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STABILITY AND BIFURCATION OF CIRCULAR KIRCHHOFF ELASTIC RODS

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1. Introduction

Imagine a thin elastic rod like a piano wire. We assume its unstressed state is straight and it can be bent and twisted but inextensible. Certain particular case of the Kirchhoff elastic rod is one of the mathematical models of such an elastic rod. We consider the equilibrium states of the elastic rod when it is bent and twisted and both ends are welded together to form a smooth loop. Then, the simplest equilibrium state is a circular state with uniformly distributed torsion. The purpose of this paper is to investigate the stability and the bifurcation of such circular Kirchhoff elastic rods in equilibrium. The theory of elastic rods has been studied since the age of Leonhard Euler and Daniel Bernoulli. They initiated the theory of elastica. An elastica is a mathematical model of the equilibrium states of an elastic rod when it is assumed to be subjected to bending only. It is characterized by the critical curve of the bending energy or the total squared curvature functional. One of the mathematical models of an elastic rod with bending and twisting was considered by G. Kirchhoff. In this paper, we are mainly interested in closed objects. In the case of closed elasticae, Langer and Singer completely classified the closed elasticae and determined their knot types ([6]). Now, the integrability of the Euler-Lagrange equation of the stored energy of bending and twisting was essentially known by Kirchhoff ([4]). In recent years, Y. Shi and J. Hearst ([9]) have given the explicit expressions of the solutions of the Euler-Lagrange equation in cylindrical coordinates and investigated certain closed solutions to study supercoiled DNA.

We use the model known as the uniform and symmetric case of Kirchhoff elastic rods, which is characterized by the following energy functional. We consider the totality of curves with unit normal$\{\gamma, M\}$, that is, $\gamma = \gamma(s) : [s_1, s_2] \to \mathbf{R}^3$, is a unit-speed curve, and $M(s)$ is a unit normal vector field along $\gamma(s)$, and we define the *torsional elastic energy* $\mathfrak{T}$ on it by

$$\mathfrak{T}(\{\gamma, M\}) = \int_{s_1}^{s_2} k^2 ds + \varepsilon \int_{s_1}^{s_2} |\nabla_s^{\perp} M|^2 ds.$$

Here, k is the curvature of γ and ε is a positive constantdetermined by the material of the elastic rod. In Section 2, we shall calculate the first variation formula for the torsional elastic energy $\mathfrak{T}$ and derive the Euler-Lagrange equation. In this paper, we shall call a solution for the Euler-Lagrange equation a *torsional elastica*. (In [8], Langer and Singer used the terms *Kirchhoff elastic rod in equilibrium* or simply *elastic rod*.) If $\{\gamma, M\}$ is a torsional elastica, then the twist is uniformly distributed over the curve in the following sense. That is, M is expressed as $M(s) = \mathcal{R}(as+b)\,W(s)$ with some real constants a, b. Here, W is a vector field along γ which is parallel with respect to the normal connection $\nabla^{\perp}$ along γ, and $\mathcal{R}(\varphi)$ is the rotation on the normal vector space by angle φ. We shall call the constant a, the Here, N is the unit principal normal of the circle and a is an arbitrary real number.

In Section 3, we calculate the second variation formula for $\mathfrak{T}$, and determine the stability and instability of the above circle $\{\gamma, M\}$ and estimate its Morse index. We state our first main theorem.

Theorem 1.1 (cf. Theorem 3.12). *For a triple (n, a, r) where n is a positive integer, and a and r are real numbers, let $\{\gamma, M\}$ be a with an n-fold circle γ of radius r and the a. Let $A(n, a, r)$ denote the number of integers m satisfying $|m| < n\sqrt{1+\varepsilon^2 a^2 r^2}$, $m \neq 0$ and $m \neq \pm n$. We denote by* $\mathrm{Ind}(\{\gamma, M\})$ *the Morse index of $\{\gamma, M\}$. Then, the following holds.*

$$A(n, a, r) \leqslant \mathrm{Ind}(\{\gamma, M\}) \leqslant 18A(n, a, r).$$

Therefore, if either $n \geqslant 2$ or $n = 1$ and $\varepsilon^2 a^2 r^2 > 3$, then $\{\gamma, M\}$ is unstable. Also, if $n = 1$ and $\varepsilon^2 a^2 r^2 \leqslant 3$, then $\{\gamma, M\}$ is weakly stable. Moreover, if $\varepsilon^2 a^2 r^2 \neq 3$, then $\{\gamma, M\}$ is stable.

Therefore, for any n and r, the Morse index can be made arbitrarily large by increasing the absolute value of the torsional parameter$a$. Also, by virtue of our theorem, we can explain the following experimentally observable fact. For $n \geqslant 2$, when a piece of a straight piano wire is formed into an n-fold circle, it cannot preserve its shape. But, when it is formed into a 1-fold circle without torsion, it preserves its shape. And, when it is sufficiently twisted, it will change its shape. In the last section of this paper, we consider the problem related to the deformed shape.

In Section 4, we derive the explicit expression of the curvature and torsion of torsional elasticain terms of Jacobi elliptic functions.

In Section 5, we give the explicit expression of torsional elasticain terms of cylindrical coordinates in the same way as Langer and Singer ([6],[8]).

In Section 6,we first show that if $\{\gamma, M\}$ is a closed torsional elasticaand γ does not pass through the axis of the cylindrical coordinates, then γ lies on a torus of revolution and forms a torus knot. Next, we consider a smooth deformation of the circle such that all the curves with unit normalare closed torsional elasticae, and show our second main theorem. That is, we classify such local deformations under certain conditions. In particular, for each integer $n \geqslant 1$, there exist countable families of the deformations bifurcating from the n-fold circle. Each of these deformations has certain symmetry. We state our second main theorem. Let r_0 be a positive real number, and n, a positive integer. We consider a C^∞ one-parameter family $\{\gamma, M\}_\lambda = \{\gamma_\lambda, M_\lambda\}$ $(\lambda_0 > 0, |\lambda| < \lambda_0)$ of closed torsional elasticaesatisfying the following three conditions:

(i) γ_0 is a circle of radius r_0, and the torsional parameterof $\{\gamma, M\}_0$ is not zero.

(ii) For $\lambda \neq 0$, γ_λ is not a circle.

(iii) For all λ, γ_λ have a period $2n\pi r_0$.

Here, we note that the angle between $M_\lambda(s)$ and $M_\lambda(s + 2n\pi r_0)$ may depend on λ.

Theorem 1.2 (cf. Theorem 6.4).
(1) *Let m be an integer greater than n. We can construct a one-parameter family of closed torsional elasticae$\{\gamma, M\}_\lambda^{m,n} = \{\gamma_\lambda^{m,n}, M_\lambda^{m,n}\}$, which is real analytic in $\lambda \in I_{m,n}$, where $I_{m,n}$ is a neighborhood of* 0, *satisfying the above conditions* (i), (ii), *and* (iii) *and the following property*: *For all $\lambda \in I_{m,n}$, $\gamma_\lambda^{m,n}$ is $G_{m,n}$-symmetric, where $G_{m,n}$ is the group generated by the rotation about the z-axis by angle $2n\pi/m$. Here z-axis is the straight line which passes the center of the circle γ_0 and is perpendicular to the plane including γ_0. Furthermore, the following holds. Let d denote the greatest common divisor of m and n, and $\tilde{m} = m/d, \tilde{n} = n/d$. If the relatively prime pairs $\tilde{m}, \tilde{n}$ are distinct, then $\{\gamma, M\}_\lambda^{m,n}$ are geometrically distinct. Also, the knot type of $\gamma_\lambda^{m,n}|_{[0,2\tilde{n}\pi r_0]}$ is the $(\tilde{m}, \tilde{n})$-torus knot for each λ $(\neq 0)$.*
(2) *Let $\{\hat{\gamma}, \hat{M}\}_\lambda$ $(\lambda_0 > 0, |\lambda| < \lambda_0)$ be a C^∞ one-parameter family of closed torsional elasticasatisfying the above* (i), (ii), *and* (iii). *Then, there exists an integer m $(> n)$ satisfying the following*: *By changing the parameter λ, $\{\hat{\gamma}, \hat{M}\}_\lambda$ is isometric to $\{\gamma, M\}_\lambda^{m,n}$ given above when $|\lambda|$ is sufficiently small.*

The shape of $\gamma_\lambda^{2,1}$ is similar to the shape of the curve which is experimentally observed when a piano wire formed into a 1-fold circle is twisted. We also get that there exist infinite relatively prime pairs m, n such that the knot type of $\gamma_\lambda^{m,n}|_{[0,2n\pi r_0]}$ cannot be represented by a closed elastica (cf. Main Theorem of [6]).

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2. Torsional elastic energy and its critical point

In this section, we shall derive the Euler-Lagrange equation for the torsional elastic energy.

2.1. The variational problem

We now introduce a mathematical formulation of the torsion of an elastic rod, and define the stored energy when the elastic rod is bent and twisted. And, we consider a variational problem of the energy under certain boundary condition. In the last of this subsection, we modify the variational problem to facilitate to derive the Euler-Lagrange equation. Unless otherwise specified, all curves, vector fields, etc., will be assumed to be of class C^∞. Let $\mathbf{R}^3$ denote the 3-dimensional Euclidean space, $\langle *, * \rangle$ the Euclidean metric, and $|*|$ the norm. We fix an orientation on $\mathbf{R}^3$, and let $\times$ denote the cross product on $\mathbf{R}^3$.

We consider a regular curve $\gamma = \gamma(t) : [t_1, t_2] \to \mathbf{R}^3$. Let $V = V(t)$ denote the tangent vector to γ, and v the speed $v(t) = |V(t)| = \langle V(t), V(t) \rangle^{1/2}$, and T the unit tangent $T(t) = (1/v(t))V(t)$. Let Tf denote $(1/v)(df/dt)$, where $f(t)$ is a function defined on $[t_1, t_2]$. We denote the length of γ by $\mathfrak{L}(\gamma) = \int_{t_1}^{t_2} v dt$. The curvature $k(t)$ of γ is given by $k = |\nabla_T T|$, where $\nabla_T = \nabla^{\gamma^{-1}T\mathbf{R}^3}_{(1/v)(\partial/\partial t)} = (1/v)(\partial/\partial t)$. We shall denote the total squared curvature of γ by

$$\mathfrak{F}(\gamma) = \int_{t_1}^{t_2} k^2 v dt.$$

We also call $\mathfrak{F}(\gamma)$ the *elastic energy* or *bending energy* of γ. It is proportional to the stored energy when the elastic rod is bent. It is clear that $\mathfrak{F}(\gamma) = 0$ if and only if γ is a straight line.

Let $M = M(t)$ be a unit normal along γ. We consider the pair of γ and M. We shall use the term *curve with unit normal* to refer to such a pair $\{\gamma, M\}$. Set $L = T \times M$. Then, (T, M, L) is an orthonormal frame along γ.

We shall denote by $\mathcal{R}_t(\varphi)$ the rotation on the normal vector space $T^\perp_{\gamma(t)}\mathbf{R}^3$ by angle φ. Its direction is determined by the requirement that $\mathcal{R}_t(\pi/2)(M(t)) = L(t)$. For a function $\psi(t)$ and a normal vector field $X(t)$, a normal vector field $\mathcal{R}(\psi) X$ is given by $(\mathcal{R}(\psi) X)(t) = \mathcal{R}_t(\psi(t))X(t)$.

DEFINITION 2.1. Let $\{\gamma, M\}$ be a curve with unit normal.The *torsional function* of $\{\gamma, M\}$ is defined by

$$h = \langle \nabla^\perp_T M, L \rangle,$$

where $\nabla^\perp$ is the normal connection on the normal bundle $T^\perp\mathbf{R}^3$ along γ.

Proposition 2.2. *Let W denote a unit normal parallel with respect to the normal connection. Suppose that M is expressed as $M = \mathcal{R}(\psi) W$, where $\psi = \psi(t)$ is a function on $[t_1, t_2]$. Then, $h = T\psi$.*

Proof. Set $Z = \mathcal{R}(\pi/2)\,W$. By using the Leibniz rule, we see that Z is also a unit normal parallel with respect to the normal connection. Thus,

$$\begin{aligned} h &= \langle \nabla_T^{\perp}(\mathcal{R}(\psi)\,W), \mathcal{R}(\psi)\,Z \rangle \\ &= \langle \nabla_T^{\perp}((\cos\psi)W + (\sin\psi)Z), -(\sin\psi)W + (\cos\psi)Z \rangle = T\psi. \end{aligned}$$

□

Proposition 2.2 shows that $h(s)$ represents the quantity of torsion of $\{\gamma, M\}$ at $\gamma(s)$. It is clear that $h \equiv 0$ if and only if M is a unit normal parallel with respect to the normal connection.

For any constant φ, h is invariant under the rotation $\mathcal{R}(\varphi)$ of M. When γ and h are given, M is uniquely determined up to the rotation by a constant angle.

DEFINITION 2.3. Let $\{\gamma, M\}$ be a curve with unit normaldefined on $[t_1, t_2]$.

(1) The *torsional energy* of $\{\gamma, M\}$ is defined by

$$\int_{t_1}^{t_2} |\nabla_T^{\perp} M|^2 v dt = \int_{t_1}^{t_2} h^2 v dt.$$

(2) Let ε be a positive constant. The *torsional elastic energy* of $\{\gamma, M\}$ with coefficient ε is defined by

$$\mathfrak{T}(\{\gamma, M\}) = \mathfrak{F}(\gamma) + \varepsilon \int_{t_1}^{t_2} |\nabla_T^{\perp} M|^2 v dt.$$

REMARK 2.4. The positive constant ε would be determined by the material of the elastic rod. In this paper, we always treat ε as a fixed constant. We simply call $\mathfrak{T}(\{\gamma, M\})$ the torsional elastic energy of $\{\gamma, M\}$.

We now consider a variational problem on the space consisting of curves with unit normalwhich satisfy certain boundary condition with respect to γ and M. Let $P_1,\ P_2 \in \mathbf{R}^3$. Let X_1, Y_1 be two orthogonal unit tangent vectors at P_1. And, let X_2, Y_2 be two orthogonal unit tangent vectors at P_2. Let $\ell \geqslant |P_1 - P_2|$. We denote by $\mathcal{B}(0, P_1, X_1, Y_1 \,; \ell, P_2, X_2, Y_2)$ or simply $\mathcal{B}$ the space of curves with unit normal $\{\gamma, M\}$ satisfying the following conditions:

$$\begin{aligned} &\gamma(0) = P_1,\ T(0) = X_1,\ M(0) = Y_1, \\ &\gamma(\ell) = P_2,\ T(\ell) = X_2,\ M(\ell) = Y_2. \end{aligned}$$

We denote by $\mathcal{UB}(0,P_1,X_1,Y_1;\ell,P_2,X_2,Y_2)$ or simply $\mathcal{UB}$ the space of all elements $\{\gamma,M\}$ of $\mathcal{B}(0,P_1,X_1,Y_1;\ell,P_2,X_2,Y_2)$ satisfying $v(t)\equiv 1$, and by $\mathcal{B}_\ell(0,P_1,X_1,Y_1;\ell,P_2,X_2,Y_2)$ or simply $\mathcal{B}_\ell$ the space of all elements $\{\gamma,M\}$ of $\mathcal{B}(0,P_1,X_1,Y_1;\ell,P_2;X_2,Y_2)$ satisfying $\mathfrak{L}(\gamma)=\ell$. We note that $\mathcal{UB}\subset\mathcal{B}_\ell\subset\mathcal{B}$.

Now we consider a variation of a curve with unit normal. Let $\{\gamma,M\}$ be a curve with unit normaldefined on $[t_1,t_2]$, and $\lambda_0>0$ a number. We consider a map $\tilde{\gamma}:(-\lambda_0,\lambda_0)\times[t_1,t_2]\to\mathbf{R}^3$. And let $\tilde{M}$ be a vector field along $\tilde{\gamma}$. We shall denote by γ_λ the curve defined by $\gamma_\lambda(t)=\tilde{\gamma}(\lambda,t)$, and by M_λ the vector field along γ_λ defined by $M_\lambda(t)=\tilde{M}(\lambda,t)$. We denote the pair $\{\gamma_\lambda,M_\lambda\}$ by $\{\gamma,M\}_\lambda$. The pair $\{\tilde{\gamma},\tilde{M}\}$ or the family $\{\gamma,M\}_\lambda$ $(|\lambda|<\lambda_0)$ is called a variation of $\{\gamma,M\}$ if $\{\gamma,M\}_\lambda$ is a curve with unit normalfor any $\lambda\in(-\lambda_0,\lambda_0)$ and $\{\gamma,M\}_0=\{\gamma,M\}$.

We shall denote $\mathcal{B},\mathcal{B}_\ell$, or $\mathcal{UB}$ by $\mathcal{D}$ hereafter.

Let $\{\gamma,M\}\in\mathcal{D}$. A variation $\{\gamma,M\}_\lambda$ $(|\lambda|<\lambda_0)$ is called a $\mathcal{D}$-variation if $\{\gamma,M\}_\lambda\in\mathcal{D}$ for any $\lambda\in(-\lambda_0,\lambda_0)$. And, $\{\gamma,M\}$ is called a $\mathcal{D}$-critical point if

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0}\mathfrak{T}(\{\gamma,M\}_\lambda)=0$$

for any $\mathcal{D}$-variation $\{\gamma,M\}_\lambda$ of $\{\gamma,M\}$.

Here, we consider the following variational problem: determine the $\mathcal{UB}$-critical points of the torsional elastic energy $\mathfrak{T}$.

Here, our configuration space is $\mathcal{UB}$, because the elastic rod is assumed to be inextensible. Now we shall not directly treat the above boundary value problem, but we shall derive the Euler-Lagrange equation in the next subsection and solve it in Section 4 and Section 5. To facilitate to derive the Euler-Lagrange equation we transform the above variational problem on $\mathcal{UB}$ to the problem on $\mathcal{B}$. Since the functional $\mathfrak{T}$ is invariant under reparametrization of t, we get the following lemma.

Lemma 2.5. *Let $\{\gamma,M\}\in\mathcal{UB}(0,P_1,X_1,Y_1;\ell,P_2,X_2,Y_2)$. Then, $\{\gamma,M\}$ is a $\mathcal{UB}$-critical point of $\mathfrak{T}$ if and only if $\{\gamma,M\}$ is a $\mathcal{B}_\ell$-critical point of $\mathfrak{T}$.*

Now we define the functional $\mathfrak{T}^\mu$, in order to apply the Lagrange multiplier method, by

$$\mathfrak{T}^\mu(\{\gamma,M\})=\mathfrak{T}(\{\gamma,M\})+\mu\mathfrak{L}(\gamma),$$

for a constant $\mu\in\mathbf{R}$.

Lemma 2.6 (Lagrange multiplier method). *Let $\{\gamma,M\}\in\mathcal{B}_\ell$. Suppose that γ is not a straight line. Then, $\{\gamma,M\}$ is a $\mathcal{B}_\ell$-critical point of $\mathfrak{T}$ if and only if there exists a constant $\mu\in\mathbf{R}$ such that $\{\gamma,M\}$ is a $\mathcal{B}$-critical point of $\mathfrak{T}^\mu$.*

By Lemma 2.5 and Lemma 2.6, we get the following.

Lemma 2.7. *Let* $\{\gamma, M\} \in \mathcal{UB}$. *Suppose that* γ *is not a straight line. Then,* $\{\gamma, M\}$ *is a* $\mathcal{UB}$*-critical point of* $\mathfrak{T}$ *if and only if there exists a constant* $\mu \in \mathbf{R}$ *such that* $\{\gamma, M\}$ *is a* $\mathcal{B}$*-critical point of* $\mathfrak{T}^{\mu}$.

2.2. The first variation formula for $\mathfrak{T}^{\mu}$

In this subsection, we introduce the "tangent space" of $\mathcal{B}$ and calculate the first variation formula for $\mathfrak{T}^{\mu}$ on $\mathcal{B}$.

Let $\{\gamma, M\} \in \mathcal{B}$ be a curve with unit normal, and $\{\gamma, M\}_{\lambda}$ $(|\lambda| < \lambda_0)$ a variation of $\{\gamma, M\}$.We denote the tangent vector, the speed, the unit tangent, and the curvature of γ_{λ} by $V_{\lambda}(t) = \tilde{V}(\lambda, t)$, $v_{\lambda}(t) = \tilde{v}(\lambda, t)$, $T_{\lambda}(t) = \tilde{T}(\lambda, t)$, and $k_{\lambda}(t) = \tilde{k}(\lambda, t)$, respectively.Then, $\tilde{k}(\lambda, t) = |(\nabla_{\tilde{T}}\tilde{T})(\lambda, t)|$, where $\nabla_{\tilde{T}} = \nabla^{\tilde{\gamma}^{-1}T\mathbf{R}^3}_{(1/\tilde{v})(\partial/\partial t)} = (1/\tilde{v})(\partial/\partial t)$. Set $\tilde{L}(\lambda, t) = L_{\lambda}(t) = T_{\lambda}(t) \times M_{\lambda}(t)$. We shall denote by $\mathcal{R}_{\lambda,t}(\varphi)$ the rotation on the normal vector space $T^{\perp}_{\gamma_{\lambda}(t)}\mathbf{R}^3$ by angle φ. Also, $\tilde{\mathcal{R}}(\varphi)X$ is defined by $(\tilde{\mathcal{R}}(\varphi)X)(\lambda, t) = \mathcal{R}_{\lambda,t}(\varphi)(X)$, where X is a vector field along $\tilde{\gamma}$ such that $X(\lambda, t)$ is a normal vector field along γ_{λ} for all λ. We write $\nabla^{\tilde{\gamma}^{-1}T\mathbf{R}^3}_{\partial/\partial\lambda} = \partial/\partial\lambda$ as $\nabla_{\tilde{\Lambda}}$, and $\nabla_{\tilde{\Lambda}}|_{\lambda=0}$ as ∇_{Λ}. We define $\tilde{\Lambda}, \tilde{U}, \tilde{f}$ as follows.

$$\tilde{\Lambda}(\lambda, t) = \frac{\partial\tilde{\gamma}}{\partial\lambda}(\lambda, t),$$
$$\tilde{U}(\lambda, t) = (\nabla_{\tilde{\Lambda}}\tilde{M})(\lambda, t),$$
$$\tilde{f}(\lambda, t) = \langle\tilde{U}(\lambda, t), \tilde{L}(\lambda, t)\rangle.$$

We shall denote the restriction of $\tilde{\Lambda}, \tilde{U}, \tilde{f}$ to $\lambda = 0$ by $\Lambda(t), U(t), f(t)$ respectively. We call the pair (Λ, U) the variation vector field of the variation $\{\gamma, M\}_{\lambda}$. By easy computation, we have the following.

$$\partial\tilde{v}/\partial\lambda = \langle\nabla_{\tilde{T}}\tilde{\Lambda}, \tilde{T}\rangle\tilde{v}, \tag{2.1}$$
$$\nabla_{\tilde{\Lambda}}\tilde{T} = \nabla_{\tilde{T}}\tilde{\Lambda} - \langle\nabla_{\tilde{T}}\tilde{\Lambda}, \tilde{T}\rangle\tilde{T}. \tag{2.2}$$

By using the Leibniz rule and (2.2), we obtain

$$\langle U, T\rangle = -\langle M, \nabla_{\Lambda}\tilde{T}\rangle = -\langle M, \nabla_T\Lambda\rangle,$$

and $\langle U, M\rangle = 0$. Hence, the components of U with respect to T and M are determined by Λ. Therefore, there is a one-to-one correspondence between (Λ, U) and (Λ, f). We also call the pair (Λ, f) the variation vector field of the variation $\{\gamma, M\}_{\lambda}$.

We denote by $T_{\{\gamma,M\}}\mathcal{B}$ the space of the pairs (Λ, f) of vector fields along γ and functions satisfying the following boundary condition.

$$\Lambda(0) = \Lambda(\ell) = 0,$$
$$(\nabla_T\Lambda)^{\perp}(0) = (\nabla_T\Lambda)^{\perp}(\ell) = 0,$$
$$f(0) = f(\ell) = 0,$$

where $(\nabla_T\Lambda)^{\perp}$ is the normal component of $\nabla_T\Lambda$, i.e. $(\nabla_T\Lambda)^{\perp} = \nabla_T\Lambda - \langle\nabla_T\Lambda, T\rangle T$. Then, the following lemma holds, so that the vector space $T_{\{\gamma,M\}}\mathcal{B}$ can be identified with the tangent space at $\{\gamma, M\}$ to $\mathcal{B}$.

Lemma 2.8. *If $\{\gamma, M\}_\lambda$ is a $\mathcal{B}$-variation of $\{\gamma, M\}$, then the variation vector field (Λ, f) belongs to $T_{\{\gamma,M\}}\mathcal{B}$. Conversely, for an arbitrary $(\Lambda, f) \in T_{\{\gamma,M\}}\mathcal{B}$ there exists a $\mathcal{B}$-variation $\{\gamma, M\}_\lambda$ whose variation vector field is (Λ, f).*

Proof. Suppose $\{\gamma, M\}_\lambda$ is a $\mathcal{B}$-variation of $\{\gamma, M\}$. Then $\gamma_\lambda(0), \gamma_\lambda(\ell)$ and $M_\lambda(0), M_\lambda(\ell)$ are independent of λ. Hence, $\Lambda(0) = \Lambda(\ell) = 0$, $f(0) = f(\ell) = 0$. And $T_\lambda(0), T_\lambda(\ell)$ are also independent of λ, so that $(\nabla_\Lambda\tilde{T})(0) = (\nabla_\Lambda\tilde{T})(\ell) = 0$. Thus, from (2.2) we get $(\nabla_T\Lambda)^{\perp}(0) = (\nabla_T\Lambda)^{\perp}(\ell) = 0$.

We show the converse. Suppose that $(\Lambda, f) \in T_{\{\gamma,M\}}\mathcal{B}$. We define $\tilde{\gamma}$ by

$$\tilde{\gamma}(\lambda, t) = \gamma_\lambda(t) := \gamma(t) + \lambda\Lambda(t). \tag{2.3}$$

Then we have

$$\frac{\partial\tilde{\gamma}}{\partial\lambda}(0, t) = \Lambda(t).$$

We note that if we take a sufficiently small $\lambda_0 > 0$, $\gamma_\lambda(t)$ is a regular curve for each $\lambda \in (-\lambda_0, \lambda_0)$. Now let us check the boundary condition. By $\Lambda(0) = 0$, we see $\gamma_\lambda(0) = \gamma(0)$ for any λ. Differentiating (2.3) by t, we have

$$V_\lambda(t) = v(t)(T(t) + \lambda(\nabla_T\Lambda)(t)). \tag{2.4}$$

Thus (2.4) and $(\nabla_T\Lambda)^{\perp}(0) = 0$ imply that $V_\lambda(0)$ is parallel to $T(0)$ for any λ. This shows that $T_\lambda(0) = T(0)$ for any λ. We can show that $\gamma_\lambda(\ell)$ and $T_\lambda(\ell)$ are independent of λ in the same way.

Now we shall define $\tilde{M}$. We consider the following first order linear ordinary differential equation with independent variable λ and parameter t:

$$\nabla_{\tilde{\Lambda}}\tilde{M} = -\langle\nabla_{\tilde{\Lambda}}\tilde{T}, \tilde{M}\rangle\tilde{T} + f\tilde{T}\times\tilde{M}. \tag{2.5}$$

We consider the solution $\tilde{M}(\lambda, t)$ of (2.5) with initial data $\tilde{M}(0, t) = M(t)$. We note that $\tilde{M}(\lambda, t)$ is of class C^∞ in (λ, t) on $(-\lambda_0, \lambda_0) \times [0, \ell]$. We show that $M_\lambda(t) := \tilde{M}(\lambda, t)$ is a unit normal along γ_λ. By (2.5) and the Leibniz rule, we see that $\frac{\partial}{\partial \lambda}\langle \tilde{M}, \tilde{T}\rangle = 0$. Thus, $\langle \tilde{M}, \tilde{T}\rangle|_{\lambda=0} = 0$ implies that $\langle \tilde{M}, \tilde{T}\rangle = 0$ for all λ, t. Also, we have $\frac{\partial}{\partial \lambda}\langle \tilde{M}, \tilde{M}\rangle = 0$. This implies $\langle \tilde{M}, \tilde{M}\rangle = 1$ for all λ, t.

By (2.5) it is clear that $f = \langle \nabla_\Lambda \tilde{M}, L\rangle$. Now let us check the boundary condition with respect to M_λ. Since $(\nabla_{\tilde{\Lambda}} \tilde{T})(\lambda, 0) = 0$ and $f(0) = 0$, we have $(\nabla_{\tilde{\Lambda}} \tilde{M})(\lambda, 0) = 0$. So $M_\lambda(0)$ is independent of λ. We can also check $M_\lambda(\ell)$ is independent of λ in the same way. □

Now we calculate the first variation formula for the functional $\mathfrak{T}^\mu$.

Proposition 2.9. *Let $\{\gamma, M\} \in \mathcal{B}$, and let $\{\gamma, M\}_\lambda$ be a $\mathcal{B}$-variation of $\{\gamma, M\}$. Let (Λ, f) denote the variation vector field of $\{\gamma, M\}_\lambda$. Then,*

$$\begin{aligned} &\left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathfrak{T}^\mu(\{\gamma, M\}_\lambda) \\ &= \int_0^\ell \langle \nabla_T [2(\nabla_T)^2 T + (3k^2 - \mu + \varepsilon h^2)T - 2\varepsilon h \mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)], \Lambda\rangle v dt \\ &\quad - 2\varepsilon \int_0^\ell (Th) f v dt. \end{aligned} \tag{2.6}$$

Proof. We shall write $\tilde{V}, \tilde{v}, \tilde{T}, \tilde{k}$, etc. as V, v, T, k, etc., unless confusions could occur. The following lemma collects some facts which facilitate the derivations of the variational formulas.

Lemma 2.10 (cf. Lemma 1.1 of [5]).
(1) $\partial v/\partial \lambda = -gv$, *where* $g = -\langle \nabla_T \Lambda, T\rangle$.
(2) $\nabla_\Lambda T = \nabla_T \Lambda + gT$.
(3) *If X is a vector field along $\tilde{\gamma}$, then*

$$\nabla_\Lambda \nabla_T X = \nabla_T \nabla_\Lambda X + g \nabla_T X.$$

In particular, $\nabla_\Lambda \nabla_T T = (\nabla_T)^2 \Lambda + 2g \nabla_T T + (Tg)T$.
(4) $\partial k^2/\partial \lambda = 2\langle (\nabla_T)^2 \Lambda, \nabla_T T\rangle + 4gk^2$.
(5) $\partial h/\partial \lambda = \langle \nabla_T \Lambda, \mathcal{R}(\pi/2)(\nabla_T T)\rangle + gh + Tf$.

Proof. (3) follows easily from (2), and (4) from (3), so we shall only show (5). By (3), we have

$$\begin{aligned} \frac{\partial h}{\partial \lambda} &= \frac{\partial}{\partial \lambda}\langle \nabla_T M, L\rangle \\ &= \langle \nabla_T \nabla_\Lambda M, L\rangle + \langle g \nabla_T M, L\rangle + \langle \nabla_T M, \nabla_\Lambda L\rangle. \end{aligned} \tag{2.7}$$

Here we see $\langle \nabla_\Lambda M, T\rangle = -\langle \nabla_\Lambda T, M\rangle = -\langle \nabla_T \Lambda, M\rangle$ by (2), so that we get

$$\nabla_\Lambda M = -\langle \nabla_T \Lambda, M\rangle T + fL. \tag{2.8}$$

In the same way, we have

$$\nabla_\Lambda L = -\langle \nabla_T \Lambda, L\rangle T - fM. \tag{2.9}$$

Substituting (2.8) and (2.9) to (2.7), we obtain

$$\begin{aligned}\frac{\partial h}{\partial \lambda} &= \Big\langle \nabla_T \Lambda, \langle \nabla_T T, M\rangle L - \langle \nabla_T T, L\rangle M\Big\rangle + gh + Tf \\ &= \Big\langle \nabla_T \Lambda, \mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)\Big\rangle + gh + Tf.\end{aligned}$$

□

Lemma 2.10 and integration by parts yield the following.

$$\begin{aligned}&\frac{d}{d\lambda}\mathfrak{T}^\mu(\{\gamma, M\}_\lambda) \\ &= \int_0^\ell \langle \nabla_T [2(\nabla_T)^2 T + (3k^2 - \mu + \varepsilon h^2)T - 2\varepsilon h \mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)], \Lambda\rangle v dt \\ &\quad - 2\varepsilon \int_0^\ell (Th) f v dt \\ &\quad + \Bigg[2\langle \nabla_T \Lambda, \nabla_T T\rangle + 2\varepsilon h f \\ &\qquad - \Big\langle \Lambda, 2(\nabla_T)^2 T + (3k^2 - \mu + \varepsilon h^2)T - 2\varepsilon h \mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)\Big\rangle \Bigg]_{t=0}^{t=\ell}.\end{aligned} \tag{2.10}$$

Since $\{\gamma, M\}_\lambda$ is a $\mathcal{B}$-variation, $(\nabla_T \Lambda)^\perp = 0$, $\Lambda = 0$ and $f = 0$ if $t = 0$ or $t = \ell$. Thus the third term of the right hand side of (2.10) vanishes. We have Proposition 2.9.

□

2.3. Definition of a torsional elastica

In this subsection, we define a torsional elasticaas a solution of the Euler-Lagrange equation for $\mathfrak{T}^\mu$. And we also define a closed torsional elastica, which is the main object in this paper. First, we describe the Euler-Lagrange equation.

Proposition 2.11. *Let $\{\gamma, M\} \in \mathcal{UB}$. Suppose that γ is not a straight line. Then $\{\gamma, M\}$ is a $\mathcal{UB}$-critical point of $\mathfrak{T}$ if and only if there exist two constants μ and a such that the following* (1) *and* (2) *hold.*

(1) $h(t) = a$.
(2) $\nabla_T[\, 2(\nabla_T)^2 T + (3k^2 - \mu + \varepsilon a^2)T - 2\varepsilon a \mathcal{R}\,(\pi/2)\,(\nabla_T T)\,] = 0$.

Also, the constants a and μ are uniquely determined.

Proof. By virtue of Proposition 2.9 and Lemma 2.7, we get that $\{\gamma, M\}$ is a $\mathcal{UB}$-critical point of $\mathfrak{T}$ if and only if there exist two constants μ and a such that (1) and (2) hold. The uniqueness of a is obvious. We show the uniqueness of μ. Suppose that (2) holds for two real numbers μ_1 and μ_2. Then $(\mu_1 - \mu_2)\nabla_T T = 0$. Since γ is not a straight line, we see $\mu_1 = \mu_2$. □

DEFINITION 2.12. Let $\{\gamma, M\}$ be a curve with unit normalsuch that γ is a C^4 unit-speed curve and M is a C^1 unit normal along γ. It is said to be a *torsional elastica* if and only if there exist two constants μ and a satisfying (1) and (2) of Proposition 2.11. The constant a is called the *torsional parameter* of $\{\gamma, M\}$. When γ is not a straight line, the constant μ in (2) of Proposition 2.11 is called the *Lagrange multiplier* of the torsional elastica$\{\gamma, M\}$.

Proposition 2.13. *If $\{\gamma(t), M(t)\}$ is a torsional elastica, then both $\gamma(t)$ and $M(t)$ are real analytic in t.*

Proof. When $\gamma(t)$ is a unit-speed curve, the equation (2) of Proposition 2.11 is expressed as the form $\gamma^{(4)}(t) = F(\gamma'(t), \gamma''(t), \gamma'''(t))$. Here, $F = (F_1, F_2, F_3)$ and each F_j $(j = 1, 2, 3)$ is a cubic polynomial of nine variable. Therefore, if $\{\gamma, M\}$ is a torsional elastica, then $\gamma(t)$ is real analytic in t on the whole interval where it is defined. Thus, a unit normal parallel with respect to the normal connection is also real analytic. And so, $M(t)$ is real analytic by Proposition 2.2. □

Next we define a *closed* torsional elastica. A closed torsional elasticais thought of a mathematical model of the equilibrium states of an elastic rod when it is bent and twisted and both ends are welded together to form a smooth loop.

Let $\ell > 0$ and $\varphi \in \mathbf{R}/(2\pi\mathbf{Z})$. Let $\{\gamma(t), M(t)\}$ denote a curve with unit normalde-fined on $\mathbf{R}$. It is called *ℓ-quasi-periodic with corrective angle φ* if $\{\gamma, M\}$ satisfies

$$\gamma(t+\ell) = \gamma(t), \tag{2.11}$$

$$M(t+\ell) = \mathcal{R}\,(\varphi)\,M(t), \tag{2.12}$$

for all $t \in \mathbf{R}$. We denote by $\mathcal{C}(\ell,\varphi)$ the space of curves with unit normalwhich are ℓ-quasi-periodic with corrective angle φ. In this situation, the length functional $\mathfrak{L}$, the total squared curvature functional $\mathfrak{F}$ and the torsional elastic energy functional $\mathfrak{T}$ are defined as the integral of the corresponding function over $[0,\ell]$. We denote by $\mathcal{C}_\ell(\ell,\varphi)$ the space of curves with unit normalin $\mathcal{C}(\ell,\varphi)$ such that $\mathfrak{L}(\gamma)=\ell$, and by $\mathcal{UC}(\ell,\varphi)$ the space of curves with unit normalin $\mathcal{C}(\ell,\varphi)$ such that γ are unit-speed.

We consider the variational problem to determine the critical points of $\mathfrak{T}$ on $\mathcal{C}(\ell,\varphi)$. Now, $\mathcal{C}(\ell,\varphi)$-variation, $\mathcal{C}(\ell,\varphi)$-critical point, etc., are also defined in the same way as $\mathcal{B}$. Then the analogous facts to Lemma 2.5, Lemma 2.6 and Lemma 2.7 hold.

Now, let $\{\gamma,M\}\in\mathcal{C}(\ell,\varphi)$. Let Λ be a vector field along γ, and f a function on $\mathbf{R}$. Denoting by $T_{\{\gamma,M\}}\mathcal{C}(\ell,\varphi)$ the space of all pairs (Λ,f) of vector fields along γ and functions satisfying

$$\Lambda(t+\ell)=\Lambda(t), \tag{2.13}$$

$$f(t+\ell)=f(t), \tag{2.14}$$

we obtain the following proposition analogous to Lemma 2.8. That is, $T_{\{\gamma,M\}}\mathcal{C}(\ell,\varphi)$ can be identified with the tangent space at $\{\gamma,M\}$ to $\mathcal{C}(\ell,\varphi)$.

Proposition 2.14. *Let $\{\gamma,M\}\in\mathcal{C}(\ell,\varphi)$. If $\{\gamma,M\}_\lambda$ is a $\mathcal{C}(\ell,\varphi)$-variation of $\{\gamma,M\}$, then the variation vector field (Λ,f) belongs to $T_{\{\gamma,M\}}\mathcal{C}(\ell,\varphi)$. Conversely, for an arbitrary $(\Lambda,f)\in T_{\{\gamma,M\}}\mathcal{C}(\ell,\varphi)$ there exists a $\mathcal{C}(\ell,\varphi)$-variation $\{\gamma,M\}_\lambda$ whose variation vector field is (Λ,f).*

Using (2.13) and (2.14), we see that the third term of the right hand side of (2.10) vanishes. Thus the fact analogous to Proposition 2.9 also holds. Therefore, we have the following.

Proposition 2.15. *Let $\{\gamma,M\}\in\mathcal{UC}(\ell,\varphi)$. Then, $\{\gamma,M\}$ is a $\mathcal{UC}(\ell,\varphi)$-critical point if and only if $\{\gamma,M\}$ is a torsional elastica.*

If a curve with unit normal$\{\gamma,M\}$ defined on $\mathbf{R}$ is a torsional elasticaand γ has a period ℓ, then M satisfies the condition (2.12) for some $\varphi\in\mathbf{R}/(2\pi\mathbf{Z})$. Therefore, a curve with unit normal$\{\gamma,M\}$ defined on $\mathbf{R}$ is a $\mathcal{UC}(\ell,\varphi)$-critical point for some $\ell>0$ and $\varphi\in\mathbf{R}/(2\pi\mathbf{Z})$ if and only if $\{\gamma,M\}$ is a torsional elasticaand γ is periodic. And, we define a closed torsional elasticaas follows.

DEFINITION 2.16. A curve with unit normal$\{\gamma,M\}$ defined on $\mathbf{R}$ is said to be a *closed torsional elastica* if $\{\gamma,M\}$ is a torsional elasticaand γ is periodic.

We note that even when $\{\gamma,M\}$ is a closed torsional elastica, M is not necessarily periodic.

2.4. Examples of torsional elasticae

In this subsection, we give some simple examples of torsional elasticae. We give all the torsional elasticaeof constant curvature. Let s denote arc length parameter. Then, $\nabla_T = d/ds$. Let $\{\gamma(s), M(s)\}$ be a curve with unit normal. Suppose that the curvature $k(s)$ of γ is positive at all points. We denote by $\tau(s)$ the torsion of γ, and by (T, N, B) the Frenet frame, where the direction of B is determined by the requirement $B = T \times N$. Then the Frenet formulas hold.

$$\begin{aligned} \nabla_T T &= kN, \\ \nabla_T N &= -kT + \tau B, \\ \nabla_T B &= -\tau N. \end{aligned}$$

Suppose that $M(s)$ is expressed as

$$M(s) = \mathcal{R}\left(\psi(s)\right) N(s), \tag{2.15}$$

where $\psi(s)$ is a function. Then the torsional function is

$$\begin{aligned} h &= \left\langle \nabla_T^{\perp}(\mathcal{R}\left(\psi\right) M), \mathcal{R}\left(\psi\right) B \right\rangle \\ &= \left\langle \psi' \mathcal{R}\left(\psi\right) B + \cos\psi \nabla_T^{\perp} N + \sin\psi \nabla_T^{\perp} B, \mathcal{R}\left(\psi\right) B \right\rangle \\ &= \psi' + \tau, \end{aligned}$$

where $'$ is the derivation with respect to s. Thus (1) of Proposition 2.11 yields that

$$\psi' + \tau = a. \tag{2.16}$$

Substituting the Frenet formulas to (2) of Proposition 2.11, we have

$$2k'' + k^3 + (\varepsilon a^2 - \mu)k - 2k\tau(\tau - \varepsilon a) = 0, \tag{2.17}$$

$$k^2\left(\tau - \frac{\varepsilon a}{2}\right) = b, \tag{2.18}$$

where b is a constant.

If the curvature k is a constant function, then (2.18) implies that the torsion τ is also constant. Its converse holds in the following sense. For any $k \neq 0, \tau$, and a, let γ denote an ordinary helix (or a circle) with curvature k and torsion τ, and let M denote a vector field satisfying $M(s) = \mathcal{R}\left((a-\tau)s\right) N(s)$. Then $\{\gamma, M\}$ is a torsional elastica. (If γ is a straight line and $M(s) = \mathcal{R}\left(as\right) N(s)$, where N is a parallel unit normal along γ, then $\{\gamma, M\}$ is also a torsional elastica.)

3. Stability of the circles

In this section, we discuss the stability and instability of the circle given in Subsection 2.4, the simplest closed torsional elastica. First, we calculate the second variation formula for $\mathfrak{T}$ on $\mathcal{UC}(\ell, \varphi)$. Then we introduce the Hessian and define the stability of a closed torsional elastica. In the last part of this section, we determine the stability and instability of the circle and estimate the Morse index (Theorem 3.12).

3.1. The second variation formula

In this subsection, we calculate the second variation formula for $\mathfrak{T}$ on $\mathcal{UC}(\ell, \varphi)$. First, we introduce the "tangent space" of $\mathcal{UC}(\ell, \varphi)$. Let $\{\gamma, M\} \in \mathcal{UC}(\ell, \varphi)$. Setting

$$(3.1) \qquad T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi) = \{(\Lambda, f) \in T_{\{\gamma,M\}}\mathcal{C}(\ell, \varphi); \langle \nabla_T \Lambda, T\rangle = 0\},$$

we see the following lemma, so that $T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi)$ is identified with the tangent space at $\{\gamma, M\}$ to $\mathcal{UC}(\ell, \varphi)$.

Lemma 3.1. *If $\{\gamma, M\}_\lambda$ is a $\mathcal{UC}(\ell, \varphi)$-variation of $\{\gamma, M\}$, then the variation vector field (Λ, f) belongs to $T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi)$. Conversely, for an arbitrary $(\Lambda, f) \in T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi)$ there exists a $\mathcal{UC}(\ell, \varphi)$-variation $\{\gamma, M\}_\lambda$ whose variation vector field is (Λ, f).*

Proof. Suppose that $\{\gamma, M\}_\lambda$ is a $\mathcal{UC}(\ell, \varphi)$-variation of $\{\gamma, M\}$. Then γ_λ is unit-speed for all λ, so that $\partial v_\lambda / \partial \lambda = 0$. Thus, by (1) of Lemma 2.10, $\Lambda = \left.\dfrac{\partial \gamma_\lambda}{\partial \lambda}\right|_{\lambda=0}$ satisfies $\langle \nabla_T \Lambda, T\rangle = 0$.

We show the converse. Let $(\Lambda, f) \in T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi)$. We shall construct a variation γ_λ of γ such that $\left.\dfrac{\partial \gamma_\lambda}{\partial \lambda}\right|_{\lambda=0} = \Lambda$ and γ_λ is a unit-speed curve with a period ℓ for all λ. In order to construct such a variation γ_λ, we shall construct a variation of "fixed length". We first consider the two parameter variation $\gamma_{\lambda,\omega}$ given by

$$\gamma_{\lambda,\omega}(t) = (1 + \omega)\gamma(t) + \lambda\Lambda(t).$$

Then $\gamma_{\lambda,\omega}(t+\ell) = \gamma_{\lambda,\omega}(t)$ for each λ, ω. We set $A(\lambda, \omega) = \mathfrak{L}(\gamma_{\lambda,\omega})$. Then $A(0,0) = \ell$ and $\frac{\partial A}{\partial \omega}(0,0) = \ell \neq 0$. Hence, by the implicit function theorem, there exists a function $\xi(\lambda)$ defined on a neighborhood of $\lambda = 0$ such that

$$A(\lambda, \xi(\lambda)) = \ell.$$

Thus,

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} A(\lambda, \xi(\lambda)) = \frac{\partial A}{\partial \lambda}(0,0) + \ell \frac{d\xi}{d\lambda}(0),$$

and

$$\frac{\partial A}{\partial \lambda}(0,0) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \int_0^\ell v_{\lambda,0}(t)dt = \int_0^\ell \langle \nabla_T \Lambda, T\rangle dt = 0,$$

so that we have

$$\frac{d\xi}{d\lambda}(0) = 0.$$

Therefore,

$$\left.\frac{\partial \gamma_{\lambda,\xi(\lambda)}}{\partial \lambda}\right|_{\lambda=0} = \left.\frac{\partial \gamma_{\lambda,\omega}}{\partial \lambda}\right|_{\lambda=\omega=0} = \Lambda.$$

From now on, we write $\gamma_{\lambda,\xi(\lambda)}(t)$ as $\gamma_\lambda(t)(= \tilde{\gamma}(\lambda,t))$. Then $\gamma_\lambda(t+\ell) = \gamma_\lambda(t)$ and $\mathfrak{L}(\gamma_\lambda) = \ell$ for all λ, t, and the variation vector field of the variation γ_λ is Λ.

Finally we reparametrize t to the arc length parameter $s = s(t) = s(t,\lambda) = \int_0^t v_\lambda(t)dt$. Denoting by $\hat{\gamma}$ the map given by $\tilde{\gamma}(\lambda,t) = \hat{\gamma}(\lambda, s(t,\lambda))$, we see that $\left|\frac{\partial \hat{\gamma}}{\partial s}\right| \equiv 1$, $\hat{\gamma}(\lambda, s+\ell) = \hat{\gamma}(\lambda,s)$. Then, we can verify that $\frac{\partial \tilde{\gamma}}{\partial \lambda} = \frac{\partial \hat{\gamma}}{\partial \lambda}$ for $\lambda = 0$, since $\frac{\partial \tilde{v}}{\partial \lambda} = 0$ for $\lambda = 0$. Thus the variation vector field of the variation $\hat{\gamma}$ still coincides with Λ. If we rewrite $\hat{\gamma}(\lambda,s)$ as $\tilde{\gamma}(\lambda,s)(=\gamma_\lambda(s))$, the variation γ_λ is what we want to construct.

For this $\gamma_\lambda(s)$, we consider the solution $\tilde{M}(\lambda,s)(= M_\lambda(s))$ of (2.5) with initial data $\tilde{M}(0,s) = M(s)$. Then, by the proof of Lemma 2.8, $\{\gamma_\lambda(s), M_\lambda(s)\}$ is a curve with unit normalfor each λ and the variation vector field of the variation $\{\gamma_\lambda(s), M_\lambda(s)\}$ corresponds to (Λ, f). Since

$$\begin{aligned}
&\frac{\partial}{\partial \lambda}\langle M_\lambda(s), M_\lambda(s+\ell)\rangle = \langle f(s)L_\lambda(s), M_\lambda(s+\ell)\rangle + \langle M_\lambda(s), f(s+\ell)L_\lambda(s+\ell)\rangle \\
&= f(s)\big(\langle L_\lambda(s), M_\lambda(s+\ell)\rangle + \langle M_\lambda(s), L_\lambda(s+\ell)\rangle\big) = 0,
\end{aligned}$$

the angle between $M_\lambda(s)$ and $M_\lambda(s+\ell)$ is independent of λ. Also, from $M_0(s+\ell) = \mathcal{R}(\varphi)\, M_0(s)$, we obtain that $M_\lambda(s+\ell) = \mathcal{R}(\varphi)\, M_\lambda(s)$ for all λ, s. Therefore, we have $\{\gamma_\lambda, M_\lambda\} \in \mathcal{UC}(\ell,\varphi)$ for each λ. □

Proposition 3.2. *Let $\{\gamma, M\} \in \mathcal{UC}(\ell,\varphi)$ be a closed torsional elasticawith torsional parametera and Lagrange multiplier μ. Let $\{\gamma, M\}_\lambda$ be a $\mathcal{UC}(\ell,\varphi)$-variation with variation vector field (Λ, f). Then,*

$$\left.\frac{d^2 \mathfrak{T}(\{\gamma, M\}_\lambda)}{d\lambda^2}\right|_{\lambda=0} = \big\langle \mathcal{K}_{\{\gamma,M\}}(\Lambda, f), (\Lambda, f)\big\rangle_{L^2}, \tag{3.2}$$

where $\mathcal{K}_{\{\gamma,M\}}$ is a linear differential operator given as follows.

$$
(3.3)\quad
\begin{aligned}
&\mathcal{K}_{\{\gamma,M\}}(\Lambda,f)\\
&\quad=\Bigg(\nabla_T\Big[2(\nabla_T)^3\Lambda+(3k^2-\mu+\varepsilon a^2)\nabla_T\Lambda-2\varepsilon a\mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T^{\perp}\nabla_T\Lambda)\\
&\qquad\qquad-2\varepsilon\left\{\left\langle\nabla_T\Lambda,\mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)\right\rangle+Tf\right\}\mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)\Big],\\
&\qquad-2\varepsilon\left\{T\left\langle\nabla_T\Lambda,\mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)\right\rangle+T^2 f\right\}\Bigg),
\end{aligned}
$$

where $\langle *,*\rangle_{L^2}$ is the L^2 inner product on $T_{\{\gamma,M\}}\mathcal{C}(\ell,\varphi)$, i.e.

$$\langle(\Lambda,f),(\Omega,g)\rangle_{L^2}=\int_0^\ell(\langle\Lambda,\Omega\rangle+fg)ds.$$

Proof. Since $\mathfrak{L}(\gamma_\lambda)=\ell$, we see

$$\frac{d\mathfrak{T}(\{\gamma,M\}_\lambda)}{d\lambda}=\frac{d\mathfrak{T}^\mu(\{\gamma,M\}_\lambda)}{d\lambda}.$$

The right hand side is expressed as (2.10). We note that the third term of the right hand side of (2.10) vanishes, because Λ and f are periodic with a period ℓ. Let us differentiate this by λ. Noting that $\{\gamma,M\}_{0,0}$ is a torsional elasticawith torsional parameter$a$ and Lagrange multiplier μ, we get

$$
\begin{aligned}
&\left.\frac{d^2\mathfrak{T}^\mu(\{\gamma,M\}_\lambda)}{d\lambda^2}\right|_{\lambda=0}\\
&=\int_0^\ell\langle\nabla_\Lambda\tilde{E},\Lambda\rangle ds-2\varepsilon\int_0^\ell(\Lambda\tilde{T}\tilde{h})f ds,
\end{aligned}
$$

where

$$\tilde{E}=\nabla_{\tilde{T}}\big[2(\nabla_{\tilde{T}})^2\tilde{T}+(3\tilde{k}^2-\mu+\varepsilon\tilde{h}^2)\tilde{T}-2\varepsilon\tilde{h}\tilde{\mathcal{R}}\Big(\frac{\pi}{2}\Big)(\nabla_{\tilde{T}}\tilde{T})\big].$$

From now on, we write $\tilde{T},\tilde{k}$, etc. as T,k, etc.. Since $\Lambda Th=T\Lambda h$, the expression of the second component of (3.3) follows from (5) of Lemma 2.10. We shall calculate $\nabla_\Lambda E$. Note that $\nabla_T\Lambda=\nabla_\Lambda T$ and $\nabla_T\nabla_\Lambda=\nabla_\Lambda\nabla_T$. Using (2.8) and (2.9), we have

$$
\begin{aligned}
\Lambda\langle\nabla_T T,M\rangle&=\big\langle(\nabla_T)^2\Lambda,M\big\rangle+f\langle\nabla_T T,M\rangle,\\
\Lambda\langle\nabla_T T,L\rangle&=\big\langle(\nabla_T)^2\Lambda,L\big\rangle-f\langle\nabla_T T,L\rangle.
\end{aligned}
$$

Thus we get

$$\nabla_\Lambda \left(\mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)\right) = \mathcal{R}\left(\frac{\pi}{2}\right)\left((\nabla_T)^2\Lambda - \langle(\nabla_T)^2\Lambda, T\rangle T\right) - \left\langle \nabla_T\Lambda, \mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)\right\rangle T. \tag{3.4}$$

(3.4) and (4), (5) of Lemma 2.10 yield the following.

$$\begin{aligned}\nabla_\Lambda E = \nabla_T\Big[&2(\nabla_T)^3\Lambda + (3k^2 - \mu + \varepsilon h^2)\nabla_T\Lambda + \beta T \\ &- 2\varepsilon h\mathcal{R}\left(\frac{\pi}{2}\right)\left((\nabla_T)^2\Lambda - \langle(\nabla_T)^2\Lambda, T\rangle T\right) \\ &- 2\varepsilon\Big\{\left\langle \nabla_T\Lambda, \mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)\right\rangle + Tf\Big\}\mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T)\Big],\end{aligned}$$

where $\beta = 6\left\langle(\nabla_T)^2\Lambda, \nabla_T T\right\rangle + 4\varepsilon h\left\langle\nabla_T\Lambda, \mathcal{R}(\pi/2)(\nabla_T T)\right\rangle + 2\varepsilon h(Tf)$. Integrating by parts, we have

$$\int_0^\ell \langle\nabla_T(\beta T), \Lambda\rangle\, ds = -\int_0^\ell \beta\langle T, \nabla_T\Lambda\rangle\, ds = 0.$$

Therefore, (3.2) follows. □

3.2. Definition of the stability of closed torsional elasticae

In this subsection, we define the stability, instability and weak stability of a closed torsional elastica. Now, we can check that

$$\langle\mathcal{K}(\Lambda, f), (\Omega, g)\rangle_{L^2} = \langle(\Lambda, f), \mathcal{K}(\Omega, g)\rangle_{L^2}$$

for all $(\Lambda, f), (\Omega, g) \in T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi)$. Let $\mathcal{H}$ be the bilinear form on $T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi)$ defined by

$$\mathcal{H}((\Lambda, f), (\Omega, g)) = \langle\mathcal{K}(\Lambda, f), (\Omega, g)\rangle_{L^2},$$

where $(\Lambda, f), (\Omega, g) \in T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi)$. Then, $\mathcal{H}$ is a symmetric bilinear form on $T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi)$, called the *Hessian* of the functional $\mathfrak{T}$ at $\{\gamma, M\}$.

The dimension of the largest subspace of $T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi)$ on which $\mathcal{H}$ is negative definite is called the *index* or *Morse index* of $\{\gamma, M\}$. We denote it by $\mathrm{Ind}(\{\gamma, M\})$. The dimension of the vector space

$$\{(\Lambda, f) \in T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi);\ \mathcal{H}((\Lambda, f), (\Omega, g)) = 0, \text{ for all } (\Omega, g) \in T_{\{\gamma,M\}}\mathcal{UC}(\ell, \varphi)\}$$

is called the *nullity* of $\{\gamma, M\}$. We denote it by $\mathrm{Null}(\{\gamma, M\})$.

DEFINITION 3.3. A curve with unit normal $\{\gamma, M\}$ is called *weakly stable* if $\mathrm{Ind}(\{\gamma, M\}) = 0$. Also, $\{\gamma, M\}$ is called *unstable* if $\mathrm{Ind}(\{\gamma, M\}) > 0$.

Next we shall define the stability of $\{\gamma, M\}$. We consider the following transformations which transform $\{\gamma(s), M(s)\}$ into

(3.5) $\{\gamma(\pm(s - s_0)), M(\pm(s - s_0))\}$, where $s_0 \in \mathbf{R}$,

(3.6) $\{S\gamma, dS(M)\}$, where S is an isometry of $\mathbf{R}^3$,

(3.7) $\{S\gamma, dS(M)\}$, where S is an isometry of $\mathbf{R}^3$ which preserves the orientation,

(3.8) $\{\gamma, \mathcal{R}(\varphi) M\}$, where φ is a constant,

respectively. Here, dS is the differential map of S.

DEFINITION 3.4. Two curves with unit normal $\{\gamma, M\}$ and $\{\tilde{\gamma}, \tilde{M}\}$ are called *congruent* if $\{\gamma, M\}$ can be transformed into $\{\tilde{\gamma}, \tilde{M}\}$ by a map of finite compositions of the above transformations (3.5), (3.7), and (3.8). Also, $\{\gamma, M\}$ and $\{\tilde{\gamma}, \tilde{M}\}$ are called *isometric* if $\{\gamma, M\}$ can be transformed into $\{\tilde{\gamma}, \tilde{M}\}$ by a map of finite compositions of the above transformations (3.5), (3.6), and (3.8).

DEFINITION 3.5. Let $\{\gamma, M\}_\lambda$ $(|\lambda| < \lambda_0)$ be a $\mathcal{UC}(\ell, \varphi)$-variation of $\{\gamma, M\}$. It is called a *trivial variation* if $\{\gamma, M\}_\lambda$ is congruent with $\{\gamma, M\}_0 (= \{\gamma, M\})$ for all λ. Also, let $(\Lambda, f) \in T_{\{\gamma, M\}}\mathcal{UC}(\ell, \varphi)$. A variation vector field (Λ, f) is called *trivial* if there exists a trivial variation of $\{\gamma, M\}$ whose variation vector field coincides with (Λ, f).

DEFINITION 3.6. A curve with unit normal $\{\gamma, M\}$ is called *stable* if $\mathcal{H}((\Lambda, f), (\Lambda, f)) > 0$ for any nontrivial variation vector field $(\Lambda, f) \in T_{\{\gamma, M\}}\mathcal{UC}(\ell, \varphi)$.

3.3. Jacobi operators

In this subsection, we give two Jacobi operators for the Hessian $\mathcal{H}$. First, we consider the L^2 inner product $\langle *, * \rangle_{L^2}$ and introduce the Jacobi operator $\mathcal{J}$ with respect to $\langle *, * \rangle_{L^2}$. Next, we introduce another inner product and give the Jacobi operator $\tilde{\mathcal{J}}$ with respect to it. The latter Jacobi operator $\tilde{\mathcal{J}}$ shall be used in the next subsection to show the first main theorem.

Now we give the Jacobi operator $\mathcal{J}$ with respect to $\langle *, * \rangle_{L^2}$. Note that $\mathcal{K}$ in (3.3) is not the Jacobi operator, because $\mathcal{K}(\Lambda, f) = (\mathcal{K}_1(\Lambda, f), \mathcal{K}_2(\Lambda, f))$ does not necessarily satisfy the condition $\langle \nabla_T(\mathcal{K}_1(\Lambda, f)), T \rangle = 0$. Let $\mathcal{P}$ be the orthogonal projection on the L^2 completion of $T_{\{\gamma, M\}}\mathcal{C}(\ell, \varphi)$ to the closure of $T_{\{\gamma, M\}}\mathcal{UC}(\ell, \varphi)$ with respect to the L^2 inner product. We define $\mathcal{J} = \mathcal{P}\mathcal{K}$. Then, $\mathcal{J}$ is the Jacobi operator with respect to the L^2 inner product $\langle *, * \rangle_{L^2}$.

To investigate the stability of the circles, we want to use the expansion theorem for the self-adjoint eigenvalue problem of a system of ordinary linear differential equations. But, it is not easy to apply the expansion theorem to the above Jacobi operator $\mathcal{J}$. And so we shall introduce another inner product $\langle\!\langle *, * \rangle\!\rangle$ on $T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$ and give the Jacobi operator $\tilde{\mathcal{J}}$ with respect to it. From now on we assume that γ has no vanishing curvature. We denote by (T, N, B) the Frenet frame for γ. By (3.1),

$$T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi) = \{(j_1 T + (j_1'/k)N + j_3 B, f);\ j_1, j_3, f \in C_\ell^\infty(\mathbf{R})\},$$

where $C_\ell^\infty(\mathbf{R})$ is the space of all smooth functions on $\mathbf{R}$ with period ℓ. We denote by $(C_\ell^\infty(\mathbf{R}))^3$, the space of all triplets of elements in $C_\ell^\infty(\mathbf{R})$. Then, we can identify $T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$ with $(C_\ell^\infty(\mathbf{R}))^3$. We define the symmetric bilinear form $\langle\!\langle *, * \rangle\!\rangle$ on $T_{\{\gamma,M\}}\mathcal{C}(\ell,\varphi)$ by

$$\langle\!\langle (\Lambda, f), (\Omega, g) \rangle\!\rangle = \langle (\Lambda - \langle \Lambda, N \rangle N, f), (\Omega - \langle \Omega, N \rangle N, g) \rangle_{L^2}, \tag{3.9}$$

where $(\Lambda, f), (\Omega, g) \in T_{\{\gamma,M\}}\mathcal{C}(\ell,\varphi)$. We note that

$$\langle\!\langle (\Lambda + \alpha_1 N, f), (\Omega + \alpha_2 N, g) \rangle\!\rangle = \langle\!\langle (\Lambda, f), (\Omega, g) \rangle\!\rangle, \tag{3.10}$$

for any $\alpha_1, \alpha_2 \in C_\ell^\infty(\mathbf{R})$. The bilinear form $\langle\!\langle *, * \rangle\!\rangle$ can be viewed as a positive definite inner product on $T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$ which corresponds to the L^2 inner product on $(C_\ell^\infty(\mathbf{R}))^3$ through the above correspondence. We often identify $T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$ with $(C_\ell^\infty(\mathbf{R}))^3$ as a vector space with inner product.

Proposition 3.7. *Let $(\Lambda, f) \in T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$. Set*

$$\mathcal{L}(\Lambda, f) = \left(\mathcal{K}_1(\Lambda, f) - \nabla_T\left(\frac{\langle \mathcal{K}_1(\Lambda, f), N\rangle}{k} T\right), \mathcal{K}_2(\Lambda, f)\right),$$

where $\mathcal{K}_1(\Lambda, f)$ and $\mathcal{K}_2(\Lambda, f)$ are the first and second components of $\mathcal{K}(\Lambda, f)$. And let $\tilde{\mathcal{J}}$ be the linear differential operator on $T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$ defined by

$$\tilde{\mathcal{J}}(\Lambda, f) = \left(\mathcal{L}_1(\Lambda, f) + \left(\frac{\langle \mathcal{L}_1(\Lambda, f), T\rangle'}{k}\right) N,\ \mathcal{K}_2(\Lambda, f)\right),$$

where $\mathcal{L}_1(\Lambda, f)$ is the first component of $\mathcal{L}(\Lambda, f)$. Then

$$\mathcal{H}((\Lambda, f), (\Omega, g)) = \langle\!\langle \tilde{\mathcal{J}}(\Lambda, f), (\Omega, g) \rangle\!\rangle.$$

That is, the differential operator $\tilde{\mathcal{J}} : T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi) \to T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$ is just the Jacobi operator with respect to the inner product $\langle\!\langle *, * \rangle\!\rangle$.

Proof. Let (Λ, f), $(\Omega, g) \in T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$. By integration by parts, we have

$$\langle \mathcal{L}(\Lambda, f) - \mathcal{K}(\Lambda, f), (\Omega, g)\rangle_{L^2} = 0.$$

Also, we can check that $\langle \mathcal{L}_1(\Lambda, f), N\rangle = 0$. Thus,

$$\mathcal{H}((\Lambda, f), (\Omega, g)) = \langle \mathcal{L}(\Lambda, f), (\Omega, g)\rangle_{L^2} = \langle\!\langle \mathcal{L}(\Lambda, f), (\Omega, g)\rangle\!\rangle .$$

By (3.10), we have $\langle\!\langle \mathcal{L}(\Lambda, f), (\Omega, g)\rangle\!\rangle = \langle\!\langle \tilde{\mathcal{J}}(\Lambda, f), (\Omega, g)\rangle\!\rangle$. Furthermore, we can check that $\tilde{\mathcal{J}}(\Lambda, f)$ belongs to $T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$, because $\langle \mathcal{L}_1(\Lambda, f), N\rangle = 0$. Therefore, $\tilde{\mathcal{J}}$ is just the Jacobi operator with respect to the inner product $\langle\!\langle *, *\rangle\!\rangle$. □

3.4. The first main theorem on stability of the circles

In this subsection, we show the first main theorem (Theorem 3.12) on the stability and instability of the circles. First we calculate all the trivial variation vector fields (see Definition 3.2.) in the case of the circles and define the essential nullity. Next we describe the expansion theorem for the eigenvalue problem of a system of ordinary linear differential equations of higher order. Last, we show the first main theorem by examining the eigenvalue problem of the Jacobi operator $\tilde{\mathcal{J}}$.

Throughout this subsection, let $\{\gamma, M\}$ denote a closed torsional elasticasuch that γ is a circle of radius r and the torsional parameteris a. Let n be a positive integer. We set $\ell = 2n\pi r, \varphi = 2na\pi r$. We think of $\{\gamma, M\}$ as an element of $\mathcal{UC}(\ell,\varphi)$. Physically, it means an n-fold circular wire. We consider the stability of $\{\gamma, M\}$ in $\mathcal{UC}(\ell,\varphi)$.

Since the curvature of γ is $1/r$, we see

$$T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi) = \{(j_1 T + r j_1' N + j_3 B, f);\ j_1, j_3, f \in C^\infty_\ell(\mathbf{R})\}. \tag{3.11}$$

Now we shall express all the trivial variation vector fields explicitly. Let $\{\gamma, M\}_\lambda$ be a trivial variation of $\{\gamma, M\}$. Any two of the transformations (3.5), (3.7), (3.8) are commutative and (3.5) is, in this situation, expressed as the composition of (3.7) and (3.8). So there exists a smooth one-parameter family of Euclidean motions S_λ and a smooth function $\varphi(\lambda)$ for $|\lambda| < \lambda_0$ such that $S_0 = \mathrm{id}$, $\varphi(0) = 0$ and

$$\{\gamma_\lambda, M_\lambda\} = \{S_\lambda \circ \gamma, \mathcal{R}(\varphi(\lambda))(dS_\lambda M)\}.$$

Lemma 3.8. $(\Lambda, f) = (j_1 T + r j_1' N + j_3 B, f) \in T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$ *is a trivial variation vector field if and only if there exist seven constants* $c_3, c_4, \dots, c_9$ *such that*

j_1, j_3, f *are expressed as follows.*

$$j_1(s) = c_3 r \cos\frac{s}{r} + c_4 r \sin\frac{s}{r} + c_7, \tag{3.12}$$

$$j_3(s) = c_5 r \cos\frac{s}{r} + c_6 r \sin\frac{s}{r} + c_8, \tag{3.13}$$

$$f(s) = -c_5 \cos\frac{s}{r} - c_6 \sin\frac{s}{r} + c_9. \tag{3.14}$$

Proof. To show this lemma, we need the following proposition, which also play an essential role in Section 5.

Proposition 3.9 ([5]). *Let $\gamma = \gamma(s)$ be a unit-speed curve in $\mathbf{R}^3$ whose curvature $k(s)$ is positive everywhere. Let W be a vector field along γ. Then, W can be extended to a Killing vector field on $\mathbf{R}^3$ if and only if W satisfies the following three differential equations. Moreover the Killing vector field is uniquely determined.*

$$\langle \nabla_T W, T \rangle = 0, \tag{3.15}$$

$$\langle (\nabla_T)^2 W, N \rangle = 0, \tag{3.16}$$

$$\left\langle (\nabla_T)^3 W - \frac{k'}{k}(\nabla_T)^2 W + k^2 \nabla_T W, B \right\rangle = 0, \tag{3.17}$$

where (T, N, B) is the Frenet frame. Such a vector field W is said to be a Killing vector field along γ.

Let $\{\gamma, M\}_\lambda$ ($|\lambda| < \lambda_0$) be a trivial variation of $\{\gamma, M\}$ and (Λ, f) its variation vector filed. Then there exists a one-parameter family of Euclidean motions S_λ and functions $\varphi(\lambda)$ such that $S_0 = \mathrm{id}, \varphi(0) = 0$ and

$$\{\gamma_\lambda, M_\lambda\} = \{S_\lambda \gamma, \mathcal{R}(\varphi(\lambda))(dS_\lambda M)\}.$$

Let X be a Killing vector field associated with S_λ, that is, $X = \left.\dfrac{dS_\lambda}{d\lambda}\right|_{\lambda=0}$. Since $\gamma_\lambda(s) = S_\lambda(\gamma(s))$, we see $\Lambda(s) = X(\gamma(s))$. Thus Λ is a Killing vector field along γ. Therefore, (3.16) and (3.17) yield

$$j_1''' + \frac{j_1'}{r^2} = 0, \tag{3.18}$$

$$j_3''' + \frac{j_3'}{r^2} = 0. \tag{3.19}$$

By solving these differential equations, we obtain (3.12) and (3.13). Also, $M_\lambda(s)$ is expressed as

$$M_\lambda(s) = \mathcal{R}(\varphi(\lambda))\,\mathcal{R}(as + \psi)\,N_\lambda(s),$$

where ψ is a constant. Then

$$\begin{aligned} f &= \Big\langle \nabla_\Lambda(\mathcal{R}\,(\varphi(\lambda)+\psi+as)\,\tilde{N}), \mathcal{R}\,(\varphi(\lambda)+\psi+as)\,B\Big\rangle \\ &= \langle \nabla_\Lambda \tilde{N}, B\rangle + \frac{d\varphi}{d\lambda}(0). \end{aligned}$$

Since $\tilde{N} = r\nabla_{\tilde{T}}\tilde{T}$, we see

$$\nabla_\Lambda \tilde{N} = r\nabla_\Lambda \nabla_{\tilde{T}}\tilde{T} = r(\nabla_T)^2 T.$$

Hence

$$f = r\big\langle (\nabla_T)^2 T, B\big\rangle + \frac{d\varphi}{d\lambda}(0) = rj_3'' + \frac{d\varphi}{d\lambda}(0).$$

Therefore we obtain (3.14) by setting $c_9 = \dfrac{d\varphi}{d\lambda}(0)$.

We shall show the converse. Suppose that j_1, j_3, f are expressed as (3.12), (3.13) and (3.14). Since $\Lambda = j_1T + rj_1'N + j_3B$ is a Killing vector field along γ, there exists a Killing vector field X on $\mathbf{R}^3$ such that $\Lambda(s) = X(\gamma(s))$. Let S_λ denote a one-parameter family of Euclidean motions generated by X. We set $\gamma_\lambda = S_\lambda \circ \gamma$ and $M_\lambda(s) = \mathcal{R}\,(c_9\lambda)\,dS_\lambda(M(s))$. Then, $\{\gamma, M\}_\lambda = \{\gamma_\lambda, M_\lambda\}$ is a trivial variation and its variation vector field corresponds to (Λ, f). □

Let $(\Lambda, f) = (j_1T + rj_1'N + j_3B, f) \in T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$. Substituting $k = 1/r$ and $\tau = 0$ to (2.17), we obtain that the Lagrange multiplier of $\{\gamma, M\}$ is $\mu = \varepsilon a^2 + (1/r^2)$. Then, $\tilde{\mathcal{J}}(\Lambda, f)$ is expressed as follows.

$$\begin{aligned} \tilde{\mathcal{J}}(\Lambda,f) = \Bigg(&\Big[-2r^2\big(j_1''+\frac{j_1}{r^2}\big)^{(4)} - 2\big(j_1''+\frac{j_1}{r^2}\big)'' - 2\varepsilon ar\big(j_3''+\frac{j_3}{r^2}\big)''\Big]T \\ &+\Big[-2r^3\big(j_1''+\frac{j_1}{r^2}\big)^{(5)} - 2r\big(j_1''+\frac{j_1}{r^2}\big)^{(3)} - 2\varepsilon ar^2\big(j_3''+\frac{j_3}{r^2}\big)^{(3)}\Big]N \\ &+\Big[2\big(j_3''+\frac{j_3}{r^2}\big)'' - 2\varepsilon ar\big(j_1''+\frac{j_1}{r^2}\big)'' - \frac{2\varepsilon}{r}\big(\frac{j_3}{r}+f\big)''\Big]B, \\ &-2\varepsilon\big(\frac{j_3}{r}+f\big)''\Bigg). \end{aligned} \tag{3.20}$$

It follows from Lemma 3.8 that the space of all trivial variation vector fields is a 7-dimensional vector subspace of $T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$. Also, by using (3.18), (3.19), and $f - rj_3'' =$ const., we can check that a trivial variation vector field (Λ, f) satisfies $\tilde{\mathcal{J}}(\Lambda, f) = 0$. Thus the dimension of the eigenspace of $\tilde{\mathcal{J}}$ with eigenvalue zero is greater than or equal to 7. Since $\mathrm{Null}(\{\gamma, M\})$ is equal to the dimension of the eigenspace of $\tilde{\mathcal{J}}$ with eigenvalue zero, we have $\mathrm{Null}(\{\gamma, M\}) \geqslant 7$.

DEFINITION 3.10. Let $\mathrm{Null_k}(\{\gamma, M\})$ denote the dimension of the space of all the trivial variation vector fields (see Definition 3.2.). (In the present case, $\mathrm{Null_k}(\{\gamma, M\}) = 7$.) The number $\mathrm{Null_e}(\{\gamma, M\}) = \mathrm{Null}(\{\gamma, M\}) - \mathrm{Null_k}(\{\gamma, M\})$ is called the *essential nullity* of $\{\gamma, M\}$.

Now we shall give the expansion theorem for the eigenvalue problem of a system of ordinary linear differential equations of higher order. Let $n_1, n_2, \dots, n_p \geqslant 1$ be p integers. Let $P(s)$ be a p-by-p matrix of class C^∞ in s for $s_1 \leqslant s \leqslant s_2$. Also, let $A(s)$ be a p-by-$(n_1+n_2+\cdots+n_p)$ matrix of class C^∞ in s for $s_1 \leqslant s \leqslant s_2$. Suppose $\det P(s) \neq 0$ on $[s_1, s_2]$. Let $x(s)$ be the column vector function with p components $x_1(s), x_2(s), \dots, x_p(s)$. Let

$$L : C^{n_1}([s_1, s_2]) \times C^{n_2}([s_1, s_2]) \times \cdots \times C^{n_p}([s_1, s_2]) \to (C^0([s_1, s_2]))^p$$

be the differential operator given by

$$(Lx)(s) = P(s) \begin{bmatrix} x_1^{(n_1)}(s) \\ x_2^{(n_2)}(s) \\ \vdots \\ x_p^{(n_p)}(s) \end{bmatrix} + A(s)\tilde{x}(s),$$

where $\tilde{x}(s)$ is the column vector function with $n_1 + n_2 + \cdots + n_p$ components $x_1(s)$, $x_1'(s), \dots, x_1^{(n_1-1)}, x_2(s), x_2'(s), \dots, x_2^{(n_2-1)}, \dots, x_p(s), x_p'(s), \dots, x_p^{(n_p-1)}$.

Let C_1 and C_2 be $(n_1+n_2+\cdots+n_p)$-by-$(n_1+n_2+\cdots+n_p)$ real constant matrices. For $x(s)$, let Ux denote the column vector with $n_1+n_2+\cdots+n_p$ components defined by

$$Ux = C_1\tilde{x}(s_1) + C_2\tilde{x}(s_2). \tag{3.21}$$

Suppose that

$$(Lx, y)_{L^2} = (x, Ly)_{L^2} \tag{3.22}$$

for all $x(s), y(s) \in C^{n_1}([s_1, s_2]) \times C^{n_2}([s_1, s_2]) \times \cdots \times C^{n_p}([s_1, s_2])$ which satisfy the boundary conditions

$$Ux = Uy = 0. \tag{3.23}$$

Here $(*,*)_{L^2}$ is the L^2 inner product of the vector function on $[s_1, s_2]$ with p components. We consider the self-adjoint eigenvalue problem

$$Lx = \lambda x, \quad Ux = 0, \tag{3.24}$$

where λ is a real parameter. Then the following holds. (The proof is parallel to the case of the single equation ([2]), so we shall omit the proof.)

Lemma 3.11. *The set of all eigenvalues of the problem* (3.24) *consists of a discrete sequence of real numbers. Each eigenspace is finite dimensional and eigenspaces corresponding to distinct eigenvalues are orthogonal. Furthermore, the direct sum of all the eigenspaces is dense in* $(L^2([s_1, s_2]))^p$ *with respect to the* L^2 *norm.*

Now we obtain the first main theorem.

Theorem 3.12.
(1) *Let* $m_0 = n\sqrt{1+\varepsilon^2 a^2 r^2}$. *If* m_0 *is an integer and* $m_0 \neq n$, *then* $\mathrm{Null}_e(\{\gamma, M\}) = 2$. *If* m_0 *is not an integer or* $m_0 = n$, *then* $\mathrm{Null}_e(\{\gamma, M\}) = 0$. *Here, see Definition 3.4. for* $\mathrm{Null}_e(\{\gamma, M\})$.
(2) *For all* $\{\gamma, M\}$, *the index* $\mathrm{Ind}(\{\gamma, M\})$ *is finite. Let* $A(n,a,r)$ *denote the number of integers* m *satisfying* $|m| < m_0$, $m \neq 0$, $m \neq \pm n$. *Then,*

$$A(n,a,r) \leqslant \mathrm{Ind}(\{\gamma, M\}) \leqslant 18A(n,a,r).$$

Therefore, if either $n \geqslant 2$ *or* $n = 1$ *and* $\varepsilon^2 a^2 r^2 > 3$, *then* $\{\gamma, M\}$ *is unstable. Also, if* $n = 1$ *and* $\varepsilon^2 a^2 r^2 \leqslant 3$, *then* $\{\gamma, M\}$ *is weakly stable.*
(3) *If* $n = 1$ *and* $\varepsilon^2 a^2 r^2 < 3$, *then* $\{\gamma, M\}$ *is stable.*

Proof. The differential operator $\tilde{\mathcal{J}}$ on $T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$ corresponds to the following differential operator L on $(C^\infty_\ell(\mathbf{R}))^3$.

$$L\begin{bmatrix} j_1 \\ j_3 \\ f \end{bmatrix} = \begin{bmatrix} -2r^2\left(j_1'' + \frac{j_1}{r^2}\right)^{(4)} - 2\left(j_1'' + \frac{j_1}{r^2}\right)'' - 2\varepsilon ar\left(j_3'' + \frac{j_3}{r^2}\right)'' \\ 2\left(j_3'' + \frac{j_3}{r^2}\right)'' - 2\varepsilon ar\left(j_1'' + \frac{j_1}{r^2}\right)'' - \frac{2\varepsilon}{r}\left(\frac{j_3}{r} + f\right)'' \\ -2\varepsilon\left(\frac{j_3}{r} + f\right)'' \end{bmatrix} \tag{3.25}$$

The operator $\tilde{\mathcal{J}}$ has an eigenvector with eigenvalue $\rho(\in \mathbf{R})$ if and only if the following system of linear ordinary differential equations has a non-zero solution with period $2n\pi r$.

$$L\begin{bmatrix} j_1 \\ j_3 \\ f \end{bmatrix} = \rho \begin{bmatrix} j_1 \\ j_3 \\ f \end{bmatrix} \tag{3.26}$$

First we show (1). The nullity $\mathrm{Null}(\{\gamma, M\})$ is equal to the dimension of the eigenspace of $\tilde{\mathcal{J}}$ with eigenvalue zero. We set $\rho = 0$ in (3.26). Then the solutions of (3.26) with period $2n\pi r$ are expressed as follows: If m_0 is an integer, then

$$\begin{aligned} j_1(s) &= r\left(c_1 \cos\frac{m_0 s}{nr} + c_2 \sin\frac{m_0 s}{nr}\right) + \left[c_3 r\cos\frac{s}{r} + c_4 r\sin\frac{s}{r} + c_7\right], \\ j_3(s) &= \varepsilon a r^2\left(c_1 \cos\frac{m_0 s}{nr} + c_2 \sin\frac{m_0 s}{nr}\right) + \left[c_5 r\cos\frac{s}{r} + c_6 r\sin\frac{s}{r} + c_8\right], \\ f(s) &= -\varepsilon a r\left(c_1 \cos\frac{m_0 s}{nr} + c_2 \sin\frac{m_0 s}{nr}\right) + \left[-c_5 \cos\frac{s}{r} - c_6 \sin\frac{s}{r} + c_9\right], \end{aligned} \tag{3.27}$$

where $c_1, \dots, c_9$ are arbitrary real constants. If m_0 is not an integer, then the solutions are expressed as (3.27), where $c_1 = c_2 = 0$ and $c_3, \dots, c_9$ are arbitrary real constants. Thus, if m_0 is an integer and $m_0 \neq n$, then the space of all eigenvectors of $\tilde{\mathcal{J}}$ with eigenvalue zero is a 9-dimensional vector space. Then, $\mathrm{Null}(\{\gamma, M\}) = 9$, and so $\mathrm{Null}_e(\{\gamma, M\}) = 2$. Also, if m_0 is not an integer or $m_0 = n$, then the space of all eigenvectors of $\tilde{\mathcal{J}}$ with eigenvalue zero is a 7-dimensional vector space. Then, $\mathrm{Null}(\{\gamma, M\}) = 7$, and so $\mathrm{Null}_e(\{\gamma, M\}) = 0$.

Next we shall show (2). By definition, the index $\mathrm{Ind}(\{\gamma, M\})$ is equal to the number of negative eigenvalues (with multiplicity) of $\tilde{\mathcal{J}}$. By normalizing the system (3.26), we get the following system.

$$\begin{aligned} j_1^{(6)} &= \frac{-\rho}{2r^2} j_1 - \left(\frac{1}{r^4} + \frac{\varepsilon^2 a^2}{r^2}\right) j_1'' - \left(\frac{2}{r^2} + \varepsilon^2 a^2\right) j_1^{(4)} - \frac{\varepsilon a \rho}{2r} j_3 + \frac{\varepsilon a \rho}{2r^2} f, \\ j_3^{(4)} &= \frac{\varepsilon a}{r} j_1'' + \varepsilon a r j_1^{(4)} + \frac{\rho}{2} j_3 - \frac{1}{r^2} j_3'' - \frac{\rho}{2r} f, \\ f'' &= -\frac{1}{r} j_3'' - \frac{\rho}{2\varepsilon} f. \end{aligned} \tag{3.28}$$

We denote by $F(\rho)$ the coefficient matrix of the first order system which is associated with the system (3.28). Since $F(\rho)$ is constant in s, we have the following. If the first order system has a non-zero solution with period $2n\pi r$, then the characteristic equation $\det(\xi I - F(\rho)) = 0$, where I is the unit matrix of size 12, has a root of the form $\xi = m\sqrt{-1}/(nr)$ $(m \in \mathbf{Z})$. Substituting $\xi = m\sqrt{-1}/(nr)$ to the characteristic equation, we obtain

$$\begin{aligned} & r^{10} n^{12} \rho^3 - 2r^6 n^6 m^2 \left[\varepsilon(r^2+1)n^4 + m^2(m^2 - n^2)\right]\rho^2 \\ & \quad + 4r^2 n^2 m^4 (m^2 - n^2)\left[-\varepsilon^2 a^2 r^2 n^2 (m^2 - n^2) \right. \\ & \qquad\qquad \left. + \varepsilon n^2\{(r^2+1)m^2 - n^2\} + (m^2 - n^2)^2\right]\rho \\ & \quad - 8\varepsilon m^6 (m^2 - n^2)^2 \left[m^2 - (1 + \varepsilon^2 a^2 r^2) n^2\right] = 0. \end{aligned} \tag{3.29}$$

Conversely, if there exist ρ and m satisfying (3.29), then there exists an eigenvector of $\tilde{\mathcal{J}}$ with the eigenvalue ρ such that each of j_1, j_2, j_3, f is a linear combination of

$\cos(ms/(nr))$, $\sin(ms/(nr))$. Let us denote by $\Phi(\rho, m)$ the left hand side of (3.29).

The dimension of the eigenspace of $\tilde{\mathcal{J}}$ corresponding to the eigenvalue ρ coincides with the sum of the dimensions of the eigenspaces of the matrix $F(\rho)$ corresponding to the characteristic roots of the form $m\sqrt{-1}/(nr)$, where m is an integer. Furthermore, the dimension of the eigenspace of $F(\rho)$ corresponding to $\xi = m\sqrt{-1}/(nr)$ is less than or equal to the multiplicity of the root ξ of the characteristic equation $\det(\xi I - F(\rho)) = 0$. If $m = 0$, then the root of $\Phi(\rho, m) = 0$ is $\rho = 0$. Thus, if $\rho < 0$, then the multiplicity of the root $\xi = m\sqrt{-1}/(nr)$ is less than or equal to 6. We denote by $N(\rho, m)$ the number of the pairs (ρ, m) satisfying $\Phi(\rho, m) = 0$ and $\rho < 0$. Then, the number of negative eigenvalues (with multiplicity) of $\tilde{\mathcal{J}}$ is more than or equal to $N(\rho, m)$ and less than or equal to $6N(\rho, m)$. Thus,

$$N(\rho, m) \leqslant \operatorname{Ind}(\{\gamma, M\}) \leqslant 6N(\rho, m). \tag{3.30}$$

We shall investigate the negative roots of the cubic equation $\Phi(\rho, m) = 0$ in ρ. If $m = 0$, then the root of $\Phi(\rho, m) = 0$ is $\rho = 0$. Also, if $m = \pm n$, then the roots are $\rho = 0,\ 2\varepsilon(r^2 + 1)/r^4$. From now on, we assume $m \neq 0,\ m \neq \pm n$. Suppose that $|m| \geqslant m_0$. We show the cubic equation $\Phi(\rho, m) = 0$ in ρ has no negative roots. Since

$$\Phi(0, m) = -8\varepsilon m^6 (m^2 - n^2)^2 \big[m^2 - (1 + \varepsilon^2 a^2 r^2) n^2\big],$$

we see $\Phi(0, m) \leqslant 0$. So, it suffices to show that the cubic polynomial $\Phi(\rho, m)$ in ρ is monotone increasing on $\rho \leqslant 0$. By $m^2 - n^2 > 0$,

$$\varepsilon(r^2 + 1)n^4 + m^2(m^2 - n^2) > 0.$$

And so the coefficient of ρ on the quadratic function $\partial\Phi/\partial\rho$ is negative. Also, by $m^2 \geqslant m_0^2$, we have

$$\begin{aligned}&(m^2 - n^2)\big[-\varepsilon^2 a^2 r^2 n^2 (m^2 - n^2) + \varepsilon n^2\{(r^2 + 1)m^2 - n^2\} + (m^2 - n^2)^2\big] \\ &\geqslant (m^2 - n^2)\varepsilon n^2 \{(r^2 + 1)m^2 - n^2\} > 0.\end{aligned}$$

Thus the constant term of $\partial\Phi/\partial\rho$ in ρ is positive. Therefore if the quadratic equation $\partial\Phi/\partial\rho = 0$ has a real root, it must be positive. Hence, $\Phi(\rho, m)$ is monotone increasing on $\rho \leqslant 0$. Next we suppose that $|m| < m_0$. Then $\Phi(0, m) > 0$, and so the equation $\Phi(\rho, m) = 0$ has one, two or three negative roots. Thus,

$$A(n, a, r) \leqslant N(\rho, m) \leqslant 3A(n, a, r). \tag{3.31}$$

Therefore (3.30) and (3.31) yield $A(n, a, r) \leqslant \operatorname{Ind}(\{\gamma, M\}) \leqslant 18A(n, a, r)$. Next we shall show the stability of $\{\gamma, M\}$ for $n = 1$ and $\varepsilon^2 a^2 r^2 < 3$. In this case, $A(n, a, r) =$

0. If we set $C_1 = I$ and $C_2 = -I$, where I is the unit matrix of size 12, then (3.22) holds for all $x(s), y(s) \in C^6([0,\ell]) \times C^4([0,\ell]) \times C^2([0,\ell])$ satisfying (3.23). Thus, by Lemma 3.11, there exists a complete orthonormal system $\{X_i\}_{i=1}^{\infty}$ of $T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$ such that $\tilde{\mathcal{J}}X_i = \rho_i X_i$, where $\{\rho_i\}_{i=1}^{\infty}$ are the eigenvalues of $\tilde{\mathcal{J}}$ counting multiplicity. Since $\mathrm{Null}(\{\gamma, M\}) = 7$, we may assume $\rho_1 = \cdots = \rho_7 = 0$. Also, by $A(n,a,r) = 0$, we see $\rho_i > 0$ for $i \geqslant 8$. Then, $X_1, X_2, \ldots, X_7$ are trivial variation vector fields. Since for any $Y \in T_{\{\gamma,M\}}\mathcal{UC}(\ell,\varphi)$,

$$Y = \sum_{i=1}^{\infty} \langle\!\langle Y, X_i \rangle\!\rangle X_i,$$

we see

$$\begin{aligned}\langle\!\langle \tilde{\mathcal{J}}Y, Y \rangle\!\rangle &= \left\langle\!\!\left\langle \tilde{\mathcal{J}}Y, \sum_{i=1}^{\infty} \langle\!\langle Y, X_i \rangle\!\rangle X_i \right\rangle\!\!\right\rangle = \sum_{i=1}^{\infty} \langle\!\langle Y, X_i \rangle\!\rangle \langle\!\langle \tilde{\mathcal{J}}Y, X_i \rangle\!\rangle \\ &= \sum_{i=1}^{\infty} \langle\!\langle Y, X_i \rangle\!\rangle \langle\!\langle Y, \tilde{\mathcal{J}}X_i \rangle\!\rangle = \sum_{i=8}^{\infty} \rho_i \langle\!\langle Y, X_i \rangle\!\rangle^2 \geqslant 0.\end{aligned}$$

Therefore, if $\langle\!\langle \tilde{\mathcal{J}}Y, Y \rangle\!\rangle = 0$, then $\langle\!\langle Y, X_i \rangle\!\rangle = 0$ for all $i \geqslant 8$, and so Y is a trivial variation vector field. □

4. The curvature and torsion of a torsional elastica

In this section, we discuss the integration of the equations (2.17), (2.18). (cf. [5]). We refer the reader to [5], [1] about the Jacobi elliptic functions $\mathrm{sn}(x,p), \mathrm{cn}(x,p)$.

Lemma 4.1. *The space of all isometry classes of torsional elasticaedefined on* $\mathbf{R}$ *corresponds to the parameter space of quadruplets* $(a_*, \alpha_1, \alpha_2, \alpha_3)$ *satisfying* $-\alpha_1 \leqslant 0 \leqslant \alpha_2 \leqslant \alpha_3, \alpha_3 > 0$, *and* $a_* \in \mathbf{R}$. *(Here,* $(a_*, \alpha_1, \alpha_2, \alpha_3)$ *is identified with* $(-a_*, \alpha_1, \alpha_2, \alpha_3)$ *if either* $\alpha_1 = 0$ *or* $\alpha_2 = 0$.*) The parameter* $(a_*, \alpha_1, \alpha_2, \alpha_3)$ *corresponds to the isometry class of torsional elasticaewith torsional parameter*$\pm a_*$ *whose curvature and torsion are expressed as follows.*

$$u(s) = \alpha_3(1 - q^2 \mathrm{sn}^2(y(s - s_0), p)), \tag{4.1}$$

$$\tau(s) = \pm\left(\frac{\sqrt{\alpha_1\alpha_2\alpha_3}}{2u(s)} + \frac{\varepsilon a_*}{2}\right), \tag{4.2}$$

where $s_0 \in \mathbf{R}$ *is a constant and*

$$p = \sqrt{\frac{\alpha_3 - \alpha_2}{\alpha_3 + \alpha_1}}, \quad q = \sqrt{\frac{\alpha_3 - \alpha_2}{\alpha_3}}, \quad y = \frac{\sqrt{\alpha_1 + \alpha_3}}{2}.$$

Here, we make the convention that the double signs of a_* *and that of the right hand side of* (4.2) *are in the same order.*

Proof. Let $\{\gamma(s), M(s)\}$ be a torsional elasticadefined on $\mathbf{R}$. Suppose that γ is not a straight line. Let a and μ denote the torsional parameterand Lagrange multiplier of $\{\gamma, M\}$ respectively. We prove this lemma by dividing into three cases.

Case 1. Nonvanishing and nonconstant curvature case. We first consider the case that the curvature k is not constant and $k(s) > 0$ for all s. As we mentioned in Section 2, the curvature and torsion of γ satisfy the equations (2.17) and (2.18). Using the substitution $\tau = b/k^2 + \varepsilon a/2$ and multiplication by k' and integration, we have

$$(k')^2 + \frac{k^4}{4} + \frac{1}{2}\left(\varepsilon a^2 - \mu + \frac{\varepsilon^2 a^2}{2}\right) k^2 + \frac{b^2}{k^2} = c, \tag{4.3}$$

where c is a constant. We shall make the change of variable $u = k^2$. Then we get

$$(u')^2 = -u^3 - 2\left(\varepsilon a^2 - \mu + \frac{\varepsilon^2 a^2}{2}\right) u^2 + 4cu - 4b^2. \tag{4.4}$$

We denote by $P(u)$ the right hand side of (4.4). Since $P(0) = -4b^2 \leqslant 0$, the minimum real root of the cubic equation $P(u) = 0$ is nonpositive. We denote the minimum root by $-\alpha_1$. Furthermore, since $u' \not\equiv 0$, there exists some $u > 0$ such that the cubic polynomial $P(u)$ is positive. Therefore the equation $P(u) = 0$ has at least one positive root. We denote by α_3 the maximum of these positive roots, and by α_2 the other root of the cubic equation. Since $P(0) \leqslant 0$ implies $\alpha_2 \geqslant 0$, and there exists some $u > 0$ such that $P(u) > 0$, we see $\alpha_2 \neq \alpha_3$. Therefore

$$-\alpha_1 \leqslant 0 \leqslant \alpha_2 < \alpha_3. \tag{4.5}$$

The real numbers $\alpha_1, \alpha_2, \alpha_3$ are related to the parameters μ, a, b, c by

$$2\left(\varepsilon a^2 - \mu + \frac{\varepsilon^2 a^2}{2}\right) = \alpha_1 - \alpha_2 - \alpha_3, \tag{4.6}$$

$$-4c = -\alpha_1\alpha_2 + \alpha_2\alpha_3 - \alpha_1\alpha_3, \tag{4.7}$$

$$4b^2 = \alpha_1\alpha_2\alpha_3. \tag{4.8}$$

The solutions of the ordinary differential equation $(u')^2 = P(u)$ is expressed in terms of Jacobi sn function and the parameters $\alpha_1, \alpha_2, \alpha_3$ as follows:

$$u(s) = \alpha_3(1 - q^2 \mathrm{sn}^2(y(s - s_0), p)), \tag{4.9}$$

where

$$p=\sqrt{\frac{\alpha_3-\alpha_2}{\alpha_3+\alpha_1}},\quad q=\sqrt{\frac{\alpha_3-\alpha_2}{\alpha_3}},\quad y=\frac{\sqrt{\alpha_1+\alpha_3}}{2}\left(=\frac{q\sqrt{\alpha_3}}{2p}\right), \tag{4.10}$$

and s_0 is a constant. We set $a_*=a$ if $b\geqslant 0$, and $a_*=-a$ if $b<0$. Then it follows from (2.18) and (4.8) that

$$\tau(s)=\pm\left(\frac{\sqrt{\alpha_1\alpha_2\alpha_3}}{2u(s)}+\frac{\varepsilon a_*}{2}\right), \tag{4.11}$$

where the sign $+$ holds if $b\geqslant 0$, and the one $-$ holds if $b<0$.

Case 2. Constant curvature case. Next we consider the case that $k(s)$ is constant. In this case the torsion is also constant. We formally define the parameters $\alpha_1,\alpha_2,\alpha_3$ in terms of the squared curvature u and the torsion τ as follows.

$$\alpha_1=4\left(\tau-\frac{\varepsilon a}{2}\right)^2,\quad \alpha_2=\alpha_3=u.$$

We define a_* in the same way as Case 1. Then u,τ are also expressed as (4.9) and (4.11). Here we note that it follows from (2.17) and (4.3) that

$$\begin{aligned}\mu&=u+\varepsilon a^2-2\tau(\tau-\varepsilon a),\\ c&=-\frac{u^2}{4}+2u\left(\tau-\frac{\varepsilon a}{2}\right)^2,\end{aligned}$$

and $\alpha_1,\alpha_2,\alpha_3$ defined as above satisfy the relation (4.6), (4.7) and (4.8).

Case 3. The case that γ has inflection points.

We consider the case that γ has inflection points, that is, the points at which $k(s)$ vanishes. In this case we restrict the argument of Case 1 to an open interval I on which $k(s)$ is positive. Then there exist $\alpha_1,\alpha_2,\alpha_3$ satisfying (4.5), and the squared curvature u is expressed as (4.9) on I. This expression is valid for the whole $\mathbf{R}$, because u is real analytic on $\mathbf{R}$ by Proposition 2.13. In particular, we can verify that the parameters $\alpha_1,\alpha_2,\alpha_3$ are determined not depending on the open interval I. Also, since there exists a point such that $u(s)=0$, we see $q=1$, and $\alpha_2=0$. Thus $b=0$ by (4.8). Therefore the torsion τ is $\varepsilon a/2$ except at the inflection points. We define a_* in the same way as the other cases, that is $a_*=a$.

For a given $\{\gamma,M\}$, we determine the parameters $(a_*,\alpha_1,\alpha_2,\alpha_3)$ as above. In any cases, $k(s)$ is invariant under the transformation (3.6) of $\{\gamma,M\}$. And, $\tau(s),a$, and b are also invariant if S preserves the orientation of $\mathbf{R}$. They are multiplied by -1

if S does not preserve the orientation. Thus, if $b \neq 0$, then $(a_*, \alpha_1, \alpha_2, \alpha_3)$ is invariant. If $b = 0$, that is, $\alpha_1 = 0$ or $\alpha_2 = 0$, then it is transformed to $(-a_*, \alpha_1, \alpha_2, \alpha_3)$. Also, it is obvious that $(a_*, \alpha_1, \alpha_2, \alpha_3)$ is invariant under the transformations (3.5) and (3.8).

Consequently, $(a_*, \alpha_1, \alpha_2, \alpha_3)$ is determined only by an isometry class. (Note that $(a_*, \alpha_1, \alpha_2, \alpha_3)$ is identified with $(-a_*, \alpha_1, \alpha_2, \alpha_3)$ if either $\alpha_1 = 0$ or $\alpha_2 = 0$.) Next we shall show the bijectivity of our correspondence. We take $a_*, \alpha_1, \alpha_2, \alpha_3$ satisfying

$$-\alpha_1 \leqslant 0 \leqslant \alpha_2 \leqslant \alpha_3, \quad \alpha_3 > 0, \tag{4.12}$$

$$a_* \geqslant 0 \quad \text{if either } \alpha_1 = 0 \text{ or } \alpha_2 = 0. \tag{4.13}$$

Let $u(s)$ denote the function defined by the right hand side of (4.9) with $s_0 = 0$, and set $k(s) = \sqrt{u(s)}$. Also, let $\tau(s)$ denote the function defined by the right hand side of (4.11) with the sign $+$. To complete the proof, it suffices to show that there exists a unique torsional elastica$\{\gamma, M\}$ (up to congruent transformations) such that the curvature of γ is $k(s)$, and the torsion is $\tau(s)$ except at the inflection points, and the torsional parameteris a_*. We first consider the case $\alpha_2 > 0$ or $\alpha_1 = \alpha_2 = 0$. Then, $k(s) > 0$ for all $s \in \mathbf{R}$. (Note that $u(s) = \alpha_3 \mathrm{sech}^2(ys)$ if $\alpha_1 = \alpha_2 = 0$.) Thus, there exists a curve $\gamma(s)$ in $\mathbf{R}^3$ whose curvature is $k(s)$ and torsion is $\tau(s)$. The curve $\gamma(s)$ is uniquely determined up to congruent transformations on $\mathbf{R}^3$. Let $M(s)$ be a unit normal along γ such that the torsional functionof $\{\gamma, M\}$ agrees with a_*. That is, we define $M(s)$ as follows. Take a function $\psi(s)$ satisfying

$$\psi'(s) = a_* - \tau(s). \tag{4.14}$$

Let $M(s)$ be the unit normal along γ defined by (2.15). If $\gamma(s)$ is determined, then $M(s)$ is uniquely determined up to the transformation (3.8). We set $a = a_*$ and define b $(\geqslant 0)$ and μ by (4.6) and (4.8). Then, (2.17) and (2.18) hold, and $\{\gamma, M\}$ is a torsional elasticawith curvature $k(s)$, torsion $\tau(s)$, and torsional parametera_*. We also see that $\{\gamma, M\}$ is uniquely determined up to congruent transformations. Next we consider the case $\alpha_2 = 0$ and $\alpha_1 \neq 0$. In this case,

$$k(s) = \sqrt{\alpha_3}\,|\mathrm{cn}(ys, p)|,$$
$$\tau(s) = \varepsilon a_*/2.$$

We note that $k(s)$ has periodic zeros. In this case, we set

$$\hat{k}(s) = \sqrt{\alpha_3}\mathrm{cn}(ys, p).$$

Let γ be a curve with "signed curvature" $\hat{k}$ and torsion τ. That is, we define γ in the following way. Let (T_0, N_0, B_0) be a positive orthonormal frame at the origin of

$\mathbf{R}^3$. We denote by $T(s), \hat{N}(s), \hat{B}(s)$ the solutions of the system of ordinary differential equations

$$\begin{aligned} T' &= \hat{k}\hat{N} \\ \hat{N}' &= -\hat{k}T + (\varepsilon a_*/2)\hat{B} \\ \hat{B}' &= -(\varepsilon a_*/2)\hat{N} \end{aligned}$$

with initial data $T(0) = T_0, \hat{N}(0) = N_0, \hat{B}(0) = B_0$. Then, from the Leibniz rule, $(T(s), \hat{N}(s), \hat{B}(s))$ is a positive orthonormal frame for all $s \in \mathbf{R}$. We define the curve $\gamma(s)$ by $\gamma(s) = \int_0^s T(s)ds$. Then, at the points such that $\hat{k}(s) > 0$ (resp. $\hat{k}(s) < 0$), the Frenet frame for γ is $(T, \hat{N}, \hat{B})$ (resp. $(T, -\hat{N}, -\hat{B})$). Thus, the curvature of γ is k, and the torsion is τ except at the inflection points. We set $M(s) = \mathcal{R}(\psi(s))\hat{N}(s)$, where $\psi(s)$ is a function satisfying (4.14). Since the torsional functionof the curve with unit normal$\{\gamma, \hat{N}\}$ is τ, the torsional functionof $\{\gamma, M\}$ is a_*. We define a, b, μ in the same way as the former case. Then γ satisfies (ii) of Proposition 2.11 except at the inflection points. However, since γ is real analytic, the left hand side of (ii) of Proposition 2.11 is continuous. Hence, γ satisfies (ii) of Proposition 2.11, and $\{\gamma, M\}$ is a torsional elasticawith torsional parametera_* on the whole of $\mathbf{R}$. Let $\{\gamma, M\}$ and $\{\tilde{\gamma}, \tilde{M}\}$ be torsional elasticaesuch that the curvature are $k(s)$ and the torsion are $\tau(s)$ and the torsional parameterare a_*. Then, the restriction of $\{\gamma, M\}$ and $\{\tilde{\gamma}, \tilde{M}\}$ to an interval on which $k(s)$ is positive are congruent. Therefore, from the real analyticity of $\{\gamma, M\}$ and $\{\tilde{\gamma}, \tilde{M}\}$, these are congruent on the whole of $\mathbf{R}$. □

5. Explicit expression in terms of cylindrical coordinates

We now introduce the parameters α, η, p and w defined by

$$\alpha = \alpha_3, \quad \eta = \frac{a_*}{\sqrt{\alpha_3}}, \quad p = \sqrt{\frac{\alpha_3 - \alpha_2}{\alpha_3 + \alpha_1}}, \quad w = \sqrt{\frac{\alpha_3}{\alpha_3 + \alpha_1}}. \tag{5.1}$$

Then,

$$q = \frac{p}{w}, \quad y = \frac{\sqrt{\alpha}}{2w}. \tag{5.2}$$

Also,

$$\alpha_1 = \frac{\alpha(1 - w^2)}{w^2}, \quad \alpha_2 = \frac{\alpha(w^2 - p^2)}{w^2}, \quad \alpha_3 = \alpha. \tag{5.3}$$

Under the relations (5.1) and (5.3), the set of all $(a_*, \alpha_1, \alpha_2, \alpha_3)$ satisfying (4.12) is in one-to-one correspondence with the set of all (α, η, p, w) satisfying the following.

$$\alpha > 0, \quad 0 \leqslant p \leqslant w \leqslant 1, \quad w > 0. \tag{5.4}$$

Then the space consisting of all (α, η, p, w), where (α, η, p, w) is identified with $(\alpha, -\eta, p, w)$ if either $p = w$ or $w = 1$, is in one-to-one correspondence with the set of the all isometry classes of torsional elastica.Also, if $\zeta \neq 0$, the (isometry class of) torsional elasticacorresponding to $(\zeta^2\alpha, \eta, p, w)$ is the similar extension by factor $\pm 1/\zeta$ of that corresponding to (α, η, p, w). Thus (η, p, w) determines a similarity class of torsional elastica. Now let $\{\gamma, M\}$ be a torsional elasticadefined on $\mathbf{R}$. We use the same notations of parameters in the previous section. If γ is expressed explicitly, then so is the Frenet frame. Therefore, by (2.16), M is expressed explicitly in terms of the torsional parameter$a$.Now we shall give an explicit expression for γ. We may assume $s_0 = 0$ in Lemma 4.1 without loss of generality. First we consider the generic case. From now on we assume $p \neq w$. Then the curvature is positive everywhere. We shall define the two vector fields J_0, H along γ by

$$J_0 = 2(\nabla_T)^2 T + (3k^2 - \mu + \varepsilon a^2)T - 2\varepsilon a \mathcal{R}\left(\frac{\pi}{2}\right)(\nabla_T T), \tag{5.5}$$

$$H = \varepsilon a T + kB. \tag{5.6}$$

Then we have the following lemma.

Proposition 5.1. *The vector fields J_0 and H are Killing vector fields along γ.*

Proof. By the definition of torsional elastica, $\nabla_T J_0 = 0$. Thus J_0 satisfies (3.15), (3.16), and (3.17). Also, by the Frenet formulas, we see that $\langle \nabla_T H, T\rangle = 0$ and

$$\langle (\nabla_T)^2 H, N\rangle = -\frac{1}{k}\left[k^2\left(\tau - \frac{\varepsilon a}{2}\right)\right]'.$$

Thus, by (2.18), H satisfies (3.16). Also, by (2.17), we see

$$\langle (\nabla_T)^3 H, B\rangle = \frac{-3k^2k'}{2} - \frac{(\varepsilon a^2 - \mu)k'}{2}.$$

Thus,

$$\begin{aligned}&\left\langle (\nabla_T)^3 H - \frac{k'}{k}(\nabla_T)^2 H + k^2 \nabla_T H, B\right\rangle \\ &= \frac{-k'}{2k}[k^3 + (\varepsilon a^2 - \mu)k + 2k'' - 2k\tau(\tau - \varepsilon a)] = 0.\end{aligned}$$

□

Therefore, J_0 can be uniquely extended to a Killing vector field on $\mathbf{R}^3$. We denote it by $\tilde{J}_0$. This is a constant vector field on $\mathbf{R}^3$. Now, by using the Frenet frame, J_0 is expressed as follows.

$$J_0 = (k^2 + \varepsilon a^2 - \mu)T + 2k'N + 2k(\tau - \varepsilon a)B.$$

Then, by (2.18), (4.3), (4.6), (4.7), (4.8),

$$\begin{aligned} |J_0|^2 &= 4c - 4\varepsilon ab + (\varepsilon a^2 - \mu)^2 \\ &= (\alpha + \varepsilon a^2 - \mu)^2 + \frac{4}{\alpha}\left(b - \frac{\varepsilon a \alpha}{2}\right)^2. \end{aligned} \tag{5.7}$$

From now on, we assume $|J_0| \neq 0$. Define

$$J_1 = H - \frac{2d}{|J_0|^2} J_0,$$

where we set $d = \langle J_0, H \rangle /2 = b + \varepsilon a(\varepsilon a^2 - \mu)/2$. Then, $\langle J_0, J_1 \rangle = 0$. Also, by the linearity of the equations of Proposition 3.9, J_1 is a Killing vector field along γ. Denote by $\tilde{J}_1$ the unique extension of J_1 as a Killing vector field on $\mathbf{R}^3$. From now on we shall assume that $\tilde{J}_1$ is not a constant vector field on $\mathbf{R}^3$ and γ is not planar. Then we can verify that the following proposition holds.

Proposition 5.2. *The vector field $\tilde{J}_1$ is a rotation vector field perpendicular to $\tilde{J}_0$, namely, there exist a constant $\zeta \neq 0$ and a constant vector A such that $\langle \tilde{J}_0, A \rangle = 0$ and*

$$\tilde{J}_1(x) = \zeta \tilde{J}_0 \times (x - A), \quad x \in \mathbf{R}^3.$$

Thus we can take a system of cylindrical coordinates (r, θ, z) such that

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{|\tilde{J}_0|} \tilde{J}_0, \\ \frac{\partial}{\partial \theta} &= -Q \tilde{J}_1, \end{aligned}$$

where $Q > 0$ is a constant such that $|\partial / \partial \theta| = r$.

Proposition 5.3. *Suppose that $\gamma(s)$ does not pass through the z-axis. Let $r(s)$,*

$\theta(s), z(s)$ denote the r, θ, z components of $\gamma(s)$. Then,

$$r(s) = \frac{2}{|J_0|^2}\sqrt{(u(s)+\varepsilon^2a^2)|J_0|^2 - 4d^2}, \tag{5.8}$$

$$z(s) = \frac{1}{|J_0|}\int_0^s (u(s)+\varepsilon a^2 - \mu)ds + z(0), \tag{5.9}$$

$$\theta(s) = \int_0^s \frac{1}{|J_0|}\left[d + \left(\frac{|J_0|^2\sigma}{(\alpha+\varepsilon^2a^2)|J_0|^2 - 4d^2}\right) \times \left(\frac{1}{1-\xi^2\mathrm{sn}^2(ys,p)}\right)\right]ds + \theta(0), \tag{5.10}$$

where

$$\sigma = (-\varepsilon^2a^2 + \varepsilon a^2 - \mu)d + \frac{4d^3}{|J_0|^2} - \frac{\varepsilon a|J_0|^2}{2}, \tag{5.11}$$

$$\xi = \sqrt{\frac{\alpha q^2|J_0|^2}{(\alpha+\varepsilon^2a^2)|J_0|^2 - 4d^2}}. \tag{5.12}$$

Proof. (5.9) follows from

$$z' = \left\langle T, \frac{\partial}{\partial z}\right\rangle = \frac{u+\varepsilon a^2-\mu}{|J_0|}. \tag{5.13}$$

Also, $|H|^2 = u + \varepsilon^2a^2$ yields $|J_1|^2 = u+\varepsilon^2a^2 - (4d^2/|J_0|^2)$. Therefore we have

$$r = Q|J_1| = \frac{Q\sqrt{(u+\varepsilon^2a^2)|J_0|^2 - 4d^2}}{|J_0|}.$$

So r attains the maximum (resp. minimum) value $r_{\max}$ (resp. $r_{\min}$) if and only if $u=\alpha$ (resp. $u=\alpha_2$). Here we shall seek for Q. First we consider the case $b \neq \varepsilon a\alpha/2$. Suppose that $u(s_1)=\alpha$. We shall compare $r(s_1)$ $(= r_{\max})$ with $|J_1(s_1)|$. Denote by $T_h(s_1)$ the orthogonal projection of $T(s_1)$ to the direction $\partial/\partial\theta$. By (5.13),

$$T_h(s_1) = T(s_1) - \left(\frac{\alpha+\varepsilon a^2-\mu}{|J_0|}\right)\frac{\partial}{\partial z}. \tag{5.14}$$

Thus, by (5.7),

$$|T_h(s_1)| = \frac{2}{|J_0|\sqrt{\alpha}}\left|b - \frac{\varepsilon ar}{2}\right| > 0. \tag{5.15}$$

Therefore the curvature of the circle of radius $r(s_1)$ is expressed as

$$\frac{1}{r(s_1)} = \frac{1}{|T_h(s_1)|}\left|\left(\nabla_T \frac{J_1}{|J_1|}\right)(s_1)\right| = \frac{|(\nabla_T J_1)(s_1)|}{|T_h(s_1)|\,|J_1(s_1)|},$$

because $T(|J_1|) = 0$ at $s = s_1$. Here, since $|\nabla_T J_1| = |k(\varepsilon a - \tau)N + k_s B|$, we see $|(\nabla_T J_1)(s_1)| = (1/\sqrt{\alpha})\,|b - (\varepsilon a\alpha/2)|$, and $r(s_1) = 2\,|J_1(s_1)|\,/|J_0|$. Thus,

$$Q = \frac{r(s_1)}{|J_1(s_1)|} = \frac{2}{|J_0|}.$$

Next we consider the case $b = \varepsilon a\alpha/2$. Suppose that $u(s_2) = \alpha_2$. We shall compare $r(s_2)$ $(= r_{\min})$ with $|J_1(s_2)|$. By $|J_0|^2 = (\alpha + \varepsilon a^2 - \mu)^2$, $|J_1(s)|^2 = u(s)$. Thus $|J_1(s_2)| = \sqrt{\alpha_2}$. By the assumption $p \neq w$, $\sqrt{\alpha_2}$ is positive, and so is $r(s_2)$. Also, by (4.6), (4.7), and (4.8), $|J_0|^2$ is expressed as

$$|J_0|^2 = (\alpha_2 + \varepsilon a^2 - \mu)^2 + \frac{4}{\alpha_2}\left(b - \frac{\varepsilon a\alpha_2}{2}\right)^2. \tag{5.16}$$

Therefore, by the similar calculations of the former case,

$$|T_h(s_2)| = \frac{2}{|J_0|\sqrt{\alpha_2}}\left|b - \frac{\varepsilon a\alpha_2}{2}\right|. \tag{5.17}$$

If $b - (\varepsilon a\alpha_2/2) = 0$, then $a = 0$ or $\alpha = \alpha_2$. If $a = 0$, then $b = 0$, and so γ is a planar elastica. Thus $\alpha = \alpha_2$. But, by (4.6) and (4.8), $\alpha + \varepsilon a^2 - \mu = 0$. This implies $|J_0| = 0$, which contradicts the assumption. Therefore $|T_h(s_2)| > 0$. In the same way as the former case, we have $r(s_2) = 2\,|J_1(s_2)|\,/\,|J_0|$, and so $Q = 2/\,|J_0|$. Finally, we shall show the expression for θ'.

$$\begin{aligned}\theta'(s) &= \frac{\langle T, \partial/\partial\theta\rangle}{|\partial/\partial\theta|^2} = \frac{-\langle T, J_1\rangle}{Q\,|J_1|^2}\\ &= \frac{|J_0|\,[2d(u(s) + \varepsilon a^2 - \mu) - \varepsilon a\,|J_0|^2]}{2[|J_0|^2\,(u(s) + \varepsilon^2 a^2) - 4d^2]}\\ &= \frac{1}{|J_0|}\left[d + \left(\frac{|J_0|^2\,\sigma}{(\alpha + \varepsilon^2 a^2)\,|J_0|^2 - 4d^2}\right)\left(\frac{1}{1 - \xi^2 \mathrm{sn}^2(ys, p)}\right)\right].\end{aligned} \tag{5.18}$$

Then we get (5.10). □

From now on we mainly use the parameters α, η, p, w. Here we shall collect the relation between α, η, p, w and other parameters. We introduce the following notations:

$$V = \sqrt{1-w^2}, \qquad X = \sqrt{w^2-p^2}, \qquad Y_1 = 1+p^2-(1+\varepsilon^2\eta^2)w^2,$$
$$R = VX - \varepsilon\eta w^2, \qquad Y_2 = Y_1 - 2\varepsilon\eta R, \qquad Z = \sqrt{Y_1^2+4R^2},$$
$$S_1 = X(1-p^2-(1-\varepsilon^2\eta^2)w^2) - 2\varepsilon\eta w^2 V,$$
$$S_2 = V(1-p^2+(1-\varepsilon^2\eta^2)w^2) - 2\varepsilon\eta w^2 X.$$

Then the following relations hold.

(5.19) $$a = \pm a_* = \pm\eta\sqrt{\alpha},$$

(5.20) $$b = \pm\frac{\alpha^{3/2}}{2w^2}VX,$$

(5.21) $$\varepsilon a^2 - \mu = \frac{\alpha}{2w^2}[Y_1 - 2w^2],$$

(5.22) $$d = \pm\frac{\alpha^{3/2}}{4w^2}(\varepsilon\eta Y_1 + 2R),$$

(5.23) $$|J_0| = \frac{\alpha}{2w^2}Z,$$

(5.24) $$(1+\varepsilon^2\eta^2)|J_0|^2 - \frac{4d^2}{\alpha} = \frac{\alpha^2}{4w^4}Y_2^2,$$

(5.25) $$w^2Y_2^2 - p^2Z^2 = S_1^2,$$

(5.26) $$Z^2R - Y_2(\varepsilon\eta Y_1 + 2R)w^2 = S_1S_2,$$

(5.27) $$r_{\min} = \frac{4w|S_1|}{\sqrt{\alpha}Z^2},$$

(5.28) $$|J_0|^2\sigma = \pm\frac{\alpha^{9/2}}{16w^8}Y_2S_1S_2.$$

Proposition 5.4. *The orientation of the frame $(\partial/\partial r, \partial/\partial\theta, \partial/\partial z)$ is positive, that is, $\partial/\partial r \times \partial/\partial\theta = \partial/\partial z$.*

Proof. It suffices to show that ζ in Proposition 5.2 is negative. Suppose that $r(s_1) = r_{\max}$. Then, $N(s_1)$ is perpendicular to $J_0(s_1)$ and $J_1(s_1)$. Also, since $r''(s_1) < 0$, $N(s_1) = -\partial/\partial r|_{\gamma(s_1)}$. Therefore the sign of ζ corresponds to that of $\langle J_0 \times J_1, N\rangle|_{s=s_1}$. By calculation, we have

$$\langle J_0 \times J_1, N\rangle|_{s=s_1} = \frac{-\alpha^{3/2}Y_2}{2w^2} < 0.$$

□

Here, we shall recall the assumption in constructing the cylindrical coordinates.

Proposition 5.5. *If* $|J_0| = 0$ *or* $\tilde{J}_1$ *is a constant vector field on* $\mathbf{R}^3$*, then* γ *is an ordinary helix (or a circle) with curvature* $\sqrt{\alpha}$ *and torsion* $\pm\varepsilon\eta\sqrt{\alpha}$*. And the torsional parameterof* $\{\gamma, M\}$ *is* $\pm\eta\sqrt{\alpha}$.

Proof. Since Y_2 is expressed as

$$Y_2 = (1 - V^2)(\varepsilon\eta)^2 - 2VX\varepsilon\eta + 1 - X^2, \tag{5.29}$$

we see $Y_2 \geqslant 0$. Here the equality holds if and only if

$$\eta \geqslant 0, \quad p = 0, \quad w = \sqrt{\frac{1}{1 + \varepsilon^2\eta^2}}. \tag{5.30}$$

If $|J_0| = 0$, then $Z = 0$, and so $Y_2 = 0$. Therefore, (5.30) holds. By (4.1) and (4.2), γ is an ordinary helix (or a circle) with curvature $\sqrt{\alpha}$ and torsion $\pm\varepsilon\eta\sqrt{\alpha}$. And the torsional parameterof $\{\gamma, M\}$ is $\pm\eta\sqrt{\alpha}$. Conversely, if (5.30) holds, then $|J_0| = 0$. Also, we can verify that $\tilde{J}_1$ is a constant vector field on $\mathbf{R}^3$ if and only if the parameters satisfy (5.30). □

Proposition 5.6. *Suppose that* γ *is planar. If the torsional parametera* $\neq 0$*, then* γ *is a circle. Also, if* $a = 0$*, then* γ *is a planar elastica.*

Proof. If $a \neq 0$, then the curvature is constant and $b = -\varepsilon a\alpha/2$ by (2.18).Then we see that

$$\eta < 0, \quad p = 0, \quad w = \sqrt{\frac{1}{1 + \varepsilon^2\eta^2}}. \tag{5.31}$$

Therefore, γ is a circle. If $a = 0$, then $\eta = 0$ and $p = w$ or $\eta = 0$ and $w = 1$. In this case, γ is a planar elastica (cf. [5], [7], [10]). □

Consequently, if $|J_0| = 0$ or $\tilde{J}_1 = \text{const.}$ or γ is planar, $\{\gamma, M\}$ is a relatively trivial object. Next, we shall recall the assumption in Proposition 5.3.

Proposition 5.7. *Suppose that* γ *passes through the z-axis. Then* $r(s), z(s)$ *are expressed as* (5.8), (5.9) *respectively, and* $\theta(s)$ *as the following.*

$$\theta(s) = \frac{ds}{|J_0|} + m\pi, \quad (2m-1)K/y < s < (2m+1)K/y, \quad m \in \mathbf{Z} \tag{5.32}$$

Proof. In the same way as the proof of Proposition 5.3, we get the expression for $r(s)$ and $z(s)$. By (5.27), we see $S_1 = 0$. Thus, (5.28) yields $|J_0|^2 \sigma = 0$. Also, we can verify that (5.18) holds at the points where $r(s)$ does not vanish. Therefore, at such points we have

$$\theta'(s) = \frac{d}{|J_0|}.$$

Also, we can check that $T(s)$ is not parallel to the z-axis at the points s such that $u(s) = \alpha_2$. Therefore, (5.32) follows. □

Next, we shall consider the case that $p = w$. If $a = 0$, then γ is a planar elastica ([5], [6]). We assume $a \neq 0$.

Proposition 5.8. *Suppose that $p = w, a \neq 0$. Then, the cylindrical coordinates can be constructed and γ is also expressed in the same way as Proposition 5.3.*

Proof. If $p = w = 1$, then γ has no vanishing curvature. In this case, we can construct the cylindrical coordinates and γ is also expressed in the same way as Proposition 5.3. If $p = w \neq 1$, then γ has isolated inflection points. In this case, we take an interval I on which the curvature does not vanish. We define J_0, H on I and construct the cylindrical coordinates in the same way. Then, γ is expressed in the same way as Proposition 5.3 on I. We can check that γ does not intersect the z-axis and the expressions for r, θ, z are valid on the whole of $\mathbf{R}$ by the real analyticity of γ. □

6. Closed torsional elasticae

In this section, we investigate closed torsional elasticaeand show the second main theorem on bifurcation from the circles (Theorem 6.4). We first show that if $\{\gamma, M\}$ is a closed torsional elasticaand γ does not pass through the axis of the cylindrical coordinates, then γ lies on a torus of revolution and forms a torus knot. Next, we consider a smooth deformation of the circle such that all the curves with unit normalare closed torsional elasticae, and show the second main theorem.

We denote by $\Delta z, \Delta\theta$ the changes in $z(s), \theta(s)$, respectively, through the primitive period of k, that is,

$$\Delta z = z(s + 2K/y) - z(s), \qquad \Delta\theta = \theta(s + 2K/y) - \theta(s).$$

Here $z(s) - (ys\Delta z/(2K))$ and $\theta(s) - (ys\Delta\theta/(2K))$ are functions with primitive period $2K/y$. (Except the case $S_1 = 0$. If $S_1 = 0$, then $\theta(s) - (ys\Delta\theta/(2K))$ is a constant function.) Thus both $z(s)$ and $\theta(s)$ are expressed as sums of linear functions and functions with primitive period $2K/y$.

We consider the case that γ is periodic, that is, $\{\gamma, M\}$ is a closed torsional elastica. Suppose that γ has constant curvature, that is $p = 0$. Then, $\{\gamma, M\}$ is a closed torsional elasticaif and only if γ is a circle. In this case the parameters satisfy

$$\eta \leqslant 0, \quad p = 0, \quad w = \sqrt{\frac{1}{1+\varepsilon^2\eta^2}}.$$

On the other hand, a planar elastica $\{\gamma, M\}$ is a closed torsional elasticaif and only if γ is a circle or "figure eight" curve (cf. [6], [7]). If γ is neither a curve with constant curvature nor a plane curve, then the equivalent condition that γ is periodic is the following:

(6.1) $$\Delta z = 0 \quad \text{and} \quad \Delta\theta/2\pi \ \text{ is a rational number.}$$

Now we shall calculate Δz and $\Delta\theta$. We denote by $E = E(p)$ the complete elliptic integral of the second kind (cf. [1]). Since

$$\int_0^K \operatorname{sn}^2(x,p)dx = \int_0^{\pi/2} \frac{\sin^2\theta d\theta}{\sqrt{1-p^2\sin^2\theta}} = \frac{1}{p^2}(K-E),$$

we have

$$\Delta z = 2\int_0^{K/y} z' ds = \frac{2}{|J_0|\, y}\left[(\alpha+\varepsilon a^2-\mu)K - \frac{\alpha}{w^2}(K-E)\right].$$

Thus, by (5.21), the equivalent condition for $\Delta z = 0$ is

(6.2) $$w = \sqrt{\frac{1}{1+\varepsilon^2\eta^2}\left(\frac{2E(p)}{K(p)} + p^2 - 1\right)}.$$

Suppose that $\Delta z = 0$. Then z is a periodic function with primitive period $2K/y$. Also, $z(s)$ has just two critical points in each primitive period. Furthermore, $r(s)$ is critical at $s = mK/y$ for each integer m. Also, $z(s)$ is not critical at $s = mK/y$. Consequently, $(r(s), z(s))$ draws a simple closed curve with primitive period $2K/y$ in rz half plane. This curve is symmetric with respect to the straight line $z = z(0)$, since $r(s)$ is even and $z(s)$ is odd.

If $S_1 \neq 0$, then the simple closed curve is a real analytic curve which does not intersect the z-axis. Thus, $\gamma(t)$ has no self-intersection points, that is, $\gamma(t)$ forms a torus knot.

If $S_1 = 0$, then γ passes through the z-axis and the simple closed curve drawn by $(r(s), z(s))$ passes one point on the z-axis at $s = (2m+1)K/y \quad (m \in \mathbf{Z})$, which is a corner point of the curve. The point $\gamma((2m+1)K/y)$ is a self-intersection point of γ.

Therefore, we have the following theorem. We shall use below, the term torus of revolution to refer to the torus obtained by revolving a smooth simple closed curve in a half plane about its boundary line. (Here, the simple closed curve is assumed not to intersect the boundary line.) When the closed curve intersects one point on the boundary line, we call the surface of revolution obtained as above a degenerate torus of revolution. (Topologically, it is not a torus.)

Theorem 6.1. *Let $\{\gamma, M\}$ be a closed torsional elastica. Suppose that γ is neither a circle nor a planar "figure eight" elastica. If $S_1 \neq 0$ (so γ does not pass through the z-axis), then γ lies on a torus of revolution and forms a torus knot. If $S_1 = 0$, then γ lies on a degenerate torus of revolution and has a self-intersection point on the z-axis.*

Here we express $\Delta\theta$ in terms of the parameters η, p, w.

Proposition 6.2. *If $S_1 \neq 0$, then*

$$\Delta\theta = \pm\left[\frac{2w(\varepsilon\eta Y_1 + 2R)}{Z}K + \frac{2S_1S_2}{wY_2Z}\int_0^K \frac{dx}{1 - p^2Z^2\mathrm{sn}^2(x,p)/(w^2Y_2^2)}\right]. \tag{6.3}$$

Proof. By using (5.10), we have

$$\begin{aligned}\Delta\theta &= 2\int_0^{K/y}\theta'(s)ds \\ &= \frac{4w}{|J_0|\sqrt{\alpha}}\left(Kd + \frac{|J_0|^2\sigma}{(\alpha+\varepsilon^2a^2)|J_0|^2 - 4d^2}\int_0^K \frac{dx}{1-\xi^2\mathrm{sn}^2(x,p)}\right)\end{aligned}$$

Substituting (5.22), (5.23), (5.24), (5.28), we get

$$\Delta\theta = \pm\left[\frac{2w(\varepsilon\eta Y_1 + 2R)}{Z}K + \frac{2S_1S_2}{wY_2Z}\int_0^K \frac{dx}{1-\xi^2\mathrm{sn}^2(x,p)}\right].$$

By using (5.12), (5.23), (5.24), we see $\xi^2 = p^2Z^2/(w^2Y_2^2)$. Thus we obtain the expression. □

Next we consider a smooth deformation of the circle consisting of closed torsional elasticae. Let r_0 be a positive real number, and n, a positive integer. Let $\{\gamma, M\}_\lambda$ $(\lambda_0 > 0, |\lambda| < \lambda_0)$ be a C^∞ one-parameter family of closed torsional elasticaesatisfying the following three conditions:

(i) γ_0 is a circle of radius r_0, and the torsional parameterof $\{\gamma, M\}_0$ is not zero.
(ii) For $\lambda \neq 0$, γ_λ is not a circle.
(iii) For all λ, γ_λ have a period $2n\pi r_0$.
Here, we note that the angle between $M_\lambda(s)$ and $M_\lambda(s + 2n\pi r_0)$ may depend on λ.

Now let $a(\lambda)$ be the torsional parameterof $\{\gamma, M\}_\lambda$. We denote other parameters of $\{\gamma, M\}_\lambda$ by $b(\lambda), p(\lambda), w(\lambda)$, and so on. Then we have the following lemma.

Lemma 6.3. *There exists an integer $m(> n)$ such that*

(6.4)
$$\eta(0) = \frac{-1}{\varepsilon}\sqrt{\frac{m^2}{n^2} - 1},$$

and

(6.5)
$$\Delta\theta(\{\gamma, M\}_\lambda) \equiv \pm\frac{2n\pi}{m},$$

on a neighborhood of $\lambda = 0$ except at 0.

Proof. For any $\lambda(\neq 0)$, the primitive period of the curvature and torsion of γ_λ is $2K(p(\lambda))/y(\lambda) = 4K(p(\lambda))w(\lambda)/\sqrt{\alpha(\lambda)}$. Thus, by the condition (iii), there exists a positive integer $m(\lambda)$ such that

$$\left(\frac{4K(p(\lambda))w(\lambda)}{\sqrt{\alpha(\lambda)}}\right) m(\lambda) = 2n\pi r_0.$$

Therefore,

(6.6)
$$m(\lambda) = \frac{n\pi r_0\sqrt{\alpha(\lambda)}}{2K(p(\lambda))w(\lambda)}.$$

Since the right hand side of (6.6) is continuous with respect to λ on a neighborhood of 0, the integer $m(\lambda)$ is independent of λ. (We shall check the continuity below.) We simply write $m(\lambda)$ as m. By $\alpha(0) = 1/r_0^2, K(0) = \pi/2$, we see

$$m = n\sqrt{1 + \varepsilon^2\eta(0)^2}.$$

In particular, $m > n$. Since γ_0 is a circle, $\eta(0) \leqslant 0$. Then we have (6.4). Now we check the right hand side of (6.6) is continuous on a neighborhood of $\lambda = 0$. We may assume $k(\lambda, s) > 0$ for all $s \in \mathbf{R}$ if $|\lambda|$ is sufficiently small. From now on, we consider on a sufficiently small neighborhood of $\lambda = 0$. Since $M(\lambda), k(\lambda)$, and $\tau(\lambda)$ are C^∞ with respect to λ on a neighborhood of $\lambda = 0$, so are $a(\lambda)$ and $b(\lambda)$. Then, by (2.17) and (4.3), we see that $\mu(\lambda), c(\lambda)$ and the coefficients of the cubic polynomial $P(u)$ are also C^∞. By the condition (ii), $\alpha_2(\lambda) < \alpha_3(\lambda)$. Also, if $-\alpha_1(\lambda) = \alpha_2(\lambda) = 0$, then $p(\lambda) = w(\lambda) = 1$, which contradicts the assumption that γ_λ is periodic. Therefore if $\lambda \neq 0$, then $P(u) = 0$ has three distinct real roots $-\alpha_1(\lambda), \alpha_2(\lambda)$, and $\alpha_3(\lambda)$. Thus, by the implicit function theorem, we see that $\alpha_1(\lambda), \alpha_2(\lambda)$, and $\alpha_3(\lambda)$ are of class C^∞ except at $\lambda = 0$. Since $\alpha_2(\lambda) = \min_{s\in\mathbf{R}} k^2(\lambda, s)$ and $\alpha_3(\lambda) = \max_{s\in\mathbf{R}} k^2(\lambda, s)$, $\alpha_2(\lambda)$ and $\alpha_3(\lambda)$ are continuous at $\lambda = 0$. By (4.6), $\alpha_1(\lambda)$ is also continuous at $\lambda = 0$. And so, by (5.1), $p(\lambda)$ and $\eta(\lambda)^2$ are continuous, and C^∞ except at $\lambda = 0$. Thus, by the periodicity condition (6.2), $w(\lambda)$ is C^∞ except at $\lambda = 0$, and so is the right hand side of (6.6).

Next we shall show that $\Delta\theta(\{\gamma, M\}_\lambda)$ is continuous on a neighborhood of $\lambda = 0$. By the condition (i), $a(0) \neq 0$ and $b(0) \neq 0$. Thus, if $|\lambda|$ is sufficiently small, then both the signs of $a(\lambda)$ and $b(\lambda)$ are independent of λ. From now on, we assume $a(\lambda) < 0$ and $b(\lambda) > 0$. Then the $\pm$ sign of the right hand side of (6.3) is $+$. Viewing the contents of the bracket of (6.3) as a formal function of η, p, and w, we see this function is real analytic on a neighborhood of

$$(\eta, p, w) = \left(\frac{-1}{\varepsilon}\sqrt{\frac{m^2}{n^2} - 1}, 0, \frac{n}{m} \right). \tag{6.7}$$

Also, since $a_*(\lambda) = a(\lambda)$, $\eta(\lambda)$ is continuous on a neighborhood of $\lambda = 0$ and C^∞ except at $\lambda = 0$. Also, $S_1 \neq 0$ on a neighborhood of (6.7). Therefore, $\Delta\theta(\{\gamma, M\}_\lambda)$ is continuous on a neighborhood of $\lambda = 0$. Now, $\Delta\theta = 2n\pi/m$ at $\lambda = 0$, and $\Delta\theta(\{\gamma, M\}_\lambda)/(2\pi)$ is a rational number for all λ. Thus

$$\Delta\theta(\{\gamma, M\}_\lambda) \equiv \frac{2n\pi}{m}$$

on a neighborhood of $\lambda = 0$. In the case of $a(\lambda) > 0$ and $b(\lambda) < 0$, we have $\Delta\theta(\{\gamma, M\}_\lambda) \equiv -2n\pi/m$ in the same way as the above case. □

Finally we obtain the second main theorem.

Theorem 6.4.
(1) *Let m be an integer greater than n. We can construct a one-parameter family of closed torsional elasticae* $\{\gamma, M\}_\lambda^{m,n} = \{\gamma_\lambda^{m,n}, M_\lambda^{m,n}\}$, *which is real analytic in $\lambda \in I_{m,n}$, where $I_{m,n}$ is a neighborhood of* 0, *satisfying the above conditions* (i), (ii), *and* (iii) *and the following property: For all $\lambda \in I_{m,n}$, $\gamma_\lambda^{m,n}$ is $G_{m,n}$-symmetric, where $G_{m,n}$ is the group generated by the rotation about the z-axis by angle $2n\pi/m$. Here z-axis is the straight line which passes the center of the circle γ_0 and is perpendicular to the plane including γ_0. Furthermore, the following holds. Let d denote the greatest*

common divisor of m and n, and $\tilde{m} = m/d, \tilde{n} = n/d$. If the relatively prime pairs $\tilde{m}, \tilde{n}$ are distinct, then $\{\gamma, M\}_\lambda^{m,n}$ are geometrically distinct. Also, the knot type of $\gamma_\lambda^{m,n}|_{[0,2\tilde{n}\pi r_0]}$ is the $(\tilde{m}, \tilde{n})$-torus knot for each λ $(\neq 0)$.

(2) *Let $\{\hat{\gamma}, \hat{M}\}_\lambda$ $(\lambda_0 > 0, |\lambda| < \lambda_0)$ be a C^∞ one-parameter family of closed torsional elasticasatisfying the above* (i), (ii), *and* (iii). *Then, there exist an integer m $(> n)$ and a continuous function $p(\lambda)$ on a neighborhood U of $\lambda = 0$ satisfying the following: $p(\lambda) \geqslant 0$, $p(0) = 0$, $p(\lambda)$ is C^∞ on a neighborhood of $\lambda = 0$ except at $\lambda = 0$, and $\{\hat{\gamma}, \hat{M}\}_\lambda$ is isometric to $\{\gamma, M\}_{p(\lambda)}^{m,n}$ for each $\lambda \in U$.*

Proof. We show there exists a C^∞ one-parameter family of closed torsional elasticaesatisfying (i), (ii), and (iii). Let m be an arbitrary integer satisfying $m > n$. We denote by C_+ the $\Delta\theta$ in (6.3) with the sign $+$. We assume below, that (6.2) holds. We write the right hand side of (6.2) as $w(\eta, p)$. We think of C_+ as a formal function of η and p. Then $C_+(\eta, p)$ is an even function with respect to p. Let η_0 be a negative number. Then C_+ is real analytic on a neighborhood of $(\eta_0, 0)$ since $\xi^2 \neq 1, Y_2 \neq 0, Z \neq 0$. By calculation, we get

$$C_+(\eta_0, 0) = \frac{2\pi}{\sqrt{1 + \varepsilon^2\eta_0^2}}.$$

Thus,

$$\frac{\partial C_+}{\partial \eta}(\eta_0, 0) = \frac{-2\pi\varepsilon^2\eta_0}{(1 + \varepsilon^2\eta_0^2)^{3/2}} \neq 0.$$

So we can apply the implicit function theorem to $C_+(\eta, p)$ at $(\eta, p) = \left(\frac{-1}{\varepsilon}\sqrt{\frac{m^2}{n^2} - 1}, 0\right)$. Therefore there exist a neighborhood I $(= I_{m,n})$ of $p = 0$ and a real analytic function $\eta(p)$ $(= \eta_{m,n}(p))$ such that

$$C_+(\eta(p), p) = \frac{2n\pi}{m}. \tag{6.8}$$

Here we define $\alpha(p), a(p)$ by the following relations.

$$\frac{4mK(p)w(\eta(p), p)}{\sqrt{\alpha(p)}} = 2n\pi r_0, \tag{6.9}$$

$$a(p) = \eta(p)\sqrt{\alpha(p)}. \tag{6.10}$$

For each $p \in I_{m,n}$ we define a closed torsional elastica$\{\gamma, M\}_p = \{\gamma_p, M_p\}$ in the following way. Let (r, θ, z) be a system of cylindrical coordinates on $\mathbf{R}^3$ such that the orientation of the frame $(\partial/\partial r, \partial/\partial\theta, \partial/\partial z)$ is positive. We substitute $(\alpha, \eta, p, w) = (\alpha(p), \eta(p), |p|, w(\eta(p), p))$ to (5.8), (5.10), and (5.9) expressed in terms of the parameters α, η, p, w. We write them as $r(p,s), \theta(p,s), z(p,s)$. Let $\gamma_p(s)$ be the curve defined by $(r(p,s), \theta(p,s), z(p,s))$ in terms of the above cylindrical coordinates. Also, we define the unit normal $M_p(s)$ along $\gamma_p(s)$ by

$$M_p(s) = \mathcal{R}\left(a(p)s - \int_0^s \tau(p,s)ds\right) N_p(s).$$

Then the torsional functionof $\{\gamma, M\}_p$ agrees with $a(p)$. By (6.10), $\Delta\theta(\{\gamma, M\}_p)$ corresponds to $C_+(\eta(p), p)$. Thus, by (6.8), $\{\gamma, M\}_p$ satisfies the condition (6.1), and so $\{\gamma, M\}_p$ is a closed torsional elasticafor each p. Also, γ_p has a period $m(2K(p)/y(p))$ because $\Delta\theta(\{\gamma, M\}_p) = 2n\pi/m$. By (6.9), we have $m(2K(p)/y(p)) = 2n\pi r_0$. Therefore γ_p has a period $2n\pi r_0$. Furthermore, the primitive period of γ_p is $\tilde{m}(2K(p)/y(p)) = 2\tilde{n}\pi r_0$, because $\Delta\theta(\{\gamma, M\}_p) = 2\tilde{n}\pi/\tilde{m}$ and $\tilde{m}, \tilde{n}$ is a relatively prime pair. Therefore, the knot type of $\gamma_p|_{[0, 2\tilde{n}\pi r_0]}$ is the $(\tilde{m}, \tilde{n})$-torus knot for each p $(\neq 0)$. We shall denote below, the above $\{\gamma, M\}_p$ by $\{\gamma, M\}_p^{m,n}$.

We show (2). Let $\{\hat{\gamma}, \hat{M}\}_\lambda$ $(\lambda_0 > 0, |\lambda| < \lambda_0)$ be a C^∞ one-parameter family of closed torsional elasticasatisfying the conditions (i), (ii) and (iii). Suppose that $a(0) < 0$. By Lemma 6.3, there exists an integer m $(> n)$ such that $\eta(0) = \frac{-1}{\varepsilon}\sqrt{\frac{m^2}{n^2} - 1}$ and $\Delta\theta(\{\gamma, M\}_\lambda) \equiv 2n\pi/m$. Thus, the parameters $p(\lambda), \eta(\lambda)$ for $\{\hat{\gamma}, \hat{M}\}_\lambda$ satisfy the following equation:

$$C_+(\eta(\lambda), p(\lambda)) = \frac{2n\pi}{m}.$$

Therefore, by the implicit function theorem, $\eta(\lambda) = \eta_{m,n}(p(\lambda))$ when $|\lambda|$ is sufficiently small. (This also holds in the case of $a(0) > 0$.) Consequently, the parameters α, η, p, w for $\{\hat{\gamma}, \hat{M}\}_\lambda$ equal to those for $\{\gamma, M\}_{p(\lambda)}^{m,n}$, and so $\{\hat{\gamma}, \hat{M}\}_\lambda$ is isometric to $\{\gamma, M\}_{p(\lambda)}^{m,n}$. □

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