind}_K^G \delta_{10} &= \xi_{10} + \xi_{12}
\end{align*}
\]
Table 2. The complex characters of $G = S_5 \times C_2$.

<table>
<thead>
<tr>
<th>1a</th>
<th>2a</th>
<th>2b</th>
<th>2c</th>
<th>3a</th>
<th>6a</th>
<th>2d</th>
<th>2e</th>
<th>4a</th>
<th>4b</th>
<th>6b</th>
<th>6c</th>
<th>5a</th>
<th>10a</th>
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<tr>
<td>$\xi_{1C}$</td>
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<tr>
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<td>-1</td>
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<td>-1</td>
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<td>-1</td>
<td>1</td>
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</tr>
<tr>
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</tr>
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<td>-1</td>
<td>1</td>
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<td>1</td>
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<td>$\xi_{14C}$</td>
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<td>0</td>
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<td>0</td>
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</tr>
</tbody>
</table>

hold, we obtain $\text{Ind}_K^G(x_1) = \text{Ind}_K^G(x_2) = -y$, which determines the induction map $\text{Ind}_K^G : \text{Sm}(K) \to \text{Sm}(G)$. 

\[ \text{Ind}_K^G (\text{RO}(K)_{\mathcal{P}(K)}) \subseteq \text{RO}(G)_{\mathcal{P}(G)} \]

4. Induction and restriction

Let $G$ be a finite group.

**Lemma 4.1.** If $K \leq G$, then

\[ \text{Ind}_K^G (\text{RO}(K)_{\mathcal{P}(K)}) \subseteq \text{RO}(G)_{\mathcal{P}(G)} \]

**Proof.** By definition,

\[ \text{RO}(G)_{\mathcal{P}(G)} = \text{RO}(G)_{\mathcal{P}(G)} \cap \text{RO}(G)^{\mathcal{L}(G)} \]

So we will prove following (1) and (2).

1. $\text{Ind}_K^G (\text{RO}(K)_{\mathcal{P}(K)}) \subseteq \text{RO}(G)_{\mathcal{P}(G)}$.
2. $\text{Ind}_K^G (\text{RO}(K)^{\mathcal{L}(K)}) \subseteq \text{RO}(G)^{\mathcal{L}(G)}$.

(1) Let $x = V - W \in \text{RO}(K)_{\mathcal{P}(K)}$ where $V$ and $W$ are real $K$-modules. It suffices to prove

\[ \text{Res}_K^G (\text{Ind}_K^G V) \cong \text{Res}_K^G (\text{Ind}_K^G W) \]
for all \( P \in \mathcal{P}(G) \). By the Mackey decomposition, we have

\[
\text{Res}_P^G(\text{Ind}_K^G V) = \bigoplus_{P \in G \cap K \mid P \in \mathcal{P}(G)} \text{Ind}_P^{P \cap K} g (g \ast \text{Res}_K^{P \cap K} V),
\]

\[
\text{Res}_P^G(\text{Ind}_K^G W) = \bigoplus_{P \in G \cap K \mid P \in \mathcal{P}(G)} \text{Ind}_P^{P \cap K} g (g \ast \text{Res}_K^{P \cap K} W).
\]

Since \( V \rightarrow W \in \text{RO}(K) \), it follows that

\[
\text{Res}_K^{P \cap K} V \simeq \text{Res}_K^{P \cap K} W.
\]

(2) Let \( x = V \rightarrow W \in \text{RO}(K)\mathcal{L}(K) \) where \( V \) and \( W \) are \( \mathcal{L}(K) \)-free real \( K \)-modules. By definition, \( V^{\mu(p)} = 0 = W^{\nu(p)} \) for all primes \( p \). By the Mackey decomposition, we have

\[
\text{Res}_K^{G \cap G} (\text{Ind}_K^G V) = \bigoplus_{G \cap G \in G^{G \cap G} / G \cap K} \text{Ind}_{G \cap G}^{G \cap G} g (g \ast \text{Res}_{G \cap G}^{G \cap G} V).
\]

Clearly \([ K : (K \cap G^{G \cap G}) ]\) is a \( p \)-power. Thus we have

\[
V^{K \cap G^{G \cap G}} = (V^{K \cap G^{G \cap G}})^{(K \cap G^{G \cap G}) / G^{G \cap G}}
\]

\[
= 0^{(K \cap G^{G \cap G}) / G^{G \cap G}}
\]

\[
= 0.
\]

Similarly, \( W^{K \cap G^{G \cap G}} = 0 \). Thus \((\text{Ind}_K^G V)^{G^{G \cap G}} = 0 = (\text{Ind}_K^G W)^{G^{G \cap G}}\). \(\square\)

**Lemma 4.2.** If \( K \leq G \), then

\[
\text{Res}_K^G (\text{RO}(G) \mathcal{P}(G)) \subseteq \text{RO}(K) \mathcal{P}(K).
\]

Moreover, if \( G^{G \cap G} = G \) \((p \neq 2)\) and \( K \supseteq G^{G \cap G} \) then

\[
\text{Res}_K^G (\text{RO}(G) \mathcal{L}(K)) \subseteq \text{RO}(K) \mathcal{L}(K).
\]

**Proof.** Let \( V \rightarrow W \in \text{RO}(G) \mathcal{P}(G) \) where \( V \) and \( W \) are real \( G \)-modules. So

\[
\text{Res}_P^G V \simeq \text{Res}_P^G W
\]

for all \( P \in K \subseteq \mathcal{P}(G) \). In general, \( \text{Res}_P^K (\text{Res}_K^G V) = \text{Res}_P^G V \). Therefore

\[
\]

Thus,

\[
\text{Res}_K^G (\text{RO}(G) \mathcal{P}(G)) \subseteq \text{RO}(K) \mathcal{P}(K).
\]
Suppose $G^{(p)} = G$ ($p \neq 2$) and $K \supseteq G^{(2)}$. Since $G^{(2)} = K \cap G^{(2)} \subseteq K$, we have $K^{(2)} \subseteq G^{(2)}$. Since $K^{(2)} \subseteq K$ and $G^{(2)} \subseteq K$, we get $K^{(2)} \subseteq G^{(2)}$. For all $g \in G$, we obtain

$$gK^{(2)}g^{-1} \subseteq gG^{(2)}g^{-1} = G^{(2)}.$$ 

Let $a \in G^{(2)}$. Then

$$a(gK^{(2)}g^{-1})a^{-1} = g(g^{-1}ag)K^{(2)}(g^{-1}ag)^{-1}g^{-1}.$$ 

Since $g^{-1}ag \in G^{(2)}$ and $K^{(2)} \subseteq G^{(2)}$, we get $(g^{-1}ag)K^{(2)}(g^{-1}ag)^{-1} = K^{(2)}$. Thus

$$a(gK^{(2)}g^{-1})a^{-1} = gK^{(2)}g^{-1}.$$ 

That is $gK^{(2)}g^{-1} \subseteq G^{(2)}$. Set

$$S = \bigcap_{g \in G} gK^{(2)}g^{-1}.$$ 

Clearly, $S \subseteq G$. We know $G^{(2)}/K^{(2)}$ is a subgroup of $K/K^{(2)}$. Since $K/K^{(2)}$ is a 2-group, it follows that $G^{(2)}/K^{(2)}$ is a 2-group. It is easy to show that $G^{(2)}/S$ is a 2-group. Since $G/G^{(2)} \cong (G/S)/(G^{(2)}/S)$, $G/S$ is a 2-group. Therefore $S = G^{(2)} = K^{(2)}$.

Let $U_1 - U_2 \in \text{RO}(G)^{2(G)}$ where $U_1$ and $U_2$ are $\mathcal{L}(G)$-free real $G$-modules. We obtain

$$(\text{Res}^G_K U_1)^{K^{(2)}} = U_1^{G^{(2)}} = 0$$

and

$$(\text{Res}^G_K U_2)^{K^{(2)}} = U_2^{G^{(2)}} = 0.$$ 

Let $G = S_3 \times X_2$ and $K = A_3 \times X_2$ where $X_2 = C_2 \times \cdots \times C_2$.

Proof of Lemma 1.7. The conjugacy classes of the maximal elementary subgroups of $G$ not belonging to $K$ are represented by $E_1 = D_8 \times X_2$ and $E_2 = C_6 \times X_2$. As $E_2 \subseteq H_2 = D_{12} \times X_2$, by Brauer’s theorem [29, p. 78]

$$\text{RO}(G) = \text{Ind}_{E_1}^G \text{RO}(E_1) + \text{Ind}_{H_2}^G \text{RO}(H_2) + \text{Ind}_K^G \text{RO}(K).$$

Thus we have

$$1_G = \text{Ind}_{E_1}^G t + \text{Ind}_{H_2}^G u + \text{Ind}_K^G v$$

for some $t \in \text{RO}(E_1)$, $u \in \text{RO}(H_2)$ and $v \in \text{RO}(K)$. Let $x$ be an arbitrary element of $\text{RO}(G)^{2(G)}$. Then we have

$$x = \text{Ind}_{E_1}^G (t \cdot \text{Res}_E^G x) + \text{Ind}_{H_2}^G (u \cdot \text{Res}_H^G x) + \text{Ind}_K^G (v \cdot \text{Res}_K^G x).$$
Since $E_1$ is a 2-group, $\text{Res}_{E_1}^G x = 0$ and hence 

$$x = \text{Ind}_{H_1}^G (u \cdot \text{Res}_{H_1}^G x) + \text{Ind}_K^G (v \cdot \text{Res}_K^G x).$$

Let $H \leq G$ and $a \in \text{RO}(H)$. Then $\mathcal{P}(H)$ is a subset of $\mathcal{P}(G)$. Thus for $P \in \mathcal{P}(H)$, we have

$$\text{Res}_P^H (a \cdot \text{Res}_H^G x) = \text{Res}_P^H (a) (\text{Res}_P^H (\text{Res}_H^G x))$$

$$= \text{Res}_P^H (a) (\text{Res}_P^G x)$$

$$= \text{Res}_P^H (a) \cdot 0$$

$$= 0.$$ 

Namely $a \cdot \text{Res}_H^G x \in \text{RO}(H) \mathcal{P}(H)$.

Suppose $A_5 \leq H \leq G$. Write $a = U_1 - U_2$ and $x = V_1 - V_2$ with real $H$-modules $U_1$ and $U_2$ and $\mathcal{L}(G)$-free real $G$-modules $V_1$ and $V_2$. Then note $V_1^{A_5} = V_2^{A_5} = 0$ and 

$$a \cdot \text{Res}_H^G x = \left\{ (U_1 \otimes \text{Res}_H^G V_1) \oplus (U_2 \otimes \text{Res}_H^G V_2) \right\}$$

$$- \left\{ (U_1 \otimes \text{Res}_H^G V_2) \oplus (U_2 \otimes \text{Res}_H^G V_1) \right\}.$$ 

Let $W_1 = (U_1 \otimes \text{Res}_H^G V_1) \oplus (U_2 \otimes \text{Res}_H^G V_2)$ and $W_2 = (U_1 \otimes \text{Res}_H^G V_2) \oplus (U_2 \otimes \text{Res}_H^G V_1)$. Since $A_5$ does not have subgroups with index 2, we have

$$W_1^{A_5} = (U_1 \otimes \text{Res}_H^G V_1)^{A_5} \oplus (U_2 \otimes \text{Res}_H^G V_2)^{A_5}$$

$$= (U_1^{A_5} \otimes (\text{Res}_H^G V_1)^{A_5}) \oplus (U_2^{A_5} \otimes (\text{Res}_H^G V_2)^{A_5})$$

$$= (U_1^{A_5} \otimes 0) \oplus (U_2^{A_5} \otimes 0)$$

$$= 0.$$ 

Similarly, $W_2^{A_5} = 0$. Therefore, $a \cdot \text{Res}_H^G x \in \text{RO}(H) \mathcal{P}(H)$. Consequently,

$$x = \text{Ind}_{H_1}^G (u \cdot \text{Res}_{H_1}^G x) + \text{Ind}_K^G (v \cdot \text{Res}_K^G x)$$

$$= \text{Ind}_{H_2}^G x_1 + \text{Ind}_K^G x_2$$

with $x_1 = u \cdot \text{Res}_{H_2}^G x \in \text{RO}(H_2) \mathcal{P}(H_2)$ and $x_2 = v \cdot \text{Res}_K^G x \in \text{RO}(K) \mathcal{P}(K)$.

In order to show $\text{Ind}_{H_2}^G x_1 = 0$, we regard

$$\text{RO}(H_2) = \text{RO}(D_{12}) \otimes \text{RO}(X_2)$$

in a canonical way. For each $T \leq X_2$ with $[X_2 : T] \leq 2$, there is a unique 1-dimensional real $X_2$-representation $\xi_T$ such that the kernel of $\xi_T$ is $T$. The set

$$\{ \xi_T \mid T \leq X_2, [X_2 : T] \leq 2 \}$$
is a \( \mathbb{Z} \)-basis of \( \text{RO}(X_2) \). Thus we can regard

\[
\text{RO}(H_2) = \text{RO}(D_{12}) \xi_{T_1} \oplus \text{RO}(D_{12}) \xi_{T_2} \oplus \cdots \oplus \text{RO}(D_{12}) \xi_{T_\nu}.
\]

We can write \( x_1 \) above in the form

\[
x_1 = \sum_{i=1}^{2^n} u_{T_i} \cdot \xi_{T_i}
\]

with \( u_{T_i} \in \text{RO}(D_{12}) \). Since \( x \in \text{RO}(G)^{L(G)} \) and \( \text{Ind}_{H_2}^G x_2 \in \text{RO}(G)^{L(G)} \), we get \( \text{Ind}_{H_2}^G x_1 \in \text{RO}(G)^{L(G)} \). Since \( x_1 \in \text{RO}(H_2) \), it must hold that

\[
u_{T_i} \in \text{RO}(D_{12})_{\mathcal{P}_2(D_{12})}
\]

for each \( i \) and

\[
\sum_{i=1}^{2^n} u_{T_i} \in \text{RO}(D_{12})_{\mathcal{P}_1(D_{12})}
\]

where \( \mathcal{P}_p(H) := \{ P \leq H \mid P \text{ is a } p\text{-group} \} \).

Regard \( D_{12} = \langle a, b, c \rangle \) with \( a = (1, 2, 3), b = (1, 2), c = (4, 5) \). By a straightforward computation [33], we can check that \( \{U_1 - U_2, U_3 - U_4\} \) is a basis of \( \text{RO}(D_{12})_{\mathcal{P}_1(D_{12})} \), where \( U_i \) are real \( D_{12} \)-modules of dimension 2 with action:

\[
U_1: a \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad c \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

\[
U_2: a \mapsto \begin{bmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad c \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

\[
U_3: a \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad c \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

\[
U_4: a \mapsto \begin{bmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad c \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

and

\[
(\text{Ind}_{D_{12}}^{S_5}(U_1 - U_2))^{S_5} = \mathbb{R}, \quad (\text{Ind}_{D_{12}}^{S_5}(U_3 - U_4))^{S_5} = 0, \\
(\text{Ind}_{D_{12}}^{S_5}(U_1 - U_2))^{A_5} = \mathbb{R}, \quad (\text{Ind}_{D_{12}}^{S_5}(U_3 - U_4))^{A_5} = \mathbb{R}.
\]
Then we can write \( \text{Ind}^G_{H_2} x_1 \) in the form

\[
\text{Ind}^G_{H_2} x_1 = \sum_{i=1}^{2^n} \{ m_T, \text{Ind}^{S_3}_{D_{12}} (U_1 - U_2) + n_T, \text{Ind}^{S_3}_{D_{12}} (U_3 - U_4) \} \cdot \xi_T.
\]

Note

\[
(\text{Ind}^G_{H_2} x_1)^G = \{ m_X, (\text{Ind}^{S_3}_{D_{12}} (U_1 - U_2))^{S_3} + n_X (\text{Ind}^{S_3}_{D_{12}} (U_3 - U_4))^{S_3} \} \cdot \xi_X
\]

\[
= m_X \mathbb{R} \cdot \xi_X,
\]

This shows \( m_X = 0 \). Next note

\[
(\text{Ind}^G_{H_2} x_1)^{S_3 \times X_2} = \{ m_T, (\text{Ind}^{S_3}_{D_{12}} (U_1 - U_2))^{S_3} + n_T (\text{Ind}^{S_3}_{D_{12}} (U_3 - U_4))^{S_3} \} \cdot \xi_T
\]

\[
+ n_X (\text{Ind}^{S_3}_{D_{12}} (U_3 - U_4))^{S_3} \cdot \xi_X
\]

\[
= m_T \mathbb{R} \cdot \xi_T
\]

This shows \( m_T = 0 \). Therefore we get the equality

\[
\text{Ind}^G_{H_2} x_1 = \sum_{i=1}^{2^n} n_T, \text{Ind}^{S_3}_{D_{12}} (U_3 - U_4) \cdot \xi_T.
\]

The equalities

\[
(\text{Ind}^G_{H_2} x_1)^{A_{1} \times X_2} = n_X (\text{Ind}^{S_3}_{D_{12}} (U_3 - U_4))^{A_{1}} \cdot \xi_X
\]

\[
= n_X \mathbb{R} \cdot \xi_X
\]

conclude \( n_X = 0 \). Similarly, we can show \( n_T = 0 \). Thus we have established \( \text{Ind}^G_{H_2} x_1 = 0 \). \( \square \)

5. Orientation triviality

We use the following notation.

\( \mathcal{H}(G, 2) := \{(H, P) \in \mathcal{H}(G) \mid [H : P] = 2\} \),

\( \mathcal{H}(G, 2)_0 := \{(H, P) \in \mathcal{H}(G, 2) \mid \langle H, G^{(2)} \rangle : \langle P, G^{(2)} \rangle \} = 2 \)

and \( \langle P, G^{(q)} \rangle = G \) for any odd prime \( q \),

\( \mathcal{A}(G) := \{(H, g) \in S(G) \times G \mid g \in N_G(H), \exists P \triangleleft H \text{ satisfying } (H, P) \in \mathcal{H}(G, 2)\} \),

\( \mathcal{B}(G) := \{(H, g) \in S(G) \times G \mid g \in N_G(H), \exists P \triangleleft H \text{ satisfying } (H, P) \in \mathcal{H}(G, 2)_0\} \).
For each element \( x = V - W \in \text{RO}(G) \), we define a map
\[
\psi : \mathcal{A}(G) \times \text{RO}(G) \to \mathbb{Z}_2
\]
by
\[
\psi((H, g), x) = \text{Ori}(g, V^H) - \text{Ori}(g, W^H)
\]
where
\[
\text{Ori}(g, V^H) = \begin{cases} 
0 & \text{if } g : V^H \to V^H \text{ is orientation preserving}, \\
1 & \text{if } g : V^H \to V^H \text{ is orientation reversing}.
\end{cases}
\]
The value \( \psi((H, g), x) \) is also written as \( \text{Ori}(g, x^H) \).

**Lemma 5.1.** For a real \( G \)-module \( V \) and \( (H, g) \in \mathcal{A}(G) \), \( \text{Ori}(g, V^H) = \dim V^{(H, g^2)} - \dim V^{(H, g)} \) (mod 2).

For a subset \( C \subset \mathcal{A}(G) \), \( x \in \text{RO}(G) \) is called orientation trivial on \( C \) if \( \text{Ori}(g, x^H) = 0 \) for all \( (H, g) \in C \).

In the following, we always invoke the next hypothesis.

**Hypothesis 5.2.** Let \( K \) be a gap subgroup of \( G \) of index 2.

Let \( U \) be a gap \( K \)-module and set \( V = \text{Ind}_K^G U \). If \( H \subseteq K \) then we have
\[
\text{Res}_H^G V = \bigoplus_{H \subseteq K \subseteq G} \text{Ind}_H^K \text{Res}_K^K g_* (\text{Res}_K^K g H g^{-1} U)
\]
where \( g \) is an arbitrary element in \( G \setminus K \).

**Lemma 5.3.** Let \( U \) be a gap \( K \)-module. Then for \( V = \text{Ind}_K^G U \) and \( (H, P) \in \mathcal{H}\mathcal{P}(K) \), the inequality \( \dim V^P > 2 \dim V^H \) holds.

**Proof.** By the formula above, we get \( \dim V^H = \dim U^H + \dim U^g Hg^{-1} \) and \( \dim V^P = \dim U^P + \dim U^g P g^{-1} \). Since \( U \) is a gap \( K \)-module, we have \( \dim U^P > 2 \dim U^H \) and \( \dim U^g P g^{-1} > 2 \dim U^g H g^{-1} \). These imply \( \dim V^P > 2 \dim V^H \). \( \square \)

**Lemma 5.4.** Let \( V_0 - W_0 \in \text{RO}(K) \) and \( U_0 \) a gap \( K \)-module. Then \( V_1 = V_0 \oplus (\dim V_0 + 1)U_0 \) and \( W_1 = W_0 \oplus (\dim V_0 + 1)U_0 \) are gap \( K \)-modules. For \( V = \text{Ind}_K^G V_1 \), \( W = \text{Ind}_K^G W_1 \) and \( U = 2(\dim V_1 + 1) V(G) \), the real \( G \)-modules \( V \oplus U \) and \( W \oplus U \) fulfill the gap condition for any \( (H, P) \in \mathcal{H}\mathcal{P}(G) \) whenever \( H \subseteq K \) or \( (H, P) \not\in \mathcal{H}\mathcal{P}(G, 2)_0 \).
Proof. Let \((H, P) \in \mathcal{HP}(G)\). First observe the computation

\[
\dim V_1^P - 2 \dim V_1^H = \dim V_0^P - 2 \dim V_0^H + (\dim U_0^P - 2 \dim U_0^H) \\
\geq \dim V_0^P - 2 \dim V_0^H + (\dim V_0 + 1) \\
\geq (\dim V_0 + 1) - \dim V_0^H \\
> 0.
\]

Thus \(V_1\) is a gap \(K\)-module. Similarly, \(W_1\) is a gap \(K\)-module. By Lemma 5.3 \(V\) and \(W\) fulfill the gap condition for the pair \((H, P)\) whenever \(H \leq K\).

Now assume \((H, P) \notin \mathcal{HP}(G, 2)_0\). By Lemma 2.1 (2), the inequality \(\dim(V(G))^P > 2 \dim V(G)^H\) holds. Thus we get

\[
\dim(V \oplus U)^P - 2 \dim(V \oplus U)^H = \dim(\text{Ind}_K^G V_1)^P - 2 \dim(\text{Ind}_K^G V_1)^H \\
+ 2(\dim V_1 + 1)(\dim V(G)^P - 2 \dim V(G)^H) \\
\geq 2(\dim V_1 + 1) - \dim(\text{Ind}_K^G V_1)^H \\
\geq 2(\dim V_1 + 1) - \dim \text{Ind}_K^G V_1 \\
> 0.
\]

This shows that \(V \oplus U\) fulfills the gap condition for the pair \((H, P)\). Similarly, \(W \oplus U\) fulfills the gap condition for the pair \((H, P)\). \(\square\)

To apply Morimoto’s surgery result for \(y \in \text{Ind}_K^G (\text{RO}(K)_{\mathcal{P}(K)})\), we need to show that \(y\) is orientation trivial on the set

\[
\overline{B(G)}_2 := \{(H, g) \in B(G) \mid \text{Ord}(g) = 2^l \text{ for some } l \in \mathbb{N} \text{ and } H \not\subseteq K\}.
\]

In Proposition 3.1, we checked the orientation triviality holds for the group \(G = S_5 \times C_2\). In order to show that the orientation triviality holds for \(G = S_5 \times X_2\) with \(X_2 = C_2 \times \cdots \times C_2\) (\(n\)-fold) such that \(n \geq 2\), we introduce the notation

\[
\overline{B(G)}_{2\text{even}} := \{(H, g) \in \overline{B(G)}_2 \mid |H| = 2^k \text{ for some } k \in \mathbb{N}\}
\]

and

\[
\overline{B(G)}_{2\text{odd}} := \overline{B(G)}_2 \setminus \overline{B(G)}_{2\text{even}}.
\]

We can prove the following two lemmas without difficulties.

\textbf{Lemma 5.5.} Let \(G = S_5 \times X_2\) and \(a = (\sigma, b) \in G\) with \(\sigma \in S_5 \setminus A_5\) and \(b \in X_2\). Then there exists an isomorphism \(\varphi : G \to G\) such that

\begin{enumerate}
\item \(\varphi(\sigma) = a\),
\end{enumerate}
(2) \( \varphi(x) = x \) for all \( x \in A_5 \cup X_2 \), and
(3) \( \varphi \circ \varphi = \text{id}_G \).

**Lemma 5.6.** Let \( G = S_5 \times X_2 \). Then the implication

\[
\overline{\mathcal{B}(G)}_{2\text{even}} \subset \bigcup_{Y \leq G \text{; } Y \text{ not a 2-group}} \mathcal{A}(Y)
\]

holds.

Then we have the next lemma.

**Lemma 5.7.** The implication

\[
\overline{\mathcal{B}(G)}_{2\text{odd}} \subset \bigcup_{T \leq G \text{; } T \cong S_5 \times C_2} \mathcal{A}(T)
\]

holds.

Proof. Let \((H, g) \in \overline{\mathcal{B}(G)}_{2\text{odd}}\). By definition, we get \( H \not\subseteq A_5 = G^{[2]} \) as well as \( H \not\subseteq K \). It is easy to show the following.

(1) \( |H| = 2p \) for \( p = 3 \) or 5.
(2) \( H \) has a unique (normal) Sylow \( p \)-subgroup \( P = \langle u \rangle \) such that the order of \( u \) is \( p \).
(3) \( P \) is a unique (normal) Sylow \( p \)-subgroup of \( L = \langle H, g \rangle \) (\( \subseteq G \)).
(4) \( P \subseteq A_5 \).

Since \( p = 3 \) or 5, \( H \) is isomorphic to \( C_{2p} \) or \( D_{2p} \). Thus, we can take \( a \in H \setminus A_5 \) of order 2. Write

\[
a = (\sigma, b)
\]

and

\[
g = (\tau, c)
\]

with \( \sigma, \tau \in S_5 \) and \( b, c \in X_2 \). Since \( H \not\subseteq K \), \( \sigma \notin A_5 \). In addition, since the order of \( g \) is a power of 2 by definition, the order is 2 or 4. There exists an isomorphism \( \varphi : G \to G \) such that \( \varphi(H) \subseteq S_5 \) and \( \varphi|_{X_2} = \text{id}_{X_2} \). Then \( \varphi(L) = \langle \varphi(H), \varphi(g) \rangle \) is a subgroup of \( S_5 \times \langle c \rangle \). Thus \( (L, g) \) belongs to \( \mathcal{A}(T) \) for some \( T \leq G \) such that \( T \cong S_5 \times C_2 \).

**Lemma 5.8.** Let \( G = S_5 \times X_2 \) and \( K = A_5 \times X_2 \). For an arbitrary element \( x \in \text{RO}(K)_{P(K)} \), \( y = \text{Ind}_K^G x \) is orientation trivial on \( \overline{\mathcal{B}(G)}_2 \).
Proof. By Lemmas 5.6 and 5.7, the implication

$$\bar{B}(G)_2 \subset \bigcup_{T \leq G} A(T) \cup \bigcup_{Y \leq G} A(Y)$$

holds. Clearly, $y$ is orientation trivial on $A(Y)$ because $Y$ is a 2-group. In the proof of Proposition 3.1, we saw that for the basis element $y = V - W$ of $\text{RO}(T)_{\mathcal{P}(T)}$, $V \oplus 2U$ and $W \oplus 2U$ satisfy the weak gap condition. Thus each element of $\text{RO}(T)_{\mathcal{P}(T)}$ is orientation trivial on $A(T)$. \qed

6. Completion of proofs of Theorems A and B

In this section, we proceed as follows. Firstly, we give proofs of Lemmas 1.4 and 1.6. Secondly, for $G = S_5 \times X_2$ and $A_5 \times X_2$, we compute the rank of the Smith set of $G$.

Proof of Lemma 1.4. Let $K = A_5 \times X_2$. Since $A_5$ is a simple group, it follows that $K^{[2]} = A_5$ and $K^{[p]} = K$ ($p \neq 2$). Thus $K^{\text{nil}} = A_5$, and $K/K^{\text{nil}} \cong X_2$. Clearly $K$ contains no elements of 8. $K$ is an Oliver group, because $K$ is non-solvable. Clearly $\mathcal{P}(K) \cap \mathcal{L}(K) = \emptyset$. Since $A_5 \times C_2$ is a gap group (see the proof of Proposition 3.1), by [17, Theorem 0.4], it follows that $K$ is a gap group. \qed

Proof of Lemma 1.6. For arbitrary $x \in \text{RO}(G)_{\mathcal{P}(G)}$, there exists an element $y \in \text{RO}(K)_{\mathcal{P}(K)}$ such that $x = \text{Ind}_K^G y$. Let $y = V_0 - W_0$ such that $V_0$ and $W_0$ are $\mathcal{L}(K)$-free real $K$-modules, and $U_0$ $\mathcal{L}(K)$-free gap $K$-module. Then $V_1 = V_0 \oplus (\dim V_0 + 1)U_0$ and $W_1 = W_0 \oplus (\dim V_0 + 1)U_0$ are $\mathcal{L}(K)$-free gap $K$-modules. Set $V = \text{Ind}_K^G V_1$, $W = \text{Ind}_K^G W_1$ and $U = \max\{6, 2(\dim V_1 + 1)\}V(G)$.

For subgroups $H$, $K$ of $G$ and a real $G$-module $X$,

$$\text{Res}_H^G (\text{Ind}_K^G X) = \bigoplus_{HgK \in H \backslash G / K} \text{Ind}_H^{H \cap gKg^{-1}} (g_* \text{Res}_K^g X_{H \cap g^{-1}Hg})$$

$$= \begin{cases} \text{Res}_H^G X \oplus g_* \text{Res}_K^g X & \text{if } H \leq K \text{ (here } g \in S_5 \setminus A_5), \\ \text{Ind}_H^{H \cap K} (\text{Res}_K^g X) & \text{if } H \not\leq K. \end{cases}$$

Hence

$$\dim(\text{Ind}_K^G X)_H = \begin{cases} \dim X_H + \dim X_{g^{-1}Hg} & \text{if } H \leq K \text{ (here } g \in S_5 \setminus A_5), \\ \dim X_{H \cap K} & \text{if } H \not\leq K. \end{cases}$$

Let $(H, P) \in \mathcal{HP}(G, 2)$.

**Case** $H \leq K$. By Lemma 5.4, $V \oplus U$ and $W \oplus U$ satisfy the gap condition for $(H, P)$.
CASE $P \not\subseteq K$. We obtain
\[
\dim V^P - 2 \dim V^H = \dim V_1^{P \cap K} - 2 \dim V_1^{H \cap K}.
\]
Note $[H \cap K : P \cap K] = 2$, because $[P : P \cap K] = 2$ and $[H : H \cap K] = 2$. Thus
\[
\dim V_1^{P \cap K} - 2 \dim V_1^{H \cap K} > 0.
\]
By Lemma 2.1 (1), $\dim U^P \geq 2 \dim U^H$. Thus $V \oplus U$ satisfies the gap condition for $(H, P)$. Similarly $W \oplus U$ satisfies the gap condition for $(H, P)$.

CASE $P \leq K$, $H \not\subseteq K$. For an element $g \in H \setminus P$, we obtain
\[
\dim V^P - 2 \dim V^H = \dim V_1^P + \dim V_1^{g^{-1}Pg} - 2 \dim V_1^{H \cap K}.
\]
Since $P \triangleleft H$ and $H \cap K = P$, it follows that
\[
\dim V_1^P + \dim V_1^{g^{-1}Pg} - 2 \dim V_1^{H \cap K} = 2 \dim V_1^P - 2 \dim V_1^P = 0.
\]
By Lemma 5.8, $V - W$ is orientation trivial on $B(G)_2$. Thus $V \oplus U$ satisfies (WG6). By [8, Corollary 3.5], $6V(G)$ satisfies (WG1)–(WG6). Hence $V \oplus U$ satisfies (WG1), (WG2), (WG4), (WG5). By [12, Theorem 2.5], $V \oplus U$ satisfies (WG3). Similarly $W \oplus U$ satisfies the weak gap condition.

Let $H$ be a normal subgroup of $G$. We denote by $b_{G,H}$ the number of real conjugacy classes $(gH)^\pm$ in $G/H$ of cosets $gH$ containing elements of $G$ not of prime power order.

**Lemma 6.1.** If $G^{\text{nil}} = G^{[p]}$ for some prime $p$, then

\[
\text{Rank}_G (RO(G)_{P(G)}) = a_G - b_{G, G^{\text{nil}}}.\]

**Proof.** By [21, p.858, Subgroup Lemma], we have
\[
RO(G)^{[G^{\text{nil}}]}_{P(G)} \subseteq RO(G)_{P(G)} \subseteq RO(G)^{[G^{[p]}]}_{P(G)}.
\]
Since $G^{\text{nil}} = G^{[p]}$, it follows $RO(G)^{[G^{\text{nil}}]}_{P(G)} = RO(G)_{P(G)}$. By [21, p.856, Second Rank Lemma],
\[
\text{Rank}_G (RO(G)_{P(G)}) = \text{Rank}_G (RO(G)^{[G^{\text{nil}}]}_{P(G)}) = a_G - b_{G, G^{\text{nil}}}.\]

\qed
Proposition 6.2. Let $G = S_5 \times X_2$ and $K = A_5 \times X_2$ where $X_2 = C_2 \times \cdots \times C_2$ (n-folds). Then the following hold.

(1) $a_G = 1 + 3(2^n - 1)$ and $b_{G,G^{nil}} = 2^{n+1} - 1$.
(2) $a_K = 3(2^n - 1)$ and $b_{K,K^{nil}} = 2^n - 1$.

The proof is straightforward.

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References

THE SMITH SET OF $S_3 \times C_2 \times \cdots \times C_2$


Department of Mathematics
Graduate School of Natural Science and Technology
Okayama University
Okayama, 700–8530
Japan
e-mail: juxianmeng@gmail.com

Current address:
Department of Mathematics
College of Sciences
Shanghai University
99 Shangda Road, Baoshan District, Shanghai
200444
China