

Title	The space of pseudo-metrics on a complete uniform space
Author(s)	Shirota, Taira
Citation	Osaka Mathematical Journal. 1953, 5(2), p. 147-153
Version Type	VoR
URL	https://doi.org/10.18910/11212
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The Space of Pseudo-Metrics on a Complete Uniform Space

By Taira SHIROTA

1. In a paper, B. H. Arnold¹⁾ considered the class of all upper semi-continuous decompositions of a T_1 space and showed that it is possible to construct a space homeomorphic to the given space from the partially ordered set of decompositions. In another paper²⁾, M. E. Shanks obtained results on the semi-linear space of all metrics compatible with the topology on a compactum.

In this paper we will show that the complete metric space as well as the lattice ordered semi-additive-group of all bounded pseudo-metrics compatible with the uniformity on a complete uniform space determine the given uniform space.

2. Let X ³⁾ be a uniform space. Then the set $\mathfrak{M}(X)$ of all bounded pseudo-metrics⁴⁾ compatible with its uniformity for X is a complete metric space with the distance $(\rho, \sigma) = \sup_{x, y \in X} |\rho(x, y) - \sigma(x, y)|$ and it is a lattice ordered semi-group⁵⁾ with the ordinary addition and order considered as a subsystem of the system of all continuous functions from the product space $X \times X$ into the reals.

Moreover for $\rho \in \mathfrak{M}(X)$ let $X_{[\rho]}$ be a metrizable uniform space whose points are equivalence classes $[x]_\rho$ with respect to the equivalence relation $\rho(x, y) = 0$ and whose metric is defined by the distance $d_\rho([x]_\rho, [y]_\rho) = \rho(x, y)$. Then we write $X_{[\rho]} \geq X_{[\sigma]}$ when the mapping $F_{[\rho], [\sigma]} : [x]_\rho \rightarrow [x]_\sigma$ is uniformly continuous from $X_{[\rho]}$ onto $X_{[\sigma]}$, and $X_{[\rho]} > X_{[\sigma]}$ if $X_{[\rho]} \geq X_{[\sigma]}$ but not $X_{[\sigma]} \geq X_{[\rho]}$ and we denote by $\mathfrak{D}(X)$ the partially ordered set of all such metrizable uniform space $X_{[\rho]}$ with the above order.

1) Cf. B. H. Arnold: Decompositions of a T_1 space, Bull. Amer. Math. Soc., 46 (1943).

2) Cf. M. E. Shanks: The space of metrics on a compact metrizable space, Amer. Jour. Math., 66 (1944).

3) In the present note we may assume that the potency of X is greater than 4, since otherwise our results are trivial.

4) We say that ρ is a pseudo-metric compatible with the uniformity for X if it satisfies the following conditions, i) $\rho(x, x) = 0$, ii) $\rho(x, y) = \rho(y, x)$, iii) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ and iv) for any $\varepsilon > 0$ there exists a neighbourhood V such that $\rho(x, y) < \varepsilon$ for $x \in V(y)$.

5) We say that S is a lattice ordered semi-group if it satisfies the following conditions i) S is a lattice and semi-group and ii) $a \vee b + c = (a + c) \vee (b + c)$ for any a, b and $c \in S$. Cf. G. Birkhoff, Lattice theory, (1949), p. 201.

3. The partially ordered set $\mathfrak{D}(X)$. In this section we will show that $\mathfrak{D}(X)$ determines the space X whenever X is a complete uniform space. For this purpose we shall prove the following lemmas.

Lemma 1. *For two $X_{\{\rho_1\}}$ and $X_{\{\rho_2\}}$ of a uniform space X the following conditions are equivalent:*

- (i) $X_{\{\rho_1\}} \geq X_{\{\rho_2\}}$
- (ii) *for two disjoint subsets A and B of X , $\rho_1(A, B)^{6)} = 0$ implies $\rho_2(A, B)^{6)} = 0$.*

Proof. Since obviously (i) implies (ii), we only prove that (ii) implies (i). Suppose that $X_{\{\rho_1\}}$ and $X_{\{\rho_2\}}$ satisfy the condition (ii). Then evidently $F_{\{\rho_1\}, \{\rho_2\}}$ is a continuous mapping from $X_{\{\rho_1\}}$ onto $X_{\{\rho_2\}}$. Now we assume that there exist subsets $\{x_n\}$ and $\{y_n\}$ such that $\rho_1(x_n, y_n) < \frac{1}{n}$ and $\rho_2(x_n, y_n) \geq \varepsilon > 0$. If $\{[x_n]_{\rho_2}\}$ contains a Cauchy subsequence $\{[x'_n]_{\rho_2}\}$ whose limit point in the completion $\bar{X}_{\{\rho_2\}}$ is \bar{x} , then $\rho_1(A, B) = 0$ and $\rho_2(A, B) \geq \frac{\varepsilon}{2}$, where $A = \{x | [x]_{\rho_2} \in \{[x'_n]_{\rho_2}\} \cap S_{\rho_2}(\bar{x}, \varepsilon/4)\}$ and $B = \{y | [y]_{\rho_2} \in \{[y'_n]_{\rho_2}\} \& [x'_n]_{\rho_2} \in S_{\rho_2}(\bar{x}, \varepsilon/4)\}$. Hence we see that both $\{[x_n]_{\rho_2}\}$ and $\{[y_n]_{\rho_2}\}$ contain no Cauchy subsequences, so that there exists subsets $\{[x'_n]_{\rho_2}\}$ and $\{[y'_n]_{\rho_2}\}$ of $\{[x_n]_{\rho_2}\}$ and $\{[y_n]_{\rho_2}\}$ respectively such that for some δ $d_{\rho_2}([x'_m]_{\rho_2}, [x'_n]_{\rho_2}) > \delta$ and $d_{\rho_2}([y'_m]_{\rho_2}, [y'_n]_{\rho_2}) > \delta$ if $m \neq n$. Then we can construct two infinite subsets \bar{A} and \bar{B} such that $\bar{A} \subset \{[x'_n]_{\rho_2}\}$ and $\bar{B} \subset \{[y'_n]_{\rho_2}\}$ and $d_{\rho_2}(\bar{A}, \bar{B}) > 0$. Let $A = \{x | [x]_{\rho_2} \in \bar{A}\}$ and let $B = \{y | [y]_{\rho_2} \in \bar{B}\}$. Then $\rho_1(A, B) = 0$ and $\rho_2(A, B) > 0$. Hence we see that $F_{\{\rho_1\}, \{\rho_2\}}$ is uniformly continuous.

Lemma 2. *For two $X_{\{\rho_1\}}$ and $X_{\{\rho_2\}}$ such that $X_{\{\rho_1\}} > X_{\{\rho_2\}}$, the following conditions are equivalent:*

- (i) $X_{\{\rho_1\}}$ covers $X_{\{\rho_2\}}$
- (ii) a) *there exists a unique pair of different points \bar{x} and \bar{y} of the completion $\bar{X}_{\{\rho_1\}}$ such that $\bar{d}_{\rho_2}(\bar{x}, \bar{y}) = 0$, where \bar{d}_{ρ_2} is an extension of the pseudo metric $d_{\rho_2} : d_{\rho_2}([x]_{\rho_1}, [y]_{\rho_1}) = \rho_2(x, y)$, and b) if, for three subsets A_1, A_2 and B of X , $\rho_1(A_1, A_2) > 0$ and $\rho_1(A_1 \cup A_2, B) > 0$, then either $\rho_2(A_1, B) > 0$ or $\rho_2(A_2, B) > 0$.*

Proof. we show that (i) implies (ii). Let $X_{\{\rho_1\}}$ cover $X_{\{\rho_2\}}$. Now suppose that there exist three subsets A_1, A_2 and B such that they do not satisfy b). Then there exists a continuous function f of X such that $f(A_1 \cup B) = 0$ and $f(A_2) = 1$ and such that for any $\varepsilon > 0$ $\rho_1(x, y) < \delta$

6) $\rho(A, B) = \inf_{x \in A, y \in B} \rho(x, y)$

7) $S_{\rho}(x, \varepsilon) = \{y | \rho(x, y) < \varepsilon\}$

implies $|f(x) - f(y)| < \varepsilon$ for some δ . Let ρ_f be a pseudo-metric of X such that $\rho_f(x, y) = |f(x) - f(y)|$ and let $\tau = \sigma_2 + \rho_f$. Then obviously for $\varepsilon > 0$ there exists δ such that $\rho_1(x, y) < \delta$ implies $\tau(x, y) < \varepsilon$ and hence $X_{\tau} \supseteq X_{\rho_1}$ and in fact $X_{\tau} \supset X_{\rho_1}$ since $\tau(A_1, B) = 0$ and $\rho_1(A_1, B) > 0$. On the other hand $X_{\tau} \supseteq X_{\rho_2}$, and moreover $X_{\tau} \supset X_{\rho_2}$, since $\rho_2(A_2, B) = 0$ and $\tau(A_2, B) > 0$. Hence X_{ρ_1} and X_{ρ_2} satisfy b). Furthermore if $\bar{d}_{\rho_2}(\bar{x}, \bar{y}) = 0$ implies $\bar{x} = \bar{y}$ for any \bar{x} and \bar{y} in \bar{X}_{ρ_1} , since \bar{d}_{ρ_2} is not a metric of \bar{X}_{ρ_1} and since \bar{X}_{ρ_1} is a complete metrizable, by the same method used in the proof of Lemma 1, there exists three subsets A_1' , A_2' and B' of \bar{X}_{ρ_1} such that $\bar{d}_{\rho_1}(A_1', A_2') > 0$, $\bar{d}_{\rho_1}(A_1' \cup A_2', B') > 0$, $\bar{d}_{\rho_2}(A_1', B') = 0$ and $\bar{d}_{\rho_2}(A_2', B') = 0$. Let $A_i = \{x | [x]_{\rho_1} \in U(A_i') \cap X_{\rho_1}\}$ and $B = \{x | [x]_{\rho_1} \in U(B') \cap X_{\rho_1}\}$, where $U(A_i')$ and $U(B')$ are suitable neighbourhoods of A_i' and B' respectively. Then A_i and B do not satisfy b). Hence there exists a pair of different points \bar{x} and \bar{y} of \bar{X}_{ρ_1} such that $\bar{d}_{\rho_2}(\bar{x}, \bar{y}) = 0$. Furthermore we easily see that such pair of points is uniquely determined.

The proof of the converse will be omitted, as it can be done by the method used above.

DEFINITION. We say that a proper ideal I of $\mathfrak{D}(X)$ is p -ideal if it satisfies the following property: for any X_{ρ_1} there exists an $X_{\rho_1'}$ such that $X_{\rho_1'} \in I$ and $X_{\rho_1} = X_{\rho_1'}$ or X_{ρ_1} covers $X_{\rho_1'}$, and if $X_{\rho_1} \supseteq X_{\rho_2}$, $X_{\rho_1'} \supseteq X_{\rho_2'}$.

Lemma 3. *Let X be a complete uniform space and let I be a p -ideal. Then there exists a unique pair of different points x and y of X such that $I = \{X_{\rho} | \rho(x, y) = 0\}$.*

Proof. First of all we remark that if $X_{\rho_1} \notin I$ then $X_{\rho_1'} \in I$ is uniquely determined. For if two different elements $X_{\rho_1'}$ and $X_{\rho_2'} \in I$ are covered by X_{ρ_1} , since I is an ideal, $X_{\rho_1'} \vee X_{\rho_2'} \in I$, but $X_{\rho_1} = X_{\rho_1'} \vee X_{\rho_2'}$, which is a contradiction. Now let $X_{\rho_0} \notin I$ be fixed and let \bar{x}_{ρ_0} and \bar{y}_{ρ_0} be two points of \bar{X}_{ρ_0} such that $\bar{d}_{\rho_0}(\bar{x}_{\rho_0}, \bar{y}_{\rho_0}) = 0$. Moreover let \bar{x}_1 and \bar{x}_2 be two points of \bar{X}_{ρ_1} such that $\bar{d}_{\rho_1}(\bar{x}_1, \bar{x}_2) = 0$, where $X_{\rho_1} \supset X_{\rho_0}$. Then $\bar{F}_{\rho_1, \rho_0}(\{\bar{x}_1, \bar{x}_2\}) = \{\bar{x}_{\rho_0}, \bar{y}_{\rho_0}\}$ where \bar{F}_{ρ_1, ρ_0} is a mapping from X_{ρ_1} into \bar{X}_{ρ_0} such that it is an extension of F_{ρ_1, ρ_0} . For if $\bar{d}_{\rho_0}(\bar{x}_1, \bar{x}_2) = 0$, by Lemma 1 and 2 we see that $X_{\rho_0} \leq X_{\rho_1}$ and $X_{\rho_0} \in I$, which is a contradiction. Hence $\bar{d}_{\rho_0}(\bar{x}_1, \bar{x}_2) \neq 0$, accordingly $\bar{d}_{\rho_0}(\bar{F}_{\rho_1, \rho_0}(\bar{x}_1), \bar{F}_{\rho_1, \rho_0}(\bar{x}_2)) \neq 0$, but $\bar{d}_{\rho_1}(\bar{x}_1, \bar{x}_2) = 0$ and $X_{\rho_1} \supseteq X_{\rho_0}$, hence $\bar{d}_{\rho_0}(\bar{F}_{\rho_1, \rho_0}(\bar{x}_1), \bar{F}_{\rho_1, \rho_0}(\bar{x}_2)) = 0$. By the uniqueness of such pair of points we see that $\bar{F}_{\rho_1, \rho_0}(\{\bar{x}_1, \bar{x}_2\}) = \{\bar{x}_{\rho_0}, \bar{y}_{\rho_0}\}$. Let \bar{x}_ρ be the point such that $\bar{x}_\rho \in \{\bar{x}_1, \bar{x}_2\}$ and $\bar{F}_{\rho_1, \rho_0}(\bar{x}_\rho) = \bar{x}_{\rho_0}$ and let \bar{y}_ρ be the other point of $\{\bar{x}_1, \bar{x}_2\}$. More-

over let $A_{\rho, n} = \left\{ x \mid \bar{d}_\rho([x]_\rho, \bar{x}_\rho) \leq \frac{1}{2n} \right\}$ and $B_{\rho, n} = \left\{ x \mid \bar{d}_\rho([x]_\rho, \bar{y}_\rho) \leq \frac{1}{2n} \right\}$.

Then $\{A_{\rho, n}\}$ and $\{B_{\rho, n}\}$ are both Cauchy closed family of X . Since X is complete, there exists a pair of points x_0 and y_0 of X such that $x_0 = \coprod A_{\rho, n}$ and $y_0 = \coprod B_{\rho, n}$. Then obviously $[x_0]_\rho = \bar{x}_\rho$ and $[y_0]_\rho = \bar{y}_\rho$ for $X_\rho \geq X_{\rho_0}$. Furthermore if $\rho(x_0, y_0) = 0$, then $X_{\tau\rho} \in I$, for let $X_{\tau\rho} = X_{\tau\rho} \vee X_{\tau\rho_0}$, then $X_{\tau\rho} > X_{\tau\rho_0}$ and $\bar{d}_\rho(\bar{x}_\tau, \bar{y}_\tau) = 0$, hence $X_{\tau\rho} \geq X_{\tau\rho_0}$, which implies $X_{\tau\rho} \in I$. Since evidently $\rho(x_0, y_0) > 0$ implies $X_{\tau\rho} \notin I$, $I = \{X_{\tau\rho} \mid \rho(x_0, y_0) = 0\}$.

Theorem 1. *If X is a complete uniform space, the partially ordered set $\mathfrak{D}(X)$ determines the uniform space X .*

Proof (1). Let $I\{x, y\}$ be the p -ideal which corresponds to a pair of points x and y of X as in Lemma 3. Moreover for two p -ideals I_1, I_2 we denote by $I_1 \sim I_2$ the relation: $I_1 = I_2$ or $I_1 \wedge I_2 \subset I_3$ for some I_3 . Then by the triangle axiom of pseudo-metrics, $I\{x, y\}$ and $I\{u, v\}$ are equivalent if and only if $\{x, y\} \cap \{u, v\} \neq \phi$. Furthermore we say that a subset P of the set of all p -ideals is a maximal collection if it satisfies the following conditions: i) P contains at least four p -ideals, ii) any two p -ideals $\in P$ are equivalent and iii) it is maximal with respect to i) and ii). Then for a maximal collection P there exists a unique point x of X such that $P = \{I\{x, y\} \mid y \in X \text{ \& } y \neq x\}$, which is denoted by $P(x)$. Conversely any $P(x)$ is a maximal collection. Let \tilde{X} be the set of all maximal collections. Then we see that the correspondence: $x \rightarrow P(x)$ is a one-to-one mapping from X to \tilde{X} . Furthermore let $\tilde{A} = \{P(x) \mid x \in A\}$.

(II) We say that a subset \tilde{A} with potency ≥ 2 is *basic-closed* if there exists a $X_{\tau\rho} \in \mathfrak{D}(X)$ such that for any $P \in \tilde{A}$, $\tilde{A} = \{Q \mid Q \cap P \ni I \ni X_{\tau\rho}\} \cup \{P\}$. Then a subset \tilde{A} of \tilde{X} is basic-closed if and only if A is a closed G_δ -set which is a zero-set of a uniformly continuous function of X and the potency $|A| \geq 2$. Let the set of all basic-closed sets of \tilde{X} be a closed basis for \tilde{X} . Then we see that \tilde{X} is a topological space which is homeomorphic to X by the mapping P .

(III) Now we define the uniformity for \tilde{X} by pseudo-metrics. For this purpose we define the uniformity for \tilde{X} by pseudo-metrics. For this purpose we say that two disjoint basic-closed subsets \tilde{A}_i : ($i = 1, 2$) are ρ -separated if there exists $X_{\tau\rho_1} \subset X_{\tau\rho_2}$ such that $\tilde{A}_i = \{Q \mid Q \cap P_i \ni I \ni X_{\tau\rho_1}\} \cup \{P_i\}$ for any $P_i \in \tilde{A}_i$ and that two subsets \tilde{A}_i are ρ -separated if they are contained respectively in two ρ -separated disjoint basic-closed subsets. Furthermore we define that a pseudo-metric $\tilde{\rho}$ of \tilde{X} is compatible with the uniformity for \tilde{X} if there exists $X_{\tau\rho} \in \mathfrak{D}(X)$ such that if $\tilde{\rho}(\tilde{A}, \tilde{B}) > 0$, \tilde{A} and \tilde{B} are ρ -separated.

Now let $\tilde{\rho}$ be compatible with the uniformity for \tilde{X} and let ρ_1 be a pseudo-metric of X such that $\rho_1(x, y) = \tilde{\rho}(P(x), P(y))$. Then if $\rho_1(A_1, A_2) > 0$, then $\tilde{\rho}(\tilde{A}_1, \tilde{A}_2) > 0$, hence $\tilde{A}_i (i = 1, 2)$ are ρ -separated, accordingly there exists subsets $\tilde{A}'_i (i = 1, 2)$ and $X_{\{\sigma\}} < X_{\{\rho\}}$ such that $\tilde{A}'_i = \{Q \mid Q \wedge P_i \ni I \ni X_{\{\sigma\}} \cup \{P_i\}$ and $\tilde{A}'_i > \tilde{A}_i$. This means that $A_i < \{x \mid \sigma(x, y) = 0$ for a fixed $y_i\}$ and $[y_1]_\sigma \neq [y_2]_\sigma$, so that $\rho(A_1, A_2) > 0$. Thus we see by Lemma 1 that $\rho_1 \in \mathfrak{SM}(X)$ and $X_{\{\rho_1\}} \leq X_{\{\rho\}}$. Conversely for any $\rho \in SM(X)$ let $\tilde{\rho}$ be a pseudo-metric of \tilde{X} such that $\tilde{\rho}(P(x), P(y)) = \rho(x, y)$. Then $\tilde{\rho}(\tilde{A}, \tilde{B}) > 0$ if and only if $\rho(A, B) > 0$, i.e., \tilde{A} and \tilde{B} are ρ -separated. Thus we see that the mapping P is a uniform homeomorphism.

REMARK. Let $\mathfrak{D}'(X)$ be the partially ordered set whose elements are equivalence relations on X : $\rho(x, y) = 0, \rho \in \mathfrak{SM}(X)$. Then if X is a complete uniform space, $\mathfrak{D}'(X)$ determined the given topological space X , but does not determine the uniform space X .

For example we consider the space $X = \bigcup_{n=1}^{\infty} X_n$ where X_n are mutually disjoint the n -dimensional cubes and whose relative topology on X_n is a usual one. Let X_1 be the coarsest uniform space⁸⁾ over X for which all continuous functions are uniformly continuous and let X_2 be the uniform space⁹⁾ over X with the uniformity made up of all countable normal coverings. Then two space are complete and $\mathfrak{D}'(X_1) = \mathfrak{D}'(X_2)$. For there exists $\rho' \in \mathfrak{SM}(X_i)$ for any $\rho \in \mathfrak{SM}(X_i)$ such that $\rho(x, y) = 0$ and $\rho'(x, y) = 0$ are the same equivalence relation on X and is totally bounded, and so $\mathfrak{D}'(X_1)$ and $\mathfrak{D}'(X_2)$ are determined by the totally bounded-pseudo metrics which are identical on both X_1 and X_2 . But X_1 and X_2 are not uniformly homeomorphic. For let \mathfrak{B}_n be the finite open covering of X_n such that any refinement of \mathfrak{B}_n has order $\geq n+1$ and let $\mathfrak{U} = \{U \mid U \in \mathfrak{B}_n$ for some $n\}$, then \mathfrak{U} is contained in the uniformity for X_2 . Suppose that there exists a uniform homeomorphism F from X_1 onto X_2 . Then $F^{-1}(\mathfrak{U})$ is contained in the uniformity for X_1 , hence there must exist a finite number of continuous functions $\{f_1, f_2, \dots, f_n\}$ and a real number $\varepsilon > 0$ such that

$$\mathfrak{U}_1 = \{\{y \mid |f_i(x) - f_i(y)| < \varepsilon \text{ for any } i\} \mid x \in X\}$$

is a refinement of $F^{-1}(\mathfrak{U})$. But since the mapping f from X into the n -dimensional Euclidean space $E: f(x) = \{f_i(x) \mid i = 1, 2, \dots, n\}$ is continuous, by the extended Lebesgue's covering theorem¹⁰⁾ \mathfrak{U}_1 has a refinement

8) Cf. E. Hewitt: Rings of real valued continuous functions, Trans. Amer. Math. Soc 64 (1948).

9) Cf. T. Shirota: A class of topological spaces, Osaka Math. J. 4 (1952).

10) C. H. Dowker: Lebesgue dimension of a normal space, Bull. of Amer. Math. Soc. 52 (1946). K. Morita: On the dimension theory of normal space I, Japanese Journ. Math. 20 (1950).

\mathfrak{U}_2 with order $\leq n+2$. Hence $F(\mathfrak{U}_2) \leq \mathfrak{U}$ and the order of $F(\mathfrak{U}_2)$ is $\leq n+2$. Accordingly the order of $F(\mathfrak{U}_2)|X_{n+1}$ is $\leq n+2$ and $F(\mathfrak{U}_2)|X_{n+1} \leq \mathfrak{B}_{n+1}$. But by the property of \mathfrak{B}_{n+1} the order of $F(\mathfrak{U}_2)|X_{n+1}$ is $\geq n+2$, which is a contradiction.

4. The complete metric space $\mathfrak{SM}(X)$. We remark first that the zero 0 of the semi-linear space $\mathfrak{SM}(X)$ is determined by the property that it can not be the middle point¹¹⁾ of two different points. Accordingly we can characterize the norm of an element ρ of $\mathfrak{SM}(X)$ as $(0, \rho)$ and we write it by $\|\rho\|$.

Definition. For any real $\gamma > 0$ and $\rho \in \mathfrak{SM}(X)$ we denote the surface $\{\rho' | (\rho', \rho) = \gamma\}$ by $S_\gamma(\rho)$ and in particular, when $\rho = 0$, by S_γ . Then for two ρ_1 and ρ_2 we write $\rho_1 \gg \rho_2$ if $S_\gamma(\rho_1) \cap S_\gamma \subset S_\gamma(\rho_2) \cap S_\gamma$ whenever $r > \|\rho_1\| \vee \|\rho_2\|$.

Lemma 4. For a uniform space X following conditions are equivalent:

- (i) $\rho_1 \gg \rho_2$,
- (ii) $X_{[\rho_1]} \geq X_{[\rho_2]}$.

Proof. We have only to prove that i) implies ii). Suppose that there exist two subsets A and B such that $\rho_1(A, B) = 0$, but $\rho_2(A, B) > 0$. Then for $r > \|\rho_1\| \vee \|\rho_2\|$ if $\rho = r/\rho_2(A, B) (\rho_2 \wedge \rho_2(A, B))$, we see that $\|\rho - \rho_1\| = r = \|\rho\|$, but that $\|\rho - \rho_2\| < r$. For there exists subsets $\{x_n\}$ and $\{y_n\}$ of A and B respectively such that $\rho_1(x_n, y_n) \rightarrow 0$ and $\rho_2(x_n, y_n) \geq \gamma$, hence $(\rho - \rho_1)(x_n, y_n) \rightarrow \gamma$ and so $\|\rho - \rho_1\| = r$. Furthermore for $\varepsilon < \rho_2(A, B)$ if $\rho_2(x, y) < \varepsilon$, then $|\rho(x, y) - \rho_2(x, y)| \leq \varepsilon \vee \varepsilon r/\rho_2(A, B) < r$ and if $\rho_2(x, y) \geq \varepsilon$, then $|\rho(x, y) - \rho_2(x, y)| \leq (r - \varepsilon) \vee \|\rho_2\| < r$. Thus $\|\rho - \rho_2\| < r$, hence $S_\gamma(\rho_1) \cap S_\gamma \subset S_\gamma(\rho_2) \cap S_\gamma$, i.e., $\rho_1 \gg \rho_2$.

Theorem 2. For a complete uniform space X , the complete metric space $\mathfrak{SM}(X)$ determines the uniform space X .

Proof. Let $\rho_1 \sim \rho_2$ if $\rho_1 \gg \rho_2$ and $\rho_2 \gg \rho_1$. Then obviously it is an equivalence relation and we denote by $[\rho_1]$ the equivalence class containing ρ_1 and let $[\rho_1] \geq [\rho_2]$ if $\rho_1 \gg \rho_2$. Then the partially ordered set obtained above is isomorphic to $\mathfrak{D}(X)$ which determines by Theorem 1 the uniform space X .

REMARKS. It will be easily seen by Lemma 1 and 2 that a metrizable uniform space X is determined by the semi-linear topological space $\mathfrak{SM}_0(X)$ whose elements are pseudo-metrics compatible with the uniformity and vanishing only on the diagonal of the product space $X \times X$ and that

11) We say that a point x of metric space X is a middle point of y and z of X if $(x, y) = (x, z) = \frac{1}{2}(y, z)$. Cf. Menger: Untersuchung über allgemeiner Metrik, Math. Ann. 100.

a completely metrizable uniform space X is determined by the semi-linear topological space $\mathfrak{M}(X)$ whose elements are metrics compatible with the uniformity.

5. The lattice ordered semi-additive-group $\mathfrak{SM}(X)$.

Lemma 5. For a uniform space X the following conditions are equivalent:

- (i) $X_{\{\rho_1\}} \geq X_{\{\rho_2\}}$,
- (ii) there exists a sequence $\{\rho_n' | n = 0, 1, 2, \dots\}$ such that for any n a) $n\rho_n' \leq \rho_0'$ and b) $\rho_2 \leq \rho_n' \vee m_n\rho_1$ for some integer m_n .

Proof. Let $X_{\{\rho_1\}} \geq X_{\{\rho_2\}}$, $\rho_n' = \rho_2 \wedge \frac{1}{n^3}$ and $\rho_0' = \sum (n\rho_2 \wedge \frac{1}{n^2})$. Then $\rho_n' (n = 0, 1, 2, \dots) \in \mathfrak{SM}(X)$ and $n\rho_n' = n\rho_2 \wedge \frac{1}{n} \leq \rho_0'$. Moreover since $X_{\{\rho_1\}} \geq X_{\{\rho_2\}}$, there exists $\delta > 0$ such that $\rho_2(x, y) \geq \frac{1}{n^3}$ implies $\rho_1(x, y) \geq \delta$.

Accordingly if $\rho_2(x, y) \geq \frac{1}{n^3}$, $\rho_2(x, y) \leq \|\rho_2\| \leq \frac{\|\rho_2\|}{\delta} \rho_1(x, y)$. Hence for $m_n > \frac{\|\rho_2\|}{\delta}$, $\rho_2 < \rho_n' \vee m_n\rho_1$.

Conversely let there exist a sequence $\{\rho_n'\}$ such that it satisfies a) and b). Then from a) $\|\rho_n'\| < \frac{1}{n} \|\rho_0'\|$. Furthermore for any $\varepsilon > 0$ let n be an integer such that $\frac{1}{n} \|\rho_0'\| < \varepsilon$ and let δ be a positive number such that $m_n\delta < \varepsilon$. Then if $\rho_1(x, y) < \delta$, $\rho_2(x, y) < \frac{1}{n} \|\rho_0'\| \vee m_n\delta < \varepsilon$, which implies $X_{\{\rho_1\}} \geq X_{\{\rho_2\}}$.

By the same method used in the proof of Theorem 2 we obtain the following

Theorem 3. If X is a complete uniform space, the lattice ordered semi-additive-group $\mathfrak{SM}(X)$ determines the uniform space X .

REMARK. By a well known theorem obtained by several authors and by the method used by the author¹²⁾ we see easily that for a completely metrisable uniform space X , the system $\mathfrak{C}_u(X)$ of all (bounded) uniformly continuous real valued function on X determines the uniform space X considering $\mathfrak{C}_u(X)$ as ring, lattice or Banach space.

But for complete uniform spaces we can obtain from \mathfrak{C}_u almost nothing, even for complete uniform space whose base space is separable metrizable. For example we consider the space X_1 and X_2 of the example in the section 3. The complete uniform space X_1 and X_2 are not uniformly homeomorphic, but $\mathfrak{C}_u(X_1)$ and $\mathfrak{C}_u(X_2)$ coincide.

(Received September 17, 1953)

12) Cf. T. Shirota: A generalization of a theorem of I. Kaplansky, Osaka Math. J. 4 (1952).

