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1. Introduction

Let $G/K$ be a Hermitian symmetric space where $G$ is a connected non-compact semisimple Lie group and $K \subseteq G$ is a maximal compact subgroup. We fix a discrete subgroup $\Gamma$ of $G$ which acts freely on $G/K$ and for which the quotient $X = \Gamma \backslash G/K$ is compact. Let $E, \rightarrow G/K$ be a homogeneous $C^\infty$ vector bundle over $G/K$ induced by a finite-dimensional irreducible representation $\tau$ of $K$. Then $E, \rightarrow G/K$ has a holomorphic structure and one can define a presheaf by assigning to an open set $U$ in $X$ the abelian group of $\Gamma$-invariant holomorphic sections of $E, \rightarrow G/K$ on the inverse image (under the map $G/K \rightarrow X$) of $U$. Let $\Theta \rightarrow X$ be the sheaf generated by this presheaf and let $H^q(X, \Theta,)$ denote the $q$th cohomology space of $X$ with coefficients in $\Theta,\cdot$. In this paper we continue the program initiated in [23] of obtaining some general vanishing theorems for the spaces $H^q(X, \Theta,)$ by the application of recent representation-theoretic results. This allows for a unified viewpoint and one by which, in particular, the classical vanishing theorems of [3], [4], [5], [6], [7], [12], and [13] may be deduced.

Following Hotta and Murakami [4] we represent $H^q(X, \Theta,)$ as a space of automorphic forms. Then its dimension can be expressed by a formula of Matsushima and Murakami [14] in terms of certain irreducible unitary representations $\pi$ of $G$, the multiplicity of $\pi$ in $L^2(\Gamma \backslash G)$, and the $K$ intertwining number of $\pi$ with $\text{Ad}^q \otimes \tau$ where $\text{Ad}^q$ is the $q$th exterior power of the adjoint representation of $K$ on the space of holomorphic tangent vectors at the origin of $G/K$. Based on results of Kumaresan [9], Parthasarathy [17], and Vogan [21], we have been able to obtain in [23] and [24] a clearer understanding of the structure of the unitary representations $\pi$ of $G$ in the Matsushima-Murakami formula; also see Theorem 3.3 of the present paper. We apply this new knowledge in conjunction with the Matsushima-Murakami formula to deduce the main result of this paper, which is Theorem 4.3. We can deduce, in particular, results of [23] from Theorem 4.3 without assuming the linearity of $G$. Thus we drop the linearity assumption in the present paper, which was enforced in [23].
2. Unitary representations intertwining $\chi^\pm \otimes \tau_{\Lambda+\delta_s}$

In this section $G$ will denote a non-compact connected semisimple Lie group with finite center and $K \subset G$ will denote a maximal compact subgroup of $G$. However, proceeding more generally, we shall not assume that $G/K$ is Hermitian symmetric (until later). Let $g_0 = \mathfrak{t}_0 + \mathfrak{p}_0$ be a Cartan decomposition of the Lie algebra $g_0$ of $G$, where $\mathfrak{t}_0$ is the Lie algebra of $K$ and $\mathfrak{p}_0$ is the orthogonal complement of $\mathfrak{t}_0$ relative to the Killing form $(\ , \ )$ of $g_0$. Let $\mathfrak{g}, \mathfrak{t}, \mathfrak{p}$ denote, respectively, the complexifications of $g_0, \mathfrak{t}_0, \mathfrak{p}_0$. We shall assume throughout that $\mathfrak{t}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; i.e. we assume $G$ and $K$ have the same rank. This will be the case in particular when $G/K$ is Hermitian. Let $\Delta$ be the set of non-zero roots of $(\mathfrak{g}, \mathfrak{h})$, let $\Delta_\text{cs}, \Delta_\text{n}$ denote the compact, non-compact roots respectively in $\Delta$, let $\Delta^+_c=\Delta^+ \cap \Delta_\text{cs}, \Delta^+_n=\Delta^+ \cap \Delta_\text{n}$, and let $2\delta=\langle \Delta^+_c \rangle, 2\delta_n=\langle \Delta^+_n \rangle, 2\delta_s=\langle \Delta^+_s \rangle$, where we write $\langle \Phi \rangle = \sum_{\alpha \in \Phi} \alpha$ for $\Phi \subset \Delta$. Let $\mathcal{S}$ denote the integral linear forms $\Lambda$ on $\mathfrak{h}$; i.e. $\Lambda \in \mathfrak{h}^*$ (the dual space of $\mathfrak{h}$) satisfies $2(\Lambda, \alpha)$ is an integer for each $\alpha$ in $\Delta$. We define

\begin{equation}
\mathcal{S}'_0 = \{\Lambda \in \mathcal{S} | (\Lambda+\delta, \alpha) \neq 0 \text{ for } \alpha \in \Delta \text{ and } (\Lambda+\delta, \alpha)>0 \text{ for } \alpha \in \Delta^+_s\}.
\end{equation}

Let $g_\alpha$ be the (one dimensional) root space of $\alpha \in \Delta$. Given $\Lambda \in \mathcal{S}'_0 \Lambda+\delta_s$ is the highest weight with respect to $\Delta^+_s$ of an irreducible representation $\tau_{\Lambda+\delta_s}$ of $\mathfrak{t}$. The Killing form of $g_0$ induces a real inner product on $\mathfrak{p}_0$ and since $\mathfrak{p}_0$ is even-dimensional (because $G$ and $K$ are of equal rank) the spin representation $\sigma$ of $\mathfrak{so}(\mathfrak{p}_0)$ has a decomposition $\sigma=\sigma^+ \oplus \sigma^-$ into two irreducible representations $\sigma^\pm$. Let

\begin{equation}
\mathcal{S}' = \sigma^+ \otimes \tau_{\Lambda+\delta_s} \mid \mathfrak{p}_0
\end{equation}

where $(\text{ad}_{\mathfrak{p}_0})|_{\mathfrak{p}_0}$ is the adjoint representation of $\mathfrak{t}_0$ on $\mathfrak{p}_0$. Then $\chi^\pm \otimes \tau_{\Lambda+\delta_s}$ always integrates to a representation of $K$ (which we shall denote by the same symbol) for $\Lambda \in \mathcal{S}'_0$ even though $\tau_{\Lambda+\delta_s}$ may not. Let $\Omega$ denote the Casimir operator of $G$ and let $\hat{G}$ denote the equivalence classes of irreducible unitary representations $(\pi, H_\pi)$ of $G$ on a Hilbert space $H_\pi$. Given $\Lambda \in \mathcal{S}'_0$ we shall want to pin down the structure of a $(\pi, H_\pi) \in \hat{G}$ such that $\pi(\Omega)=\langle \Lambda, \Lambda+2\delta \rangle 1$ and such that $\text{Hom}_\pi(\pi, \chi^+ \otimes \tau_{\Lambda+\delta_s}) \neq 0$. Here $H_\pi$ also denotes the space of $K$ finite vectors in $H_\pi$ which is regarded as a $U\mathfrak{g}$ module where $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$; thus $\pi(\Omega)$ is well-defined. We shall need the following additional notation. If $\theta \subset \mathfrak{g}$ is a parabolic subalgebra we shall write $\theta=\mathfrak{m}+\mathfrak{n}$ for its Levi decomposition where $\mathfrak{m}$ and $\mathfrak{n}$ denote the reductive and nilpotent parts respectively of $\theta$, $\Delta(\mathfrak{m})$ for the roots of $\mathfrak{m}$, $\theta_{s,n}$ for the set of non-compact roots in the nilpotent radical $\mathfrak{n}$, $M$ for the closed Lie subgroup of $G$ whose complexified Lie algebra is $\mathfrak{m}$, and we shall write $2\delta_{s,n}=\langle \theta_{s,n} \rangle$. Let $c: g_0 \to g_0$ denote the Cartan
involution for the Cartan decomposition \( \Theta = \mathfrak{g}_0 + \mathfrak{h}_0 \) above. Let \( F \) be a finite-dimensional irreducible \( \mathfrak{g} \) module and let \( \Theta = \mathfrak{m} + \mathfrak{n} + \mathfrak{h} \) be a stable parabolic subalgebra of \( \mathfrak{g} \) such that the space \( F^\Theta \) of \( \Theta \) invariants is a one-dimensional unitary \( M \) module. If \( \lambda \in \mathfrak{m}^* \) is the differential of \( F^\Theta \) then \( \lambda(\Delta(\mathfrak{m})) = 0 \) and we shall write \( A_\Theta(\lambda) \) for the unique (up to equivalence) irreducible \( \mathfrak{g} \) module with minimal \( \Theta \) type \( \lambda \). This means that \( A_\Theta(\lambda) \) is the only irreducible \( \mathfrak{g} \) module such that (i) \( A_\Theta(\lambda) \) contains the irreducible \( \mathfrak{g} \) module with \( \Delta \)-highest weight \( \lambda | \mathfrak{g} + 2\delta \mathfrak{n} \mathfrak{m} \) and (ii) the \( \Delta \)-highest weight of any irreducible \( \mathfrak{g} \) submodule of \( A_\Theta(\lambda) \) is of the form \( \lambda | \mathfrak{g} + 2\delta \mathfrak{n} \mathfrak{m} + \sum p \beta \) where \( n_p \geq 0 \). For the existence and construction of the \( \mathfrak{g} \) modules \( A_\Theta(\lambda) \) the reader may consult [16], [25]. One knows that the special \( \Theta \) type \( \lambda | \mathfrak{g} + 2\delta \mathfrak{n} \mathfrak{m} \) occurs exactly once in \( A_\Theta(\lambda) \).

Proposition 2.6. Let \( W \in K \) and let \( w \in W \) such that \( \Delta \subset \Delta P \). Then \( \Phi^P_{\tau w} = \Phi^\tau \subset \Phi^\tau \Phi^\tau - \Phi^\tau - \Phi^\tau - \Phi^\tau \). Also \( \Phi^P_{\tau w} \subset \Phi^P_{\tau w} \).

Proof. If \( \alpha \in \Phi^\tau - 1 \) then \( \alpha \in \Delta^+ \subset \Delta^P \) and \( \tau \alpha = -\Delta^+ \subset -\Delta^P \Rightarrow \Phi^\tau \subset \Phi^\tau \subset \Phi^\tau \subset \Phi^\tau - \Phi^\tau - \Phi^\tau - \Phi^\tau \). If \( \alpha \in \Phi^\tau \subset \Phi^\tau \subset \Phi^\tau \) then \( \alpha \in P^\tau \), \( w^{-1} \tau \alpha \in -P^\tau \), and we claim \( \alpha \in \Delta^+ \). For otherwise \( \tau \alpha = -\Delta^+ \) since \( \alpha \in \Phi^\tau - 1 \). Then \( \tau \alpha \in \mathfrak{w} P^\tau \Rightarrow w^{-1} \tau \alpha \in \mathfrak{w} P^\tau \) is a contradiction. Thus we must have \( \alpha \in P^\tau \subset \Delta^+ \); i.e. \( \Phi^\tau \subset \Phi^\tau \subset \Phi^\tau \subset \Phi^\tau - \Phi^\tau - \Phi^\tau - \Phi^\tau \). Conversely \( \alpha \in \mathfrak{w} P^\tau \subset \mathfrak{w} P^\tau \subset \Phi^\tau \subset \Phi^\tau \subset \Phi^\tau - \Phi^\tau - \Phi^\tau - \Phi^\tau \). Since \( \Phi^\tau - 1 \subset \Delta^+ \) and since \( \Delta \mathfrak{n} \mathfrak{m} = \Phi^\tau \), clearly \( \Phi^\tau \subset \mathfrak{w} P^\tau \subset \Delta^+ \subset \mathfrak{w} P^\tau \) since \( \Delta^+ \subset \mathfrak{w} P^\tau \). Q.E.D.

Using Proposition 2.6 we can now state the following theorem whose proof is given in [24] (see Theorem 2.15 there).

Theorem 2.7. Let \( \Lambda \in \mathbb{F}_\delta \) in (2.1), let \( P^\tau \) be the corresponding positive system in (2.3), and let \( \sigma \in W \) be the unique Weyl group element such that \( \sigma \Delta^+ = P^\tau \).
Let \((\pi, H_{\sigma}) \in \hat{G}\) be such that \(\pi(\Omega) = (\Lambda, \Lambda + 2\delta)\Omega + 1\) and such that \(\Hom_K(\pi, X^* \otimes \tau_{\Lambda + s_\delta}) \neq 0\). Then there is a pair \((\tau, w) \in W_K \times W\) and a \(c\) stable parabolic subalgebra \(\theta = m + u\) of \(g\) containing a Borel subalgebra \(\mathfrak{b} + \sum_{\alpha \in \Delta^+_\sigma} g_{\alpha}\) where \(\Delta^+_\sigma \supset \Delta^+_\pi\) such that

(i) \(H_{\sigma} = A_{\delta}(\lambda)\) and the minimal \(\mathfrak{t}\) type \(\lambda \mid g_{\delta + 2\delta_{\sigma, \Lambda}}\) (which characterizes \(H_{\sigma}\)) has the form \(\lambda \mid \delta + 2\delta_{\sigma, \Lambda} = \Lambda + \delta + \tau^{-1}(w\delta(\Lambda) - \delta_{\Lambda})\)

(ii) \((\tau, w)\) satisfy \(\Delta^+_\pi \subseteq w\mathcal{P}(\Lambda), \tau(\Lambda + \delta - \delta(\Lambda)) = \omega(\Lambda + \delta - \delta(\Lambda)) = \Lambda + \delta - \delta(\Lambda), \Phi_{w, \Lambda}, \Phi_{\pi, \Lambda} \circ \Phi_{\tau^{-1}}\) and \(\{\alpha \in \mathcal{P}(\Lambda) \mid \tau \alpha \in -\mathcal{P}(\Lambda)\}\) are identically zero \(\{\alpha \in \mathcal{P}(\Lambda) \mid \tau \alpha \in -\mathcal{P}(\Lambda)\}\) which characterizes \(H_{\sigma}\); \((\Lambda + \delta - \delta(\Lambda), \alpha) = 0\), and \((-1)^{\phi_{\pi, \Lambda}} = \pm (-1)^{\phi_{\pi, \Lambda}} = \pm (-1)^{n + |\theta_{\sigma, \Lambda}|}\) where \(|S|\) denotes the cardinality of a set \(S\) and \(n = \frac{1}{2} \dim_K G/K^{1}\) (see (2.5)); also \(\Phi_{\tau^{-1}} \subset \{\alpha \in \Delta^+_\pi \mid \tau \alpha \in \mathcal{P}(\Lambda)\}\)

(iii) the relative Lie algebra cohomology \(H^i(m, m \cap \mathfrak{t}, \mathbb{C})\) (for the trivial module \(\mathbb{C}\)) is non-zero for \(j = n - |\theta_{\sigma, \Lambda}| - |\{\alpha \in \mathcal{P}(\Lambda) \mid \tau \alpha \in -\mathcal{P}(\Lambda)\}|\). Hence the latter number is even.

REMARKS. (i) If \(F\) is the finite-dimensional irreducible \(g\) module with \(\mathcal{P}(\Lambda)\)-highest weight \(\Lambda + \delta - \delta(\Lambda)\) then \(H_{\sigma}\) in Theorem 2.7 satisfies

\(\Hom_K(H_{\sigma}, \wedge^i \mathbb{C} \otimes F) = H^i(\mathfrak{g}, \mathfrak{t}, \mathfrak{h} \otimes F^*) = H^i - |\theta_{\sigma, \Lambda}|(m, m \cap \mathfrak{t}, \mathbb{C})\) for \(i \geq 0\)

(ii) \((\Lambda + \delta + \tau^{-1}(w\delta(\Lambda) - \delta_{\Lambda})\) is the only \(\mathfrak{t}\) type which occurs both in \(\pi | K\) and in \(\mathfrak{h} \otimes \tau_{\Lambda + s_\delta}\)

(iii) If \(\sigma_1 \in W\) is the unique Weyl group element such that \(\sigma_1 \Delta^+_\pi = \mathcal{P}(\Lambda)\) then \(\sigma_1 \Delta^+_\pi = \Lambda + \delta - \delta(\Lambda)\) (see [24]).

(iv) The proof of Theorem 2.7 leans heavily on the recent unpublished results of D. Vogan [21]. Vogan's results depend in part on the important theorem of S. Kumaresan [9] which specifies the structure of an irreducible \(\mathfrak{f}\) component of \(\Delta \mathfrak{p}\) that can occur in an irreducible unitary \(g\) module \(H_{\sigma}\) when \(\pi(\Omega) = 0\).

(v) \(\Phi_{\tau} = \Delta^+_\pi - Q_{\Lambda}\).

3. Unitary representations intertwining \(\text{Ad}^*_K \otimes \tau_{\Lambda}\)

We now assume that for \(G, K\) in section 2, the quotient \(G/K\) admits a \(G\) invariant complex structure; i.e. \(G/K\) is a Hermitian symmetric domain. We choose the positive system \(\Delta^+_\pi\) above to be compatible with the complex structures on \(G/K\). This means that

\[
\mathfrak{p}^\pm = \sum_{\pm \alpha \in \Delta^+_\pi} g_{\alpha}
\]

where \(\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-\) is the splitting of \(\mathfrak{p}\) into the spaces of holomorphic and anti-
holomorphic tangent vectors $p^+, p^-$ respectively at the origin in $G/K$. The spaces $p^\pm$ are $K$ and $\mathfrak{f}$ stable abelian subalgebras of $\mathfrak{g}$. The condition of the compatibility of $\Delta^+$ with a $G$ invariant complex structure is equivalent to the following: every $\alpha \in \Delta^*_+$ is totally positive; i.e. for each $\alpha$ in $\Delta^*_+$ we have $\alpha + \beta \in \Delta^*_+$ for any $\beta \in \Delta_h$ such that $\alpha + \beta \in \Delta$. If $\mu \in \mathfrak{h}^*$ is integral and $\Delta^*_+$ dominant we write $(\tau_\mu, V_\mu)$ for the corresponding irreducible of representation of $\mathfrak{f}$ (or of $K$ if $(\tau_\mu, V_\mu) \in \mathcal{K}$). Let $L^\pm$ denote the representation space of $\chi^\pm$. Then we have

$$\sum_{(-1)^j = \pm 1} \bigoplus \Lambda^{* j} \otimes p^+ = L^\pm \otimes V_{\delta_a}$$

as $K$ modules. Here note that dim $V_{\delta_a} = 1$ by Weyl's formula since $(\delta_a, \alpha) = 0$ for $\alpha \in \Delta^*_+$ in the Hermitian symmetric case. Again $n = \frac{1}{2}$ dim$_K G/K = \dim_c G/K = |\Delta^*_+|$. We now prove the following Hermitian analogue of Theorem 2.7.

**Theorem 3.3.** Let $\Lambda, P^{(\Lambda)} \sigma$ be as in Theorem 2.7 where $\Lambda$ is the $\Delta^*_+$-highest weight of $(\tau_\Lambda, V_\Lambda) \in \mathcal{K}$. Let $(\pi, H_\sigma) \in \mathcal{G}$ be such that $\pi(\Omega) = (\Lambda, \Lambda + 2\delta)^1$ and such that Hom$_K (H_\sigma, \Lambda^* p^+ \otimes V_\Lambda) = 0$ where $q \geq 0$ is fixed. Then there is a pair $(\tau, w) \in W_K \times W$ and a $\mathfrak{c}$ stable parabolic subalgebra $\mathfrak{b} = \mathfrak{m} + \mathfrak{u}$ of $\mathfrak{g}$ containing a Borel subalgebra $\mathfrak{b} + \sum_{\alpha \in \Delta^*_+} \mathfrak{g}_\alpha$ where $\Delta^*_+ \supset \Delta^*_+$ such that $H_\sigma(\tau, w, \theta)$ satisfy conditions (i), (ii), (iii) of Theorem 2.7 where in (ii) $\pm$ is chosen according as $(-1)^{\ast - q} = \pm 1$.

If $A_{\lambda, \tau, w} = \{ \alpha \in P^{(\Lambda)} \otimes w^{-1} \tau \alpha \in - P^{(\Lambda)} \}$ (see Proposition 2.6), then $q$ satisfies $q = |A_{\lambda, \tau, w}| = 2 |Q_\Lambda \cap A_{\lambda, \tau, w}| + |Q_\Lambda|$ where $Q_\Lambda$ is given by (2.4).

Proof. Suppose that Hom$_K (H_\sigma, \Lambda^* p^+ \otimes V_\Lambda) = 0$. Writing $q = n - (n - q)$ and using (3.2) we have for $(-1)^{\ast - q} = \pm 1$ the $K$ module inclusion $\Lambda^* p^+ \otimes V_\Lambda \subset L^+ \otimes V_{\delta_a} \otimes V_\Lambda = L^+ \otimes V_{\Lambda + \delta_a}$ so that Hom$_K (H_\sigma, L^+ \otimes V_{\Lambda + \delta_a}) = 0$ since $H_\sigma \in K$ and $\Lambda^* p^+ \otimes V_\Lambda$ contain a common $K$ type $V_\Lambda$. Thus Theorem 2.2 applies. The $\Delta^*_+$-highest weight $\mu$ satisfies $\mu = \Lambda + Q_\Lambda$ where $Q_\Lambda \subset \Delta^*_+$ such that $|Q_\Lambda| = q$.

Let $Q_2 = \Delta^*_+ - Q_1$ so that $\mu = \Lambda + 2\delta_a - <\rho_1>$. Define $Q_3 = (Q_\Lambda - Q_2) \cap (Q_2 \cap Q_\Lambda)$ $\subset P^{(\Lambda)} = Q_\Lambda \cup Q_\Lambda$ where $Q_\Lambda = \Delta^*_+ - Q_\Lambda$. Then one easily checks that

$$|Q_3| = |Q_2| = 2|Q_2 \cap Q_\Lambda| + |Q_\Lambda| \quad \text{and} \quad <Q_2> = <Q_\Lambda> - <Q_\Lambda>.$$

Let $Q_4 = P^{(\Lambda)} - Q_3$. One has $\delta_a + \delta^A = <Q_\Lambda>$ so that using (3.4) $\mu = \Lambda + \delta_a + \delta_a - <Q_2> = \Lambda + \delta_a - <Q_\Lambda>$. On the other hand by remark (ii) above $\Lambda + \delta_a + \tau^{-1} (wo \delta^A - \delta_b)$ is the only $\mathfrak{f}$ type occurring both in $\pi|_K$ and $\chi^\otimes \tau_{\Lambda + \delta_a}$ which means that $\mu = \Lambda + \delta_a + \tau^{-1} (wo \delta^A - \delta_b) = \Lambda + \delta_a + \delta^A - <Q_\Lambda>$ and hence $\tau^{-1} (wo \delta^A - \delta_b) = \delta^A$. Therefore $<Q_4, \Phi_{\tau^A}^{-1}>$ (see (2.5)) = $<Q_4> + <\Phi_{\tau^A}^{-1}> = <Q_2> + \delta_a - \tau^{-1} \delta_b = \delta_a + \delta^A - <\Phi_{\tau^A}^{-1}>$. Thus by (5.10.2) of Kostant [8] $Q_4 = \Phi_{\tau^A}^{-1} - \Phi_{\tau^A}^{-1} = \Phi_{\tau^A}^{-1}$. Then $Q_4 = \Phi_{\tau^A}^{-1} - \Phi_{\tau^A}^{-1} = \Phi_{\tau^A}^{-1}$.
$A_{\Lambda, \tau, w}$ (by Proposition 2.6) and since $Q_4 = P_{\nu}^{(A)} - Q_3$, $Q_2 = \Delta^+_\tau - Q_1$ we get $|A_{\Lambda, \tau, w}| = n - |Q_3| = n - |Q_2| + 2 |Q_2 \cap Q_\Lambda| - |Q_\Lambda| \quad \text{(by (3.4))} = |Q_1| + 2 |Q_2 \cap Q_\Lambda| - |Q_\Lambda| = q + 2 |Q_2 \cap Q_\Lambda| - |Q_\Lambda|$. But by definition of $Q_3$ we have $Q_2 \cap Q_\Lambda = Q_\Lambda - Q_3 = Q_\Lambda \cap Q_\Lambda = Q_\Lambda \cap A_{\Lambda, \tau, w}$ and hence $|A_{\Lambda, \tau, w}| = q + 2 |Q_\Lambda \cap A_{\Lambda, \tau, w}| - |Q_\Lambda|$. This proves Theorem 3.3.

In the statement of Theorem 3.3 no conditions are imposed on $\Lambda \in \mathcal{F}_\theta$. However suppose for example that we impose the following condition: we assume every $\alpha \in P_{\nu}^{(A)}$ is totally positive. Then we have the following refinement of Theorem 3.3.

**Corollary 3.5.** Let $(\tau_\Lambda, V_\Lambda), P^{(A)}, \sigma, (\pi, H_\sigma)$ be as in Theorem 3.3 with $q$ fixed. Suppose in addition that $P^{(A)}$ is compatible with a $G$ invariant complex structure on $G/K$; i.e. assume every non-compact root in $P^{(A)}$ is totally positive. Then there is a Weyl group element $w$ and a $c$ stable parabolic subalgebra $\theta = m + u$ satisfying the conditions of Theorem 2.7 where in (i), (ii), (iii) $\tau \in W_K$ may be assumed equal to the identity element (thus for example $H_\sigma$ is characterized by the minimal $\mathfrak{f}$ type $\Lambda + \delta_s + w\delta^{(A)} - \delta_k$ and $j = n - |\theta_{u, w}| - |\Phi^{(A)}_w|$) and in (ii) $\pm$ is chosen according as $(-1)^{n-q} = \pm 1$. $q$ satisfies $q = |\Phi^{(A)}_w| - 2 |Q_\Lambda \cap \Phi^{(A)}_w| + |Q_\Lambda|$. 

Proof. Choose $(\tau, w), \theta = m + u$ as in Theorem 2.7 or Theorem 3.3. Since every non-compact root in $P^{(A)}$ is totally positive and since $\tau \in W_K$ we have $\tau P^{(A)} = P_{\nu}^{(A)}$. This implies that

$$A_{\Lambda, \tau, w} = \tau^{-1} \Phi^{(A)}_w$$

Also one has $\tau Q_\Lambda = Q_\Lambda$ and hence by (3.6)

$$\tau (Q_\Lambda \cap A_{\Lambda, \tau, w}) = Q_\Lambda \cap \Phi^{(A)}_w.$$
Proposition 3.8. Suppose in Theorem 3.3 the parabolic subalgebra \( \theta = m + u \) is \( g \) itself. Then \( \Lambda = \delta^{(A)} - \delta \) and \( q = n - |Q_\Lambda| \).

Proof. \( \theta = g \) means that \( u = 0 \), \( m = g \). Then \( \theta_{u,n} = \phi \) and \( \Delta(m) = \Delta \).

Recalling that \( \lambda(\Delta(m)) = 0 \) (see section 2) we have \( \lambda(\Delta) = 0 \) and hence \( \lambda|_{\delta} = 0 \).

By remark (iii) following Theorem 2.7 \( \sigma_{\Lambda} \lambda|_{\delta} = \Lambda + \delta - \delta^{(A)} \); hence \( \Lambda + \delta - \delta^{(A)} = 0 \Rightarrow \Lambda = \delta^{(A)} - \delta \). Also since \( \theta_{u,n} = \phi \) the equality of \( \mathfrak{f} \) types \( \lambda|_{\delta} + 2\delta_{u,n} = \Lambda + \delta + \tau^{-1}(w\delta^{(A)} - \delta) \) in (i) of Theorem 2.7 reduces to \( 0 = \delta^{(A)} + \tau^{-1}(w\delta^{(A)} - \delta) \), since \( \Lambda = \delta^{(A)} - \delta = \delta^{(A)}_{\Lambda} - \delta \) and so \( \Lambda + \delta = \delta^{(A)}_{\Lambda} \). But this says that \( \langle \Phi^{(A)}_{\Lambda} \rangle = \delta^{(A)} - \tau^{-1}w\delta^{(A)} = \delta^{(A)} + \delta^{(A)}_{\Lambda} = \tau^{-1}\delta = \langle P^{(A)}_w \rangle = \langle \Phi^{(A)}_{\Lambda} \rangle = \langle \Phi^{(A)}_{\Lambda} \rangle \) (see (2.5)) and hence \( \Phi^{(A)}_\Lambda = P^{(A)}_w \cup \Phi^{(A)}_{\Lambda} \cap P^{(A)}_w \) or \( A_{\Lambda,\tau,w} = P_{\Lambda}^{(A)} \) by Proposition 2.6. Then by Theorem 3.3 \( q = |A_{\Lambda,\tau,w}| - 2|Q_\Lambda| \cap A_{\Lambda,\tau,w} = n - 2|Q_\Lambda| + |Q_\Lambda| = n - |Q_\Lambda| \).

Proposition 3.9. Let \( \Lambda \in \mathcal{Z}_0 \) be such that every non-compact root in \( P(\Lambda) \) is totally positive. Let

\[
\mathfrak{p}^{(A)} = \sum_{\alpha \in P^{(A)}_w} \mathfrak{g}_\alpha
\]

be the \( \mathfrak{f} \) module of holomorphic tangent vectors for the corresponding \( G \) invariant complex structure on \( G/K \) compatible with \( P^{(A)}_w \); cf. (3.1). Suppose \( w \in W \) is a Weyl group element such that \( \Delta_+ \subset wP^{(A)} \).

Then we have a \( \mathfrak{f} \) module inclusion

\[
V_{\delta^{(A)} + w\delta^{(A)} - \delta} \subset \wedge^{n-|\Phi^{(A)}_w|} \mathfrak{p}^{(A)}.
\]

Proof. In the proof of Corollary 3.5 we observed that indeed \( \delta^{(A)} + w\delta^{(A)} = \delta \) is \( \Delta_+ \)-dominant. Of course

\[
\delta^{(A)} + w\delta^{(A)} - \delta = 2\delta^{(A)} - (\delta^{(A)} - w\delta^{(A)}) = \langle P^{(A)}_w - \Phi^{(A)}_w \rangle.
\]

Write \( P^{(A)}_w - \Phi^{(A)}_w = \{ \alpha_1, \ldots, \alpha_t \} \), \( t = n - |\Phi^{(A)}_w| \), and let

\[
\chi = \chi_{\alpha_1} \wedge \cdots \wedge \chi_{\alpha_t} \text{ where } \chi_{\alpha_j} \in \mathfrak{g}_{\alpha_j} - \{0\}.
\]

We claim that \( \chi \in \wedge^{n-|\Phi^{(A)}_w|} \mathfrak{p}^{(A)} \) is a \( \Delta_+ \)-highest weight vector. By (3.11) \( \chi \) is clearly a weight vector of the weight \( \delta^{(A)} + w\delta^{(A)} - \delta \). Let \( \beta \in \Delta_+ \) be arbitrary and choose \( \chi_{\beta} \in \mathfrak{g}_{\beta} - \{0\} \). We must show that

\[
\text{ad}_{\chi_{\beta}} \chi = \sum_{j=1}^t \chi_{\alpha_j} \wedge \cdots \wedge [\chi_{\beta}, \chi_{\alpha_j}] \wedge \cdots \wedge \chi_{\alpha_t} = 0.
\]

If \( \beta + \alpha_j \) is not a root \( [\chi_{\beta}, \chi_{\alpha_j}] = 0 \). Assume \( \beta + \alpha_j \) is a root. Then \( \beta + \alpha_j \in P^{(A)}_w \) since \( \alpha_j \in P^{(A)}_w \) is totally positive. On the other hand \( \alpha_j \in \Phi^{(A)}_w \) implies \( w^{-1}\alpha_j \in P^{(A)} \). Also by hypothesis \( \Delta_+ \subset wP^{(A)} \) so \( w^{-1}\beta \in P^{(A)} \). Hence \( w^{-1}(\beta + \alpha_j) = \).
Corollary 3.14. Let \( \Lambda, P(\Lambda), \) and \( w \) be as in Proposition 3.9. Then we have the \( k \) module inclusion \( V_{\Lambda+\delta_n+w\delta(\Lambda)-\delta_\Phi} \subset V_{\Lambda+\delta_n+w\delta(\Lambda)-\delta_\Phi} \subset V_{\Lambda+\delta_n+w\delta(\Lambda)-\delta_\Phi} \). Then 

\[
\beta+\alpha_j \in P(\Lambda); \quad \text{i.e. } \beta+\alpha_j \in P(\Lambda)-\Phi
\]

which implies that \( \beta+\beta_i = \text{some } \alpha_i, i \neq j \). Then \( [\chi_\beta, \chi_\alpha] = \text{a multiple of } \chi_\alpha \). We conclude that (3.13) is valid and \( U(\mathfrak{f})\chi \) is a \( \mathfrak{f} \) submodule of \( \wedge^t \mathfrak{p}^{(\Lambda)+} \)-equivalent to \( V_{\delta^A+\omega\delta(\Lambda)^-\delta_\Phi} \).

**Proof.**

\[
\Lambda+\delta_n+w\delta(\Lambda)-\delta_\Phi = \Lambda+\delta_n-\delta_\Phi^A+\delta(\Lambda)+w\delta(\Lambda)-\delta_\Phi
\]

\[
= \Lambda+\delta-\delta^A+\delta(\Lambda)+w\delta(\Lambda)-\delta_\Phi.
\]

Corollary 3.15. Let \( (\tau, \Lambda, V_\Lambda) \in \hat{K} \) where \( \Lambda \in \mathcal{D}_0 \) and every non-compact root in \( P(\Lambda) \) is totally positive. Let \( \mu = \Lambda+\delta_n+w\delta(\Lambda)-\delta_\Phi \) be the minimal \( \mathfrak{f} \) type of \( H_\Phi \) given by Corollary 3.5. Then relative to the positive system \( P(\Lambda) = P(\Lambda)+\delta(\Lambda)^- \) and \( -P(\Lambda) = \Delta^+ \cup \Delta^- \), \( H_\Phi \) is a highest weight \( \mathfrak{g} \) module with highest weight \( \mu \).

**Proof.** We have \( \mathfrak{f} \) module inclusions \( V_\mu \subset H_\Phi \) and (by Corollary 3.14) \( V_\mu \subset V_{\Lambda+\delta_n+w\delta(\Lambda)-\delta_\Phi} \wedge^t \mathfrak{p}^{(\Lambda)+} \) where \( t = n - |\Phi(\Lambda)| \) and where \( \Lambda+\delta-\delta(\Lambda) \) is \( P(\Lambda) \)-dominant. Since \( |(\Lambda+\delta-\delta(\Lambda)) + \delta(\Lambda)|^2 - |(\delta(\Lambda), \delta(\Lambda))|^2 = |\Lambda+\delta|^2 - |(\delta, \delta)|^2 = \pi(\Omega) \) Corollary 3.15 follows from Lemma 3.7 of [6] or from the proof of Lemma 2 of [4].

The fact that any \( (\pi, H_\Phi) \in \hat{G} \) as in Corollary 3.15 has to be a \( P(\Lambda) \)-highest weight \( \mathfrak{g} \) module is also proved in [23] (see the proof of Lemma 2.4 there) by different means.

### 4. Vanishing theorems

In this section we again assume, as in section 3, that \( G/K \) is a Hermitian symmetric domain and that the positive system \( \Delta^+ \) is compatible with the \( G \)-invariant complex structure on \( G/K \). We fix a discrete subgroup \( \Gamma \) of \( G \) which acts freely on \( G/K \) and for which the quotient \( X=\Gamma \backslash G/K \) is compact. Let \( \tau=\tau_\Lambda \in \hat{K} \) be a fixed finite-dimensional irreducible representation of \( K \) acting on a complex vector space \( V_\Lambda \) where \( \Lambda \in \mathcal{D}_0 \) is the \( \Delta^+ \)-highest weight of \( \tau \). The induced \( C^\infty \) vector bundle \( E_\tau \to G/K \) has a holomorphic structure. To prove this one usually assumes that \( G \) is a real form of a complex Lie group \( G^C \) (i.e. \( G \) is linear). Since we are not imposing the latter assumption on \( G \) we appeal to the more general criteria of [19], [20] for the existence of holomorphic structures on homogeneous bundles. The induced sheaf \( \theta_\tau \to X \) of abelian groups over \( X \) given in the introduction will also be denoted by \( \theta_\Lambda \). Let \( \text{Ad}_\Lambda^+ \) denote the adjoint representation of \( K \) on \( \wedge^t \mathfrak{p}^+ \). Then as in [4] the sheaf cohomology \( H^t(X, \theta_\Lambda) \) can be identified with the space \( \mathcal{A}(\text{Ad}_\Lambda^+ \otimes \tau_\Lambda, (\Lambda, \Lambda+2\delta), \Gamma) \) of automorphic forms of type \( (\text{Ad}_\Lambda^+ \otimes \tau_\Lambda, (\Lambda, \Lambda+2\delta), \Gamma) \); i.e.
By the formula of Matsushima-Murakami [14] we therefore have

\[ (4.2) \dim H^q(X, \Theta_A) = \sum_{\pi(\Omega) = (\Lambda, \Lambda + 2\delta)} m_\pi(\Gamma) \dim \Hom_K(H_\sigma, \Lambda^q \otimes V_\Lambda) \]

where \( m_\pi(\Gamma) \) is the multiplicity of \( \pi \) in the right regular representation of \( G \) on \( L^2(\Gamma \backslash G) \). Using (4.2) we immediately deduce from Theorem 3.3 the following main theorem.

**Theorem 4.3.** Let \( \Lambda \in \mathcal{T}_G \) in (2.1) be the \( \Delta^+ \)-highest weight of \((\tau_\Lambda, V_\Lambda) \in \hat{K} \). Let \( \sigma \in W \) be the unique Weyl group element such that \( \sigma \Delta^+ = P(\Lambda) \) where \( P(\Lambda) \) is the system of positive roots in (2.3). Suppose that \( H^q(\Gamma \backslash G, \theta_\Lambda) = 0 \). Then there is a pair \((\tau, w) \in W_K \times W \) and a stable parabolic subalgebra \( \theta = m + u \) of \( g \) containing the Borel subalgebra \( \mathfrak{h} + \sum g_\alpha \) for some positive system \( \Delta^- \supset \Delta^+ \) (cf. earlier notation) such that

(i) \( q = |A_{\Lambda, \tau, w} - 2|Q_\Lambda \cap A_{\Lambda, \tau, w} + |Q_\Lambda| \) where \( A_{\Lambda, \tau, w} = \{ \alpha \in P^+(\Lambda) \mid w^{-1} \tau \alpha \in -P(\Lambda) \} \) and where \( Q_\Lambda \) is given by (2.4)

(ii) \( \Delta^+ \subset wP(\Lambda) \) (so that by Proposition 2.6 \( A_{\Lambda, \tau, w} = \Phi^+(\Lambda) - \Phi^+, \tau(\Lambda + \delta - \delta(\Lambda)) = w(\Lambda + \delta - \delta(\Lambda)) = \Lambda + \delta - \delta(\Lambda) \) and \( A_{\Lambda, \tau, w}, \Phi^+(\Lambda) \), and \( \{ \alpha \in P^+(\Lambda) \mid \tau \alpha \in -P(\Lambda) \} \) are all contained in \( \{ \alpha \in P^+(\Lambda) \mid (\Lambda + \delta - \delta(\Lambda), \alpha) = 0 \} ; \Phi^+_1 \subset \{ \alpha \in \Delta^+_1 \mid (\Lambda + \delta - \delta(\Lambda), \alpha) = 0 \} \); and \( \Phi^+_1 \) is given by notation of (2.5)

(iii) the relative Lie algebra cohomology \( H^j(m, m \cap \mathfrak{k}, C) = 0 \) for \( j = n - |\theta_{\ast, u} - |A_{\Lambda, \tau, w}| \) (hence the latter is an even number) where, as above, \( \theta_{\ast, u} \) is the set of non-compact roots in the nilradical \( u \) of \( \theta \) and \( n = \frac{\dim_K G}{2} \).

(iv) For \( (-1)^{n-q} = \pm 1 \) we have \((-1)^{|\Phi_\sigma|} = \pm (-1)^{|\Phi^+_\Lambda|} = \pm (-1)^{n+|\theta_{\ast, u}|} \).

As has been noted \( \Phi_\sigma = \Delta^+ \cup \mathfrak{h} \), and if \( \sigma \in W \) is the unique Weyl group element such that \( \sigma \Delta^+ = P(\Lambda) \) then \( \Lambda + \delta - \delta(\Lambda) \), \( \sigma_{1}(\Delta(m)) = 0 \) where \( \Delta(m) \) is the set of roots for the reductive part \( m \) of \( \theta \). From Corollary 3.4 we obtain

**Corollary 4.4.** Let \( \Lambda \in \mathcal{T}_G \) in Theorem 4.3 satisfy the condition that every non-compact root in \( P(\Lambda) \) is totally positive. Then if \( H^q(\Gamma \backslash G, \theta_\Lambda) = 0 \) we can choose \( w \in W \) satisfying \( \Delta^+ \subset wP(\Lambda) \) and a stable parabolic subalgebra \( \theta = m + u \supset \mathfrak{h} + \sum g_\alpha \) such that

(i) \( q = |\Phi^+(\Lambda)| - 2|Q_\Lambda \cap \Phi^+(\Lambda)| + |Q_\Lambda| \)

(ii) \( H^{n-|\theta_{\ast, u} - |\Phi^+(\Lambda)|} (m, m \cap \mathfrak{k}, C) = 0 \)

(iii) \( \Phi^+(\Lambda) \subset \{ \alpha \in P^+(\Lambda) \mid (\Lambda + \delta - \delta(\Lambda), \alpha) = 0 \} \).
Statement (iv) of Theorem 4.3 holds.

Consider for example the special case when $\Lambda$ is actually $\Delta^+$-dominant. Then $P(\Lambda)=\Delta^+$ so that $\Lambda$ indeed satisfies Corollary 4.4. Also in this case $Q(\Lambda)=\Delta^+$ so that $Q(\Lambda) \cap \Phi(\Lambda)=\Phi(\Lambda)$. Thus by (i) of Corollary 4.4 $H^q=0 \Rightarrow q=|\Phi(\Lambda)|-2|\Phi(\Lambda)|+n-n-|\Phi(\Lambda)|$ and hence by (ii) $H^q=0 \Rightarrow (m, m \cap \mathfrak{h}, \mathfrak{C})=0$. Thus we have proved the following conjecture of R. Parthasarathy.

**Corollary 4.5.** Suppose the $\Delta^+$-highest weight $\Lambda$ of $\tau$ is actually $\Delta^+$-dominant. Then if $H^q(\Gamma \setminus G/K, \partial \Lambda)=0$ so is $H^q=0 \Rightarrow (m, m \cap \mathfrak{h}, \mathfrak{C})$ for some $\partial$ stable parabolic subalgebra $\partial=m+\mathfrak{h}$ of $\mathfrak{g}$.

Our argument shows moreover that in Corollary 4.5 $q=n-|\mathfrak{w}(\Delta^+) \cap \Delta^+|$ for some $\mathfrak{w} \in \mathfrak{W}$ with $\Delta^+ \subset \mathfrak{w} \Delta^+$, $\mathfrak{w}(\Delta^+) \cap \Delta^+ \subset \{\alpha \in \Delta^+ | (\Lambda, \alpha)=0\}$; $\mathfrak{w} \Lambda=\Lambda$.

Let $l(\mathfrak{w})=|\mathfrak{w}(\Delta^+) \cap \Delta^+|$ (=length of $\mathfrak{w}$) and let

\begin{equation}
(4.6) \quad n_\Lambda = |\{\alpha \in \Delta^+ | (\Lambda, \alpha)>0\}|.
\end{equation}

Then $|\{\alpha \in \Delta^+ | (\Lambda, \alpha)=0\}|=n-n_\Lambda$ so that by (b.) $l(\mathfrak{w}) \leq n-n_\Lambda$ and by (a.) $q=n-n_\Lambda$.

**Corollary 4.7** (Hotta-Murakami [4]). Suppose $\Lambda$ is $\Delta^+$-dominant. Then $H^q(\Gamma \setminus G/K, \partial \Lambda)=0$ for $q<n_\Lambda$ in (4.6). More generally for $H^q(\Gamma \setminus G/K, \partial \Lambda)=0$ $q=n-l(\mathfrak{w})$ for some $\mathfrak{w} \in \mathfrak{W}$ satisfying $\mathfrak{w}(\Delta^+) \cap \Delta^+ \subset \{\alpha \in \Delta^+ | (\Lambda, \alpha)=0\}$, $\mathfrak{w} \Lambda=\Lambda$.

We define

\begin{equation}
(4.8) \quad R=R(\mathfrak{g})=\min \{|\partial|, |\partial|=\partial \text{ stable parabolic subalgebra of } \mathfrak{g}, \partial=\mathfrak{g}\}.
\end{equation}

Again note that for $\partial=\mathfrak{g}$ $\mathfrak{u}=\mathfrak{h}$ and hence $|\partial|, \mathfrak{u}|=\dim \mathfrak{u} \cap \mathfrak{p}=0$. The values $R(\mathfrak{g})$ have been computed by Vogan for general symmetric spaces. Specializing his results to the Hermitian case we have the following table for the irreducible Hermitian symmetric spaces.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$R(G)$</th>
<th>real rank of $G/K$</th>
<th>$\frac{1}{2} \dim R$ $G/K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(n,m)$, $n \geq m$</td>
<td>$m$</td>
<td>$m$</td>
<td>$nm$</td>
</tr>
<tr>
<td>$Sp(n,R)$</td>
<td>$n$</td>
<td>$n$</td>
<td>$\frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>$SO_0(n,2)$, $n &gt; 2$</td>
<td>$2$</td>
<td>$2$</td>
<td>$n$</td>
</tr>
<tr>
<td>$SO^*(2n)$, $n &gt; 3$</td>
<td>$n-1$</td>
<td>$\begin{bmatrix} n \end{bmatrix}$</td>
<td>$\frac{n(n-1)}{2}$</td>
</tr>
<tr>
<td>real form of $E_6$</td>
<td>$8$</td>
<td>$2$</td>
<td>$16$</td>
</tr>
<tr>
<td>real form of $E_7$</td>
<td>$11$</td>
<td>$3$</td>
<td>$17$</td>
</tr>
</tbody>
</table>
In Theorem 4.3 $H^j(m, m \cap TF, C) \neq 0$ for $j = n - |A_{\theta, w}|$ by (iii); hence $j > 0$. That is $|A_{\Lambda, \tau, w}| \leq n - |A_{\theta, w}|$ and if $\theta \neq g$ $|A_{\Lambda, \tau, w}| \leq n - R(G)$. Thus applying Proposition 3.8 we get

**Proposition 4.10.** Suppose in Theorem 4.3 that either $\Lambda = \delta^{(A)} - \delta$ or $q = n - |Q_\Lambda|$. Then $A_{\Lambda, \tau, w}$ there satisfies $|A_{\Lambda, \tau, w}| \leq n - R(G)$. Similarly $w$ in Corollary 4.4 satisfies $|\Phi_{\theta}^{(A)}| \leq n - R(G)$.

Note that, in general, by Theorem 4.3 we always have $|A_{\Lambda, \tau, w}|, |\Phi_{\theta}^{(A)}| \leq \{\alpha \in P_+^{(A)} \cap (\Delta + \delta - \delta^{(A)}, \alpha) = 0\}$. In Corollary 4.7 $q = n - l(w)$ for $H^q \neq 0$. By Proposition 4.10. $l(w) \leq n - R(G)$ if either $\Lambda = 0$ or $q = 0$; i.e. $q = n - l(w) > R(G)$ which establishes

**Corollary 4.11.** Suppose $\Lambda$ is $\Delta^+$-dominant. If $\Lambda = 0$ then $H^q(\Gamma \backslash G/K, \theta_\Lambda) = 0$ for $0 \leq q < R(G)$. If $\Lambda = 0$ then $H^q(\Gamma \backslash G/K, \theta_\Lambda) = 0$ for $1 \leq q < R(G)$.

In particular we see that since for $G$ in Table 4.9 rank of $G/K < R(G)$ the following weaker version of Corollary 4.11 holds.

**Corollary 4.12.** If $G/K$ is irreducible then $H^q(\Gamma \backslash G/K, \theta_\Lambda) = 0$ for $0 \leq q < \text{rank } G/K$, $\Lambda \Delta^+$-dominant, $\Lambda = 0$. The $(0, q)$ Betti number of $\Gamma \backslash G/K$ vanishes for $1 \leq q < \text{rank } G/K$.

Corollary 4.12 is of course well-known; see Theorem 4.2 of [6] and Theorem 4 of [4]. In the case where $G/K$ is irreducible a slight improvement of Corollary 4.11 is given by Theorem 3.5 of [23]. Another extreme case is the case $Q_\Lambda = \phi$; i.e. $(\Lambda + \delta, \alpha) < 0$ for $\alpha \in \Delta^*_+, \ P^{(A)} = \Delta^*_+ = \Delta^*_+ \cup -\Delta^*_+$. If $H^q \neq 0$ then from Corollary 4.4 $q = |\Phi_{\theta}^{(A)}|$ for some $w \in W$ such that $\Delta^*_+ \subset w \Delta^*_+, \Phi_{\theta}^{(A)} \subset \{\alpha \in -\Delta^*_+ \cap (\Lambda + 2\delta_n, \alpha) = 0\}$ and (by (ii) of Corollary 4.4) $H^{n-q-10^m, \ast\ast}(m, m \cap TF, C) \neq 0$ for some $e$ stable parabolic $\theta = m + u$. By Proposition 3.8 $\theta \neq g$ unless $\Lambda = -2\delta_n$ or $q = n$. Barring the latter two cases we have $|\Phi_{\theta}^{(A)}| \leq n - R(G)$ by Proposition 4.10 so that $q \leq n - R(G)$. This gives

**Corollary 4.13.** Suppose $(\Lambda + \delta, \alpha) < 0$ for $\alpha$ in $\Delta^*_+$. If $\Lambda = -2\delta_n$ then $H^q(\Gamma \backslash G/K, \theta_\Lambda) = 0$ for $q > n - R(G)$. If $\Lambda = -2\delta_n$ then $H^q(\Gamma \backslash G/K, \theta_\Lambda) = 0$ for $n - R(G) < q < n$. In any case we always have $H^q(\Gamma \backslash G/K, \theta_\Lambda) = 0$ for $q \geq \{\alpha \in -\Delta^*_+ \cap (\Lambda + 2\delta_n, \alpha) = 0\}$.

The last statement of Corollary 4.13 is statement (i) of Theorem 3.12 of [23]. However in [23] $G$ is assumed to be linear. We now indicate how the main result of [23] (Theorem 2.3) can be deduced with the aid of Corollary 3.5; see Theorem 4.16.

**Proposition 4.14** Let $\Lambda \in \mathcal{I}_0$ and let $w \in W$ be a Weyl group element which
satisfies $Δ_\alpha \subset wP(Λ), w(Λ+δ-δ(A))=Λ+δ-δ(A)$, and $Φ_\alpha(\alpha) \subset \{α ∈ P(Δ)(Λ+δ-δ(A), α)=0\}$ (cf. (ii) of Theorem 4.3) Then $Λ+δ-δ(A)+wδ(A)$ is a regular element (i.e. $(Λ+δ-δ(A)+wδ(A), α)=0$ for every $α$ in $Δ$) and the corresponding positive system

(4.15) $P' = \{α ∈ Δ \mid (Λ+δ-δ(A)+wδ(A), α)>0\}$ coincides with $wP(Λ)$.

Also $P(Λ) = P(Λ)+P' \subset P(Δ)$. 

Proof. For $α ∈ Δ \mid (Λ+δ-δ(A)+wδ(A), α)=0 \supset w^-1P(Λ)$ Suppose $α ∈ P(Λ)$. If $(Λ+δ-δ(A), α)=0$ then $(Λ+δ-δ(A)+wδ(A), α)=(δ(A), w^-1α)=0$. Assume $(Λ+δ-δ(A), α)>0$. Then $α ∈ Φ(Δ)$ since by hypothesis $Φ(Δ) \subset \{α ∈ P(P(Δ))(Λ+δ-δ(A), α)=0\}$. Thus we must have $w^-1α ∈ P(Λ)$. Since $Λ+δ-δ(A)$ is $P(Λ)$-dominant $(Λ+δ-δ(A), α)+δ(A), w^-1α)>0$. Thus we have shown $(Λ+δ-δ(A)+wδ(A), α)=0$ for $α ∈ P(Λ)$ which proves $Λ+δ-δ(A)+wδ(A)$ is normal. Let $α ∈ P(Λ)$ be arbitrary. Then $(Λ+δ-δ(A)+wδ(A), wα)=(w^-1(Λ+δ-δ(A)+wδ(A)), α)=(Λ+δ, α)$ (since $w^-1(Λ+δ-δ(A))=Λ+δ-δ(A)$ which is positive). That is $wα ∈ P' ⇔ wP(Λ) ⊂ P' ⇔ wP(Λ)=P'$. 

Now $Φ(Δ) = w(−P(Λ)) \cap P(Λ) = −P' \cap P(Λ)$ and since $Φ(Δ) ⊂ P(Λ)$ the last equation implies that $P(Λ) = P(Δ) \cap P(Λ)$ since $Δ = P' \cup −P'$. 

**Remark.** In Proposition 4.14 (and hence in Theorem 4.3) the condition $Φ(Δ) \subset \{α ∈ P(Δ)(Λ+δ-δ(A), α)=0\}$ is automatically satisfied. Indeed for $α ∈ Φ(Δ) ⊂ P(Δ) 0 ≤ (Λ+δ-δ(A), α) = (w^-1(Λ+δ-δ(A)), w^-1α) = (Λ+δ, α)$ (since $w^-1(Λ+δ-δ(A))=Λ+δ-δ(A)$) which is positive. That is $wα ∈ P' ⇔ wP(Λ) ⊂ P' ⇔ wP(Λ)=P'$. 

**Theorem 4.16.** Assume that $G$ is linear and its complexification $G^C$ is simply connected. (In particular if $Λ ∈ h^*$ is $Δ_\alpha$-dominant integral the irreducible finite-dimensional representation of $\mathfrak{f}$ defined by $Λ$ integrates to a representation of $K$.) Let $Λ ∈ Ξ^*$ be such that every non-compact root in $P(Λ)$ is totally positive. If $H^q(G\backslash G/K, θ_Λ)=0$ then there is a parabolic subalgebra $\theta_Λ=\theta_ι=\theta_ι\cap \gamma$ such that $Λ+δ-δ(A)=\Lambda_ι\cap Q(Λ)=0$. Also $(Λ+δ-δ(A), Δ(\alpha_ι))=0, |Δ_\alpha|+|Q(Λ)|+|Q(Λ)|=|Δ_\alpha|+|Q(Λ)|=0$. 

Proof. If $H^q(G\backslash G/K, θ_Λ)=0$ then by (4.2) $\text{Hom}_κ(\mathcal{H}_κ, \wedge^p V(Λ))=0$ for some $(κ, \mathcal{H}_κ)∈ H$ such that $π(Δ)=(Λ, Λ+2δ)$. By Corollary 3.5 $H_κ$ has minimal $\mathfrak{h}$ type $μ=Λ+δ_κ-wδ(Λ)−δ_κ$ for some Weyl group element $w$ such that $Δ_κ \subset wP(Λ)$ and $q = |Δ_κ|−2|Q(Λ)|+|Q(Λ)|; w(Λ+δ−δ(A))=Λ+δ−δ(A)$. By Corollary 3.15 $H_κ$ is a highest weight $g$ module with highest weight $μ$ relative to the positive system $P(Λ)=P(Δ)∪ P(Δ)=P(Δ)∪ P(Δ)$. Also $μ_κ=δ_κ−δ(A)=Λ+δ_κ−δ(Λ)+wδ(A)=Λ+δ−δ(A)+wδ(A)$ is regular by Proposition 4.14 (see remark following Proposition 4.14). Thus since $G$ is assumed to be linear we can apply Parthasarathy's Theorem A of [17] to conclude the following:
Vanishing Theorems for Type $(0, q)$ Cohomology II

**Remark.** If additional information on the Weyl group element $\sigma_1$ above (where $\sigma_1 \Delta^*_\Lambda = P(\Lambda)$) were available the preceding proof might not require the appeal to Theorem A of [17]. For example if it were known that $<P^\Lambda(\Lambda) - \sigma_1 \Delta(\Lambda)> = \delta^\Lambda + w \delta^\Lambda - \delta_k$ for $\theta = m+u$ in Theorem 4.3 then Theorem 4.16 would follow (even for $G$ non-linear) by taking $\theta_1 = \sigma_1 \theta$. However $\otimes$ is true only when certain additional restrictions on $\Lambda$ are imposed.

Another classical vanishing theorem for the spaces $H^q(\Gamma \backslash G/K, \theta_\Lambda)$ is the following one of Hotta and Parthasarathy; see Proposition 1 of [5].

**Theorem 4.17.** Let $\Lambda \in \mathcal{D}_\Lambda$ be the $\Delta^*_\Lambda$-highest weight of $(\tau_\Lambda, V_\Lambda) \in \mathcal{K}$. Suppose that $(\Lambda + \delta - \delta^\Lambda, \alpha) > 0$ for every $\alpha$ in $P_\Lambda^\Lambda$. Then $H^q(\Gamma \backslash G/K, \theta_\Lambda) = 0$ for $q \neq |Q_{\Lambda}|$.

Here $G$ is not assumed to be linear. Theorem 4.17 follows from a trivial application of Theorem 4.3. Namely if $H^q(\Gamma \backslash G/K, \theta_\Lambda) = 0$ then $q = |A_{\Lambda, \tau, w}| - 2|Q_{\Lambda} \cap A_{\Lambda, \tau, w}| + |Q_{\Lambda}|$ where $A_{\Lambda, \tau, w} \subset \{ \alpha \in P^\Lambda_{\Lambda} | (\Lambda + \delta - \delta^\Lambda, \alpha) = 0 \}$. But $(\Lambda + \delta - \delta^\Lambda, \alpha) > 0$ for $\alpha \in P^\Lambda_{\Lambda}$ by hypothesis so $A_{\Lambda, \tau, w} = \phi$. Thus $q = |Q_{\Lambda}|$.

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