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Osaka University
THE STRUCTURE OF THE COHOMOLOGY OF MORAVA STABILIZER ALGEBRA S(3)

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Introduction.

Let $X$ be a space and let $p$ be a prime number. The $E_2$-term of the Adams-Novikov spectral sequence associated with $BP$-theory at $p$ converging to the $p$-localized homotopy group of $X$ is given by $\text{Ext}_{BP_*BP}(BP_*, BP_*(X))$ ([1], [4]). This motivates to study $\text{Ext}_{BP_*BP}(BP_*, M)$ for a $BP_*BP$-comodule $M$. If $X$ is a finite complex, $BP_*(X)$ is a finitely presented $RP^\infty$-module ([3]). Recall ([1], [14]) that $BP_* = \mathbb{Z}_p[v_1, v_2, \ldots]$, $\deg v_n = 2(p^n - 1)$, and $I_n$ denotes an invariant prime ideal $(p, v_1, v_2, \ldots, v_{n-1})$ of $BP_*$. Landweber proved the following theorem.

Theorem ([6]). Let $M$ be a $BP_*BP$-comodule which is finitely presented as a $BP_*$-module. Then, $M$ has a finite filtration by $BP_*BP$-subcomodules $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ such that for $1 \leq i \leq k$, $M_i/M_{i-1}$ is isomorphic to $BP_*/I_n$, for some $n_i \geq 0$ as a $BP_*/BP$-comodule up to shifting degrees.

By virtue of the above and a spectral sequence $E_2^{p,q} = \text{Ext}_{BP_*BP}(BP_*, BP_q BP(I_n))$ (see section 2) for $M$ as above, we can relate $\text{Ext}_{BP_*BP}(BP_*, BP_*(I_n)) (n=0, 1, 2, \ldots)$ with $\text{Ext}_{BP_*BP}(BP_*, M)$. Hence it is necessary to know $\text{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ before we study the general case.

For small $n$, $\text{Ext}_{BP_*BP}(BP_*, BP/I_n)$ also has a geometric significance since there is a spectrum $V(n)$ whose $BP$-homology is isomorphic to $BP_*/I_{n+1}$, generalizing the Moore spectrum, if $p > 2n$ and $n=0, 1, 2, 3$ ([2], [15]). Hence $\text{Ext}_{BP_*BP}(BP_*, BP_*/I_{n+1})$ is the $E_2$-term of the Adams-Novikov spectral sequence converging to the homotopy group of $V(n)$. We note that $V(n)$ is a ring spectrum if $p > 2n + 2$ ([15]).

Since multiplication by $v_n$ on $BP_*/I_n$ is a $BP_*BP$-comodule homomorphism, $v_n^{-1}BP_*/I_n$ is a $BP_*BP$-comodule and $\text{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ is a module over $F_p[v_n]$ if $n > 0$. In fact, $\text{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ is a graded commutative algebra and $\text{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ is isomorphic to $F_p[v_n]$ if $n > 0$ ([5], [11]). Thus $v_n^{-1}\text{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$ makes sense and it is obviously isomorphic to $\text{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$.
We put $K(n)_* = F_p[v_n, v_n^{-1}]$ and $\Sigma(n) = K(n)_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} K(n)_*$, then $\Sigma(n)$ is a Hopf algebra over $K(n)_*$. It is shown in [9] that $\text{Ext}_{BP_* BP}(BP_*, v_n^{-1} BP_*/I_n)$ is isomorphic to $\text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)$. Regarding $F_p$ as a $K(n)_*$-algebra by $p: K(n)_* \rightarrow F_p$, $p(v_n) = 1$, we put $S(n) = S(n) \otimes_{K(n)_*} F_p$. $S(n)$ is a $Z/2(p^n-1)$-graded Hopf algebra over $F_p$ and the dual Hopf algebra of $S(n)$ is called Morava stabilizer algebra. A functor from the category of graded $\Sigma(n)$-comodules to the category of $Z/2(p^n-1)$-graded $S(n)$-comodules which assigns $M$ to $M \otimes_{K(n)_*} F_p$ is an equivalence of these categories. Thus $\text{Ext}^2 S(n)(F_p, F_p)$ can be recovered from $\text{Ext}^2 S(n)(F_p, F_p)$. Therefore, the $v_n$-torsion free part of $\text{Ext}_{BP_* BP}(BP_*, BP_*/I_n)$ can be detected by the cohomology of Morava stabilizer algebra $\text{Ext}_{\Sigma(n)}(F_p, F_p)$ by the preceding argument.

Moreover, the chromatic spectral sequence $E^{1, t}_i(n) \Rightarrow \text{Ext}^{1, t}_{BP_* BP}(BP_*, BP_*/I_n)$ is constructed in [10], having the following properties; $E^{0, t}_i(n)$ is isomorphic to $\text{Ext}^{0, t}_{\Sigma(n)}(K(n)_*, K(n)_*)$ and the edge homomorphism $\text{Ext}^{0, t}_{BP_* BP}(BP_*, BP_*/I_n) \rightarrow E^{1, t}_i(n)$ can be identified with the localization map away from $v_n$. There is a Bockstein long exact sequence $\cdots \rightarrow E^{1, t-1}(n+1) \rightarrow E^{1, t}(n) \xrightarrow{v_n} E^{1, t}(n) \rightarrow E^{1, t+1}(n) \cdots$. Hence $\text{Ext}^{2, m}_{\Sigma(n)}(K(m)_*, K(m)_*)$ for $m = n, n+1, \cdots$ relate with $\text{Ext}^{0, t}_{BP_* BP}(BP_*, BP_*/I_n)$ through the Bockstein spectral sequences and the chromatic spectral sequence.

The cohomology of Morava stabilizer algebra $\text{Ext}_{\Sigma(n)}(F_p, F_p)$ is calculated for $n = 1, 2$ in [13] and the Poincare series of $\text{Ext}^{s, t}_{\Sigma(n)}(F_p, F_p)$ is also given for $p \geq 5$. By the above explanation, our result is a part of the initial input for the chromatic spectral sequences and it also gives the $v_n$-localization of the $E_2$-term of the Adams-Novikov spectral sequence converging to $\pi_*(V(2))$. Since $v_3 \in E_2^{p, 2(p^n-1)}$ is known to be a permanent cycle and $V(2)$ is a ring spectrum if $p > 7$, $\pi_*(V(2))$ is a module over $F_p[v_3]$. Thus our result is expected to give some information on the $v_3$-torsion free part of $\pi_*(V(2))$.

For the calculation, we apply the method of May and Ravenel ([13]), namely, define a certain filtration on $S(3)$ such that the dual of the associated graded Hopf algebra is primitively generated. By the theorem of Milnor-Moore [8], $(E_0 S(3))^* \otimes_{E_0 S(3)}$ is isomorphic to the universal enveloping algebra of a restricted Lie algebra $L(3) = P(E_0 S(3))^*$. Let $L^*(3)$ denote the unrestricted Lie algebra obtained by forgetting the restriction of $L(3)$. We use the following spectral sequences which we review in section 2; $E_2^{s, t} = \text{Ext}^{s+t}_{E_0 S(3)}(F_p, F_p) \Rightarrow \text{Ext}^{s+t}_{E_0 S(3)}(F_p, F_p)$ and $E_2^{s, t} = \text{Ext}^{s+t}_{E_0 S(3)}(F_p, F_p) \otimes P(\pi_* L(3)^*) \Rightarrow \text{Ext}^{s+t}_{E_0 S(3)}(F_p, F_p) = \text{Ext}^{s+t}_{E_0 S(3)}(F_p, F_p)$. The unrestricted Lie algebra $L^*(3)$ turns out to be a product of a nine-dimensional Lie algebra $M(3)$ and an abelian Lie algebra $I(3)$, and the edge homomorphism of the latter spectral sequence gives an isomorphism $\text{Ext}^{s+t}_{E_0 S(3)}(F_p, F_p) \rightarrow E_2^{s, t} = E_2^{s, t} = \text{Ext}^{s+t}_{E_0 S(3)}(F_p, F_p)$. After we calculate the cohomology of $M(3)$, by show-
ing that the former spectral sequence collapses and the extension is trivial, we prove that the cohomology of \( S(3) \) is isomorphic to that of \( M(3) \).

In section 1, we review how to construct an economical resolution for the universal enveloping algebra of a restricted Lie algebra according to [4], [7]. We also summarize a part of Ravenel's work [13] needed for our calculation. In section 2, we set up two kinds of spectral sequences we mentioned above. Sections 3 and 4 are devoted to calculate the cohomology of a certain nine-dimensional Lie algebra \( M(3) \), applying a sort of Cartan-Eilenberg spectral sequence. In section 5, we show that the cohomology of \( S(3) \) is isomorphic to that of \( M(3) \).

1. Recollections

First, we recall from [4] and [7] how to construct economical resolutions.

**NOTATIONS.** For a graded vector space \( V \) and an integer \( / \), we denote by
\[
(s^iV)_t = \begin{cases} 0 & \text{if } i \neq t, \\ V_t & \text{if } i = t, \end{cases}
\]
denotes a graded vector space given by
\[
(s^iV)_t = \begin{cases} V_t & \text{if } i \neq t, \\ 0 & \text{if } i = t. \end{cases}
\]
We denote by \( X \) and \( \% \) the elements of \((sV)_t\) and \((s^2\pi V)_{2t}\) corresponding to an element \( x \) of \( V \). We also denote by \( E(V) \), \( P(V) \) and \( T(V) \) the exterior algebra, the polynomial algebra and the divided polynomial algebra generated by \( V \), respectively. For an element \( x \) of \( V \), let \( \langle x \rangle \) and \( \gamma_i(x) \) be typical generators of \( E(V) \) and \( T(V) \), respectively. For a bigraded vector space \( W \) and \( t \in \mathbb{Z} \), we put \( W_t = \sum_j W_{ij} \).

Let \( K \) be a field of characteristic \( p \neq 0 \), and let \( L = \sum_{i \geq 0} L_i \) be a graded restricted Lie algebra over \( K \) with restriction \( \xi \). We denote by \( L^\ast \) the unrestricted Lie algebra obtained from \( L \) by forgetting the restriction of \( L \). We put \( L^+ = \sum_{i \geq 0} L_{2i}, \ L^- = \sum_{i \geq 0} L_{2i+1} \) if \( p > 2 \), and \( L^+ = L, \ L^- = 0 \) if \( p = 2 \). Let us denote by \( U(L^\ast) \) and \( V(L) \) the universal enveloping algebras of an unrestricted Lie algebra \( L^\ast \) and a restricted Lie algebra \( L \), respectively. J.P. May ([7], see also [4]) constructed a \( U(L^\ast) \)-free resolution \( Y(L^\ast) \) of \( K \) as follows; \( Y(L^\ast) = U(L^\ast) \otimes E(sL^+) \otimes \Gamma(sL^-) \) as a left \( U(L^\ast) \)-module. Give \( Y(L^\ast) \) a \( K \)-algebra structure such that the canonical inclusions of \( U(L^\ast) \) and \( \Gamma(sL^-) \) into each factor of \( Y(L^\ast) \) are morphisms of \( \lambda \)-algebras and that the following relations hold for \( x > y \in L^\ast \):

\[
\gamma_i(x) y = y \gamma_i(x) + \gamma_i([x, y]) \gamma_{i-1}(x), \quad \gamma_i(x) x_2 = x_2 \gamma_i(x) + ([x_1, x_2]) \gamma_{i-1}(x), \quad \gamma_i(x) \langle y \rangle = \langle y \rangle \gamma_i(x).
\]

We note that \( Y(L^\ast) \) is a bigraded algebra and, for \( w \in L \), the bidegrees of \( w \in U(L^\ast) \), \( \langle w \rangle \in E(sL^+) \) and \( \gamma_i(w) \in \Gamma(sL^-) \) are \((0, \deg w)\), \((1, \deg w)\) and \((t, t \deg w)\) respectively. The differential \( d \) of \( Y(L^\ast) \) is given by \( du = \varepsilon(u) \) for \( u \in U(L^\ast) \), \( d\langle y \rangle = y \) for \( y \in L^\ast \), \( d\gamma_i(x) = x \gamma_{i-1}(x) + \frac{1}{i}[[x, x]] \gamma_{i-1}(x) \) for \( x \in L^- \) satisfying the Leibniz formula \( d(xy) = (dx)y + (-1)^{|x|} xyd \), where \( \varepsilon : U(L^\ast) \rightarrow K \) is the augmentation and \( |x| \) is the total degree of \( x \in Y(L^\ast) \). We also define a coproduct \( \varphi : Y(L^\ast) \rightarrow Y(L^\ast) \otimes Y(L^\ast) \) by \( \varphi(x) = 1 \otimes x + x \otimes 1 \) for \( x \in L \).
for $y \in L^+$, $\varphi(\gamma_i(x)) = \sum_{i+j=i} \gamma_i(x) \otimes \gamma_j(x)$ for $x \in L^-$ so that $Y(L^*)$ has a structure of differential Hopf algebra.

Put $W(L) = Y(L) \otimes E(sL^+) \otimes \Gamma(sL^-)$ and let $q: Y(L^*) \rightarrow W(L)$ be the canonical projection induced by $U(L^*) \rightarrow Y(L^*)/(y^\delta - \xi(y)) | y \in L)$, then $W(L)$ has a unique structure of differential Hopf algebra over $K$ such that $q$ is a morphism of differential Hopf algebras. The following is obvious.

**Lemma 1.1.** $q$ induces an isomorphism $\text{Hom}^*_K(W(L), K) \rightarrow \text{Hom}^*_K(Y(L^*), K)$ of chain complexes. Hence the cohomology of $\{\text{Hom}^*_K(W(L), K), \delta^*\}$ is isomorphic to $\text{Ext}^*(K, K)$.

We choose a $K$-basis $\{y_\alpha | \alpha \in \Lambda\}$ of $L^+$. For each index $\alpha \in \Lambda$, let $\mathcal{Z}_\alpha^+$ be a copy of a monoid of non-negative integers. For an element $R = (t_\alpha)_{\alpha \in \Lambda} \in \bigoplus_{\alpha \in \Lambda} \mathcal{Z}_\alpha^+$, we set $\gamma(R) = \prod_{\alpha \in \Lambda} \gamma_{t_\alpha}(y_\alpha)$. Then $\{\gamma(R) | R \in \bigoplus_{\alpha \in \Lambda} \mathcal{Z}_\alpha^+\}$ forms a $K$-basis of $\Gamma(s^2\pi L^+)$. The bidegree of $\gamma(R)$ is $(2|R|, \sum_{\alpha \in \Lambda} t_\alpha)$, where we put $|R| = \sum_{\alpha \in \Lambda} t_\alpha$ for $R = (t_\alpha)_{\alpha \in \Lambda}$.

**Lemma 1.2 ([4], [7]).** 1) There exists a twisting cochain $\theta = (\theta_2), \theta_2: \Gamma(s^2\pi L^+)_{21} \rightarrow W(L)_{21-1}$ satisfying $\theta_2(\gamma_i(y)) = y^{\delta - 1} \langle y \rangle = \langle \xi(y) \rangle$ for $y \in L^+$.
2) There exists a twisting diagonal cochain $\lambda = (\lambda_2), \lambda_2: \Gamma(s^2\pi L^+)_{21} \rightarrow (W(L) \otimes W(L))_{21}$ satisfying $\lambda(1) = 1 \otimes 1$, $\lambda_2(\gamma_i(y)) = \sum_{\alpha \in \Lambda} (-1)^{\alpha} y^{\delta - 1} \langle y \rangle \otimes y^{\delta - 1} \langle y \rangle$.

Consider a left $V(L)$-module $X(L) = W(L) \otimes \Gamma(s^2\pi L^+)$ and define a differential $d_\theta$ and a coproduct $D$ by $d_\theta(\gamma_i(y)) = d \gamma_i(y) + (-1)^{|w|} \sum_{S+T = \alpha} w \cdot \theta(S) \otimes \gamma(T)$, $D(w \otimes \gamma(S)) = \phi(w) \cdot \sum_{S+T = \alpha} \lambda(S) \cdot \nabla \gamma(T)$, where $\phi$ is the coproduct of $W(L)$ and $\nabla$ is the standard coproduct of $\Gamma(s^2\pi L^+)$. We set $\overline{Y}(L) = E(sL^+) \otimes \Gamma(sL^-)$ and define a filtration on $X(L)$ by $F_{m+1}X(L) = V(L) \otimes F_m \overline{Y}(L)$ where $F_m \overline{Y}(L) = \bigoplus_{i \leq m} \overline{Y}(L)_{m-i} \otimes \Gamma(s^2\pi L^+)_i$. We state the following obvious fact for later use.

**Theorem 1.3 ([4], [7]).** The complex $\{X(L), d_\theta\}$ is a $V(L)$-free resolution of $K$. It is also a differential coalgebra with coproduct $D$.

We set $\overline{Y}(L) = E(sL^+) \otimes \Gamma(sL^-)$ and define a filtration on $X(L)$ by $F_m X(L) = V(L) \otimes F_m \overline{Y}(L)$ where $F_m \overline{Y}(L) = \bigoplus_{i \leq m} \overline{Y}(L)_{m-i} \otimes \Gamma(s^2\pi L^+)_i$. We state the following obvious fact for later use.

**Proposition 1.4.** 1) $0 = F_{-1} X(L) \subset W(L) = F_0 X(L) \subset \cdots \subset F_1 X(L) = X(L)$, $F_2 X(L) = F_{2m} X(L)$.
2) The inclusion $F_{m-1} X(L) \hookrightarrow F_m X(L)$ is a split monomorphism of $V(L)$-modules.
3) $\{X(L), d_\theta, D\}$ is a filtered differential coalgebra and for each $w \otimes \gamma(S) \in F_{2|S|+1} X(L)$, $d_\theta(w \otimes \gamma(S)) = d \gamma(S) \otimes \nabla \gamma(T)$ modulo $F_{2|S|+2} X(L)$, $D(w \otimes \gamma(S)) = \phi(w) \cdot \nabla \gamma(T)$ modulo $F_{2|S|+2} (X(L) \otimes X(L))$.

Next we recall some facts on Morava stabilizer algebra from [12].

Let $(BP_*, BP_+ BP)$ be the Hopf algebroid associated with $BP$-theory at a fixed prime $p$ (See [14], for example). Put $K(n)_* = F_p[v_n, v_n^{-1}]$ and regard this as
a $BP_*$-algebra by $\pi: BP_* \to K(n)_*$, $\pi(v_i) = 0$ if $i \neq n$, $\pi(v_n) = v_n$. We set $\Sigma(n) = K(n)_* \otimes_{BP_*} BP_* \otimes_{BP_*} K(n)_*$, then it is known that $\Sigma(n)$ is isomorphic to $K(n)_*[t_1, t_2, \cdots]/(v_nt^n\gamma - v_n\gamma t^n)$ (deg $t_n = 2(p^i - 1)$) as a $K(n)_*$-algebra. Note that Hopf algebroid $(K(n)_*, \Sigma(n))$ is in fact a Hopf algebra. Regarding $F_p$ as a $K(n)_*$-algebra by $p: K(n)_* \to F_p$, $\psi(\epsilon_i^* \epsilon_j^*) = x_{ij}^*$ then it is known that $\Sigma(n)$ is isomorphic to $K(n)_* \otimes_{F_p} F_p$ which is isomorphic to $F_p[t_1, t_2, \cdots]/(t_n^* - t_i)$ as an $F_p$-algebra.

Theorem 1.5 ([12]). Define integers $d_i$ for $i \in \mathbb{Z}$ recursively by $d_i = 0$ if $i \leq 0$, $d_i = \max\{i, pd_i\}$ if $i > 0$. Then, there is a unique increasing Hopf algebra filtration on $S(n)$ with $t^i_i \in F_{d_i}S(n) - F_{d_i-1}S(n)$.

Instead of considering the above filtration, we consider a new filtration $\{\bar{F}_iS(n)\}$ defined by $\bar{F}_iS(n) = F_{i+1}S(n) - F_iS(n)$ so that the associated $(\mathbb{Z} \times \mathbb{Z}/2(p^i - 1))$-graded Hopf algebra $E^0S(n)$ becomes graded commutative. Let $t_{i,j}$ denote the element of $E^0_{i,j}S(n)_{p^i(p^i - 1)}$ corresponding to $t^i_i$, where $j \in \mathbb{Z}/n$, then $E^0S(n)$ is isomorphic to $F_p[t_{i,j}|i \geq 1, j \in \mathbb{Z}/n]/(t^i_i)$ as an algebra. Consider the $(\mathbb{Z} \times \mathbb{Z}/2(p^i - 1))$-graded dual ($E^0S(n)\ast$) of $E^0S(n)$, and let $x_{i,j} \in (E^0S(n))_{-d_i, 2d_j}$ be the dual of $t_{i,j}$ with respect to the monomial basis. We set $L(n) = P(E^0S(n))\ast$, then $\{x_{i,j}|i \geq 1, j \in \mathbb{Z}/n\}$ spans $L(n)$. We note that this $L(n)$ is different from the one in [12], [13], [14], which coincides with an unrestricted Lie algebra $M(n)$ defined in the next section if $n < p - 1$. Since the $p$-th power map on $E^0S(n)$ is trivial, it follows from [8] that $(E^0S(n)\ast$ is the universal enveloping algebra of the restricted Lie algebra $L(n)$. The bracket and the restriction are given by the following.

Theorem 1.6 ([12]). \([x_{i,j}, x_{k,l}] = \delta_{i+k}^{l+i}x_{i+k,j} - \delta_{i+k}^{l+i}x_{i+l,j} \text{ if } i + k \leq pn/(p - 1), \text{ otherwise the bracket is trivial, where } \delta_i^t = 1 \text{ if } s \equiv t \text{ mod } n, \delta_i^t = 0 \text{ otherwise.} \]

Remark 1.7. The $p$-th power map $\eta$ on $S(n)$ is an automorphism of order $n$. Since $\eta$ preserves the filtration, it induces an automorphism of $E^0S(n)$ which we also denote by $\eta$. We note that $\eta$ maps $t_{i,j}$ to $t_{i,j+1}$ and that $\eta$ induces an automorphism $\eta^*$ of $L(n)$.

2. Spectral Sequences

Let $R$ be a graded commutative ring. For a filtered $R$-modules $M$ and $N$, we filter $M \otimes_R N$ by $F_i(M \otimes_R N) = \text{Im}(\sum_{i+j-s} F_j M \otimes_R F_i N \to M \otimes_R N)$ as usual. Then there is a natural epimorphism $\Sigma_{i+j-s} F_j M \otimes_R F_i N \to E^0(M \otimes_R N)$, where we put $E^0_0M = F_1M/F_{i-1}M$. Note that this epimorphism is an isomorphism if $F_iM$ is flat over $R$ for any $i$.

Let $C$ be a filtered $R$-coalgebra which is flat over $R$, and let $M$ be a filtered
left $E^0C$-comodule. By the above remark, $E^0C$ is an $R$-coalgebra and $E^0M$ is a left $E^0C$-comodule. We give a decreasing filtration on the cobar complex $\Omega^*(C; M)$ ([9], [10], [14]) by $F^s\Omega^*(C; M)=\text{Im}(\sum_{i_0+\ldots+i_{s-1}=s} E^1_i C \otimes R E^0_{i_0} C \otimes \ldots \otimes R E^0_{i_{s-1}} C \otimes R E^0_{i_s} M \to \Omega^*(C; M))$. Then the differential $d$ of $\Omega^*(C; M)$ maps $F_s \Omega^*(C; M)$ into $F_{s+1} \Omega^*(C; M)$ and $E_i'=F^s \Omega^*(C; M)/F_{s+1} \Omega^*(C; M)$ is isomorphic to $\Omega^*(E^0C; E^0M)$.

We rather call $t$ of $E^*$ the filtration degree below.

Let $H_s+t(F_s \Omega^*(C; M)) \to E^*_s+t(C; M)$ and let $i*: D_1^* \to D_1^{*+1}$ and $j*: D_1^{*+1} \to D_1^*$ be the maps induced by inclusion $i: F_{s+1} \Omega^*(C; M) \to F^s \Omega^*(C; M)$ and projection $j: F^s \Omega^*(C; M) \to E^0 \Omega^*(C; M)$ respectively. $\partial: E^*_s+t \to D_1^{*+1}_s$ denotes the boundary homomorphism associated with a short exact sequence of complexes $0 \to F^s \Omega^*(C; M) \to F^s \Omega^*(C; M) \to E^0 \Omega^*(C; M) \to 0$. Consider the spectral sequence associated with an exact couple $\langle D_1^*, i^*, j^*, \partial \rangle$. Then $E^*_s+t=E^s_0-t$ and the $E_2$-term is given by $E^s_1=H^{s+t}(\Omega^*(E^0C; E^0M))=\text{Ext}^s_\mathbb{Z}(R, E^0M)$. Filter $H^*(\Omega^*(C; M))=\text{Ext}^s_\mathbb{Z}(R, M)$ by putting $F^s \Omega^*(C; M) \to \text{Im}(H^s \Omega^*(C; M) \to H^{s+t}(\Omega^*(C; M)))$. We assume that $C=\bigcup_s F_s C$ and $M=\bigcup_s F_s M$ hold and that $F_s C=F_s M=0$ for sufficiently small $s$. Then the above spectral sequence converges to $\text{Ext}^s_\mathbb{Z}(R, M)$.

Applying the above spectral sequence to the case $R=M=F^p$, $C=S(n)$, we have a spectral sequence

$$E^*_s+t=\text{Ext}^s_\mathbb{Z}(F^p, F^p)_s \Rightarrow \text{Ext}^s_\mathbb{Z}(F^p, F^p).$$

Let $A$ be a graded algebra (not necessarily commutative) over a commutative ring. Let $X$ be a filtered $A$-complex with differential $d: X_1 \to X_1$. Put $E^s_0=\text{Hom}_A(X_1, X_1)$. Let $M$ be a graded $A$-module. Consider a complex $\{C_*, d_*, \}$ given by $C_1=\text{Hom}_A(X, X)$. Filter $C_*$ by $F^n C_*=\text{Ker}(\text{Hom}_A(X_1, X) \to \text{Hom}_A(X, X))$. We assume that the inclusions $F_s X/F_{s+1} X \to X/F_{s+1} X$ are split monomorphism of $A$-modules for any $s$. Then we have short exact sequences of complexes $0 \to F^s \Omega^*(C; M) \to F^s \Omega^*(C; M) \to E^0 \Omega^*(C; M) \to 0$, where we set $E^0_0=\text{Hom}_A^{s+t}(F^0 X, X)$.

Let $F^s \Omega^*(C; M)$ be the boundary homomorphism. Putting $D_1^*+t=H^{s+t}(F^s \Omega^*(C; M))$, we consider a spectral sequence associated with an exact couple $\langle D_1^*, i^*, j^*, \partial \rangle$. We define a filtration on $H^*(C_*)$ by $F^s_0=\text{Im}(H^{s+t}(F^s \Omega^*(C; M)) \to H^{s+t}(\Omega^*(C_*)$. Suppose that, for each integer $m$, there exist integers $a(m)$ and $b(m)$ such that $F_s X_m=X_m$ if $s>a(m)$, $F_s X_m=0$ if $s<b(m)$. Then the spectral sequence converges to $H^*(C_*)$.

By 1) and 2) of (1, 4), we can apply the above spectral sequence to the case $A=V(L)$, $X=X(L)$, $M=K$, and obtain a spectral sequence converging to $H^*(\text{Hom}_{V(L)}(X(L), K))=\text{Ext}^s_\mathbb{Z}(L, K)$. We note that the coproduct $D(X)$ makes this spectral sequence multiplicative. Identifying $E^0_0 \otimes W(L) \otimes \Gamma(\pi L^+)_n$ it follows from (1, 4), 3) that $d_\partial$ and $D$ induce $d\otimes 1: E^0_0 \to E^0_0$. and
Therefore the $E_1$-term is isomorphic to 
\[ P^s\pi(L^*)^* \otimes \text{Ext}_{\iota(L^*)}(K, K) \] as an algebra by (1.1), where $(L^*)^*$ denotes the graded $K$-dual of $L^*$. Since $E_1^{s,t}=0$ if $s$ is odd or $s<0$ or $t<0$, $E_1^{s,t}=E_1^{s,t}$ holds and we have the edge homomorphism. Thus we have shown

**Theorem 2.2** ([7]). There is a multiplicative spectral sequence $E_2^{s,t}$
\[ = P^s\pi(L^*)^* \otimes \text{Ext}_{\iota(L^*)}(K, K) \Rightarrow \text{Ext}_{\iota(L^*)}^t(K, K), \] whose edge homomorphism $E_2^{s,t} \otimes \text{Ext}_{\iota(L^*)}^t(K, K) \Rightarrow \text{Ext}_{\iota(L^*)}^{t+s}(K, K)$ is induced by the composite $Y(L^*) \rightarrow W(L) \rightarrow X(L)$.

In particular, in the case $K=F_p$, $L=L(n)$, let $(m)$ and $I(n)$ be subspaces of $L^*(n)$ spanned by \{\[ x_{i,j} | i \leq pn/(p-1), j \in \mathbb{Z}/n \]$ and \{\[ x_{i,j} | i > pn/(p-1), j \in \mathbb{Z}/n \] respectively. It follows from (1.6) that $M(n)$ is a Lie subalgebra of $L^*(n)$ and $I(n)$ is an ideal of $L^*(n)$. Obviously, $I(n)$ is an abelian Lie algebra and $L^*(n)$ is isomorphic to $M(n) \otimes I(n)$ as a Lie algebra. Therefore $U(L^*(n))$ is isomorphic to $U(M(n)) \otimes P(I(n))$. This implies that $\text{Ext}_{\iota(L^*(n))}(F_p, F_p)$ is isomorphic to $\text{Ext}_{\iota(M(n))}(F_p, F_p) \otimes E(\iota, i, j) | i > pn/(p-1), j \in \mathbb{Z}/n$ where $\deg(\iota, i, j) = (1, 2d, 2p(p-1)$. Hence the $E_2$-term of the spectral sequence is isomorphic to $P(\iota, i, j) = 0 \otimes \text{Ext}_{\iota(M(n))}^t(F_p, F_p) \otimes E(\iota, i, j) | i > pn/(p-1), j \in \mathbb{Z}/n$. By (1.2) and (1.6), we have the following fact on the differential $d_2$.

**Lemma 2.3.** ([13]). $d_2(\iota, i, j) = -\iota_{i-1, j-1}$ for $i \leq pn/(p-1)$. Thus if $n < p-1$, $E_3^{s,t}=E_3^{s,t}$ unless $s=0$, and the edge homomorphism maps $\text{Ext}_{\iota(L^*)}^t(F_p, F_p)$ bijectively onto $E_3^{s,t}$. Hence $H^*(E_3^{s,t} \otimes L \rightarrow K$ is the canonical pairing. It is straightforward to verify the following.

**3. Auxiliary Calculation**

Let $L$ be a graded unrestricted Lie algebra over a field $K$ of finite type such that $L^*=0$, and let \{\[ x_{i,j} | i \in \Lambda \}$ be a totally ordered basis of $L$. $L^*$ denotes the graded dual of $L$. Take the graded basis \{\[ x_{i,j} | i \in \Lambda \}$ of \{\[ x_{i,j} | i \in \Lambda \}. Define $\delta: E(sL^*)_t \rightarrow E(sL^*)_t+1$ by $\delta(\langle x_{i,j}^* \rangle) = -\sum_{k \leq s} \langle x_{i,k}^* \rangle \langle x_{i,j}^* \rangle$ satisfying the Leibniz formula, where $\langle \cdot, \cdot \rangle : L^* \otimes L \rightarrow K$ is the canonical pairing. It is straightforward to verify the following.

**Lemma 3.1.** \{\[ E(sL^*), \delta \} is a differential algebra isomorphic to \{\[ \text{Hom}_{\iota(L^*)}(Y(L), K), d^* \}. Hence $H^*(E(sL^*); \delta)$ is isomorphic to $\text{Ext}_{\iota(L^*)}^t(K, K)$.

Now we concentrate on the computation of $\text{Ext}_{\iota(M(3))}^t(F_p, F_p)$ for $p \geq 5$. $M(3)$ is spanned by \{\[ x_{i,j} | i=1, 2, 3, j \in \mathbb{Z}/3 \}$ over $F_p$. Then $E(sM(3)^*)=E(t_{i,j}) | i=1, 2, 3, j \in \mathbb{Z}/3$ where we put $t_{i,j}=\langle x_{i,j}^* \rangle$. It follows from (1.6) and (3.1) that $\delta$ is given by $\delta(t_{i,j})=0$, $\delta(t_{i,j})=-t_{i-1,j}$, $\delta(t_{i,j})=t_{i-1,j-1}-t_{i,j}$ for $j \in \mathbb{Z}/3$. Let $A$ be an ideal of $M(3)$ spanned by \{\[ x_{3,0}, x_{3,1}, x_{3,2} \}$ and we regard $E(s(M(3)/A)^*)$, $\delta$ as a subcomplex of $E(sM(3)^*), \delta$. We remark that $M(3)$
and $M(3)/A$ are denoted by $L(3, 3)$ and $L(3, 2)$ respectively in [13], [14]. We can manage to compute the cohomology of $\{E(s(M(3)/A)^*), \delta\}$ directly by hand and the structure of $\Ext^e_{(M(3)/A)}(F_p, F_p)$ is described below.

For a cocycle $z$ of $E(s(M(3)/A)^*)$, we denote by $[z]$ the cohomology class represented by $z$.

**Lemma 3.2.** 1) $\Ext^e_{(M(3)/A)}(F_p, F_p)$ is generated by the following seventeen elements as an algebra:

$$
\begin{align*}
&h_j = [t_1, j_0, g_j = [t_1, j_2, j_1], f_j = [t_1, j - t_2, j - t_1, j_2],
&d_j = [t_1, j_2, j_1, j_2],
&e_j = [t_1, j_2, j_1 + t_1, j_2 - t_2, j_1],
\end{align*}
$$

for $i, j \in \mathbb{Z}/3$, $i \neq 2$.

2) $\Ext^e_{(M(3)/A)}(F_p, F_p) = 0$ for $s > 6$. A basis of $\Ext^e_{(M(3)/A)}(F_p, F_p)$ ($0 < s \leq 6$) is given as follows:

- $s = 0$: 1.
- $s = 1$: $h_0, h_1, h_2$.
- $s = 2$: $g_0, g_1, g_2, g_0', g_1', g_2', f_0, f_1$.
- $s = 3$: $h_0, h_2, g_1, g_2, h_0 g_1, h_1 g_2, d_0, d_1, d_2, e_0, e_1, e_2$.
- $s = 4$: $g_0, g_1, g_2, g_0', g_1', g_2', g_0 g_1', g_0 g_2', g_1 g_2'$.
- $s = 5$: $g_1 d_0, g_2 d_1, g_0 d_2$.
- $s = 6$: $g_0 g_1 g_2$.

The operator $\eta^*$ of $L(3)$ induces an algebra automorphism of $\Ext^e_{(M(3)/A)}(F_p, F_p)$ of order three, which we denote by $\eta^*$. Obviously, we have $\eta^* h_0 = h_1, \eta^* g_0 = g_1, \eta^* g_0' = g_1', \eta^* f_0 = f_1, \eta^* d_0 = d_1, \eta^* e_0 = e_1$ for $i \in \mathbb{Z}/3$ where we put $f_2 = -f_0$.

**Lemma 3.3.** Relations of $\Ext^e_{(M(3)/A)}(F_p, F_p)$ are given by the following and the relations obtained by applying $\eta^*$ ($i = 1, 2$) to them:

$$
\begin{align*}
h_0 h_j &= 0, f_0 d_1 = 0, d_0 d_1 = d_0 e_1 = e_0 e_1 = 0 \text{ for } i \in \mathbb{Z}/3, h_0 g_0 = 0, h_0 g_0' = -h_1 g_0 \\
h_0 g_0' &= -h_1 g_2, h_0 g_2' = 0, h_0 f_1 = -h_2 g_0, g_0 = g_0^2 = g_0 g_0' = g_0 g_2' = g_0 f_1 = 0 \\
g_0 f_1 &= g_1 g_0, g_0 f_1 = g_0 g_1, g_0 g_1 = -g_0 g_1' - g_0 g_2', f_0 = 2g_0 g_1', f_0 f_1 = 2g_0 g_2', \\
h_0 d_0 &= 0, h_0 d_1 = -g_0 g_1, h_0 d_2 = g_1 g_2', h_0 e_1 = -g_1 g_0', h_0 f_2 = g_0 g_2, g_0 d_0 = g_0 d_1 = g_0 e_1 = 0, \\
g_0 d_0 = g_0 d_1 = g_0 e_1 = 0, &g_0 d_2 = -g_0 d_1, g_0 e_1 = -g_0 d_2, g_0 d_0 = g_0 d_1 = g_0 e_1 = 0, \\
g_0 e_2 = g_0 d_0, f_0 e_1 = g_0 d_1, f_0 e_1 = -2g_0 d_2, g_0 g_1 g_2' = g_0 g_2 g_1.
\end{align*}
$$

4. Main Calculation

Let $D$ be a subcomplex of $E(sM(3)^*)$ generated by $\{t_i, j | i = 1, 2, 3, j \in \mathbb{Z}/3, j \neq 2 \text{ if } i = 3\}$. We put $\xi_3 = t_{3, 0} + t_{3, 1} + t_{3, 2}$.

**Lemma 4.1.** $\delta(\xi_3) = 0$ and $\{E(sM(3)^*), \delta\}$ is isomorphic to $\{D \otimes E(\xi_3), \delta \otimes 1\}$. Therefore $\Ext^e_{(M(3)/A)}(F_p, F_p)$ is isomorphic to $H^*(D) \otimes E(\xi_3)$ as an algebra.

We filter $E(sM(3)^*)$ and $D$ by $F^e E(sM(3)^*)^m = \sum_{i \leq m-s} E(s(M(3)/A)^*)^{m-i} \otimes$
The internal degree of an element of Ext_{U(M(3))}(F_p, F_p) is the degree coming
from the grading of \(S(3)\) which takes value in \(\mathbb{Z}/2(p^3-1)\). We denote by \(i\)-deg \(x\) the internal degree of \(x\). Put \(q=2(p-1)\). Noting that \(p^2q \equiv - (p-1)q \mod 2(p^3-1)\), the internal degrees of the generator are given as follows; \(i\)-deg \(h_j = i\)-deg \(a_j = p^j q\), \(i\)-deg \(g_j = i\)-deg \(b_j = p^j (p+2)q\), \(i\)-deg \(g_j^i = i\)-deg \(b_j^i = p^j (2p+1)q\), \(i\)-deg \(u_j = i\)-deg \(w_j = 2p^j q\) for \(j \in \mathbb{Z}/3\).

These two kinds of degrees play an important role in the next section.

Let \(\eta_*\) denote the operator on \(\operatorname{Ext}_{\mathcal{O}(M)}(F_*, F_*)\) induced by \(\eta^* : L(3) \rightarrow L(3)\). By the definition of the generators in (4, 2), it is easy to verify the following

**Proposition 4.3.** \(h_j = \eta_* h_0, g_j = \eta_* g_0, g_j^i = \eta_* g_0^i\) for \(j = 1, 2, \xi_3 = \eta_* \xi_3, a_1 = \eta_* a_1 + \eta_* h_0^2, a_2 = \eta_* a_1 - \eta_* h_0^2, b_1 = \eta_* b_1, b_2 = \eta_* b_1 - \eta_* g_0^2, b_3 = \eta_* b_1 + \eta_* g_0^2 + \eta_* g_0^3, c = \eta_* c, u_1 = \eta_* u_0, u_2 = \eta_* u_0 - h_0 a_1, h_0 w_0 - u_1 \xi_3, w_2 = \eta_* w_0 + u_2 \xi_3 - a_1^2 \xi_3 / 2\).

Since \(\eta_* \xi_3 = \xi_3, \eta_*\) induces an automorphism \(\eta_*\) of \(H^*(D) = \operatorname{Ext}_{\mathcal{O}(M)}(F_*, F_*)/\langle \xi_3 \rangle\) which maps \(x_j\) to \(x_{j+1}\) for \(x = h, g, g', a, b, b', u, w\) and \(j \in \mathbb{Z}/3\).

**Theorem 4.4.** Relations of \(H^*(D)\) are given by the following and the relations obtained by applying \(\eta_*^j (j = 1, 2)\) to them;

\[
\begin{align*}
&h_j = 0 \text{ for } j \in \mathbb{Z}/3; \\
h_0 g_0 = h_0 g_1 = h_0 g_1^i = h_0 g_2 = 0, h_0 g_3 = - h_1 g_0, h_0 a_2 = h_0 a_0; \\
g_0 g_0^j = g_0 g_1^i = g_0 g_2 = 0 \text{ for } j \in \mathbb{Z}/3, h_0 b_0 = h_0 b_2 = 0, h_0 b_1 = - h_1 b_2, h_0 c = - 3h_2 b_1, a_0 g_0 = 0, a_0 g_1 = - 3h_1 b_1, a_0 g_2 = 2h_2 b_1 - h_2 b_2, a_0 g_3 = h_2 b_0 - h_0 b_0, a_0 g_4 = - 3h_3 b_1; \\
h_0 a_1 a_1 = h_0 a_3 a_3 = h_0 a_2 a_1, h_0 a_3 a_0 = h_0 a_2 a_0 = 0, h_0 a_4 = h_0 a_1 a_1, h_0 a_5 = h_0 a_1 a_2; \\
g_0 b_2 = 0, g_0 b_1 = 3 - h_1 a_1, a_0 a_1 = 3h_1 b_2, a_0 a_2 = - 3h_2 b_2, a_0 a_3 = 3a_1^2 b_2, a_0 a_4 = 3a_1 b_2 = 0, a_0 b_6 = - a_1 b_0, a_0 b_7 = - a_1 b_2, a_0 b_8 = - 3a_2 b_2; \\
h_0 a_1 b_2 = 0, h_0 a_1 b_0 = h_0 a_2 b_0, g_0 a_1 = g_0 a_1 = g_0 a_1 = 0, g_0 a_2 = - h_1 a_1 b_2, g_0 a_3 = - h_2 a_2 b_2, h_0 w_0 = 0, a_0 a_1 a_1 = 3h_2 a_2, a_0 a_1 = - 6h_2 w_1, a_0 a_2 = 6h_2 a_3, a_0 a_3 = - h_1 w_1, a_0 a_4 = 2h_2 w_2, b_1 b_1 = 0 \text{ for } j \in \mathbb{Z}/3, b_1 b_1 = - h_0 w_1, b_0 b_0 = - h_2 w_2, b_2 b_2 = h_0 w_0, b_3 b_3 = h_0 w_0, b_4 b_4 = h_0 w_0, b_5 b_5 = h_0 w_0; \\
a_0 w_0 = b_0 w_1 = 0 \text{ for } j \in \mathbb{Z}/3, g_0 w_0 = g_0 w_1 = g_0 w_2 = 0, g_0 w_3 = - g_1 w_1, b_0 a_1 = - 2g_2 w_1, b_0 a_2 = 2g_2 w_2, b_0 a_3 = 2g_2 w_3, b_0 a_4 = 2g_2 w_4; \\
h_0 g_0 w_0 = h_0 g_0 w_2, c_0 w_0 = 0, b_0 w_1 b_1 w_1 = 0 \text{ for } j \in \mathbb{Z}/3.
\end{align*}
\]

This completes a description of \(\operatorname{Ext}_{\mathcal{O}_3}(F_*, F_*)\) by virtue of (2.3) and (4.1).

**5. The Algebra Structure of the Cohomology of \(S(3)\)**

We consider the spectral sequence (2.1) for \(n=3, p \geq 5\). This spectral sequence is \((\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2(p^3-1))\)-graded and we denote \(E_{r,t,u}\) and \(F_{r,t,u}\) as the subspaces of \(E^r_{t,u}\) and \(F^r_{t,u}\) spanned by elements of internal degree \(u\). From the calculation of the previous section, we have the following table of the \(E_2\)-term, where the numbers in the parentheses in the table indicate the filtration degree.
The following facts are immediately verified from the table.

**Lemma 5.1.** If $E_2^{m-t,t} \neq 0$, $E_2^{m+1-s,t} = 0$ holds for $s < t$. Therefore the spectral sequence of $(2,1)$ collapses, that is, $E_2^{t,t} = E_{\infty}^{t,t}$.

**Lemma 5.2.** If $\sum_{s+t=m} E_2^{s,t} = 0$ for given $m \in \mathbb{Z}$ and $u \in \mathbb{Z}/2(p^3-1)$, then $E_2^{m-t,t,u} = 0$ for all but only one $t$. Hence if $\text{Ext}^m_{S(3)}(F_p, F_p) \neq 0$, there is a unique $t = \tau(m,u)$ such that $F^{m-t,t-1,u} = 0$ and $F^{m-t,t,u} = \text{Ext}^m_{S(3)}(F_p, F_p)$.

Thus there are unique elements $h_j, \zeta_3, g_j, g'_j, a_j, b_j, b'_j, c, u_j, w_j (j \in \mathbb{Z}/3)$ of $\text{Ext}^m_{S(3)}(F_p, F_p)$ corresponding to the elements of the $E_2$-term denoted by the same symbols. Let $\tilde{B}$ be a set of monomials of the above elements which corresponds to the $E(\zeta_3)$-basis of the $E_2$-term given in the previous section. We put $B = \tilde{B} \cup \ldots$

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\[ \{x^t \mid x \in \overline{B}\}, \text{ then } B \text{ is a basis of } \text{Ext}_{S(3)}^*(F_p, F_p). \] For \( x \in \text{Ext}_{S(3)}^*(F_p, F_p) \), we denote by \( \bar{x} \) the element of \( E^*_{p-t-1}(F_p, F_p) \) corresponding to \( x \) where \( t = \tau(m, u) \). For \( x \in B \cap \text{Ext}_{S(3)}^*(F_p, F_p), y \in B \cap \text{Ext}_{S(3)}^*(F_p, F_p) \), suppose that \( \bar{x} \bar{y} = \sum_i x_i z_i \) holds for \( x_i \in F_p, z_i \in B \) in \( E^*_{p-t-1}(F_p, F_p) \) where \( t = \tau(m, u) \), \( t = \tau(l, v) \), in other words, \( xy = \sum_i x_i z_i \) holds modulo \( F^{m+t-1+t', l+t'-1, u+v} \). If \( \bar{x} \bar{y} = 0 \), \( (5.2) \) implies that \( F^{m+t-1-t', l+t'-1, u+v} = 0 \). Hence \( xy \) exactly equals to \( \sum_i x_i z_i \) in this case. In the case \( \bar{x} \bar{y} = 0 \), we can verify \( xy = 0 \) case by case. In fact, it suffices to deal with the case \( F^{m+t-1-t', l+t'-1, u+v} = 0 \). Then we only have to check the cases \( (x, y) = (a_j, b_j), (a_j, b_{j-1}), (a_{j+1}, a_j), (a_j, h_j), (h_j, w_j), (a_j, u_j), (b_j, b'_j), (a_j, w_j), (b_j, w_j), (b'_j, w_j), (e, w_j), (u_j, u_j) \) for \( i, j \in \mathbb{Z}/3 \). In any of these cases, since \( \text{Ext}_{S(3)}^{p+1}(F_p, F_p) = 0 \), the assertion follows. Similarly, for \( x \in B \), \( \eta_8 x = \sum_i \mu_i x_i \) \( (\mu_i \in F_p, y_i \in B) \) implies \( \eta_8 x = \sum_i \mu_i y_i \) where the latter \( \eta_8 \) is the operation of \( \text{Ext}_{S(3)}^*(F_p, F_p) \) induced by the \( p \)-th power map of \( S(3) \). Thus we have shown

**Theorem 5.3.** \( \text{Ext}_{S(3)}^*(F_p, F_p) \) is isomorphic to \( \text{Ext}_{U(M(3))}^*(F_p, F_p) \) as an algebra over \( F_p \) and the isomorphism commutes with the operations induced by the \( p \)-th power map of \( S(3) \).


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