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Mass Distributions on the Ideal Boundaries of Abstract Riemann Surfaces, III

By Zenjiro Kuramochi

In the previous paper we defined a function $N(z, p)$ and ideal boundary points and studied some properties of superharmonic functions in $\hat{\mathbb{R}}$, but the mass distributions are only slightly discussed. In the present article, we rewrite pages from 174 to 176 of II in more precise form and continue the previous work. We use the same notations and definitions as in II.

**Theorem 1.** Let $p$ be a minimal point and $v(p)$ be a neighbourhood of $p$. Let $V^M(z)$ be a harmonic function in $v(p)$ such that $V^M(z) = \min(M, N(z, p))$ on $\partial v(p)$ and $V^M(z)$ has M.D.I. over $v(p)$. Put $V(z) = \lim_{k \to \infty} V^M(z) : M' = \sup N(z, p)$. Then $N(z, p) - V(z) = N'(z, p) > 0$ and $N'(z, p)$ has the same properties as $N(z, p)$.

Suppose $\sup N(z, p) = \infty$, i.e. $p$ is of capacity zero. Assume $V(z) = N(z, p)$. Then $N(z, p) = \int_{\hat{\mathbb{R}} - v(p)} N(z, q) d\mu(q)$. Since $N(z, p)$ is harmonic in $\hat{\mathbb{R}}$, $V(z) = \int_{\hat{\mathbb{R}} - v(p)} N(z, q) d\mu(q)$. If $\mu$ is a point mass, $N(z, p) = N(z, q) : q \notin v(p)$, which implies $p = q \notin v(p)$. This is a contradiction. Hence $\mu$ is not a point mass. Therefore there exist two positive mass distributions $\mu_i$ and $\mu_2$ such that $\mu = \mu_1 + \mu_2$ and both $V_1(z) = \int N(z, q) d\mu_1(q)$ and $V_2(z) = \int N(z, q) d\mu_2(q)$ are not multiples of $N(z, p)$. Because, if every $\mu_i$ presents a multiple of $N(z, p)$ and whose kernel $k_i$ tends to a point $q \notin v(p)$. Then $\lim_{i \to \infty} \frac{\mu_i}{\text{total mass of } \mu_i}$ represents $N(z, p) = N(z, q) : q \notin v(p)$. This is also a contradiction. Therefore $N(z, p) - V_1(z) (\geq 0)$ and $V_1(z) (\geq 0)$ are superharmonic in $\hat{\mathbb{R}}$, whence $N(z, p)$ is not minimal. Hence $V(z) < N(z, p)$. Next we show that $V(z)$ has no mass at $p$ in any canonical mass distribution. To the contrary, suppose $V(z)$ has a positive mass at $p$. Then

---


2) At present we cannot prove the uniqueness of canonical mass distributions.
where 0 < K < 1 and U(z) is superharmonic in $\tilde{R}$. Then $U(z) = (1 - K)N(z, p)$ on $\partial v(p)$ and superharmonic, whence $V_{R - v(p)}(z) \leq V(z)$. Now $V^M(z)$ has M.D.I. over $v(p)$ with value $V^M(z) = \min(M, N(z, p))$ on $\partial v(p)$, hence

$$V(z) \geq \left[ V_{R - v(p)}^M(z) \right] = V^M(z)$$

and

$$V(z) \geq V_{R - v(p)}(z) = \lim_{M \to \infty} V_{R - v(p)}^M(z) \geq \lim_{M \to \infty} V^M(z) = V(z),$$

whence

$$V_{R - v(p)}(z) = V(z) = KN_{R - v(p)}(z, p) + U_{R - v(p)}(z) \quad \text{in } v(p)$$

i.e.

$$V(z) = KV(z) + U_{R - v(p)}(z) \quad \text{in } v(p).$$

But $V(z) \leq N(z, p) + U_{R - v(p)}(z) \leq U(z)$. Hence

$$V(z) = KN(z, p) + U(z) > KV(z) + U_{R - v(p)}(z) = V(z).$$

This is a contradiction. Hence $V(z)$ has no mass at $p$.

Put $N'(z, p) = N(z, p) - V(z)$. Then $N'(z, p) = 0$ on $\partial v(p)$. Let $G' = \{ z \in R : N'(z, p) \geq M \}$ and let $v'(p)$ be a neighbourhood of $p$ such that $v'(p) \subset v(p)$. Then

$$N(z, p) \leq N_{v'(p) \cap G' M}(z, p) + N_{v'(p) \cap CG' M}(z, q)$$

$$= N_{v'(p) \cap G' M}(z, p) + V_{v'(p) \cap G' M}(z)$$

$$+ N_{v'(p) \cap CC' M}(z, p).$$

In page 158 (II), we proved that if $p$ is of capacity zero, $V(z) - V_{v(p)}(z)$ is superharmonic. If $V_{v(p)}(z) > 0$, then $V(z)$ has a positive mass at $p$. This is a contradiction. Hence $V_{v(p)}(z) = 0$.

Since $N'(z, p) = M$ on $\partial (v(p) \cap CG' M)$ and $p$ is of capacity zero,

$$\lim_{v'(p) \to p} N_{v'(p) \cap CC' M}(z, p) \leq \lim_{v'(p) \to p} (M \omega(v'(p), z) + V_{v'(p)}(z)) = 0,$$

where $\omega(v'(p), z)$ is C.P. (equilibrium potential) of $v'(p)$.

Hence

$$N(z, p) \leq \lim_{v'(p) \to p} N_{v'(p) \cap CC' M}(z, p) \leq N(z, p), \quad \text{whence}$$

$$N(z, p) = N_{v'(p) \cap CC' M}(z, p) \leq N_{v'(p)}(z, p) \leq N(z, p), \quad (1)$$

$$N(z, p) = N_{v'(p) \cap CC' M}(z, p) \leq N_{CC' M}(z, p) \leq N(z, p). \quad (2)$$

Suppose $S \subset v(p)$ and let $*V_{v(p)}^M(z)$ be a harmonic function in $v(p) - S$ such
that \( \*V_M^v(z) = \min (M, V(z)) \) on \( \partial S + \partial v(p) \) and has M.D.I. over \( v(p) - S \). Then by the definition of \( V(z) \), we have \( V(z) = \lim_{M \to -\infty} \*V_M^v(z) \) and, for \( N(z, p) \) we have by (1) and (2) the following

\[
N_{v'(p) \cap C_M'} (z, p) = N(z, p) \quad \text{and} \quad \lim_{M \to -\infty} N_{v(p) \cap C_M'} (z, p) = N'(z, p),
\]

where \( N_{v(p) \cap C_M'} (z, p) \) is a harmonic function in \( v(p) - (v(p) \cap G'M) \) such that \( N_{v(p) \cap C_M'} (z, p) = \min (M, N'(z, p)) \) on \( \partial v(p) + \partial (v(p) \cap G'M) \) and \( N_{v(p) \cap C_M'} (z, p) \) has M.D.I. over \( v(p) - (v(p) \cap G'M) \). Hence we have the following

**Property 1.** \( N_{v'(p) \cap C_M'} (z, p) = N'(z, p) \). As in page 153 (II) \( N_{v'(p) \cap C_M'} (z, p) = N'(z, p) \) and \( N_{v'(p) \cap R^n} (z, p) \)

\[
= \lim_{n} N_{v'(p) \cap C_M'} (z, p), \quad \text{where} \quad N_{v'(p) \cap C_M'} (z, p) \quad \text{is a harmonic function in} \quad R_m - R_0 - (G'M \cap R_m) \quad (m > n) \quad \text{such that} \quad N_{v'(p) \cap C_M'} (z, p) = N'(z, p)
\]
on \( G'M \cap R_n, \quad \partial \partial N_{v'(p) \cap C_M'} (z, p) = 0 \) on \( \partial R_m \). Let \( V_n(z) \) be a harmonic function in \( v(p) \) such that \( V_n(z) = N_{v'(p) \cap C_M'} (z, p) \) on \( \partial v(p) \) and \( V_n(z) \) has M.D.I. over \( v(p) \). Then \( V_n(z) = \lim_{n} V_m(z) \), where \( V_m(z) \) is a harmonic function in \( v(p) \cap R_m \) such that \( v_m(z) = V(z) \) on \( \partial v(p) \cap R_m \) and \( \partial \partial n V_m(z) = 0 \) on \( \partial R_m \cap v(p) \).

Since \( M = N(z, p) - V(z) = N'(z, p) \) on \( \partial G'M \),

\[
\int \frac{\partial}{\partial n} N_{v'(p) \cap C_M'} (z, p) ds = \int \frac{\partial}{\partial n} N_{v'(p) \cap C_M'} (z, p) ds \geq 2\pi - \varepsilon \quad \text{and}
\]

\[
\int \frac{\partial}{\partial n} V_m(z) ds = \int \frac{\partial}{\partial n} V_m(z) ds = 0,
\]

\[
2\pi \geq \int \frac{\partial}{\partial n} (N_{v'(p) \cap C_M'} (z, p) - n V_m(z)) \geq 2\pi - \varepsilon.
\]

Thus \( D_{v(p), m} (N_{v'(p) \cap C_M'} (z, p) - n V_m(z)) \leq 2\pi (M - \varepsilon) \).

Let \( m \to \infty \) and \( n \to \infty \). Then \( \{N_{v'(p) \cap C_M'} (z, p) - n V_m(z)\} \to \{N(z, p) - V(z)\} \) and \( \varepsilon \to 0 \). Then

\[
D_{v(p), \infty} (\min (M, N'(z, p))) \leq 2\pi M.
\]
On the other hand, since \( \frac{\partial}{\partial n} N_{n,m}(z, p) = \frac{\partial}{\partial n} V_n(z) = 0 \) on \( \partial R_n \cap v(p) \),

\[
D_{\kappa,\rho} (\min (M, N_n(z, p) - nV(z)) \geq D_{\kappa,\rho} - m\delta_{M-\varepsilon} (N_{n,m}(z, p) - nV_m(z)) \geq (2\pi - \varepsilon)(M - \delta) .
\]

Hence \( D_{\kappa,\rho} (\min (M, N'(z, p)) \geq (2\pi - \varepsilon)(M - \delta) \). Thus we have

**Property 2.** \( D_{\kappa,\rho} (\min (M, N'(z, p)) = 2\pi M, M < \infty . \)

Now \( N(z, p) \) has the same properties 1 and 2 as \( N(z, p) \). Therefore we can use \( N'(z, p) \) in stead of \( N(z, p) \) in \( \bar{R} \). As in case of \( \sup N(z, p) = \infty \) we have next.

If \( \sup N(z, p) < \infty \) and minimal, we have more easily the properties 1 and 2.

**Another definition of the value of a superharmonic function at a minimal point.**

In the previous paper, we defined the value of a superharmonic function \( U(z) \) at a minimal point \( p \) by

\[
U(p) = \lim \frac{1}{\pi M} \int_{C_M} U(z) \frac{\partial}{\partial n} N(z, p) d s , \tag{1)
\]

where \( M = \sup N(z, p) \) and \( C_M = E[z \in R: N(z, p) = M] \) is regular i.e. \( \int_{C_M} \frac{\partial}{\partial n} N(z, p) d s = 2\pi . \)

Above definition is inconvenient in the sense as follows: *every regular curve \( C_M \) encloses a neighbourhood \( v(p) \) but \( v(p) \) does not necessarily contain the set \( E[z \in R: N(z, p) > M] \) for any large number \( M \). In the above definition \( U(p) \) depends on a larger set than \( v(p) \). It is better to define \( U(p) \) on the behaviour of \( U(z) \) in \( v(p) \). Therefore we shall give more useful definition of \( U(p) \). \( N'(z, p) \) in \( v(p) \) in Theorem 1 has the properties. We can prove as in case of \( N(z, p) \) that there exists a set \( E \) in the interval \((0, M)\) \( (M = \sup N'(z, p)) \) such that \( \operatorname{mes} E = 0 \) and \( E \not\in \delta \) implies that \( C_\delta = E[z \in R: N'(z, p) = \delta] \) is regular i.e. \( \int_{C_\delta} \frac{\partial}{\partial n} N'(z, p) d s = 2\pi . \)

**Theorem 2.** Let \( U(z) \) be a superharmonic function in \( \bar{R} \) and let \( v(p) \) be a neighbourhood of \( p \). Let \( N'(z, p) \) be the function in Theorem 1.

---

3) In II we defined for \( N(z, p) \), but the same facts hold for \( U(z) \). It is easily seen that \( U(p) = \int_{V_{\alpha}} N(p, q) d\mu(q) \), where \( U(z) = \int_{V_{\alpha}} N(z, p) d\mu(q) \), i.e., \( \mu \) is a canonical distribution of \( U(z) \).
Then \( U^*(z) = \lim_{\beta \to M'} \frac{1}{2\pi} \int_{\beta} U(z) \frac{\partial}{\partial n} N'(z, p) ds \) exists and

\[ U(p) = U^*(p), \]

where \( M' = \sup N'(z, p) \) and \( C_\beta \) is a regular curve of \( N'(z, p) \).

**Lemma 1.** Let \( U(z) \) be a harmonic function in \( \bar{R} \cap E[z \in R : N(z, p) > \alpha] \)

\( (=V(p)) \) with continuous boundary value on \( \partial V(p) \). Then

\[ U(p) = U^*(p). \]

Suppose \( C_\alpha = E[z \in R : N(z, p) = \alpha] \) and \( C_\beta = E[z \in R : N'(z, p) = \beta] \) are regular. Let \( U^S_n(z) \) be a harmonic function in \( R_n \cap V(p) \) such that \( U^S_n(z) = \min(U(z), S) \) on \( \partial V(p) \cap R_n \) and \( \frac{\partial}{\partial n} U^S_n(z) = 0 \) on \( \partial R_n \wedge V(p) \). Then \( U^S_n(z) \to U^S(z) \) in mean and \( U^S(z) \uparrow U(z) \) as \( S \to \infty \).

Let \( V^L_n(z) \) be a harmonic function in \( v(p) \cap R_n \) such that \( V^L_n(z) = \min(L, N(z, p)) \) on \( R_n \cap \partial v(p) \) and \( \frac{\partial}{\partial n} V^L_n(z) = 0 \) on \( \partial R_n \wedge v(p) \). Then \( V^L_n(z) \to V^L(z) \) in mean and \( V^L(z) \uparrow V(z) \). Let \( N^L_n(z, p) \) be a harmonic function in \( E[z \in R : N(z, p) > \alpha] \) such that \( N^L_n(z, p) = \alpha \) on \( C_\alpha \cap R_n \), \( N^L_n(z, p) = \beta + V^L_n(z) \) on \( C_\beta \cap R_n \) and \( \frac{\partial}{\partial n} N^L_n(z, p) = 0 \) on \( \partial R_n \cap (E[z \in R : N(z, p) > \alpha] - E[z \in R : N'(z, p) > \beta]). \)

Then \( N^L_n(z, p) - V^L_n(z) = N^L_n(z, p) \) and \( N^L_n(z, p) \to N^L(z, p), N^L_n(z, p) \to N^L_n(z, p) \) in mean and \( N^L(z, p) \uparrow N(z, p), N^L(z, p) \uparrow N(z, p) \).

Now it is proved (similarly as page 151 (II)) that \( C_\alpha \) and \( C_\beta \) are also regular for \( N^L(z, p) \) and \( N^L(z, p) \) respectively, i.e.

\[ \int_{C_\alpha} U(z) \frac{\partial}{\partial n} N^L_n(z, p) ds \to \int_{C_\alpha} U(z) \frac{\partial}{\partial n} N^L_n(z, p) ds \quad \text{and} \quad \int_{C_\beta} U(z) \frac{\partial}{\partial n} N^L_n(z, p) ds \to \int_{C_\beta} U(z) \frac{\partial}{\partial n} N^L_n(z, p) ds. \]

Apply the green's formula to \( U^S_n(z) \) and \( N^L_n(z, p) \) in \( E[z \in R : N(z, p) > \alpha] \) and

\[ \int_{C_\alpha \cap R_n} \frac{\partial}{\partial n} U^S_n(z) ds = \int_{\partial R_n \cap E[z \in R : N(z, p) > \alpha]} \frac{\partial}{\partial n} U^S_n(z) ds = 0. \]
Next
\[ \int_{(c_{\beta} \cap \partial R_n) \cap R_n} U_n^S(z) \frac{\partial}{\partial n} (N_n^L(z, p) - V_n^L(z)) \, ds = \int_{(c_{\beta} \cap \partial R_n) \cap R_n} (N_n^L(z, p) - V_n^L(z)) \, ds, \]

By
\[ \int_{(c_{\beta} \cap \partial R_n) \cap R_n} U_n^S(z) \frac{\partial}{\partial n} U_n^S(z) \, ds = 0, \]

But
\[ \int_{(c_{\beta} \cap \partial R_n) \cap R_n} V_n^L(z) \frac{\partial}{\partial n} U_n^S(z) \, ds = \int_{(c_{\beta} \cap \partial R_n) \cap R_n} U_n^S(z) \frac{\partial}{\partial n} V_n^L(z) \, ds, \]

hence the term on
the left hand side of (4) = the term on the right hand side of (3). Since \( 0 < U_n^S(z) \leq S \) and by the regularity of \( C_n \) and \( C_\beta \), we have by letting \( n \to \infty \) and then \( L \to \infty \) and then \( S \to \infty \), we have
\[ U(z) = \int_{c_{\beta} \cap R_n} U(z) \frac{\partial}{\partial n} N(z, p) \, ds = \int_{c_{\beta} \cap R_n} U(z) \frac{\partial}{\partial n} N'(z, p) \, ds = U^*(p). \]

Proof of the theorem. Let \( U(z) \) be superharmonic in \( \bar{R} \). In every \( V(p) = E[z \in R : N(z, p) > \alpha] \) there exists a \( v(p) \subset V(p) \). Let \( U^V(z) \) be a *harmonic function in \( \bar{R} \cap V(p)^\circ \) with value \( U^V(z) = U(z) \) on \( \partial V(p) \) and let \( U^v(z) \) be harmonic in \( \bar{R} \cap v(p) \) with value \( U^v(z) = U(z) \) on \( \partial v(p) \). Then \( U^v(z) \leq U^V(z) \). Hence by Lemma 1
\[ U^V(p) = U^*V(p) \leq U^*v(p). \]

Clearly
\[ U^{G\beta}(p) = \int_{c_{\beta}} U(z) \frac{\partial}{\partial n} N'(z, p) \, ds \leq \int_{c_{\beta}} U(z) \frac{\partial}{\partial n} N'(z, p) \, ds \leq U^{G\beta'}(p) \]
for regular \( C_\beta \) and \( C_{\beta'}(\beta < \beta') \) by the superharmonicity of \( U(z) \), where \( G_\beta = E[z \in R : N'(z, p) > \beta] \). Hence \( \lim_{\beta \to \alpha'} \int_{c_{\beta}} U(z) \frac{\partial}{\partial n} N'(z, p) \, ds \) exists. We
4) If \( U(z) = \lim_{M \to \infty} U^M(z) \), we say \( U(z) \) is *harmonic in \( G \), where \( U^M(z) = (\min (M, U(z)) \) on \( \partial G \) and \( U^M(z) \) has M.D.I. over \( G \).
define \( U^*(z) \) by this limit. Thus

\[
U(p) \leq U^*(p) .
\]

Next we show \( U(p) \geq U^*(p) \). We suppose \( U^*(p) < \infty \). Then by definition of \( U^*(p) \), there exist \( v(p) \), \( N'(z, p) \) and a regular curve \( C \) of \( N'(z, p) \) for any given positive number \( \varepsilon \) such that

\[
U^*(p) - \varepsilon \leq \frac{1}{2\pi} \int C U(z) \frac{\partial}{\partial n} N'(z, p) ds .
\]

Let \( U_A(z) \) be the lower envelope of superharmonic functions in \( \bar{R} \) larger than \( U(z) \) on \( A \). Then by the superharmonicity of \( U(z) \),

\[
U_{Cv(p) \cap R_n}(z) \uparrow U_{Cv(p)}(z) = U^v(p)(z) \text{ as } n \to \infty .
\]

\( V_M(p) = E[z \in R : N(z, p) > M] \) clusters at the ideal boundary as \( M \uparrow \sup N(z, p) \). Therefore we can find a number \( n_0 \) and \( M \) such that

\[
U^*(p) - 2\varepsilon \leq \frac{1}{2\pi} \int C U(z) \frac{\partial}{\partial n} N'(z, p) ds \quad (n \geq n_0)
\]

and \( (R - R_{n_0}) \supset V_M(p) \) for the same \( \varepsilon > 0 \).

Since \( (Cv(p) \cap R_{n_0}) \subset Cv(p) + CV_M(p) U_{Cv(p) \cap R_n}(z) \leq U_{Cv(p) + CV_M(p)}(z) \) and \( U^*(p) - 2\varepsilon \leq \frac{1}{2\pi} \int C U_{Cv(p) + CV_M(p)}(z) \frac{\partial}{\partial n} N'(z, p) ds \) for every \( V_M(p) \) such that \( V_{M(p)} \subset (R - R_{n_0}) \).

Now \( N'(z, p) = N(z, p) - V(z) \), where \( V(z) \) is harmonic in \( \bar{R} \cap v(p) \) such that \( V(z) = N(z, p) \) on \( \partial v(p) \), i.e. \( V(z) = N_{Cv(p)}(z, p) \) in \( v(p) \). Hence \( N_{Cv(p)}(z, p) \uparrow N_{Cv(p),+Cv(p)}(z, p) \) as \( V_M(p) = V(z) \) as \( M \uparrow 0 \).

Hence the niveau curve \( C' = E[z \in R : N(z, p) - V(z) = k] \) tends to \( E[z \in R : N(z, p) - V(z) = k] \) and further, \( \frac{\partial}{\partial n} (N(z, p) - N_{Cv(p)}(z, p)) \) on \( C' \) tends to \( \frac{\partial}{\partial n} \) \( N'(z, p) \) on \( C \) as \( M \uparrow \sup N(z, p) \). Hence there exists \( M' > M \) such that

\[
U^*(p) - 2\varepsilon \leq \frac{1}{2\pi} \int C U_{Cv(p) + CV_M(p)}(z) \frac{\partial}{\partial n} N'(z, p) ds
\]

\[
\leq \frac{1}{2\pi} \int C U_{Cv(p) + CV_M'(p)}(z) \frac{\partial}{\partial n} (N(z, p) - N_{Cv(p) + CV_M'(p)}(z, p)) ds + \varepsilon ,
\]

\[
U^*(p) - 3\varepsilon \leq \frac{1}{2\pi} \int C U_{Cv(p) + CV_M'(p)}(z) \frac{\partial}{\partial n} (N(z, p) - N_{Cv(p) + CV_M'(p)}(z, p)) ds \quad (6),
\]
where \( C^* = E[z \in R; N(z, p)-N_{C_\rho+p+C V_M'(z, p)} = k] \). Next suppose \( v(p) + V_M' > V_M''(M'' > M') \) and \( V_M'' = E[z \in R; N(z, p) > M''] \) and \( \partial V_M'' \) is regular. Then similarly it is proved that

\[
\frac{1}{2\pi} \int_{C^*} U_{C_\rho+p+C V_M'(z)}(z) \frac{\partial}{\partial n} (N(z, p)-N_{C_\rho+p+C V_M'(z, p)}) ds
\leq \frac{1}{2\pi} \int_{C''} U_{C V_M''(z)}(z) \frac{\partial}{\partial n} (N(z, p)-N_{C V_M''(z, p)}) ds ,
\]

where \( C^* \) and \( C'' \) are regular niceau curves of \( N(z, p)-N_{C_\rho+p+C V_M'(z, p)} \) and \( N(z, p)-N_{C V_M''(z, p)} \) respectively. Since \( N(z, p)-N_{C V_M''(z, p)} = N(z, p)-M'' \). Hence by letting \( \varepsilon \to 0 \), the last term of (7) = \[ \frac{1}{2\pi} \int_{C'} U(z) \frac{\partial}{\partial n} N(z, p) \leq U(p) ,
\]
whence

\[ U(p) = U^*(p) . \]

In case \( U^*(p) = \infty \), we can prove similarly.

**Properties of functiontheoretic equilibrium potential.**

Let \( G \) be a non compact domain in \( R-R_0 \) and let \( \omega_{n,n+i}(z) \) be a harmonic function in \( R_{n+i}-(G \cap (R_{n+i}-R_n)) \) such that \( \omega_{n,n+i}(z) = 0 \) on \( \partial R_n \), \( \omega_{n,n+i}(z) = 1 \) on \( \partial(G \cap (R_{n+i}-R_n)) \) and \( \frac{\partial \omega_{n,n+i}(z)}{\partial n} = 0 \) on \( \partial R_{n+i}-G \). Then it is proved (pp, 145 and 154) that \( \omega_{n,n+i}(z) \to \omega_n(z) \) in mean as \( i \to \infty \) and \( \omega_n(z) \to \omega(z) \) in mean as \( n \to \infty \) and that \( \omega(z) \) is superharmonic function in \( \overline{R} \). We call \( \omega(z) \) the (functiontheoretic) equilibrium potential of the ideal boundary \( (B \cap G) \) determined by \( G \). Let \( F \) be a closed set. Put \( F_m = E[z \in \overline{R} : \delta(z, F) < \frac{1}{m}] \) and \( \omega_m(z) \) C.P. (equilibrium potential) of \( F_m \). Then \( \omega_m(z) \to \omega(z) \) in mean.

**Lemma 2.** If \( \omega(z) \), C.P. of \( (G \cap B) \) determined by \( G \) is not zero, \( \sup_{t \in \theta} \omega(z) = 1 \). Put \( G_\delta = E[z \in R : \omega(z) < 1-\delta] \), \( \delta > 0 \). Then \( (B \cap G \cap G_\delta) \) is of capacity zero.

Since \( \omega_n(z) \to \omega(z) \) in mean, \( \omega(z) = \omega'(z) \), where \( \omega'(z) \) is a harmonic function in \( R-R_0-(G \cap (R-R_n)) \) such that \( \omega'(z) = \omega(z) \) on \( \partial(G \cap (R-R_n)) \) and \( \omega'(z) \) has M.D.I.\(^5\) Suppose \( \sup_{t \in \theta} \omega(z) \leq K < 1 \). Let \( \omega_{n,n+i}(z) \) be a harmonic function in \( R_{n+i}-R_0-(G \cap (R-R_n)) \) such that \( \omega_{n,n+i}(z) = \omega(z) \) on \( \partial R_n + \partial(G \cap (R-R_n)) \) and \( \frac{\partial}{\partial n} \omega_{n,n+i}(z) = 0 \) on \( \partial R_{n+i}-G \). Then

\[ K \omega_{n,n+i}(z) \geq \omega_{n,n+i}(z) . \]

\(^5\) We abbreviate minimal Dirichlet integral by M.D.I.
Let $i \to \infty$ and $n \to \infty$. Then

$$K \omega(z) > \omega'(z) = \omega(z),$$

whence $\omega(z) = 0$. This is a contradiction.

Next let $\omega^3(z)$ be C.P. of $(B \cap G \cap G_\delta)$. Then $\omega^3(z) \leq \omega(z)$ and $\sup_{z \in G_\delta} \omega^3(z) \leq 1 - \delta$. This implies $\omega^3(z) \equiv 0$. Hence we have Lemma 2.

Let $\omega(z)$ be C.P. of closed set $F$. Then $\omega(z)$ is superharmonic in $\bar{R}$ and the value of $\omega(z)$ is defined in $\bar{R} = (R + B)$ (see Theorem 1) and it is proved that $\omega(z)$ is lower semicontinuous in $\bar{R}$ (see II).

**Theorem 3.** Let $F$ be a closed set of positive capacity and let $\omega(z)$ be C.P. of $F$. Then $\omega(z) = 1$ except at most an $F_*$ of capacity zero.

**Lemma 3.** Let $\omega(z)$ be C.P. of $F$ of positive capacity. Then $\sup_{z \in F} \omega(z) = 1$.

Since $F = \bigcap_{n \geq 0} F_m$ and $F_m$ can be considered as a non compact domain, it is clear $\sup_{z \in F_m} \omega(z) = 1$ for every $n$, but our assertion is not clear. If $F$ has a closed subset $F'$ of $F$ of positive capacity in $R$, our assertion is trivial. Hence we suppose $F \subset B$. Put $G_K = E[z \in R: \omega(z) < K < 1]$. Then $G_K \cap R$ is an open set. Let $G'_K$ be a component of $G_K$. Assume that $G_K$ has a positive distance from $\partial R_0$, then $\omega(z) < K$ in $G'_K \cap R$ and $\omega(z) = K$ on $\partial G'_K \cap R$. But by the superharmonicity of $\omega(z)$, $\omega(z) \geq H(z) = K$, where $H(z)$ is harmonic in $R \cap G'_K$ such that $H(z) = K$ on $\partial G'_K \cap R$ and $H(z)$ has M.D.I. On the other hand, $\omega(z) < K$ in $G'_K$, whence $\omega(z) = K$ in $G'_K$. But $\omega(z)$ is a non constant in $R$. This is a contradiction. Hence $G'_K$ has a subset of $\partial R_0$ as its boundary. Now $0 < \omega(z) < \varepsilon$ in a neighbourhood of $\partial R_0$ for any positive number $\varepsilon > 0$. Therefore $G'_K$ has $\partial R_0$ in its boundary which implies that $G_K$ consists of only one component.

Assume $\omega(p) = K(\leq 1)$ and that $p \in (R + B)$ has a positive distance from $G_K$. Then there exists a neighbourhood $v(p) \subset CG_K \cap \bar{R}$. Then

$$K = \omega(p) \geq \frac{1}{2\pi} \int_{\partial} \omega(z) \frac{\partial}{\partial n} N'(z, p) ds > K,$$

by the non-constancy of $\omega(z)$ in $R$. Hence every point $p \in (R + B)$ such that $\omega(p) = K$ is a limit point of a sequence $\{z_i\}$ $(z_i \in G_K)$.

Let $p \in (R + B)$ such that $\omega(p) = K$. Then $p \in G_K \subset \bar{G}_{K+\delta}$ (closure of $G_{K+\delta}$), where $G_{K+\delta} = E[z \in R: \omega(z) < K + \delta]$ for any given positive number $\delta > 0$. Since $\omega(z)$ is lower semicontinuous, there exists a neighbourhood $v(p) \subset F_m$ such that $\omega(z) > K - \varepsilon: z \in v(p)$ for any given positive number $\varepsilon > 0$. 


\[ \omega(p) \geq \frac{1}{2\pi} \left( \int_{c \cap \partial K + \delta} \omega(z) \frac{\partial}{\partial n} N'(z, p) \, ds + \int_{c \cap \partial K + \delta} \omega(z) \frac{\partial}{\partial n} N'(z, p) \, ds \right) \quad (8) \]

Assume \[ \int_{c \cap \partial K + \delta} \frac{\partial}{\partial n} N'(z, p) \, ds > \pi. \] Then

\[ \omega(p) \geq \frac{1}{2\pi} (K - \varepsilon)\pi + (K - \delta)\pi \geq \frac{\delta - \varepsilon}{2} + K. \]

This is a contradiction for \[ \varepsilon < \frac{\delta}{3}. \] Hence we have the following assertion.

Let \[ p \in B_1 + R \text{ with } \omega(p) = K < 1. \] Then for any \( \delta > 0 \), we can find a \( v(p) \) in \( F_m \) such that whose \( N'(z, p) \) satisfies the following condition

\[ \int_{c \cap \partial K + \delta} \frac{\partial}{\partial n} N'(z, p) \, ds \geq \pi \quad (9) \]

for every regular curve \( C \) of \( N'(z, p) \).

Put \( H_K = E[z \in \mathbb{R} : \omega(z) \leq K] \). Then \( H_K \) is closed by the lower semi-continuity of \( \omega(z) \). Then \( F \cap H_K \) is also closed. We show that \( F \cap H_K \) is a set of capacity zero. Let \( \omega_m(z) \) be a superharmonic function in \( R - R_0 \) such that \( \omega_m(z) = 0 \) on \( \partial R_0 \), \( \omega_m(z) = 1 \) on \( F_m \cap G_{k+\delta} \cap (R - R_m) \) and \( \omega_m(z) \) has M.D.I. Then \( \omega_m(z) \rightarrow \omega'(z) \), where \( \omega'(z) \) is C.P. of the boundary determined by \( \bigcap_{m \geq 1} F_m \cap G_{k+\delta} \) (\( m = 1, 2, \ldots \)). Hence by Lemma 2, \( \omega'(z) = 0 \).

Choose a sequence \( m_1, m_2, \ldots \) such that \( \int_{\partial G_{k+\delta}} \frac{\partial}{\partial n} \omega_m(z) \, ds \leq \frac{1}{2^m} \). Then

\[ \omega^*(z) = \sum_{m_i} \omega_m(z) < \infty \]

and

\[ \lim \omega^*(z) = \infty \quad \text{as } z \text{ tends to } F \text{ inside of } G_{k+\delta}. \]

Let \( p \in (F \cap H_K \cap B) \). Then \( \omega^*(p) \geq \frac{1}{2\pi} \int_{c} \omega^*(z) \frac{\partial}{\partial n} N'(z, p) \, ds \) whence by (9) \( \omega^*(p) = \infty \) and the lower semicontinuity of \( \omega^*(z) \),

\[ \lim_{z \rightarrow \delta \in (F \cap H_K)} \omega^*(z) = \infty \]

\( B_0 \) (set of non minimal points) is a sum of closed sets of capacity zero. We can construct as above a superharmonic function \( \omega^{**}(z) \) such that \( \lim \omega^{**}(z) = \infty \).

Proof of Lemma 3. Suppose \( \omega(z) \leq K < 1 \). Then \( \lim_{z \rightarrow \delta \in F} \varepsilon(\omega^*(z) + \omega^{**}(z)) = \infty \) for any \( \varepsilon > 0 \). Put \( \Delta_\varepsilon = E[z \in \mathbb{R} : \varepsilon(\omega^*(z) + \omega^{**}(z)) \leq 2] \). Then \( \Delta_\varepsilon \) is also closed and \( \Delta_\varepsilon \cap F = \emptyset \), which implies \( \text{dist}(\Delta_\varepsilon, F) > d_\varepsilon > 0 \).
Mass Distributions on the Ideal Boundaries of Abstract Riemann Surfaces, III

Put $F_{dt} = E[z \in R: \delta(z, F) \leq d_t]$. Let $\omega_{\epsilon}(z)$ be C.P. of $F_{dt}$. Then

$$\epsilon(\omega^*(z) + \omega^**(z)) \geq \omega_{\epsilon}(z) \geq \omega(z).$$

By letting $\epsilon \to 0$. We have $\omega(z) = 0$.

This a contradiction. Hence $\sup_{z \in R} \omega(z) = 1$.

**Proof of Theorem 3.** Let $\omega_k(z)$ be C.P. of $E_k = \{z \in (\overline{R} \cap F): \omega(z) \leq 1 - \frac{1}{k}\} (k = 1, 2, \ldots)$. Then $\omega_k(z) \leq \omega(z)$, whence $\sup_{z \in R} \omega_k(z) \leq 1 - \frac{1}{k}$. Hence by lemma 3 $E_k$ is a set of capacity zero. Then $E = \bigcup_{k > 0} E_k$ is an $F_\sigma$ of capacity zero.

**Theorem 4.** Let $\omega(z)$ be C.P. of a closed set $F$ of positive capacity. $R - F$ consists at most enumerably infinite number of domains. Let $G$ be one containing $\partial R_0$ in its boundary. Then $\omega(z) < 1$ in $G$ except at most capacity zero.

Since $\omega(z)$ is harmonic in $R - F$, $\omega(z) < 1$ in $R \cap G$. Suppose $p$ is a point in $(B \cap G \cap B_i)$.

Then there exists a neighbourhood $v(p)$ such that $v(p) \subset G$. Then

$$\omega(p) = \frac{1}{2\pi} \int_{B \cap R} \omega(z) \frac{\partial}{\partial n} N'(z, p) ds < 1$$

because $\omega(z)$ is non constant harmonic in $G - F$, i.e. $\omega(z)$ has M.D.I. over $v(p)$. On the other hand, $B_0$ is a set of capacity zero. Hence we have the theorem.

**Mass distribution on $\overline{R}$.** We have seen that $N(z, p)$ and $N'(z, p)$ have the essential properties of the logarithmic potential: lower semicontinuity in $\overline{R}$, symmetry and superharmonicity in the sense as follow:

$$N(q, p) \geq \frac{1}{2\pi} \int_{\partial B} N(z, q) \frac{\partial}{\partial n} N'(z, p) ds$$

for every $v(p)$ of $p \in R + B_i$, where $N'(z, p)$ is the function in $v(p)$ in Theorem 1. But there exists a fatal difference between our space and the euclidean space, that is, in our space there may exist points of $B_0$ where we cannot distribute any true mass. A distribution $\mu$ on $B_0$ may be called a pseudo distribution in the sense that $\mu$ can be replaced, by Theorem 8 of II, by a canonical distribution on $B_0 + R$ without any change of $U(z) = \int N(z, p) d\mu(p)$. Hence it is sufficient to consider only canonical distributions.

---

6) $G$ is open with respect to Martin's topology, whence $G$ may contain points of the ideal boundary.
Energy integral $I(\mu)$ of a canonical mass distribution on $R + B_1$ is defined as

$$I(\mu) = \int \int N(q, p) d\mu(p) d\mu(q).$$

*Capacity (potential theoretic)* of a closed set $F$ in $\bar{R}$ is defined by

$$\frac{1}{\inf I(\mu)},$$

where $\mu$ is a canonical distribution of $F \cap (R + B_1)$ of total mass unity.

**Lemma 4.** $\text{Cap}(F) > 0$ implies $\text{Cap}(F) > 0$ for every closed subset $F$ of $\bar{R}$.

In fact, if $\text{Cap}(F) > 0$, there exists C.P. of $F$ such that $\omega(z) = \omega_F(z)$ and $\omega_F(z)$ is represented by a mass distribution $\mu$ on $F$. $\omega_{F \cap (R + B_1)}(z) \leq \omega_F(z) \leq \omega_{F \cap (R + B_1)}(z) + \omega_{B_0}(z)$. But $\omega_{B_0}(z) = 0$ by Theorem 8 in (II), hence $\omega(z) = \omega_{F \cap (R + B_1)}(z)$ and $\omega(z)$ is represented by a canonical distribution on $F \cap (R + B_1)$ and the total mass is given by $\int_{\partial B_0} \frac{\partial \omega}{\partial n}(z) ds$. Since $\sup \omega(z) = 1$, $I(\mu) < \infty$. This implies $\text{Cap}(F) > 0$.

**Theorem 5.** Let $\mu$ be a canonical distribution on a closed set $F$ of capacity zero such that its potential $U(z) = U_F(z) > 0$. Then $\sup U(z) = \infty$.

It is clear $\sup_{z \in \bar{R}} U(z) = \infty$, but our assertion is not so clear. Suppose $\sup_{z \in \partial F} U(z) \leq M$. Let $p$ be a point in $(R + B_1) \cap F$. Then

$$\frac{1}{2\pi} \int_{\Gamma} U(z) \frac{\partial}{\partial n} N'(z, p) ds \leq M$$

for every regular curve of $N'(z, p)$.

Let $p_1, p_2, \ldots, p_i$ be points in $F \cap (R + B_1)$ and put $D_\lambda = E[z \in R: \sum c_i N(z, p_i) > \lambda]$, where $c_i > 0$ and $\sum c_i =$ total mass of $\mu$.

Let $U_{D\lambda}(z)$ be a harmonic function in $D_\lambda$ such that $U_{D\lambda}(z) = U(z)$ on $\partial D_\lambda$ and $U_{D\lambda}(z)$ has M.D.I. over $D_\lambda$. Then $U_{D\lambda}(z) \leq U(z)$.

Similarly as in Theorem 2 and by

$$\frac{1}{2\pi} \int_{\partial D_\lambda} \sum c_i N(z, p_i) ds \leq 2\pi \sum \frac{c_i}{c_i},$$

$$\frac{1}{2\pi} \int_{\Gamma_i} U(z) \frac{\partial}{\partial n} N(z, p_i) ds \leq \frac{1}{2\pi} \sum_{\Gamma_i} c_i U(z) \frac{\partial}{\partial n} N(z, p_i) ds \leq M \sum c_i,$$

where $\Gamma_i$ is a regular curve of $N(z, p_i)$ and contained in $D_\lambda$. By the
continuity of $N(z, p)$ there exists a linear form $\sum c'_k N(z, p_k) : p_k \in F \cap (R + B_r)$ such that $|U(z) - \sum c'_k N(z, p_k)| < \varepsilon : z \in R_m$ for any given $R_m$ and $\varepsilon > 0$. Hence there exists a sequence $\{U_j(z)\}$ of the above linear form such that $U_j(z) \to U(z)$ in $R - R_m$. Now $U_j(z) \to U(z)$ implies $\frac{\partial}{\partial n} U_j(z) \to \frac{\partial}{\partial n} U(z)$ in $R$ and $C^\lambda = E[z \in R : U_j(z) = \lambda]$ tends to $C^\lambda = E[z \in R : U(z) = \lambda]$. Then by Fatou's lemma

$$\sum c_k M \geq \lim\sup_{j} \int U(z) \frac{\partial}{\partial n} U_j(z) ds \geq \int U(z) \frac{\partial}{\partial n} U(z) ds,$$

for every $\lambda$ (10)

On the other hand, $U(z) = U_j(z) = U_{D^\lambda}(z)$ implies $U(z) = \lambda \omega(z)$, where $\omega(z)$ is C.P. of $D^\lambda$. Hence for almost all $\lambda$

$$\int_{C^\lambda} \frac{\partial}{\partial n} U(z) ds = \text{total mass of } \mu,$$

whence

$$\lim_{\lambda \to \infty} \int_{C^\lambda} U(z) \frac{\partial}{\partial n} U(z) ds = \infty$$

(10) contradicts to (11). Hence we have the theorem.

At present, we cannot prove the uniqueness of canonical mass distribution but we shall prove

**Theorem 6.** Let $U(z)$ be a superharmonic function in $\bar{R}$ such that $U(z) = U_F(z)$. Then $U(z) = \int_{F \cap (R + B_r)} N(z, p) d\mu(p)$. The mass distribution $\mu$ cannot be replaced by any other canonical distribution on $F'$ such that dist $(F, F') > 0$ without any change of $U(z)$.

As for the part of $\mu$ on $R$, the uniqueness of mass distribution is clear. We suppose both $F$ and $F'$ are contained in $B$. We cover $F$ by a finite number of closed discs $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{i_0}$ with diameter $< \frac{1}{n}$. Put $\mu = \mu_1 + \mu_2 + \cdots + \mu_{i_0}$, where $\mu_i$ is the restriction of $\mu$ on $\mathcal{S}_i$. Hence there exist $\mu_i$ and $\mathcal{S}_i$ such that $\int_{\mathcal{S}_i} N(z, p) d\mu_i(p) = \int_{\mathcal{S}_i} N(z, p) d\mu_i(p) > 0$. We denote $\mathcal{S}_i$ and $\mu_i$ by $\mathcal{S}_i$ and $\mu_i$ respectively. As above we choose $\mathcal{S}_2$ and $\mu_2$ such that $\mu_2 > 0$, dia $\mathcal{S}_2(\subset \mathcal{S}_1) < \frac{1}{2^2}$ and $\int_{\mathcal{S}_2} \left( \int_{\mathcal{S}_2} N(z, p) d\mu_i(p) \right) = \int_{\mathcal{S}_2} N(z, p) d\mu_i(p) > 0$. Hence we can find sequences $\left\{ \frac{\mu_i}{m_i} \right\}$ and $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \cdots$ such that $\bigcap_i \mathcal{S}_i = p \in (R + B_r) \cap F$, where $m_i$ is the total mass of $\mu_i$. Since $\int_{F} \left( \int N(z, p) d\mu_i(p) \right) = \int N(z, p) d\mu_i(p), \frac{1}{m_i} \int N(z, p) d\mu_i(p)$, is represented by a mass distribution.
There exists a subsequence \( \{ \mu_i^* \} \) such that \( \{ \mu_i^* \} \) converges to \( \mu^* \) on \( F' \). On the other hand, \( \{ \mu_i \} \) tends to a point mass \( N(z, p) \). Hence

\[
N(z, p) = \int_{F'} N(z, q) d\mu^*(q) : \text{dist} (F', p) > 0.
\]

Now we can prove as in Theorem 1 that \( N(z, p) \) is not minimal. This is a contradiction. Hence we have the theorem.

**Lemma 5.** Let \( \mu \) be a mass distribution and let \( \mu_c \) be its canonical distribution (on \( R + B^+ \)), i.e., \( \int N(z, p) d\mu(p) = \int N(z, p) d\mu_c(p) \). Then \( I(\mu) = I(\mu_c) \). Hence \( I(\mu) \) does not depend on a choice of particular distribution.

Suppose \( p \) and \( q \) are not minimal. Then

\[
N(z, p) = \int N(z, \alpha) d\mu_p(\alpha)
\]

and

\[
N(z, q) = \int N(z, \beta) d\mu_q(\beta), \quad \text{where} \quad \alpha, \beta \in R + B^+.
\]

Then

\[
I(\mu) = \int \int N(p, q) d\mu(p) d\mu(q) = \int \int N(\alpha, q) d\mu_p(\alpha) d\mu(p) d\mu(q)
\]

\[
= \int \int \int N(\alpha, \beta) d\mu_p(\alpha) d\mu_q(\beta) d\mu(p) d\mu(q) = \int \int N(\beta, \alpha) d\mu_p(\beta) d\mu(p) d\mu(q) \times \int d\mu_q(\beta) d\mu(q) = \int \int N(\alpha, \beta) d\mu(\alpha) d\mu(\beta) = I(\mu_c),
\]

because (12) means that a unit mass on \( p \) is replaced by \( \mu_p(\alpha) \) on \( \alpha \), whence

\[
\int d\mu_p(\alpha) d\mu(p) = d\mu(\alpha) \quad \text{and} \quad \int d\mu_q(\beta) d\mu(q) = d\mu(\beta).
\]

**Lemma 6.** If \( \mu_n \to \mu \), then \( I(\mu) \leq \lim \lim I(\mu_n) \).

\[
I(\mu) = \lim_{n \to \infty} \int \int N^M(p, q) d\mu(p) d\mu(q) \leq \lim_{n \to \infty} \lim_{M \to \infty} \int \int N^M(p, q) d\mu(p) d\mu(q)
\]

\[
\leq \lim_{n \to \infty} \int \int N(p, q) d\mu(p) d\mu(q) = \lim I(\mu_n).
\]

**Theorem 7.** (Fundamental theorem 1). Let \( F \) be a closed set in \( \mathbb{R} \) of positive *capacity. Then there exists a unit mass canonical distribution \( \mu \) on \( F \) (on \( F \cap (R + B^+) \)) whose energy integral is minimal and its potential \( U(z) \) satisfies the following conditions:

1. \( U(z) \geq V \) in \( F \) except at most a set of *capacity zero.
2. \( U(z) \leq V \) in \( F^* \) (kernel of \( \mu \)).
3. \( U(z) \geq V_{\omega}(z) \)
4. \( U(z) = V \) on \( F^* \cap R \)
where \( V = I(\mu) \) and \( \omega(z) \) is C.P. of \( F \).

In our space, the potential \( N(z, p) \) is continuous in \( R - p \) and lower semicontinuous in \( \tilde{R} \) but \( N(z, p) \) is not necessarily continuous in \( \tilde{R} - p \) and the continuity principle cannot be proved. Therefore we cannot prove the above theorem in usual manner.

Let \( \{ \mu_n \} \) be a sequence of canonical distributions on \( F \) such that \( I(\mu_n) \downarrow V \), where \( V \) is the infimum of energy integrals of all canonical distributions on \( F \) of total mass unity. Put \( \mu = \lim_n \mu_n \). Then by Lemma 6, \( I(\mu) = V \). If \( \mu \) is not canonical, we replace by a canonical distribution \( \mu_c \). Then also by Lemma 5, \( I(\mu_c) = V \). Assume that there exists a closed set \( \mathcal{G} \) in \( CF \) such that \( \int N(z, p) d\mu''(p) > 0 \), where \( \mu'' \) is the restriction of \( \mu_c \) on \( \mathcal{G} \). Then \( \mu'' \) cannot be replaced by any canonical distribution on \( F \) by Theorem 6. Hence every canonical distribution which is equivalent to \( \mu \) has a positive mass on \( \mathcal{G} \). This contradicts to \( \mu = \lim_n \mu_n \). Hence such \( \mu_c \) has no mass except on \( F \). Thus there exists a canonical distribution \( \mu \) of total mass unity on \( F \) such that \( I(\mu) = V \).

Let \( F^* \) be the kernel of the distribution \( \mu \). Then clearly \( F^* \subset F \) and closed. By symmetry of \( N(p, q) \), \( I(\mu) = \int U(p) d\mu(p) \) and \( U(z) \leq V - \varepsilon \) on \( F^* \), because \( I(\mu) = V \). Hence there exists a point \( p_0 \in F^* \) such that \( U(p_0) > V - \varepsilon \) and there exists a neighbourhood \( v(p_0) \) such that \( U(z) > V - \varepsilon \) \( (z \in v(p_0)) \) by the lower semicontinuity of \( U(z) \) and that the restriction of \( \mu \) in \( v(p_0) \) has a positive mass \( m \) in \( v(p_0) \). Assume \( U(z) \leq V - 2\varepsilon \) on a set \( F' \) of positive capacity in \( F \). We define a new canonical mass distribution \( \hat{\mu} \) on \( F' \) whose energy integral is finite and whose total mass is \( m \). Define another distribution \( \sigma \) as follows:

\[
\sigma = -\mu \text{ on } v(p_0), \quad \sigma = \hat{\mu} \text{ on } F' \text{ and } \sigma = 0 \text{ outside of } v(p_0) \text{ and } F'.
\]

Then \( \mu + h\sigma > 0 \) for \( h < 1 \) and the total mass is unity. Then the variation

\[
\delta I = I(\mu + h\sigma) - I(\mu) \geq 0 \quad \text{and}
\]

\[
\delta I = 2h \left[ U(p) d\mu(p) + h^2 I(\sigma) \right] \leq 2h \left[ m(V - 2\varepsilon) - m(V - \varepsilon) \right] + h^2 I(\sigma)
\]

\[
= -h \left[ 2m - hI(\sigma) \right].
\]

This is a contradiction for sufficiently small \( h \). Hence by letting \( \varepsilon \to 0 \), we have (1).

Put \( F' = E[z \in F: U(z) \leq V - \varepsilon] \). Then \( F' \) is closed and \( \text{Cap}(F') = 0 \) and the restriction of \( \mu' \) on \( F' \) has no mass, because \( I(\mu') \leq I(\mu) \leq V \). Hence \( \mu \) has no mass on a set \( E[z \in F: U(z) \leq V + \varepsilon] \) for any \( \varepsilon > 0 \). Next assume \( \mu \) has a positive mass \( m \) on a set \( E[z \in F: U(z) > V + \varepsilon] \).
Then $I(\mu) > V$. This also a contradiction. Hence by letting $\varepsilon \to 0$, $U(z) = V$ where the mass is distributed. Thus $U(z) = V$ on $F^*$ by the lower semicontinuity of $U(z)$, whence we have (2).

$R - F$ consists of at most enumerably infinite number of domains $G_1, G_2, \ldots$, where $G_i$ is the domain containing $\partial R_0$ in its boundary. At first, consider $U(z)$ in $G_i$. $U(z) = V$ on $F^*$ except at most a set $F'$ of capacity zero (by Lemma 4, capacity zero). Hence the exists a superharmonic function $\omega^*(z)$ in $\tilde{R}$ (as in Theorem 3) such that $\lim_{z \to F' \in F'} \omega^*(z) = -\infty$. Hence as in case of Lemma 3, $U(z) \geq V\omega(z)$ in $G_i$. Let $G_2$ be one of other domains. Then $\partial G_2 \subset F$. $\partial G_2 \cap R$ consists of continuum boundary $\Gamma_i$ $(i = 1, 2, \ldots)$ and others $\Gamma^*_i$. Put $G_2 + \Gamma^*_i = G_i^*$. Then $G_i^*$ is also a domain. Since for every point $p \in \Gamma_i \cap R$, there exists a neighbourhood $v(p)$ which is conformally equivalent to a disc in the $z$-plane. Hence the continuity principle is valid in $v(p)$, whence $U(z) \geq V$ on $\Gamma_i \cap R$. Then by the superharmonicity of $U(z)$, $U(z) \geq U_{G_i^*}(z) = V$ in $G_i^*$, whence $U(z) \geq V\omega(z)$ in $G_i$, because $\omega(z)$ is clearly $V$ in $G_i$. Hence in every $G_i$ $(i = 3, 4, \ldots) U(z) \geq V\omega(z)$ on $F$ except a set capacity zero. Similarly as in Lemma 3, $U(z) \geq V\omega(z)$. By considering sequences $\{z_i\} : z_i \in R$ which clusters at $B$, we have $U(z) \geq V\omega(z)$ in $B$. Thus we have $U(z) \geq V\omega(z)$ in $\tilde{R}$ and we have (3). Clearly by the continuity principle, $U(z) = V$ on $F \cap R$. Hence we have (4).

We know the property of $U(z)$ very little, i.e. at present we don't know whether $U(z)$ is bounded in $\tilde{R}$ or not. We shall prove the next

**Theorem 8.** (Fundamental theorem 2)

$U(z) \equiv V\omega(z)$ in $\tilde{R}$ and

$I(\mu) = D(U(z)) = D(V\omega(z)) = V$.

**Lemma 7.** Let $U(z)$ be a function in Theorem 7. Put $G_a = E[z \in \tilde{R} : U(z) \geq V + a]$ $(a > 0)$ and $G_a^* = E[z \in \tilde{R} : \delta(z, F^*) \leq \frac{1}{n}]$. Then $g_n = G_a \cap G_a^*$ is open. Let $\omega^{g_n}(z)$ be C.P. of $g_n$. Then $\lim_{n \to \infty} \omega^{g_n}(z) = 0$.

Let $\{\tilde{G}_i\}$ be a sequence of closed subsets of $g_n$ such that $\tilde{G}_i \uparrow g_n$. Let $\omega^{\tilde{G}_i}(z)$ be C.P. of $\tilde{G}_i$. Then $\omega^{\tilde{G}_i}(z) \uparrow \omega^{g_n}(z)$ in mean (see page 154, II). Hence $\omega^{g_n}(z)$ is superharmonic in $\tilde{R}$. Put $H_a^* = E[z \in \tilde{R} : \omega^{\tilde{G}_i}(z) = 1]$ and $H_n^* = E[z \in \tilde{R} : \omega^{g_n}(z) = 1]$. Then $H_a^*$ and $H_n^*$ are closed. Clealy by Theorem 3 and 4, $\omega^{\tilde{G}_i}(z) \leq \omega^{H_n^*}(z)$, where $\omega^{H_n}(z)$ is C.P. of $H_n^*$. By the superharmonicity of $\omega^{g_n}(z)$, $\omega^{g_n}(z) \equiv \omega^{H_n^*}(z)$, where $\omega^{H_n}(z)$ is C.P. of $H_n$. On the other hand, $\omega^{\tilde{G}_i}(z) \leq \omega^{H_n}(z)$ for every $i$. Hence $\omega^{g_n}(z) \equiv \omega^{H_n}(z)$. Thus $\omega^{g_n}(z) = \omega^{H_n}(z)$.
Clearly \( \omega^g_n(z) \geq \omega^g_n(z) \) (\( g_n \) is the closure of \( g_n \)). By (4) of Theorem 7, \( F^* \cap g_n \cap R = 0 \). Hence the complementary set \( Cg_n \) of \( g_n \) consists of only one component containing \( \partial R \) in its boundary. Hence by Theorem 4 \( \omega^g_n(z) \leq \omega^g_n(z) < 1 \) in \( Cg_n \) except at most a subset of \( Cg_n \) of capacity zero, whence \( H_n \subset g_n \) except a subset of \( H_n \) of capacity zero.

Next \( \omega^g_n(z) \leq \omega^g(z) (g = \bigcap_{n>1} g_n) \) in mean as \( n \to \infty \) and \( H_n \downarrow H \) and \( H \subset g_n < F^* \).

Assume \( \omega^g(z) > 0 \). Then \( \omega^g(z) = 1 \) on \( H \) except a subset of \( H \) of capacity zero, whence by Theorem 4 there exists at least a point \( z_0 \in F^* \) such that \( \omega^g(z_0) = 1 \). Since \( F > F^* \),

\[
U(z) = U_{k^*}(z) \geq (V + a) \omega^g_n(z) \quad \text{and} \quad U(z) \geq (V + a) \omega^g(z),
\]

where \( F^* = E[z \in \bar{R} : \delta(z, F^*) \leq \frac{1}{n}] \). Hence \( U(z_0) \geq (V + a) : z_0 \in F^* \).

This contradicts to (2) of Theorem 7. Hence \( \omega^g(z) = 0 \).

**Lemma 8.** Put \( G^N = E[z \in \bar{R} : U(z) > N] \). Then \( \lim_{N \to \infty} U_{G^N}(z) = 0 \).

By the superharmonicity of \( U(z) \), \( U(z) \geq N \omega^{G^N}(z) \), where \( \omega^{G^N}(z) \) is C.P. of \( G^N \). Hence \( \lim_{N \to \infty} \omega^{G^N}(z) = 0 \), i.e. \( \cap G^N = G \) is a set of capacity zero. Assume \( \lim_{N \to \infty} U_{G^N}(z) = U^*(z) > 0 \). Then \( U^*(z) \) is represented by a canonical distribution \( \mu^* \) on \( G \) and the kernel \( k^* \) of \( \mu^* \) is closed and \( \subset G \). Hence \( k^* \) is a set of capacity zero and \( U_k^*(z) = U^*(z) \). Suppose dist \( (F^*, k^*) \geq 0 \). Then by Theorem 6, \( \mu^* \) cannot be replaced by a distribution of \( F^* \). On the other hand, \( U(z) - U_k^*(z) \) is superharmonic (see p. 158, II), whence \( U(z) \) has \( \mu^* \) on \( k^* \), which implies \( k^* \subset F^* \). Now by Theorem 5, \( \sup_{z \in F^*} U_k^*(z) = \infty \). Hence there exists a point \( z_0 \) in \( k^* \subset F^* \) for any large number \( N \) such that

\[
U(z_0) > U_k^*(z_0) > N.
\]

This contradicts to (2) of Theorem 7. Hence \( U^*(z) = 0 \).

**Proof of theorem 8.** Let \( U_{G^N}(z) \) be function in Lemma 8. Then there exists a number \( N \) such that \( U_{G^N}(z) < \varepsilon \) for given number \( \varepsilon > 0 \) and a point \( z_0 \). Put \( G_\delta = E[z \in \bar{R} : U(z) > V + \delta] \) and \( F^*_n = E[z \in \bar{R} : \delta(z, F^*) \leq \frac{1}{n}] \) and \( g_n = F^*_n \cap G_\delta \). Let \( \omega^g_n(z) \) and \( \omega^g(z) \) be C.P. of \( g_n \) and
respectively. Then since $F^* \subset F^*_n$,

$$U(z) = U_{F^*_n}(z) \leq \delta + V_\omega F^*_n(z) + N_\omega F^*_n(z) + U_\omega(z).$$

Let $n \to \infty$. Then $N_\omega F^*_n(z) \to 0$, by Lemma 8, Hence

$$U(z_0) \leq \delta + V_\omega F^*_n(z) + \epsilon \leq \delta + \delta V_\omega(z) + \epsilon.$$

Then by letting $\epsilon \to 0$ and then $\delta \to 0$, we have

$$U(z_0) \leq V_\omega(z_0).$$

On the other hand, $U(z) \geq V_\omega(z)$, hence we have $U(z) = V_\omega(z)$ and

$$D(U(z)) = D(V_\omega(z)) = V^2 \frac{1}{2\pi} \int_{\partial \Omega} \frac{\partial \omega}{\partial n}(z) ds = V^2 D(\omega(z)) = V = I(\mu).$$

By Theorem 8 we have the following

**Corollary.** $\text{Cap}^*(F) = \text{Cap}^*(F)$, and

$\text{Cap}^*(F) > 0$ implies $\text{Cap}^*(F) > 0$.

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