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Author(s)	Yamaguchi, Atsushi
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THE STRUCTURE OF THE COHOMOLOGY OF MORAVA STABILIZER ALGEBRA S(3)

ATSUSHI YAMAGUCHI

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Introduction.

Let X be a space and let p be a prime number. The E_2-term of the Adams-Novikov spectral sequence associated with BP-theory at p converging to the p-localized homotopy group of X is given by $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*(X))$ ([1], [4]). This motivates to study $\mathrm{Ext}_{BP_*BP}(BP_*, M)$ for a BP_*BP-comodule M. If X is a finite complex, $BP_*(X)$ is a finitely presented BP_*-module ([3]). Recall ([1], [14]) that $BP_*=\boldsymbol{Z}_{(p)}[v_1, v_2, \cdots]$, $deg\ v_n=2(p^n-1)$, and I_n denotes an invariant prime ideal $(p, v_1, v_2, \cdots, v_{n-1})$ of BP_*. Landweber proved the following theorem.

Theorem ([6]). *Let M be a BP_*BP-comodule which is finitely presented as a BP_*-module. Then, M has a finite filtraion by BP_*BP-subcomodules $0=M_0 \subset M_1 \subset \cdots \subset M_k=M$ such that for $1 \leq i \leq k$, M_i/M_{i-1} is isomorphic to BP_*/I_{n_i} for some $n_i \geq 0$ as a BP_*BP-comodule up to shifting degrees.*

By virtue of the above and a spectral sequence $E_2^{s,t}=\mathrm{Ext}^{s+t}_{BP_*BP}(BP_*, M_t/M_{t-1}) \Rightarrow \mathrm{Ext}^{s+t}_{BP_*BP}(BP_*, M)$ (See section 2) for M as above, we can relate $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ $(n=0, 1, 2, \cdots)$ with $\mathrm{Ext}_{BP_*BP}(BP_*, M)$. Hence it is necessary to know $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ before we study the general case.

For small n, $\mathrm{Ext}_{BP_*BP}(BP_*, BP/I_n)$ also has a geometric significance since there is a spectrum $V(n)$ whose BP-homology is isomorphic to BP_*/I_{n+1}, generalizing the Moore spectrum, if $p>2n$ and $n=0, 1, 2, 3$ ([2], [15]). Hence $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_{n+1})$ is the E_2-term of the Adams-Novikov spectral sequence converging to the homotopy group of $V(n)$. We note that $V(n)$ is a ring spectrum if $p>2n+2$ ([15]).

Since multiplication by v_n on BP_*/I_n is a BP_*BP-comodule homomorphism, $v_n^{-1}BP_*/I_n$ is a BP_*BP-comodule and $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ is a module over $\boldsymbol{F}_p[v_n]$ if $n>0$. In fact, $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ is a graded commutative algebra and $\mathrm{Ext}^0_{BP_*BP}(BP_*, BP_*/I_n)$ is isomorphic to $\boldsymbol{F}_p[v_n]$ if $n>0$ ([5], [11]). Thus $v_n^{-1}\mathrm{Ext}_{BP_*BP}(BP_*, BP/I_n)$ makes sense and it is obviously isomorphic to $\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$.

We put $K(n)_*=\boldsymbol{F}_p[v_n, v_n^{-1}]$ and $\Sigma(n)=K(n)_*\otimes_{BP_*}BP_*BP\otimes_{BP_*}K(n)_*$, then $\Sigma(n)$ is a Hopf algebra over $K(n)_*$. It is shown in [9] that $\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$ is isomorphic to $\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)$. Regarding $\boldsymbol{F}_p$ as a $K(n)_*$-algebra by $\rho: K(n)_*\to\boldsymbol{F}_p$, $\rho(v_n)=1$, we put $S(n)=\Sigma(n)\otimes_{K(n)_*}\boldsymbol{F}_p$. $S(n)$ is a $\boldsymbol{Z}/2(p^n-1)$-graded Hopf algebra over $\boldsymbol{F}_p$ and the dual Hopf algebra of $S(n)$ is called Morava stabilizer algebra. A functor from the category of graded $\Sigma(n)$-comodules to the category of $\boldsymbol{Z}/2(p^n-1)$-graded $S(n)$-comodules which assigns M to $M\otimes_{K(n)_*}\boldsymbol{F}_p$ is an equivalence of these categories. Thus $\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)\otimes_{K(n)_*}\boldsymbol{F}_p$ is isomorphic to $\mathrm{Ext}_{S(n)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ and $\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)$ can be recovered from $\mathrm{Ext}_{S(n)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$. Therefore, the v_n-torsion free part of $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ can be detected by the cohomology of Morava stabilizer algebra $\mathrm{Ext}_{S(n)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ by the preceding argument.

Moreover, the chromatic spectral sequence $E_1^{s,t}(n)\Rightarrow\mathrm{Ext}_{BP_*BP}^{s+t}(BP_*, BP_*/I_n)$ is constructed in [10], having the following properties; $E_1^{0,t}(n)$ is isomorphic to $\mathrm{Ext}_{\Sigma(n)}^t(K(n)_*, K(n)_*)$ and the edge homomorphism $\mathrm{Ext}_{BP_*BP}^t(BP_*, BP_*/I_n)\to E_1^{0,t}(n)$ can be identified with the localization map away from v_n. There is a Bockstein long exact sequence$\cdots\to E_1^{s-1,t}(n+1)\to E_1^{s,t}(n)\xrightarrow{v_n\times} E_1^{s,t}(n)\to E_1^{s-1,t+1}(n)\to\cdots$. Hence $\mathrm{Ext}_{\Sigma(m)}(K(m)_*, K(m)_*)$ for $m=n, n+1, \cdots$ relate with $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ through the Bockstein spectral sequences and the chromatic spectral sequence.

The cohomology of Morava stabilizer algebra $\mathrm{Ext}_{S(n)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ is calculated for $n=1, 2$ in [13] and the Poincare series of $\mathrm{Ext}_{S(3)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ is also given for $p\geq 5$. In this paper we determine the algebra structure of $\mathrm{Ext}_{S(3)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ for $p\geq 5$. By the above explanation, our result is a part of the initial input for the chromatic spectral sequences and it also gives the v_3-localization of the E_2-term of the Adams-Novikov spectral sequence converging to $\pi_*(V(2))$. Since $v_3\in E_2^{0,2(p^3-1)}$ is known to be a permanent cycle and $V(2)$ is a ring spectrum if $p\geq 7$, $\pi_*(V(2))$ is a module over $\boldsymbol{F}_p[v_3]$. Thus our result is expected to give some information on the v_3-torsion free part of $\pi_*(V(2))$.

For the calculation, we apply the method of May and Ravenel ([13]), namely, define a certain filtration on $S(3)$ such that the dual of the associated graded Hopf algebra is primitively generated. By the theorem of Milnor-Moore [8], $(E_0S(3))^*$ is isomorphic to the universal enveloping algebra of a restricted Lie algebra $L(3)=P(E_0(S(3))^*$. Let $L^u(3)$ denote the unrestricted Lie algebra obtained by forgetting the restriction of $L(3)$. We use the following spectral sequences which we review in section 2; $E_2^{s,t}=\mathrm{Ext}_{E^0S(3)}^{s+t}(\boldsymbol{F}_p, \boldsymbol{F}_p)_t\Rightarrow\mathrm{Ext}_{S(3)}^{s+t}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ and $E_2^{s,t}=\mathrm{Ext}_{U(L^u(3))}^s(\boldsymbol{F}_p, \boldsymbol{F}_p)\otimes P(s^2\pi L(3)^*)^t\Rightarrow\mathrm{Ext}_{V(L(3))}^{s+t}(\boldsymbol{F}_p, \boldsymbol{F}_p)=\mathrm{Ext}_{E^0S(3)}^{s+t}(\boldsymbol{F}_p, \boldsymbol{F}_p)$. The unrestricted Lie algebra $L^u(3)$ turns out to be a product of a nine-dimensional Lie algebra $M(3)$ and an abelian Lie algebra $I(3)$, and the edge homomorphism of the latter spectral sequence gives an isomorphism $\mathrm{Ext}_{E^0S(3)}^t(\boldsymbol{F}_p, \boldsymbol{F}_p)\to E_\infty^{0,t}=E_3^{0,t}=\mathrm{Ext}_{U(M(3))}^t(\boldsymbol{F}_p, \boldsymbol{F}_p)$. After we calculate the cohomololgy of $M(3)$, by showing that the former spectral sequence collapses and the extension is trivial, we prove that the cohomology of $S(3)$ is isomorphic to that of $M(3)$.

In section 1, we review how to construst an economical resolution for the universal enveloping algebra of a restricted Lie algebra according to [4], [7]. We also summarize a part of Ravenel's work [13] needed for our calculation. In section 2, we set up two kinds of spectral sequences we mentioned above. Sections 3 and 4 are devoted to calculate the cohomology a certain nine-dimensional Lie algebra $M(3)$, applying a sort of Cartan-Eilenberg spectral sequence. In section 5, we show that the cohomology of $S(3)$ is isomorphic to that of $M(3)$.

1. Recollections

First, we recall from [4] and [7] how to construct economical resolutions.

NOTATIONS. For a graded vector space V and an integer l, we denote by s^lV a bigraded vector space given by $(s^lV)_{i,j}=0$ if $i\neq l$, $(s^lV)_{i,j}=V_j$ and πV denotes a graded vector space given by $(\pi V)_i=0$ if p does not divide i, $(\pi V)_{pj}=V_j$. We denote by $\bar{x}$ and $\tilde{x}$ the elements of $(sV)_{1,j}$ and $(s^2\pi V)_{2,pj}$ corresponding to an element x of V_j. We also denote by $E(V)$, $P(V)$ and $\Gamma(V)$ the exterior algebra, the polynomial algebra and the divided polynomial algebra generated by V, respectively. For an element x of V, let $\langle x\rangle$ and $\gamma_i(x)$ be typical generators of $E(V)$ and $\Gamma(V)$, respectively. For a bigraded vector space W and $l\in\boldsymbol{Z}$, we put $W_l=\sum_j W_{l,j}$.

Let K be a field of characteristic $p\neq 0$, and let $L=\sum_{i\geqq 0}L_i$ be a graded restricted Lie algebra over K with restriction ξ. We denote by L^u the unrestricted Lie algebra obtained from L by forgetting the restriction of L. We put $L^+=\sum_{i\geqq 0}L_{2i}$, $L^-=\sum_{i\geqq 0}L_{2i+1}$ if $p>2$, and $L^+=L$, $L^-=0$ if $p=2$. Let us denote by $U(L^u)$ and $V(L)$ the universal enveloping algebras of an unresticted Lie algebra L^u and a restricted Lie algebra L, respectively. J.P. May ([7], see also [4]) constructed a $U(L^u)$-free resolution $Y(L^u)$ of K as follows; $Y(L^u)=U(L^u)\otimes E(sL^+)\otimes\Gamma(sL^-)$ as a left $U(L^u)$-module. Give $Y(L^u)$ a K-algebra structure such that the canonical inclusions of $U(L^u)$, $E(sL^+)$ and $\Gamma(sL^-)$ into each factor of $Y(L^u)$ are monomorphism of K-algebras and that the following relations hold for $x, x_i\in L^-, y, y_i\in L^+ (i=1,2)$; $\langle\bar{y}_1\rangle y_2=y_2\langle\bar{y}_1\rangle+\langle\overline{[y_1,y_2]}\rangle$, $\langle\bar{y}\rangle x=-x\langle\bar{y}\rangle+\gamma_1(\overline{[y,x]})$, $\gamma_t(\bar{x})y=y\gamma_t(\bar{x})+\gamma_t(\overline{[x,y]})\gamma_{t-1}(\bar{x})$, $\gamma_t(\bar{x}_1)x_2=x_2\gamma_t(\bar{x}_1)+\langle\overline{[x_1,x_2]}\rangle\gamma_{t-1}(\bar{x}_1)$, $\gamma_t(\bar{x})\langle\bar{y}\rangle=\langle\bar{y}\rangle\gamma_t(\bar{x})$, We note that $Y(L^u)$ is a bigraded algebra and, for $w\in L$, the bidegrees of $w\in U(L^u)$, $\langle\bar{w}\rangle\in E(sL^+)$ and $\gamma_t(\bar{w})\in\Gamma(sL^-)$ are $(0, \deg w)$, $(1, \deg w)$ and $(t, t\deg w)$ respectively. The differential d of $Y(L^u)$ is given by $du=\varepsilon(u)$ for $u\in U(L^u)$, $d\langle\bar{y}\rangle=y$ for $y\in L^+$, $d\gamma_t(\bar{x})=x\gamma_{t-1}(\bar{x})+\frac{1}{2}\langle\overline{[x,x]}\rangle\gamma_{t-2}(\bar{x})$ for $x\in L^-$ satisfying the Leibniz formula $d(xy)=(dx)y+(-1)^{|x|}xdy$, where ε: $U(L^u)\to K$ is the augmentation and $|x|$ is the total degree of $x\in Y(L^u)$. We also define a coproduct φ: $Y(L^u)\to Y(L^u)\otimes Y(L^u)$ by $\varphi(z)=1\otimes z+z\otimes 1$ for $z\in L$, $\varphi(\langle\bar{y}\rangle)=$

$1\otimes\langle\bar{y}\rangle+\langle\bar{y}\rangle\otimes 1$ for $y\in L^+$, $\varphi(\gamma_t(\bar{x}))=\sum_{i+j=t}\gamma_i(\bar{x})\otimes\gamma_j(\bar{x})$ for $x\in L^-$ so that $Y(L^u)$ has a structure of differential Hopf algebra.

Put $W(L)=V(L)\otimes E(sL^+)\otimes\Gamma(sL^-)$ and let q: $Y(L^u)\to W(L)$ be the canonical projection induced by $U(L^u)\to V(L)=U(L^u)/(y^p-\xi(y)\,|\,y\in L)$, then $W(L)$ has a unique structure of differential Hopf algebra over K such that q is a morphism of differential Hopf algebras. The following is obvious.

Lemma 1.1. *q induces an isomorphism* $\mathrm{Hom}^*_{V(L)}(W(L),K)\to\mathrm{Hom}^*_{U(L)}(Y(L^u),K)$ *of chain complexes. Hence the cohomology of* $\{\mathrm{Hom}^*_{V(L)}(W(L),K),d^*\}$ *is isomorphic to* $\mathrm{Ext}_{U(L^u)}(K,K)$.

We choose a K-basis $\{y_\alpha\,|\,\alpha\in\Lambda\}$ of L^+. For each index $\alpha\in\Lambda$, let $\boldsymbol{Z}^+_\alpha$ be a copy of a monoid of non-negative integers. For an element $R=(t_\alpha)_{\alpha\in\Lambda}\in\oplus_{\alpha\in\Lambda}\boldsymbol{Z}^+_\alpha$, we set $\gamma(R)=\prod_{\alpha\in\Lambda}\gamma_{t_\alpha}(\bar{y}_\alpha)$. Then $\{\gamma(R)\,|\,R\in\oplus_{\alpha\in\Lambda}\boldsymbol{Z}^+_\alpha\}$ forms a K-basis of $\Gamma(s^2\pi L^+)$. The bidegree of $\gamma(R)$ is $(2|R|, p\sum_{\alpha\in\Lambda}t_\alpha\, deg\, y_\alpha)$, where we put $|R|=\sum_{\alpha\in\Lambda}t_\alpha$ for $R=(t_\alpha)_{\alpha\in\Lambda}$.

Lemma 1.2 ([4], [7]). 1) *There exists a twisting cochain* $\theta=(\theta_{2l})$, θ_{2l}: $\Gamma(s^2\pi L^+)_{2l}\to W(L)_{2l-1}$ *satisfying* $\theta_2(\gamma_1(\bar{y}))=y^{p-1}\langle\bar{y}\rangle-\langle\overline{\xi(y)}\rangle$ *for* $y\in L^+$.
2) *There exists a twisting diagonal cochain* $\lambda=(\lambda_{2l})$, λ_{2l}: $\Gamma(s^2\pi L^+)_{2l}\to(W(L)\otimes W(L))_{2l}$ *satisfying* $\lambda_0(1)=1\otimes 1$, $\lambda_2(\gamma_1(\bar{y}))=\sum_{i=1}^{p-1}(-1)^i y^{i-1}\langle\bar{y}\rangle\otimes y^{p-1-i}\langle\bar{y}\rangle$.

Consider a left $V(L)$-module $X(L)=W(L)\otimes\Gamma(s^2\pi L^+)$ and define a differential d_θ and a coproduct D by $d_\theta(w\otimes\gamma(R))=dw\otimes\gamma(R)+(-1)^{|w|}\sum_{S+T=R}w\cdot\theta(\gamma(S))\otimes\gamma(T)$, $D(w\otimes\gamma(R))=\phi(w)\cdot\sum_{S+T=R}\lambda(\gamma(S))\cdot\nabla\gamma(T)$, where ϕ is the coproduct of $W(L)$ and ∇ is the standard coproduct of $\Gamma(s^2\pi L^+)$.

Theorem 1.3 ([4], [7]). *The complex $\{X(L),d_\theta\}$ is a $V(L)$-free resolution of K. It is also a differential coalgebra with coproduct D.*

We set $\bar{Y}(L)=E(sL^+)\otimes\Gamma(sL^-)$ and define a filtration on $X(L)$ by $F_mX(L)=V(L)\otimes F_m\bar{X}(L)$ where $F_m\bar{X}(L)=\sum_{i\leq m}\bar{Y}(L)_{m-i}\otimes\Gamma(s^2\pi L^+)_i$. We state the following obvious fact for later use.

Proposition 1.4. 1) $0=F_{-1}X(L)_l\subset W(L)_l=F_0X(L)_l\subset\cdots\subset F_lX(L)_l=X(L)_l$, $F_{2m}X(L)=F_{2m+1}X(L)$.
2) *The inclusion $F_{m-1}X(L)\hookrightarrow F_mX(L)$ is a split monomorphism of $V(L)$-modules.*
3) *$\{X(L),d_\theta,D\}$ is a filtered differential coalgebra and for each $w\otimes\gamma(R)\in F_{2|R|}X(L)$, $d_\theta(w\otimes\gamma(R))\equiv dw\otimes\gamma(R)$ modulo $F_{2|R|-2}X(L)$, $D(w\otimes\gamma(R))\equiv\phi(w)\cdot\nabla\gamma(R)$ modulo $F_{2|R|-2}(X(L)\otimes X(L))$.*

Next we recall some facts on Morava stabilizer algebra from [12].

Let (BP_*, BP_*BP) be the Hopf algebroid associated with BP-theory at a fixed prime p (See [14], for example). Put $K(n)_*=\boldsymbol{F}_p[v_n,v_n^{-1}]$ and regard this as a BP_*-algebra by $\pi: BP_* \to K(n)_*$, $\pi(v_i)=0$ if $i \neq n$, $\pi(v_n)=v_n$. We set $\Sigma(n)=K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$, then it is known that $\Sigma(n)$ is isomorphic to $K(n)_*[t_1, t_2, \cdots]/(v_n t_i^{p^n} - v_n^{p^i} t_i)$ $(deg\ t_i = 2(p^i-1))$ as a $K(n)_*$-algebra. Note that Hopf algebroid $(K(n)_*, \Sigma(n))$ is in fact a Hopf algebra. Regarding $\boldsymbol{F}_p$ as a $K(n)_*$-algebra by $\rho: K(n)_* \to \boldsymbol{F}_p$, $\rho(v_n)=1$, put $S(n)=\Sigma(n) \otimes_{K(n)_*} \boldsymbol{F}_p$. $S(n)$ is a $\boldsymbol{Z}/2(p^n-1)$-graded Hopf algebra over $\boldsymbol{F}_p$ which is isomorphic to $\boldsymbol{F}_p[t_1, t_2, \cdots]/(t_i^{p^n} - t_i)$ as an $\boldsymbol{F}_p$-algebra.

Thoerem 1.5 ([12]). *Define integers d_i for $i \in \boldsymbol{Z}$ recursively by $d_i=0$ if $i \leq 0$, $d_i = \max\{i, pd_{i-n}\}$ if $i>0$. Then, there is a unique increasing Hopf algebra filtration on $S(n)$ with $t_i^{p^j} \in F_{d_i}S(n) - F_{d_i-1}S(n)$.*

Instead of considering the above filtration, we consider a new filtration $\{\widetilde{F}_i S(n)\}$ defined by $\widetilde{F}_{2i}S(n)=\widetilde{F}_{2i+1}S(n)=F_iS(n)$ so that the associated $(\boldsymbol{Z} \times \boldsymbol{Z}/2(p^n-1))$-graded Hopf algebra $E^0S(n)$ becomes graded commutative. Let $t_{i,j}$ denote the the element of $E^0_{2d_i}S(n)_{2p^j(p^i-1)}$ corresponding to $t_i^{p^j}$, where $j \in \boldsymbol{Z}/n$, then $E^0S(n)$ is isomorphic to $\boldsymbol{F}_p[t_{i,j} \mid i \geq 1, j \in \boldsymbol{Z}/n]/(t_{i,j}^p)$ as an algebra. Consider the $(\boldsymbol{Z} \times \boldsymbol{Z}/2(p^n-1))$-graded dual $(E^0S(n))^*$ of $E^0S(n)$, and let $x_{i,j} \in (E^0S(n))^{2d_i, 2p^j(p^i-1)}$ be the dual of $t_{i,j}$ with respect to the monomial basis. We set $L(n)=P(E^0S(n))^*$, then $\{x_{i,j} \mid i \geq 1, j \in \boldsymbol{Z}/n\}$ spans $L(n)$. We note that this $L(n)$ is different from the one in [12], [13], [14], which coincides with an unrestricted Lie algebra $M(n)$ defined in the next section if $n<p-1$. Since the p-th power map on $E^0S(n)$ is trivial, it follows from [8] that $(E^0S(n))^*$ is the universal enveloping algebra of the restricted Lie algebra $L(n)$. The bracket and the restriction are given by the following.

Theorem 1.6 ([12]). $[x_{i,j}, x_{k,l}] = \delta^l_{i+j} x_{i+k,j} - \delta^j_{k+l} x_{i+k,l}$ *if* $i+k \leq pn/(p-1)$, *otherwise the bracket is trivial, where* $\delta^s_t = 1$ *if* $s \equiv t$ *mod* n, $\delta^s_t = 0$ *otherwise.* $\xi(x_{i,j})=0$ *if* $i \leq n/(p-1)$, $\xi(x_{i,j}) = -x_{i+n,j+1}$ *otherwise.*

REMARK 1.7. The p-th power map η on $S(n)$ is an automorphism of order n. Since η preserves the filtration, it induces an automorphism of $E^0S(n)$ which we also denote by η. We note that η maps $t_{i,j}$ to $t_{i,j+1}$ and that η induces an automorphism η^* of $L(n)$.

2. Spectral Sequences

Let R be a graded commutative ring. For a filtered R-modules M and N, we filter $M \otimes_R N$ by $F_s(M \otimes_R N) = \mathrm{Im}(\sum_{i+j=s} F_iM \otimes_R F_jN \to M \otimes_R N)$ as usual. Then there is a natural epimorphism $\sum_{i+j=s} E^0_iM \otimes_R E^0_jN \to E^0_s(M \otimes_R N)$, where we put $E^0_iM = F_iM/F_{i-1}M$. Note that this epimorphism is an isomorphism if F_iM is flat over R for any i.

Let C be a filtered R-coalgebra which is flat over R, and let M be a filtered left C-comodule. By the above remark, E^0C is an R-coalgebra and E^0M is a left E^0C-comodule. We give a decreasing filtration on the cobar complex $\Omega^*(C; M)$ ([9], [10], [14]) by $F^s\Omega^m(C; M)=\mathrm{Im}(\sum_{i_0+\cdots+i_m=m-s} F_{i_1}C\otimes_R F_{i_2}C\otimes_R\cdots\otimes_R F_{i_m}C\otimes_R F_{i_0}M\to\Omega^m(C; M))$. Then the differential d of $\Omega^*(C; M)$ maps $F^s\Omega^m(C; M)$ into $F^{s+1}\Omega^{m+1}(C; M)$ and $E_0^{s,t}=F^s\Omega^{s+t}(C; M)/F^{s+1}\Omega^{s+t}(C; M)$ is isomorphic to $\Omega^{s+t}(E^0C; E^0M)_t=\sum_{t_0+\cdots+i_{s+t}=t} E^0_{i_1}C\otimes_R E^0_{i_2}C\otimes_R\cdots\otimes_R E^i_{0_{s+t}}C\otimes_R E^0_{i_0}M$. We rather call t of $E_0^{s,t}$ the filtration degree below. Put $D_1^{s,t}=H^{s+t}(F^s\Omega^*(C; M))$. $E_1^{s,t}=H^{s+t}(E_0^{s,*})$ and let $i_*: D_1^{s+1,t-1}\to D_1^{s,t}$ and $j_*: D_1^{s,t}\to E_1^{s,t}$ be the maps induced by inclusion $i: F^{s+1}\Omega^*(C; M)\hookrightarrow F^s\Omega^*(C; M)$ and projection $j: F^s\Omega^*(M; C)\to E_0^{s,*}$ respectively. $\partial: E_1^{s,t}\to D_1^{s+1,t}$ denotes the boundary homomorphism associated with a short exact sequence of complexes $0\to F^{s+1}\Omega^*(C; M)\to F^s\Omega^*(C; M)\to E_0^{s,*}\to 0$. Consider the spectral sequence associated with exact couple $\langle D_1^{s,t}, E_1^{s,t}, i_*, j_*, \partial\rangle$. Then $E_1^{s,t}=E_0^{s,t}$ and the E_2-term is given by $E_2^{s,t}=H^{s+t}(\Omega^*(E^0C; E^0M))_t=\mathrm{Ext}^{s+t}_{E^0C}(R, E^0, M)_t$. Filter $H^*(\Omega^*(C; M))=\mathrm{Ext}^*_C(R, M)$ by putting $F^{s,t}=\mathrm{Im}(H^{s+t}(F^s\Omega^*(C; M))\to H^{s+t}(\Omega^*(C; M)))$. We assume that $C=\bigcup_s F_sC$ and $M=\bigcup_s F_sM$ hold and that $F_sC=F_sM=0$ for sufficiently small s. Then the above spectral sequence converges to $\mathrm{Ext}^*_C(R, M)$.

Applying the above spectral sequence to the case $R=M=\boldsymbol{F}_p$, $C=S(n)$, we have a spectral sequence

$$(2.1)\qquad E_2^{s,t}=\mathrm{Ext}^{s+t}_{E^0S(n)}(\boldsymbol{F}_p, \boldsymbol{F}_p)_t\Rightarrow\mathrm{Ext}^{s+t}_{S(n)}(\boldsymbol{F}_p, \boldsymbol{F}_p)\,.$$

Let A be a graded algebra (not necessarily commutative) over a commutative ring. Let X be a filtered A-complex with differential $d: X_i\to X_{i-1}$. Put $E^0_{s,t}=F_sX_{s+t}/F_{s-1}X_{s+t}$. Let M be a graded A-module. Consider a complex $\{C^*, d^*\}$ given by $C^i=\mathrm{Hom}^i_A(X, M)$. Filter C^* by $F^sC^*=\mathrm{Ker}(\mathrm{Hom}^*_A(X, M)\to\mathrm{Hom}^*_A(F_{s-1}X, M))$. We assume that the inclusions $F_sX/F_{s-1}X\hookrightarrow X/F_{s-1}X$ are split monomorphism of A-modules for any s (This holds if the inclusions $F_{s-1}X\hookrightarrow F_sX$ split for any s). Then we have short exact sequences of complexes $0\to F^{s+1}C^*\xrightarrow{i}F^sC^*\xrightarrow{j}E_0^{s,*}\to 0$, where we set $E_0^{s,t}=\mathrm{Hom}^{s+t}_A(E^0_{s,*}, M)$. Let $\Delta: H^*(E_0^{s,*})\to H^{*+1}(F^{s+1}C^*)$ be the boundary homomorphism. Putting $D_1^{s,t}=H^{s+t}(F^sC^*)$, $E_1^{s,t}=H^{s+t}(E_0^{s,*})$, we consider a spectral sequence associated with an exact couple $\langle D_1^{s,t}, E_1^{s,t}, i_*, j_*, \Delta\rangle$. We define a filtration on $H^*(C^*)$ by $F^{s,t}=\mathrm{Im}(H^{s+t}(F^sC^*)\to H^{s+t}(C^*))$. Suppose that, for each integer m, there exist integers $a(m)$ and $b(m)$ such that $F_sX_m=X_m$ if $s>a(m)$, $F_sX_m=0$ if $s<b(m)$. Then the spectral sequence converges to $H^*(C^*)$.

By 1) and 2) of (1, 4), we can apply the above spectral sequence to the case $A=V(L)$, $X=X(L)$, $M=K$, and obtain a spectral sequence converging to $H^*(\mathrm{Hom}_{V(L)}(X(L), K))=\mathrm{Ext}^*_{V(L)}(K, K)$. We note that the coproduct D of $X(L)$ makes this spectral sequence multiplicative. Identifying $E^0_{s,*}$ with $W(L)\otimes\Gamma(s^2\pi L^+)_s$, it follows from (1, 4), 3) that d_θ and D induce $d\otimes 1: E^0_{s,*}\to E^0_{s,*}$ and

$\phi\otimes\nabla: E^0_{s,*}\to\sum_{i+j=s}E^0_{i,*}\otimes E^0_{j,*}$. Therefore the E_1-term is isomorphic to $P(s^2\pi(L^+)^*)\otimes\mathrm{Ext}^*_{U(L^u)}(K,K)$ as an algebra by (1, 1), where $(L^+)^*$ denotes the graded K-dual of L^+. Since $E_1^{s,t}=0$ if s is odd or $s<0$ or $t<0$, $E_2^{s,t}=E_1^{s,t}$ holds and we have the edge homomorphism. Thus we have shown

Theorem 2.2 ([7]). *There is a multiplicative spectral sequence* $E_2^{s,t}=P(s^2\pi(L^+)^*)^s\otimes\mathrm{Ext}^t_{U(L^*)}(K,K)\Rightarrow\mathrm{Ext}^{s+t}_{V(L)}(K,K)$, *whose edge homomorphism* $\mathrm{Ext}^t_{V(L)}(K,K)=F^{0,t}\to E_\infty^{0,t}\hookrightarrow E_2^{0,t}=\mathrm{Ext}^t_{U(L^u)}(K,K)$ *is induced by the composite* $Y(L^u)\xrightarrow{q}W(L)\hookrightarrow X(L)$.

In particular, in the case $K=\boldsymbol{F}_p$, $L=L(n)$, let $M(n)$ and $I(n)$ be subspaces of $L^u(n)$ spanned by $\{x_{i,j}\mid i\leq pn/(p-1), j\in\boldsymbol{Z}/n\}$ and $\{x_{i,j}\mid i>pn/(p-1), j\in\boldsymbol{Z}/n\}$ respectively. It follows from (1, 6) that $M(n)$ is a Lie subalgebra of $L^u(n)$ and $I(n)$ is an ideal of $L^u(n)$. Obviously, $I(n)$ is an abelian Lie algebra and $L^u(n)$ is isomorphic to $M(n)\times I(n)$ as a Lie algebra. Therefore $U(L^u(n))$ is isomorphic to $U(M(n))\otimes P(I(n))$. This implies that $\mathrm{Ext}_{U(L^u(n))}(\boldsymbol{F}_p,\boldsymbol{F}_p)$ is isomorphic to $\mathrm{Ext}_{U(M(n))}(\boldsymbol{F}_p,\boldsymbol{F}_p)\otimes E(\langle\bar{t}_{i,j}\rangle\mid i>pn/(p-1),\ j\in\boldsymbol{Z}/n)$ where $deg\langle\bar{t}_{i,j}\rangle=(1,2d_i,2p^j(p^i-1))$. Hence the E_2-term of the spectral sequence is isomorphic to $P(\tilde{t}_{i,j}\mid i\geq 1,\ j\in\boldsymbol{Z}/n)\otimes\mathrm{Ext}_{U(M(n))}(\boldsymbol{F}_p,\boldsymbol{F}_p)\otimes E(\langle\bar{t}_{i,j}\rangle\mid i>pn/(p-1),\ j\in\boldsymbol{Z}/n)$. By (1, 2) and (1, 6), we have the following fact on the differential d_2.

Lemma 2.3. ([13]). $d_2\langle\bar{t}_{i,j}\rangle=-\tilde{t}_{i-n,j-1}$ *for* $i>pn/(p-1)$. *Thus if* $n<p-1$, $E_\infty^{s,t}=E_3^{s,t}=0$ *unless* $s=0$, *and the edge homomorphism maps* $\mathrm{Ext}^t_{V(L(n))}(\boldsymbol{F}_p,\boldsymbol{F}_p)$ *bijectively onto* $E_\infty^{0,t}=E_3^{0,t}=\mathrm{Ext}^t_{U(M(n))}(\boldsymbol{F}_p,\boldsymbol{F}_p)\subset E_2^{0,t}$.

3. Auxiliary Calculation

Let L be a graded unrestricted Lie algebra over a field K of finite type such that $L^-=0$, and let $\{x_\lambda\mid\lambda\in\Lambda\}$ be a totally ordered basis of L. L^* denotes the graded dual of L. Take the dual basis $\{x^*_\lambda\mid\lambda\in\Lambda\}$ of $\{x_\lambda\mid\lambda\in\Lambda\}$. Define $\delta: E(sL^*)_i\to E(sL^*)_{i+1}$ by $\delta(\langle\overline{x^*_\lambda}\rangle)=-\sum_{\mu<\nu}\langle x^*_\lambda,[x_\mu,x_\nu]\rangle\langle\overline{x^*_\mu}\rangle\langle\overline{x^*_\nu}\rangle$ satisfying the Leibniz formula, where $\langle,\rangle: L^*\otimes L\to K$ is the canonical pairing. It is straightforward to verify the following.

Lemma 3.1. $\{E(sL^*),\delta\}$ *is a differential algebra isomorphic to* $\{\mathrm{Hom}^*_{U(L)}(Y(L),K),d^*\}$. *Hence* $H^*(E(sL^*);\delta)$ *is isomorphic to* $Ext^*_{AU(L)}(K,K)$.

Now we concentrate on the computation of $\mathrm{Ext}^*_{U(M(3))}(\boldsymbol{F}_p,\boldsymbol{F}_p)$ for $p\geq 5$. $M(3)$ is spanned by $\{x_{i,j}\mid i=1,2,3, j\in\boldsymbol{Z}/3\}$ over $\boldsymbol{F}_p$. Then $E(sM(3)^*)=E(t_{i,j}\mid i=1,2,3, j\in\boldsymbol{Z}/3)$ where we put $t_{i,j}=\langle\overline{x^*_{i,j}}\rangle$. It follows from (1.6) and (3.1) that δ is given by $\delta(t_{1,j})=0$, $\delta(t_{2,j})=-t_{1,j}t_{1,j+1}$, $\delta(t_{3,j})=t_{1,j-1}t_{2,j}-t_{1,j}-t_{2,j+1}$ for $j\in\boldsymbol{Z}/3$. Let A be an ideal of $M(3)$ spanned by $\{x_{3,0},x_{3,1},x_{3,2}\}$ and we regard $\{E(s(M(3)/A)^*),\delta\}$ as a subcomplex of $\{E(sM(3)^*),\delta\}$. We remark that $M(3)$

and $M(3)/A$ are denoted by $L(3, 3)$ and $L(3, 2)$ respectively in [13], [14]. We can manage to compute the cohomology of $\{E(s(M(3)/A)^*), \delta\}$ directly by hand and the structure of $\mathrm{Ext}_{U(M(3)/A)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ is described below.

For a cocycle z of $E(s(M(3)/A)^*)$, we denote by $[z]$ the cohomology class represented by z.

Lemma 3.2. 1) *$\mathrm{Ext}^*_{U(M(3)/A)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ is generated by the following seventeen elements as an algebra;*

$h_j=[t_{1,j}], g_j=[t_{1,j}t_{2,j}], g'_j=[t_{1,j+1}t_{2,j}], f_i=[t_{1,i-1}t_{2,i}-t_{1,i}t_{2,i+1}], d_j=[t_{1,j}t_{2,j-1}t_{2,j}],$
$e_j=[t_{1,j}t_{2,j}t_{2,j+1}+t_{1,j+1}t_{2,j-1}t_{2,j}]$, *for* $i, j\in \boldsymbol{Z}/3$, $i\neq 2$.

2) $\mathrm{Ext}^s_{U(M(3)/A)}(\boldsymbol{F}_p, \boldsymbol{F}_p)=0$ *for* $s>6$. *A basis of* $\mathrm{Ext}^s_{U(M(3)/A)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ $(0\leq s\leq 6)$ *is given as follows;*

$s=0$; 1.
$s=1$; h_0, h_1, h_2.
$s=2$; $g_0, g_1, g_2, g'_0, g'_1, g'_2, f_0, f_1$.
$s=3$; $h_1g_0, h_2g_1, h_0g_2, h_2g_0, h_0g_1, h_1g_2, d_0, d_1, d_2, e_0, e_1, e_2$.
$s=4$; $g_0g_1, g_1g_2, g_2g_0, g'_2g'_0, g'_0g'_1, g'_1g'_2, g_0g'_1, g_1g'_2$.
$s=5$; g_1d_0, g_2d_1, g_0d_2.
$s=6$; $g_0g_1g_2$.

The operator $\eta^\sharp$ of $L(3)$ induces an algebra automorphism of $\mathrm{Ext}^*_{U(M(3)/A)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ of order three, which we denote by η_*. Obviously, we have $\eta^i_*h_0=h_i$, $\eta^i_*g_0=g_i$, $\eta^i_*g'_0=g'_i$, $\eta^i_*f_0=f_i$, $\eta^i_*d_0=d_i$, and $\eta^i_*e_0=e_i$ *for* $i\in\boldsymbol{Z}/3$ where we put $f_2=-f_0-f_1$.

Lemma 3.3. *Relations of* $\mathrm{Ext}^*_{U(M(3)/A)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ *are given by the following and the relations obtained by applying* η^i_* $(i=1, 2)$ *to them;*

$h_0h_i=0$, $f_0d_i=0$, $d_0d_i=d_0e_i=e_0e_i=0$ *for* $i\in\boldsymbol{Z}/3$, $h_0g_0=0$, $h_0g'_0=-h_1g_0$
$h_0g'_1=-h_1g_2$, $h_0g'_2=0$, $h_0f_0=h_0f_1=-h_2g_0$, $g_0^2=g_0'^2=g_0g'_0=g_0g'_2=g_0f_0=0$
$g_0f_1=g'_2g'_0$, $g'_0f_0=-g'_0f_1=g_0g_1$, $g_2g'_0=-g_0g'_1-g_1g'_2$, $f_0^2=2g_0g'_1$, $f_0f_1=2g_2g'_0$,
$h_0d_0=0$, $h_0d_1=-g_0g_1$, $h_0d_2=g'_1g'_2$, $h_0e_0=-g'_2g'_0$, $h_0e_1=-g_0g'_1+g_1g'_2$, $h_0e_2=g_2g_0$,
$g_0d_0=g_0d_1=g_0e_0=g_0e_2=0$, $g_0d_2=-g_2d_0$, $g_0e_1=-g_1d_0$, $g'_0d_0=g'_0d_1=g'_0e_0=g'_0e_1=0$,
$g'_0e_2=g_1d_0$, $f_0e_0=g_1d_0$, $f_0e_1=g_2d_1$, $f_0e_2=-2g_0d_2$, $g'_0g'_1g'_2=g_0g_1g_2$.

4. Main Calculation

Let D be a subcomplex of $E(sM(3)^*)$ generated by $\{t_{i,j}\,|\,i=1, 2, 3, j\in\boldsymbol{Z}/3, j\neq 2 \text{ if } i=3\}$. We put $\zeta_3=t_{3,0}+t_{3,1}+t_{3,2}$.

Lemma 4.1. $\delta(\zeta_3)=0$ *and* $\{E(sM(3)^*), \delta\}$ *is isomorphic to* $\{D\otimes E(\zeta_3), \delta\otimes 1\}$. *Therefore* $\mathrm{Ext}^*_{U(M(3))}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ *is isomorphic to* $H^*(D)\otimes E(\zeta_3)$ *as an algebra.*

We filter $E(sM(3)^*)$ and D by $F^sE(sM(3)^*)^m=\sum_{i\leq m-s}E(s(M(3)/A)^*)^{m-i}\otimes$

$E(sA^*)^i$, $F^sD^m=D^m\cap F^sE(sM(3)^*)^m$. Then $F^sE(sM(3)^*)^m=F^sD^m+F^{s-1}D^{m-1}\zeta_3$ and $0=F^{m+1}D^m\subset F^mD^m=E(s(M(3)/A)^*)^m\subset F^{m-1}D^m\subset F^{m-2}D^m=D^m$ hold. Also note that $\delta(F^sE(sM(3)^*)^{s+t})\subset F^{s+1}E(sM(3)^*)^{s+t+1}$. Thus we have spectral sequences $E_1^{s,t}\Rightarrow \mathrm{Ext}_{U(M(3))}^{s+t}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ and $\tilde{E}_1^{s,t}\Rightarrow H^*(D)$ associated with these filtrations. By the above lemma, the former spectral sequence is isomorphic to $\{\tilde{E}_r^{s,t}\otimes E(\zeta_3), \tilde{d}_r\otimes 1\}$. Hence it suffices to compute the latter. The $\tilde{E}_1$-term is given by $\tilde{E}_1^{s,t}=H^{s+t}(F^sD/F^{s+1}D)=E(s(M(3)/A)^*)^s\otimes E(t_{3,0}, t_{3,1})^t$ and $\tilde{d}_1$ coincides with $\delta\otimes 1$. Therefore $\tilde{E}_2^{s,t}=\mathrm{Ext}_{U(M(3)/A)}^*(\boldsymbol{F}_p, \boldsymbol{F}_p)\otimes E(t_{3,0}, t_{3,1})^t$ and $\tilde{d}_2$ is given by $\tilde{d}_2(t_{3,j})=f_j (j=0, 1)$. By computing the $\tilde{E}_3$-term, we find that $\tilde{E}_3=\tilde{E}_\infty$ for dimensional reasons. It is not difficult (but very tedious) to solve the extension problem and we can determine the structure of $\mathrm{Ext}_{U(M(3))}^*(\boldsymbol{F}_p, \boldsymbol{F}_p)$ as given below.

Theorem 4.2. 1) *$\mathrm{Ext}_{U(M(3))}^*(\boldsymbol{F}_p, \boldsymbol{F}_p)$ is generated by the following twenty-six elements;*
$h_j=[t_{1,j}]$, $\zeta_3=[t_{3,0}+t_{3,1}+t_{3,2}]$, $g_j=[t_{1,j}t_{2,j}]$, $g'_j=[t_{1,j+1}t_{2,j}]$, $a_0=[t_{1,0}t_{3,0}-t_{1,0}t_{3,1}+t_{2,2}t_{2,0}]$, $a_1=[t_{1,1}t_{3,0}+2t_{1,1}t_{3,1}+t_{2,0}t_{2,1}]$, $a_2=[-2t_{1,2}t_{3,0}-t_{1,2}t_{3,1}+t_{2,1}t_{2,2}]$, $b_0=[t_{1,0}t_{2,0}t_{3,0}]$, $b_1=[t_{1,1}t_{2,1}t_{3,1}]$, $b_2=[t_{1,2}t_{2,2}(-t_{3,0}-t_{3,1})]$, $b'_0=[t_{1,1}t_{2,0}(-t_{3,0}-t_{3,1})]$, $b'_1=[t_{1,2}t_{2,1}t_{3,0}]$, $b'_2=[t_{1,0}t_{2,2}t_{3,1}]$, $c=[t_{1,2}t_{2,0}t_{3,0}-t_{1,0}t_{2,1}t_{3,0}+t_{1,2}t_{2,0}t_{3,1}-t_{1,1}t_{2,2}t_{3,1}+t_{2,0}t_{2,1}t_{2,2}]$, $u_0=[t_{1,0}t_{2,2}t_{2,0}t_{3,0}]$, $u_1=[t_{1,1}t_{2,0}t_{2,1}t_{3,1}]$, $u_2=[t_{1,2}t_{2,1}t_{2,2}(-t_{3,0}-t_{3,1})]$, $w_j=[t_{1,j}t_{2,j-1}t_{2,j}t_{3,0}t_{3,1}]$ *for* $j\in \boldsymbol{Z}/3$.
$H^*(D)$ is generated by the above elements except for ζ_3.

2) *$H^s(D)=0$ for $s>8$. A basis of $H^s(D)$ $(0\leq s\leq 8)$ is given as follows;*

$s=0$; 1.

$s=1$; h_0, h_1, h_2.

$s=2$; $g_0, g_1, g_2, g'_0, g'_1, g'_2, a_0, a_1, a_1$.

$s=3$; $h_1g_0, h_2g_1, h_0g_2, h_0a_0, h_1a_1, h_2a_2, h_0a_1, h_1a_2, h_2a_0, b_0, b_1, b_2, b'_0, b'_1, b'_2, c$.

$s=4$; $h_1b_0, h_2b_1, h_0b_2, h_2b_0, h_0b_1, h_1b_2, h_0b'_0, h_1b'_1, h_2b'_2, a_0^2, a_1^2, a_2^2, a_0a_1, a_1a_2, a_2a_0, u_0, u_1, u_2$.

$s=5$; $h_0a_1a_2, h_1a_0a_1, h_2a_1a_2, h_0a_2a_0, h_0a_0a_1, h_1a_1a_2, h_2a_2a_0, a_1b_0, a_2b_1, a_0b_2, a_2b_0, a_0b_1, a_1b_2, w_0, w_1, w_2$.

$s=6$; $h_1a_2b_0, h_2a_0b_1, h_0a_1b_2, h_0w_1, h_1w_2, h_2w_0, h_1w_0, h_2w_1, h_0w_2$.

$s=7$; g_1w_0, g_2w_0, g_0w_2.

$s=8$; $h_2g_1w_0$.

REMARKS. 1) By (4, 1) the basis given above is an $E(\zeta_3)$-basis of $\mathrm{Ext}_{U(M(3))}^*(\boldsymbol{F}_p, \boldsymbol{F}_p)$.

2) An element x of $\mathrm{Ext}_{U(M(3))}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ is said to be of filtration degree t if the image of x by the isomorphism $\mathrm{Ext}_{U(M(3))}(\boldsymbol{F}_p, \boldsymbol{F}_p)\cong \mathrm{Ext}_{V(L(3))}(\boldsymbol{F}_p, \boldsymbol{F}_p)=\mathrm{Ext}_{E^0S(3)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ of (2, 3) belongs to $E_2^{*,t}$ in the spectral sequence (2, 1). We denote by *f-deg x* the filtration degree of x. Then it is easy to see *f-deg* $h_j=2$, *f-dge* $\zeta_3=$ *f-deg* $g_j=$ *f-deg* $g'_j=6$, *f-deg* $a_j=8$, *f-deg* $b_j=$ *f-deg* $b'_j=$ *f-deg* $c=12$, *f-deg* $u_j=16$, *f-deg* $w_j=22$ for $j\in \boldsymbol{Z}/3$.
The internal degree of an element of $\mathrm{Ext}_{U(M(3))}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ is the degree coming

from the grading of $S(3)$ which takes value in $\boldsymbol{Z}/2(p^3-1)$. We denote by *i-deg* x the internal degree of x. Put $q=2(p-1)$. Noting that $p^2q\equiv(-p-1)q$ modulo $2(p^3-1)$, the internal degrees of the generator are given as follows; $i\text{-}deg\ h_j=i\text{-}deg\ a_j=p^jq$, $i\text{-}deg\ \zeta_3=i\text{-}deg\ c=0$, $i\text{-}deg\ g_j=i\text{-}deg\ b_j=p^j(p+2)q$, $i\text{-}deg\ g'_j=i\text{-}deg\ b'_j=p^j(2p+1)q$, $i\text{-}deg\ u_j=i\text{-}deg\ w_j=2p^jq$ for $j\in\boldsymbol{Z}/3$.
These two kinds of degrees play an important role in the next section.

Let η_* denote the operator on $\mathrm{Ext}_{U(M(3))}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ induced by $\eta^\sharp: L(3)\to L(3)$. By the definition of the generators in (4, 2), it is easy to verify the following

Proposition 4.3. *$h_j=\eta_*^jh_0$, $g_j=\eta_*^jg_0$, $g'_j=\eta_*^jg'_0$ for $j=1, 2$, $\zeta_3=\eta_*\zeta_3$, $a_1=\eta_*a_0+h_1\zeta_3$, $a_2=\eta_*^2a_0-h_2\zeta_3$, $b_1=\eta_*b_0$, $b_2=\eta_*^2b_0-g_2\zeta_3$, $b'_1=\eta_*b'_0+g'_1\zeta_3$, $b'_2=\eta_*^2b'_0+g'_2\zeta_3$, $c=\eta_*c$, $u_1=\eta_*u_0$, $u_2=\eta_*^2u_0-h_2a_2\zeta_3$, $w_1=\eta_*w_0-u_1\zeta_3$, $w_2=\eta_*^2w_0+u_2\zeta_3-a_2^2\zeta_3/2$.*

Since $\eta_*\zeta_3=\zeta_3, \eta_*$ induces an automorphism $\tilde{\eta}_*$ of $H^*(D)=\mathrm{Ext}^*_{U(M(3))}(\boldsymbol{F}_p, \boldsymbol{F}_p)/(\zeta_3)$ which maps x_j to x_{j+1} for $x=h, g, g', a, b, b', u, w$ and $j\in\boldsymbol{Z}/3$.

Theorem 4.4. *Relations of $H^*(D)$ are given by the following and the relations obtained by applying $\tilde{\eta}_*^j$ $(j=1, 2)$ to them;*
$h_0h_j=0$ *for* $j\in\boldsymbol{Z}/3$;
$h_0g_0=h_0g_1=h_0g'_1=h_0g'_2=0$, $h_0g'_0=-h_1g_0$, $h_0a_2=h_2a_0$;
$g_0g_j=g_0g'_j=g'_0g'_j=0$ *for* $j\in\boldsymbol{Z}/3$, $h_0b_0=h_0b'_2=0$, $h_0b'_1=-h_1b_2$, $h_0c=-3h_2b_0$, $a_0g_0=a_0g'_2=0$, $a_0g_1=-3h_0b_1$, $a_0g_2=2h_2b'_2-h_0b_2$, $a_0g'_0=h_0b'_0-h_1b_0$, $a_0g'_1=-3h_1b_2$;
$h_0a_1a_2=h_1a_2a_0=h_2a_0a_1$, $h_0a_0^2=h_0u_0=h_0u_2=0$, $h_0a_1^2=h_1a_0a_1$, $h_0u_1=h_1a_0a_1/2$, $g_0b_0=g_0b'_0=g_0b'_2=0$, $g_0b_1=-h_1a_0a_1/2$, $g_0b_2=h_0a_2a_0/2$, $g_0b'_1=h_0a_1a_2/6$, $g_0c=-h_0a_0a_1/2$, $g'_0b_0=g'_0b_1=g'_0b'_0=0$, $g'_0b_2=h_0a_1a_2/6$, $g'_0c=h_2a_1a_2/2$, $g'_0b'_1=h_1a_1a_2/2$, $g'_0b'_2=-h_0a_0a_1/2$, $a_0b_0=a_0b'_2=0$, $a_0b'_0=-a_1b_0$, $a_0b'_1=-a_1b_2$, $a_0c=-3a_2b_0$;
$h_0a_0b_2=0$, $h_0a_0b_1=h_1a_2b_0$, $g_0u_0=g_0u_1=g'_0u_0=g'_0u_1=0$, $g_0u_2=-h_0a_1b_2/2$, $g'_0u_2=-h_2a_0b_1$, $h_0w_0=0$, $a_0a_1a_2=a_0^3=a_0u_0=0$, $a_0a_1^2=-6h_0w_1$, $a_0a_2^2=6h_0w_2$, $a_0u_1=-h_0w_1$, $a_0u_2=2h_0w_2$, $b_0b'_j=0$ *for* $j\in\boldsymbol{Z}/3$, $b_0b_1=-h_0w_1$, $b_2b_0=-h_2w_0$, $b_0c=h_1w_0$, $b'_0b'_1=h_2w_1$, $b'_2b'_0=h_1w_0$, $b'_0c=-h_1w_2$;
$a_0w_j=b'_0u_j=0$ *for* $j\in\boldsymbol{Z}/3$, $g_0w_0=g_0w_1=g'_0w_0=g'_0w_1=0$, $g'_0w_2=-g_2w_1$, $b_0a_1^2=b_0u_0=b_0u_1=cu_0=0$, $b_0a_2^2=2g_0w_2$, $b_0a_1a_2=2g_1w_0$, $b_0u_2=g_0w_2$;
$h_2g_1w_0=h_0g_2w_1=h_1g_0w_2$, $cw_0=0$, $b_0w_j=b'_0w_j=u_0u_j=0$ *for* $j\in\boldsymbol{Z}/3$

This completes a description of the structure of $\mathrm{Ext}^*_{E^0S(3)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ by virtue of (2.3) and (4,1).

5. The Algebra Structure of the Cohomology of $S(3)$

We consider the spectral sequence (2.1) for $n=3$, $p\geq 5$. This spectral sequence is $(\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}/2(p^3-1))$-graded and we denote $E_r^{s,t,u}$ and $F^{s,t,u}$ the subspaces of $E_r^{s,t}$ and $F^{s,t}$ spanned by elements of internal degree u. From the calculation of the previous section, we have the following table of the E_2-term, where the numbers in the parentheses in the table indicate the filtration degree

t of the elements.

$u\backslash s+t$	0	1	2	3	4	5
0	$1(0)$	$\zeta_3(6)$	—	$c(12)$	$c\zeta_3(18)$	$h_0a_1a_2(18)$
p^jq	—	$h_j(2)$	$a_j, h_j\zeta_3(8)$	$a_j\zeta_3(14)$	$h_{j+2}b_j(14)$	$a_{j+2}b_j, h_{j+2}b_j\zeta_3(20)$
$p^j(p+1)q$	—	—	—	$h_ja_{i+1}(10)$	$a_ja_{j+1}, h_ja_{j+1}\zeta_3(16)$	$a_ja_{j+1}\zeta_3(22)$
$p^j(p+2)q$	—	—	$g_j(6)$	$b_j, g_j\zeta_3(12)$	$b_j\zeta_3(18)$	$h_ja_ja_{j+1}(18)$
$p^j(2p+1)q$	—	—	$g'_j(6)$	$b'_j, g'_j\zeta_3(12)$	$b'_j\zeta_3(18)$	$h_{j+1}a_ja_{j+1}(18)$
$2p^j(p+1)q$	—	—	—	$h_{j+1}g_j(8)$	$h_{j+1}b_j, h_jb'_j, h_{j+1}g_j\zeta_3(14)$	$a_{j+1}b_j, h_{j+1}b_j\zeta_3(20)$
$2p^jq$	—	—	—	$h_ja_j(10)$	$a_j^2, u_j, h_ja_j\zeta_3(16)$	$w_j, a_j^2\zeta_3, u_j\zeta_3(22)$

$u\backslash s+t$	6	7	8	9
0	$h_0a_1a_2\zeta_3(24)$	—	$h_2g_1w_0(30)$	$h_2g_1w_0\zeta_3(36)$
p^jq	$a_{j+2}b_j\zeta_3(26)$	—	—	—
$p^j(p+1)q$	$h_{j+1}a_{j+2}b_j(22)$	$g_{j+1}w_j, h_{j+1}a_{j+2}b_j\zeta_3(28)$	$g_{j+1}w_j\zeta_3(34)$	—
$p^j(p+2)q$	$h_{j+1}w_j, h_ja_ja_{j+1}\zeta_3(24)$	$h_{j+1}w_j\zeta_3(30)$	—	—
$p^j(2p+1)q$	$h_jw_{j+1}, h_{j+1}a_ja_{j+1}\zeta_3(24)$	$h_jw_{j+1}\zeta_3(30)$	—	—
$2p^j(p+1)q$	$a_{j+1}b_j\zeta_3(26)$	—	—	—
$2p^jq$	$w_j\zeta_3(28)$	—	—	—

The following facts are immediately verified from the table.

Lemma 5.1. *If* $E_2^{m-t,t}\neq 0$, $E_2^{m+1-s,s}=0$ *holds for* $s<t$. *Therefore the spectral sequence of* (2, 1) *collapses, that is,* $E_2^{s,t}=E_\infty^{s,t}$.

Lemma 5.2. *If* $\sum_{s+t=m} E_2^{s,t,u}\neq 0$ *for given* $m\in \boldsymbol{Z}$ *and* $u\in \boldsymbol{Z}/2(p^3-1)$, *then* $E_2^{m-t,t,u}=0$ *for all but only one* t. *Hence if* $\mathrm{Ext}_{S(3)}^{m,u}(\boldsymbol{F}_p, \boldsymbol{F}_p)\neq 0$, *there is a unique* $t=\tau(m,u)$ *such that* $F^{m-t+1,t-1,u}=0$ *and* $F^{m-t,t,u}=\mathrm{Ext}_{S(3)}^{m,u}(\boldsymbol{F}_p, \boldsymbol{F}_p)$.

Thus there are unique elements $h_j, \zeta_3, g_j, g'_j, a_j, b_j, b'_j, c, u_j, w_j$ ($j\in \boldsymbol{Z}/3$) of $\mathrm{Ext}_{S(3)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ corresponding to the elements of the E_2-term denoted by the same symbols. Let $\bar{B}$ be a set of monomials of the above elements which corresponds to the $E(\zeta_3)$-basis of the E_2-term given in the previous section. We put $B=\bar{B}\cup$

$\{x\zeta_3 | x \in \bar{B}\}$, then B is a basis of $\mathrm{Ext}_{S(3)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$. For $x \in \mathrm{Ext}_{S(3)}^{m,u}(\boldsymbol{F}_p, \boldsymbol{F}_i)$, we denote by $\tilde{x}$ the element of $E_2^{m-t,t,u}$ corresponding to x where $t=\tau(m,u)$. For $x \in B \cap \mathrm{Ext}_{S(3)}^{m,u}(\boldsymbol{F}_p, \boldsymbol{F}_p)$, $y \in B \cap \mathrm{Ext}_{S(3)}^{l,v}(\boldsymbol{F}_p, \boldsymbol{F}_p)$, suppose that $\tilde{x}\tilde{y}=\sum_i \nu_i \tilde{z}_i$ holds for $\nu_i \in \boldsymbol{F}_p$, $z_i \in B$ in $E_2^{m+l-t-t',t+t',u+v}$ where $t=\tau(m,u)$, $t'=\tau(l,v)$, in other words, $xy=\sum_i \nu_i z_i$ holds modulo $F^{m+l-t-t'+1,t+t'-1,u+v}$. If $\tilde{x}\tilde{y} \neq 0$, (5.2) implies that $F^{m+l-t-t'+1,t+t'-1,u+v}=0$. Hence xy exactly equals to $\sum_i \nu_i z_i$ in this case. In the case $\tilde{x}\tilde{y}=0$, we can verify $xy=0$ case by case. In fact, it suffices to deal with the case $F^{m+l-t-t',t+t'} \neq 0$. Then we only have to ckeck the cases $(x,y)=(a_j, b_j)$, (a_j, b'_{j-1}), $(a_0 a_1, a_2)$, (a_j^2, a_j), (h_j, w_j), (a_j, u_j), (b_i, b'_j), (a_i, w_j), (b_i, w_j), (b'_i, w_j), (c, w_j), (u_i, u_j) for $i, j \in \boldsymbol{Z}/3$. In any of these cases, since $\mathrm{Ext}_{S(3)}^{m+l,u+v}(\boldsymbol{F}_p, \boldsymbol{F}_p)=0$, the assertion follows. Similarly, for $x \in B$, $\eta_* \tilde{x}=\sum_i \mu_i \tilde{y}_i (\mu_i \in \boldsymbol{F}_p, y_i \in B)$ implies $\eta_* x= \sum_i \mu_i y_i$ where the latter η_* is the operation of $\mathrm{Ext}_{S(3)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ induced by the p-th power map of $S(3)$. Thus we have shown

Theorem 5.3. $\mathrm{Ext}_{S(3)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ *is isomorphic to* $\mathrm{Ext}_{U(M(3))}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ *as an algebra over* $\boldsymbol{F}_p$ *and the isomorphism commutes with the operations induced by the p-th power map of* $S(3)$.

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Department of Mathematics
University of Osaka Prefecture
Sakai, Osaka 591
Japan