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The Space of Pseudo-Metrics on a Complete Uniform Space

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1. In a paper, B. H. Arnold[1] considered the class of all upper semi-continuous decompositions of a T_1 space and showed that it is possible to construct a space homeomorphic to the given space from the partially ordered set of decompositions. In another paper[2], M. E. Shanks obtained results on the semi-linear space of all metrics compatible with the topology on a compactum.

In this paper we will show that the complete metric space as well as the lattice ordered semi-additive-group of all bounded pseudo-metrics compatible with the uniformity on a complete uniform space determine the given uniform space.

2. Let X[3] be a uniform space. Then the set $\mathfrak{S}\mathfrak{M}(X)$ of all bounded pseudo-metrics[4] compatible with its uniformity for X is a complete metric space with the distance $(\rho, \sigma)=\sup_{x, y \in X}|\rho(x, y)-\sigma(x, y)|$ and it is a lattice ordered semi-group[5] with the ordinary addition and order considered as a subsystem of the system of all continuous functions from the product space $X \times X$ into the reals.

Moreover for $\rho \in \mathfrak{S}\mathfrak{M}(X)$ let $X_{[\rho]}$ be a metrizable uniform space whose points are equivalence classes $[x]_\rho$ with respect to the equivalence relation $\rho(x, y)=0$ and whose metric is defined by the distance $d_\rho([x]_\rho, [y]_\rho)= \rho(x, y)$. Then we write $X_{[\rho]} \geqq X_{[\sigma]}$ when the mapping $F_{[\rho], [\sigma]}: [x]_\rho \rightarrow [x]_\sigma$ is uniformly continuous from $X_{[\rho]}$ onto $X_{[\sigma]}$, and $X_{[\rho]} > X_{[\sigma]}$ if $X_{[\rho]} \geqq X_{[\sigma]}$ but not $X_{[\sigma]} \geqq X_{[\rho]}$ and we denote by $\mathfrak{D}(X)$ the partially ordered set of all such metrizable uniform space $X_{[\rho]}$ with the above order.

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1) Cf. B. H. Arnold: Decompositions of a T_1 space, Bull. Amer. Math. Soc., 46 (1943).

2) Cf. M. E. Shanks: The space of metrics on a compact metrizable space, Amer. Jour. Math., 66 (1944).

3) In the present note we may assume that the potency of X is greater than 4, since otherweise our results are trivial.

4) We say that ρ is a pseudo-metric compatible with the uniformity for X if it satisfies the following conditions, i) $\rho(x, x)=0$, ii) $\rho(x, y)=\rho(y, x)$, iii) $\rho(x, y)+\rho(y, z) \geqq \rho(x, z)$ and iv) for any $\varepsilon>0$ there exists a neighbourhood V such that $\rho(x, y)<\varepsilon$ for $x \in V(y)$.

5) We say that S is a lattice ordered semi-group if it satisfies the following conditions i) S is a lattice and semi-group and ii) $a \vee b+c=(a+c) \vee(b+c)$ for any a, b and $c \in S$. Cf. G. Birkhoff, Lattice theory, (1949), p. 201.

3. The partially ordered set $\mathfrak{D}$ (X). In this section we will show that $\mathfrak{D}(X)$ determines the space X whenever X is a complete uniform space. For this purpose we shall prove the following lemmas.

Lemma 1. *For two $X_{[\rho_1]}$ and $X_{[\rho_2]}$ of a uniform space X the following conditions are equivalent*:

(i) $X_{[\rho_1]} \geqq X_{[\rho_2]}$

(ii) *for two disjoint subsets A and B of X, $\rho_1(A, B)$[6] $= 0$ implies $\rho_2(A, B)$[6] $= 0$.*

Proof. Since obviously (i) implies (ii), we only prove that (ii) implies (i). Suppose that $X_{[\rho_1]}$ and $X_{[\rho_2]}$ satisfy the condition (ii). Then evidently $F_{[\rho_1][\rho_2]}$ is a continuous mapping from $X_{[\rho_1]}$ onto $X_{[\rho_2]}$. Now we assume that there exist subsets $\{x_n\}$ and $\{y_n\}$ such that $\rho_1(x_n, y_n) < \frac{1}{n}$ and $\rho_2(x_n, y_n) \geqq \varepsilon > 0$. If $\{[x_n]_{\rho_2}\}$ contains a Cauchy subsequence $\{[x_n']_{\rho_2}\}$ whose limit point in the completion $\bar{X}_{[\rho_2]}$ is $\bar{x}$, then $\rho_1(A, B) = 0$ and $\rho_2(A, B) \geqq \frac{\varepsilon}{2}$, where $A = \{x \mid [x]_{\rho_2} \in \{[x_n']_{\rho_2}\} \wedge S_{\rho_2}{}^{7)}(\bar{x}, \varepsilon/4)\}$ and $B = \{y \mid [y]_{\rho_2} \in \{[y_n']_{\rho_2}\}$ & $[x_n']_{\rho_2} \in S_{\rho_2}(\bar{x}, \varepsilon/4)\}$. Hence we see that both $\{[x_n]_{\rho_2}\}$ and $\{[y_n]_{\rho_2}\}$ contain no Cauchy subsequences, so that there exists subsets $\{[x_n']_{\rho_2}\}$ and $\{[y_n']_{\rho_2}\}$ of $\{[x_n]_{\rho_2}\}$ and $\{[y_n]_{\rho_2}\}$ respectively such that for some δ $d_{\rho_2}([x_n']_{\rho_2}, [x_m']_{\rho_2}) > \delta$ and $d_{\rho_2}([y_n']_{\rho_2}, [y_m']_{\rho_2}) > \delta$ if $m \neq n$. Then we can construct two infinite subsets $\bar{A}$ and $\bar{B}$ such that $\bar{A} \subset \{[x_n']_{\rho_2}\}$ and $\bar{B} \subset \{[y_n']_{\rho_2}\}$ and $d_{\rho_2}(\bar{A}, \bar{B}) > 0$. Let $A = \{x \mid [x]_{\rho_2} \in \bar{A}\}$ and let $B = \{y \mid [y]_{\rho_2} \in \bar{B}\}$. Then $\rho_1(A, B) = 0$ and $\rho_2(A, B) > 0$. Hence we see that $F_{[\rho_1], [\rho_2]}$ is uniformly continuous.

Lemma 2. *For two $X_{[\rho_1]}$ and $X_{[\rho_2]}$ such that $X_{[\rho_1]} > X_{[\rho_2]}$, the following conditions are equivalent*:

(i) *$X_{[\rho_1]}$ covers $X_{[\rho_2]}$*

(ii) a) *there exists a unique pair of different points $\bar{x}$ and $\bar{y}$ of the completion $\bar{X}_{[\rho_1]}$ such that $\bar{d}_{\rho_2}(\bar{x}, \bar{y}) = 0$, where $\bar{d}_{\rho_2}$ is an extension of the pseudo metric d_{ρ_2}: $d_{\rho_2}([x]_{\rho_1}, [y]_{\rho_1}) = \rho_2(x, y)$, and* b) *if, for three subsets A_1, A_2 and B of X, $\rho_1(A_1, A_2) > 0$ and $\rho_1(A_1 \cup A_2, B) > 0$, then either $\rho_2(A_1, B) > 0$ or $\rho_2(A_2, B) > 0$.*

Proof. we show that (i) implies (ii). Let $X_{[\rho_1]}$ cover $X_{[\rho_2]}$. Now suppose that there exist three subsets A_1, A_2 and B such that they do not satisfy b). Then there exists a continuous function f of X such that $f(A_1 \cup B) = 0$ and $f(A_2) = 1$ and such that for any $\varepsilon > 0$ $\rho_1(x, y) < \delta$

6) $\rho(A, B) = \inf\limits_{x \in A,\, y \in B} \rho(x, y)$

7) $S_\rho(x, \varepsilon) = \{y \mid \rho(x, y) < \varepsilon\}$

implies $|f(x)-f(y)|<\varepsilon$ for some δ. Let ρ_f be a pseudo-metric of X such that $\rho_f(x, y)=|f(x)-f(y)|$ and let $\tau=\sigma_2+\rho_f$. Then obviously for $\varepsilon>0$ there exists δ such that $\rho_1(x, y)<\delta$ implies $\tau(x, y)<\varepsilon$ and hence $X_{[\rho_1]}\geqq X_{[\tau]}$ and in fact $X_{[\rho_1]}>X_{[\tau]}$ since $\tau(A_1, B)=0$ and $\rho_1(A_1, B)>0$. On the other hand $X_{[\tau]}\geqq X_{[\rho_2]}$, and moreover $X_{[\tau]}>X_{[\rho_2]}$, since $\rho_2(A_2, B)=0$ and $\tau(A_2, B)>0$. Hence $X_{[\rho_1]}$ and $X_{[\rho_2]}$ satisfy b). Furthermore If $\bar{d}_{\rho_2}(\bar{x}, \bar{y})=0$ implies $\bar{x}=\bar{y}$ for any $\bar{x}$ and $\bar{y}$ in $\bar{X}_{[\rho_1]}$, since $\bar{d}_{\rho_2}$ is not a metric of $\bar{X}_{[\rho_1]}$ and since $\bar{X}_{[\rho_1]}$ is a complete metrizable, by the same method used in the proof of Lemma 1, there exists three subsets A_1', A_2' and B' of $\bar{X}_{[\rho_1]}$ such that $\bar{d}_{\rho_1}(A_1', A_2')>0$, $\bar{d}_{\rho_1}(A_1'\cup A_2', B')>0$, $\bar{d}_{\rho_2}(A_1', B')=0$ and $\bar{d}_{\rho_2}(A_2', B')=0$. Let $A_i=\{x\,|\,[x]_{\rho_1}\in U(A_i')\frown X_{[\rho_1]}\}$ and $B=\{x\,|\,[x]_{\rho_1}\in U(B')\frown X_{[\rho_1]}\}$, where $U(A_i')$ and $U(B')$ are suitable neighbourhoods of A_i' and B' respectively. Then A_i and B do not satisfy b). Hence there exists a pair of different points $\bar{x}$ and $\bar{y}$ of $\bar{X}_{[\rho_1]}$ such that $\bar{d}_{\rho_2}(\bar{x}, \bar{y})=0$. Furthermore we easily see that such pair of points is uniquely determined.

The proof of the converse will be omitted, as it can be done by the method used above.

DEFINITION. We say that a proper ideal I of $\mathfrak{D}(X)$ is *p-ideal* if it satisfies the following property: for any $X_{[\rho]}$ there exists an $X_{[\rho']}$ such that $X_{[\rho']}\in I$ and $X_{[\rho]}=X_{[\rho']}$ or $X_{[\rho]}$ covers $X_{[\rho']}$, and if $X_{[\rho_1]}\geqq X_{[\rho_2]}$, $X_{[\rho'_1]}\geqq X_{[\rho_2']}$.

Lemma 3. *Let X be a complete uniform space and let I be a p-ideal. Then there exists a unique pair of different points x and y of X such that $I=\{X_{[\rho]}\,|\,\rho(x, y)=0\}$.*

Proof. First of all we remark that if $X_{[\rho]}\notin I$ then $X_{[\rho']}\in I$ is uniquely determined. For if two different elements $X_{[\rho_1']}$ and $X_{[\rho_2']}\in I$ are covered by $X_{[\rho]}$, since I is an ideal, $X_{[\rho_1']}\vee X_{[\rho_2']}\in I$, but $X_{[\rho]}=X_{[\rho_1']}\vee X_{[\rho_2']}$, which is a contradiction. Now let $X_{\rho_0}\notin I$ be fixed and let $\bar{x}_{\rho_0}$ and $\bar{y}_{\rho_0}$ be two points of $\bar{X}_{[\rho_0]}$ such that $\bar{d}_{\rho_0'}(\bar{x}_{\rho_0}, \bar{y}_{\rho_0})=0$. Moreover let $\bar{x}_1$ and $\bar{x}_2$ be two points of $\bar{X}_{[\rho]}$ such that $\bar{d}_{\rho'}(\bar{x}_1, \bar{x}_2)=0$, where $X_{[\rho]}>X_{[\rho_0]}$. Then $\bar{F}_{[\rho][\rho_0]}(\{\bar{x}_1, \bar{x}_2\})=\{\bar{x}_{\rho_0}, \bar{y}_{\rho_0}\}$ where $\bar{F}_{[\rho],[\rho_0]}$ is a mapping from $X_{[\rho]}$ into $\bar{X}_{[\rho_0]}$ such that it is an extension of $F_{[\rho],[\rho_0]}$. For if $\bar{d}_{\rho_0}(\bar{x}_1, \bar{x}_2)=0$, by Lemma 1 and 2 we see that $X_{[\rho_0]}\leqq X_{[\rho']}$ and $X_{[\rho_0]}\in I$, which is a contradiction. Hence $\bar{d}_{\rho_0}(\bar{x}_1, \bar{x}_2)\neq 0$, accordingly $\bar{d}_{\rho_0}(\bar{F}_{[\rho],[\rho_0]}(\bar{x}_1), \bar{F}_{[\rho],[\rho_0]}(\bar{x}_2))\neq 0$, but $\bar{d}_{\rho'}(\bar{x}_1, \bar{x}_2)=0$ and $X_{[\rho']}\geqq X_{[\rho_0']}$, hence $\bar{d}_{\rho_0'}(\bar{F}_{[\rho],[\rho_0]}(\bar{x}_1), \bar{F}_{[\rho],[\rho_0]}(\bar{x}_2))=0$. By the uniquenss of such pair of points we see that $\bar{F}_{[\rho],[\rho_0]}(\{\bar{x}_1, \bar{x}_2\})=\{\bar{x}_{\rho_0}, \bar{y}_{\rho_0}\}$. Let $\bar{x}_\rho$ be the point such that $\bar{x}_\rho\in\{\bar{x}_1, \bar{x}_2\}$ and $\bar{F}_{[\rho,\rho_0]]}(\bar{x}_\rho)=\bar{x}_{\rho_0}$ and let $\bar{y}_\rho$ be the other point of $\{\bar{x}_1, \bar{x}_2\}$. Moreover let $A_{\rho,n}=\left\{x|\bar{d}_\rho([x]_\rho, \bar{x}_\rho)\leq\frac{1}{2n}\right\}$ and $B_{\rho,n}=\{x|\bar{d}_\rho([x]_\rho, \bar{y}_\rho)\leq\frac{1}{2n}\}$. Then $\{A_{\rho,n}\}$ and $\{B_{\rho,n}\}$ are both Cauchy closed family of X. Since X is complete, there exists a pair of points x_0 and y_0 of X such that $x_0=\prod A_{\rho,n}$ and $y_0=\prod B_{\rho,n}$. Then obviously $[x_0]_\rho=\bar{x}_\rho$ and $[y_0]_\rho=\bar{y}_\rho$ for $X_\rho\geqq X_{\rho_0}$. Furthermore if $\rho(x_0, y_0)=0$, then $X_{[\rho]}\in I$, for let $X_{[\tau]}=X_{[\rho]}\vee X_{[\rho_0]}$, then $X_{[\tau]}>X_{[\rho]}$ and $\bar{d}_\rho(\bar{x}_\tau, \bar{y}_\tau)=0$, hence $X_{[\tau']}\geqq X_{[\rho]}$, which implies $X_{[\rho]}\in I$. Since evidently $\rho(x_0, y_0)>0$ implies $X_{[\rho]}\notin I$, $I=\{X_{[\rho]}|\rho(x_0, y_0)=0\}$.

Theorem 1. *If X is a complete uniform space, the partially ordered set $\mathfrak{D}(X)$ determines the uniform space X.*

Proof (1). Let $I\{x, y\}$ be the p-ideal which corresponds to a pair of points x and y of X as in Lemma 3. Moreover for two p-ideals I_1 I_2 we denote by $I_1\sim I_2$ the relation: $I_1=I_2$ or $I_1\wedge I_2<I_3$ for some I_3. Then by the triangle axiom of pseudo-metrics, $I\{x, y\}$ and $I\{u, v\}$ are equivalent if and only if $\{x, y\}\frown\{u, v\}\neq\phi$. Furthermore we say that a subset P of the set of all p-ideals is a maximal collection if it satisfies the following conditions: i) P contains at least four p-ideals, ii) any two p-ideals $\in P$ are equivalent and iii) it is maximal with respect to i) and ii). Then for a maximal collection P there exists a unique point x of X such that $P=\{I\{x, y\}|y\in X \;\&\; y\neq x\}$, which is denoted by $P(x)$. Conversely any $P(x)$ is a maximal collection. Let X be the set of all maximal collections. Then we see that the correspondence: $x\rightarrow P(x)$ is a one-to-one mapping from X to $\tilde{X}$. Furthermore let $\tilde{A}=\{P(x)|x\in A\}$.

(II) We say that a subset $\tilde{A}$ with potency $\geqq 2$ is *basic-closed* if there exists a $X_{[\rho]}\in\mathfrak{D}(X)$ such that for any $P\in\tilde{A}$, $\tilde{A}=\{Q|Q\frown P\ni I\ni X_{[\rho]}\}\cup\{P\}$. Then a subset $\tilde{A}$ of $\tilde{X}$ is basic-closed if and only if A is a closed G_δ-set which is a zero-set of a uniformly continuous function of X and the potency $|A|\geqq 2$. Let the set of all basic-closed sets of $\tilde{X}$ be a closed basis for $\tilde{X}$. Then we see that $\tilde{X}$ is a topological space which is homeomorphic to X by the mapping P.

(III) Now we define the uniformity for $\tilde{X}$ by pseudo-metrics. For this purpose we define the uniformity for $\tilde{X}$ by pseudo-metrics. For this purpose we say that two disjoint basic-closed subsets $\tilde{A}_i$: $(i=1, 2)$ are ρ-separated if there exists $X_{[\rho_1]}<X_{[\rho]}$ such that $\tilde{A}_i=\{Q|Q\frown P_i\ni I\ni X_{[\rho_1]}\}\cup\{P_i\}$ for any $P_i\in\tilde{A}_i$ and that two subsets $\tilde{A}_i$ are ρ-separated if they are contained respectively in two ρ-separated disjoint basic-closed subsets. Furthermore we define that a pseudo-metric $\tilde{\rho}$ of $\tilde{X}$ is compatible with the uniformity for $\tilde{X}$ if there exists $X_{[\rho]}\in\mathfrak{D}(X)$ such that if $\tilde{\rho}(\tilde{A}, \tilde{B})>0$, $\tilde{A}$ and $\tilde{B}$ are ρ-separated.

Now let $\tilde{\rho}$ be compatible with the uniformity for $\tilde{X}$ and let ρ_1 be a pseudo-metric of X such that $\rho_1(x, y) = \tilde{\rho}(P(x), P(y))$. Then if $\rho_1(A_1, A_2) > 0$, then $\tilde{\rho}(\tilde{A}_1, \tilde{A}_2) > 0$, hence $\tilde{A}_i(i = 1, 2)$ are ρ-separated, accordingly there exists subsets $\tilde{A}_i'(i = 1, 2)$ and $X_{[\sigma]} < X_{[\rho]}$ such that $\tilde{A}_i' = \{Q \mid Q \wedge P_i \ni I \ni X_{[\sigma]}\} \cup \{P_i\}$ and $\tilde{A}_i' \supset \tilde{A}_i$. This means that $A_i \subset \{x \mid \sigma(x, y) = 0$ for a fixed $y_i\}$ and $[y_1]_\sigma \neq [y_2]_\sigma$, so that $\rho(A_1, A_2) > 0$. Thus we see by Lemma 1 that $\rho_1 \in \mathfrak{SM}(X)$ and $X_{[\rho_1]} \leq X_{[\rho]}$. Conversely for any $\rho \in SM(X)$ let $\tilde{\rho}$ be a pseudo-metric of $\tilde{X}$ such that $\tilde{\rho}(P(x), P(y)) = \rho(x, y)$. Then $\tilde{\rho}(\tilde{A}, \tilde{B}) > 0$ if and only if $\rho(A, B) > 0$, i.e., $\tilde{A}$ and $\tilde{B}$ are ρ-separated. Thus we see that the mapping P is a uniform homeomorphism.

REMARK. Let $\mathfrak{D}'(X)$ be the partially ordered set whose elements are equivalence relations on X : $\rho(x, y) = 0$, $\rho \in \mathfrak{SM}(X)$. Then if X is a complete uniform space, $\mathfrak{D}'(X)$ determined the given topological space X, but does not determine the uniform space X.

For example we consider the space $X = \bigcup_{n=1}^{\infty} X_n$ where X_n are mutually disjoint the n-dimentional cubes and whose relative topology on X_n is a usual one. Let X_1 be the coarsest uniform space[8] over X for which all continuous functions are uniformly continuous and let X_2 be the uniform space[9] over X with the uniformity made up of all countable normal coverings. Then two space are complete and $\mathfrak{D}'(X_1) = \mathfrak{D}'(X_2)$. For there exists $\rho' \in \mathfrak{SM}(X_i)$ for any $\rho \in \mathfrak{SM}(X_i)$ such that $\rho(x, y) = 0$ and $\rho'(x, y) = 0$ are the same equivalence relation on X and is totally bounded, and so $\mathfrak{D}'(X_1)$ and $\mathfrak{D}'(X_2)$ are determined by the totally bounded-pseudo metrics which are identical on both X_1 and X_2. But X_1 and X_2 are not uniformly homeomorphic. For let $\mathfrak{B}_n$ be the finite open covering of X_n such that any refinement of $\mathfrak{B}_n$ has order $\geqq n+1$ and let $\mathfrak{U} = \{U \mid U \in \mathfrak{B}_n$ for some $n\}$, then $\mathfrak{U}$ is contained in the uniformity for X_2. Suppose that there exists a uniform homeomorphism F from X_1 onto X_2. Then $F^{-1}(\mathfrak{U})$ is contained in the uniformity for X_1, hence there must exist a finite number of continuous functions $\{f_1, f_2, \ldots, f_n\}$ and a real number $\varepsilon > 0$ such that

$$\mathfrak{U}_1 = \{\{y \mid |f_i(x) - f_i(y)| < \varepsilon \text{ for any } i\} \mid x \in X\}$$

is a refinement of $F^{-1}(\mathfrak{U})$. But since the mapping f from X into the n-dimensional Euclidean space E : $f(x) = \{f_i(x) \mid i = \text{i}, 2, \ldots n\}$ is continuous, by the extended Lebesgue's covering theorem[10] $\mathfrak{U}_1$ has a refinement

8) Cf. E. Hewitt: Rings of real valued continuous functions, Trans. Amer. Math. Soc 64 (1948).

9) Cf. T. Shirota: A class of topological spaces, Osaka Math. J. 4 (1952).

10) C. H. Dowker: Lebesgue dimension of a normal space, Bull. of Amer. Math. Soc. 52 (1946). K. Morita: On the dimension theory of normal space I, Japanese Journ. Math. 20 (1950).

$\mathfrak{U}_2$ with order $\leqq n+2$. Hence $F(\mathfrak{U}_2) \leqq \mathfrak{U}$ and the order of $F(\mathfrak{U}_2)$ is $\leqq n+2$. Accordingly the order of $F(\mathfrak{U}_2)|X_{n+1}$ is $\leqq n+2$ and $F(\mathfrak{U}_2)|X_{n+1} \leqq \mathfrak{V}_{n+1}$. But by the property of $\mathfrak{V}_{n+1}$ the order of $F(\mathfrak{U}_2)|X_{n+1}$ is $\geqq n+2$, which is a contradiction.

4. **The complete metric space $\mathfrak{S}\mathfrak{M}(X)$.** We remark first that the zero 0 of the semi-linear space $\mathfrak{S}\mathfrak{M}(X)$ is determined by the property that it can not be the middle point[11] of two different points. Accordingly we can characterize the norm of an element ρ of $\mathfrak{S}\mathfrak{M}(X)$ as $(0, \rho)$ and we write it by $\| \rho \|$.

Definition. For any real $\gamma > 0$ and $\rho \in \mathfrak{S}\mathfrak{M}(X)$ we denote the surface $\{\rho' | (\rho', \rho) = \gamma\}$ by $S_\gamma(\rho)$ and in particular, when $\rho = 0$, by S_γ. Then for two ρ and ρ_2 we write $\rho_1 \gg \rho_2$ if $S_\gamma(\rho_1) \frown S_\gamma \subset S_\gamma(\rho_2) \frown S_\gamma$ whenever $r > \| \rho_1 \| \vee \| \rho_2 \|$.

Lemma 4. *For a uniform space X following conditions are equivalent*:

(i) $\rho_1 \gg \rho_2$,

(ii) $X_{[\rho_1]} \geqq X_{[\rho_2]}$.

Proof. We have only to prove that i) implies ii). Suppose that there exist two subsets A and B such that $\rho_1(A, B) = 0$, but $\rho_2(A, B) > 0$. Then for $r > \| \rho_1 \| \vee \| \rho_2 \|$ if $\rho = r/\rho_2(A, B)$ $(\rho_2 \wedge \rho_2(A, B))$, we see that $\| \rho - \rho_1 \| = r = \| \rho \|$, but that $\| \rho - \rho_2 \| < r$. For there exists subsets $\{x_n\}$ and $\{y_n\}$ of A and B respectively such that $\rho_1(x_n, y_n) \to 0$ and $\rho_2(x_n, y_n) \geqq \gamma$, hence $(\rho - \rho_1)(x_n, y_n) \to \gamma$ and so $\| \rho - \rho_1 \| = r$. Furthermore for $\varepsilon < \rho_2(A, B)$ if $\rho_2(x, y) < \varepsilon$, then $|\rho(x, y) - \rho_2(x, y)| \leqq \varepsilon \vee \varepsilon r/\rho_2(A, B) < r$ and if $\rho_2(x, y) \geqq \varepsilon$, then $|\rho(x, y) - \rho_2(x, y)| \leqq (r-\varepsilon) \vee \| \rho_2 \| < r$. Thus $\| \rho - \rho_2 \| < \gamma$, hence $S_\gamma(\rho_1) \frown S_\gamma \not\subset S_\gamma(\rho_2) \frown S_\gamma$, i.e., $\rho_1 \not\gg \rho_2$.

Theorem 2. *For a complete uniform space X, the complete metric space $\mathfrak{S}\mathfrak{M}(X)$ determines the uniform space X.*

Proof. Let $\rho_1 \sim \rho_2$ if $\rho_1 \gg \rho_2$ and $\rho_2 \gg \rho_1$. Then obviously it is an equivalence relation and we denote by $[\rho_1]$ the equivalence class containing ρ_1 and let $[\rho_1] \geqq [\rho_2]$ if $\rho_1 \gg \rho_2$. Then the partially ordered set obtained above is isomorphic to $\mathfrak{D}(X)$ which determines by Theorem 1 the uniform space X.

REMARKS. It will be easily seen by Lemma 1 and 2 that a metrizable uniform space X is determined by the semi-linear topological space $\mathfrak{S}\mathfrak{M}_0(X)$ whose elements are pseudo-metrics compatible with the uniformity and vanishing only on the diagnol of the product space $X \times X$ and that

11) We say that a point x of metric space X is a middle point of y and z of X if $(x, y) = (x, z) = \frac{1}{2}(y, z)$. Cf. Menger: Untersuchung über allgemeiner Metrik, Math. Ann. 100.

a completely metrizable uniform space X is determined by the semi-linear topological space $\mathfrak{M}(X)$ whose elements are metrics compatible with the uniformity.

5. The lattice orderd semi-additive-group $\mathfrak{S}\mathfrak{M}(X)$.

Lemma 5. *For a uniform space X the following conditions are equivalent:*

(i) $X_{[\rho_1]} \geqq X_{[\rho_2]}$,

(ii) *there exists a sequence* $\{\rho_n' | n = 0, 1, 2, \ldots\}$ *such that for any* n a) $n\rho_n' \leqq \rho_0'$ *and* b) $\rho_2 \leqq \rho_n' \vee m_n\rho_1$ *for some integer* m_n.

Proof. Let $X_{[\rho_1]} \geqq X_{[\rho_2]}$, $\rho_n' = \rho_2 \wedge \frac{1}{n^3}$ and $\rho_0' = \sum (n\rho_2 \wedge \frac{1}{n^2})$. Then $\rho_n'(n = 0, 1, 2, \ldots) \in \mathfrak{S}\mathfrak{M}(X)$ and $n\rho_n' = n\rho_2 \wedge \frac{1}{n_2} \leqq \rho_0'$. Moreover since $X_{[\rho_1]} \geqq X_{[\rho_2]}$, there exists $\delta > 0$ such that $\rho_2(x, y) \geqq \frac{1}{n^3}$ implies $\rho_1(x, y) \geqq \delta$. Accordingly if $\rho_2(x, y) \geqq \frac{1}{n^3}$, $\rho_2(x, y) \leqq \| \rho_2 \| \leqq \frac{\| \rho_2 \|}{\delta} \rho_1(x, y)$. Hence for $m_n > \frac{\| \rho_2 \|}{\delta}$, $\rho_2 < \rho_n' \vee m_n\rho_1$.

Conversely let there exist a sequence $\{\rho_n'\}$ such that it satisfies a) and b). Then from a) $\| \rho_n' \| < \frac{1}{n} \| \rho_0' \|$. Furthermoe for any $\varepsilon > 0$ let n be an integer such that $\frac{1}{n} \| \rho_0' \| < \varepsilon$ and let δ be a positive number such that $m_n\delta < \varepsilon$. Then if $\rho_1(x, y) < \delta$, $\rho_2(x, y) < \frac{1}{n} \| \rho_0' \| \vee m_n\delta < \varepsilon$, which implies $X_{[\rho_1]} \geqq X_{[\rho_2]}$.

By the same method used in the proof of Theorem 2 we obtain the following

Theorfm 3. *If X is a complete uniform space, the lattice ordered semi-additive-group $\mathfrak{S}\mathfrak{M}(X)$ determines the uniform space X.*

REMARK. By a well known theorem obtained by several authors and by the method used by the author[12] we see easily that for a completely metrisable uniform space X, the system $\mathfrak{C}_u(X)$ of all (bounded) uniformly continuous real valued function on X determines the uniform space X considering $\mathfrak{C}_u(X)$ as ring, lattice or Banach space.

But for complete uniform spaces we can obtain from $\mathfrak{C}_u$ almost nothing, even for complete uniform space whose base space is separable metrizable. For example we consider the space X_1 and X_2 of the example in the section 3. The complete uniform space X_1 and X_2 are not uniformly homeomorphic, but $\mathfrak{C}_u(X_1)$ and $\mathfrak{C}_u(X_2)$ coincide.

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12) Cf. T. Shirota: A generalization of a theorem of I. Kaplansky, Osaka Math. J. 4 (1952).