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ELEMENTARY PROOFS OF POINTWISE ERGODIC THEOREMS FOR MEASURE PRESERVING TRANSFORMATIONS

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1. Introduction

Recently Kamae and Keane [5] have obtained a simple proof of Hopf's ratio ergodic theorem using an idea due to Kamae [4] and Shields [9]. In this paper the same idea will be applied to obtain elementary proofs of pointwise ergodic theorems for superadditive processes relative to measure preserving transformations. The main tool is a maximal ergodic theorem for superadditive processes, whose proof has been motivated by Jones [3] and Akcoglu and Krengel [1]. For related results we refer the reader to [2], [6] and [10] (see also [7] and §1.5 of [8]).

2. A maximal ergodic theorem

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and $T : X \to X$ be a measure preserving transformation. As usual, two measurable functions f and g are not distinguished provided that $f(x) = g(x)$ a.e. on X. A family $\mathbf{F} = \{F_{i,k} : 0 \le i < k\}$ of measurable functions is called a *superadditive process* in $L_1(\mu)$ if it satisfies

(i) $F_{i,k} \circ T = F_{i+1,k+1}$ for $0 \le i < k$,

(ii) $F_{i,l} \ge F_{i,k} + F_{k,l}$ for $0 \le i < k < l$,

(iii) the functions $F_{i,k}$ are all integrable and $\gamma(\mathbf{F}) = \sup \{n^{-1} \int_X F_{0,n}\, d\mu : n \ge 1\} < \infty$.

It should be noted here that since $\int_X F_{0,n+m}\, d\mu \ge \int_X F_{0,n}\, d\mu + \int_X F_{0,m}\, d\mu$ by (i) and (ii), it follows easily that $\gamma(\mathbf{F}) = \lim_n n^{-1} \int_X F_{0,n} d\mu$. (This is standard. See e.g. Lemma 1.5.1 of [8].) $\mathbf{F}$ is called a *subadditive process* in $L_1(\mu)$, if $-\mathbf{F} = \{-F_{i,k}\}$ is a superadditive process in $L_1(\mu)$, and an *additive process* in $L_1(\mu)$ if $\mathbf{F}$ is a both superadditive and subadditive process in $L_1(\mu)$. $\mathbf{F}$ is called an *extended superadditive process* if it satisfies (i) and (ii), but not necessarily (iii).

The following maximal ergodic theorem is basic throughout this paper.

Theorem 1. Let $\mathbf{F} = \{F_{i,k} : 0 \le i < k\}$ be a nonnegative superadditive process in $L_1(\mu)$. If g is a nonnegative measurable function on X and

$$E = \{x : \sup_{n\geq 1} [F_{0,n}(x) - \sum_{k=0}^{n-1} g(T^k x)] > 0\}$$

then we have

(1) $$\int_E g\, d\mu \leq \gamma(\mathbf{F}) < \infty.$$

Proof. For $N \geq 1$, let us put

$$E_N = \{x : \max_{1\leq n\leq N} [F_{0,n}(x) - \sum_{k=0}^{n-1} g(T^k x)] > 0\}.$$

Since $E_N \uparrow E$, it suffices to show that inequality (1) holds for E_N instead of E. To do so, let $K > N$ and for $x \in X$, write $A(x) = \{k : 0 \leq k < K - N,\ T^k x \in E_N\}$ and

$$k_1 = \begin{cases} \min A(x) & \text{if } A(x) \neq \emptyset \\ \infty & \text{otherwise .} \end{cases}$$

If $k_1 \neq \infty$, then $T^{k_1} x \in E_N$ by definition, and there exists an n_1, $1 \leq n_1 \leq N$, such that

$$F_{0,n_1}(T^{k_1} x) - \sum_{l=0}^{n_1-1} g(T^{k_1+l} x) > 0.$$

Next, write $A_2(x) = \{k : k_1 + n_1 \leq k < K - N,\ T^k x \in E_N\}$ and

$$k_2 = \begin{cases} \min A_2(x) & \text{if } A_2(x) \neq \emptyset \\ \infty & \text{otherwise .} \end{cases}$$

If $k_2 \neq \infty$, then $T^{k_2} x \in E_N$ and there exists an n_2, $1 \leq n_2 \leq N$, such that

$$F_{0,n_2}(T^{k_2} x) - \sum_{l=0}^{n_2-1} g(T^{k_2+l} x) > 0.$$

Continuing this process, we find finite sequences $\{k_i\}_{i=1}^r$ and $\{n_i\}_{i=1}^r$ such that

(i) $0 \leq k_1 < k_2 < \ldots < k_r < K - N$,

(ii) $1 \leq n_i \leq N$ for $i = 1, 2, \ldots, r$,

(iii) the integer intervals $[k_i,\ k_i + n_i)$ $(i = 1, 2, \ldots, r)$ are pairwise disjoint and satisfy $F_{0,n_i}(T^{k_i} x) > \sum_{l=0}^{n_i-1} g(T^{k_i+l} x)$,

(iv) if $k \in A(x)$ then $k \in [k_i,\ k_i + n_i)$ for some i with $1 \leq i \leq r$.

Since $\mathbf{F}$ and g are nonnegative, it then follows that

$$F_{0,K}(x) \geq \sum_{i=1}^{r} F_{k_i,k_i+n_i}(x) = \sum_{i=1}^{r} F_{0,n_i}(T^{k_i}x)$$

$$> \sum_{i=1}^{r}\Big(\sum_{l=0}^{n_i-1} g(T^{k_i+l}x)\Big) \geq \sum_{l=0}^{K-N-1} (g\cdot\chi_{E_N})(T^l x).$$

Therefore

$$\frac{1}{K}\int_X F_{0,K}(x)\,d\mu \geq \frac{K-N}{K}\int_{E_N} g\,d\mu,$$

and by letting $K \uparrow \infty$, we have $\gamma(\mathbf{F}) \geq \int_{E_N} g\,d\mu$. The proof is complete.

3. Pointwise ergodic theorems

For the remainder let us fix an integrable function e with $0 < e(x) < \infty$ on X and write

$$C = \{x : \sum_{k=0}^{\infty} e(T^k x) = \infty\} \quad \text{and} \quad D = X \setminus C.$$

Clearly C and D are in $\mathcal{I}$, where we let $\mathcal{I} = \{E \in \mathcal{F} : T^{-1}E = E \pmod{\mu}\}$.

Lemma. If $\mathbf{F}$ is a nonnegative superadditive process in $L_1(\mu)$ then $\lim_n F_{0,n}(x) < \infty$ a.e. on D.

Proof. Since $\mathbf{F}$ is nonnegative, we have $\sup_n F_{0,n}(x) = \lim_n F_{0,n}(x)$ a.e. on X. Write $E = \{x : \lim_n F_{0,n}(x) = \infty\} \cap D$. Then it follows that for any $\alpha > 0$

$$E \subset \{x : \sup_n\,[F_{0,n}(x) - \sum_{k=0}^{n-1} \alpha\cdot e(T^k x)] > 0\},$$

and thus by Theorem 1 we get $\alpha\int_E e\,d\mu \leq \gamma(\mathbf{F}) < \infty$. Since $\alpha > 0$ was arbitrary, this yields $\int_E e\,d\mu = 0$ and hence $\mu E = 0$. The proof is complete.

Theorem 2. If $\mathbf{F}$ is a nonnegative superadditive process in $L_1(\mu)$ then the limit

$$R(\mathbf{F},e)(x) = \lim_n \frac{F_{0,n}(x)}{\sum_{i=0}^{n-1} e(T^i x)}$$

exists and is finite a.e. on X. In particular if $X = C$ then the limit function $R(\mathbf{F},e)(x)$ is invariant under T and for any $A \in \mathcal{I}$ we have

$$\int_A R(\mathbf{F},e)\cdot e\,d\mu = \lim_n \frac{1}{n}\int_A F_{0,n}\,d\mu. \tag{2}$$

Proof. By Lemma it suffices to consider the case $X = C$.

Step 1. First suppose $F_{0,n}$ has the form $F_{0,n} = \sum_{k=0}^{n-1} f \circ T^k$ for some non-negative f in $L_1(\mu)$. Since $\sum_{k=0}^{\infty} e(T^k x) = \infty$ a.e. on $X = C$, it follows that the functions

$$f^{\sim}(x) = \limsup_n \Big(\sum_{k=0}^{n-1} f \circ T^k(x)\Big) \Big/ \Big(\sum_{k=0}^{n-1} e \circ T^k(x)\Big)$$

and

$$f_{\sim}(x) = \liminf_n \Big(\sum_{k=0}^{n-1} f \circ T^k(x)\Big) \Big/ \Big(\sum_{k=0}^{n-1} e \circ T^k(x)\Big)$$

are invariant under T. As in the proof of Lemma, we see that $f^{\sim}(x) < \infty$ a.e. on X. To prove that $f^{\sim}(x) = f_{\sim}(x)$ a.e. on X, write $E = \{x : f_{\sim}(x) < a < b < f^{\sim}(x)\}$, where a and b are real numbers. It follows that $E \in \mathcal{I}$, and if $x \in E$ then

$$\sup_n \, [F_{0,n}(x) - \sum_{k=0}^{n-1} b \cdot e(T^k x)] > 0 \tag{3}$$

and

$$\sup_n \Big[\sum_{k=0}^{n-1} a \cdot e(T^k x) - F_{0,n}(x)\Big] > 0. \tag{4}$$

Thus by Theorem 1

$$b \int_E e \, d\mu \le \int_E f \, d\mu \le a \int_E e \, d\mu. \tag{5}$$

Since $a < b$, this yields $\int_E e \, d\mu = 0$ and $\mu E = 0$. Thus $R(\mathbf{F}, e)(x) = f^{\sim}(x) = f_{\sim}(x)$ a.e. on $X = C$.

To prove that (2) holds, we may assume without loss of generality that $A = X = C$, since A is an invariant set under T. Let $E(a,b) = \{x : a < R(\mathbf{F}, e)(x) < b\}$, where $0 \le a < b$. By Theorem 1 together with the fact that $E(a,b) \in \mathcal{I}$, we have

$$a \int_{E(a,b)} e \, d\mu \le \int_{E(a,b)} f \, d\mu \le b \int_{E(a,b)} e \, d\mu. \tag{6}$$

On the other hand, it is clear that

$$a \int_{E(a,b)} e \, d\mu \le \int_{E(a,b)} R(\mathbf{F}, e) \cdot e \, d\mu \le b \int_{E(a,b)} e \, d\mu. \tag{7}$$

Combining (6) and (7), we get

$$\int_{E(a,b)} R(\mathbf{F}, e) \cdot e \, d\mu \le a \int_{E(a,b)} e \, d\mu + (b-a) \int_{E(a,b)} e \, d\mu$$

$$\leq \int_{E(a,b)} f\,d\mu + (b-a)\int_{E(a,b)} e\,d\mu.$$

Similarly,

$$\int_{E(a,b)} f\,d\mu \leq \int_{E(a,b)} R(\mathbf{F},e)\cdot e\,d\mu + (b-a)\int_{E(a,b)} e\,d\mu.$$

Thus

$$\Big|\int_{E(a,b)} (f - R(\mathbf{F},e)\cdot e)\,d\mu\Big| \leq (b-a)\int_{E(a,b)} e\,d\mu. \tag{8}$$

Since given an $\epsilon > 0$ we can choose nonnegative real numbers a_n and b_n, $n \geq 1$, so that $0 < b_n - a_n < \epsilon$ for each $n \geq 1$, the open intervals $(a_n,\, b_n)$ are pairwise disjoint, and

$$X = \bigcup_{n=1}^{\infty} E(a_n, b_n) \qquad (\mathrm{mod}\ \mu),$$

it then follows from (8) that

$$\Big|\int_X f\,d\mu - \int_X R(\mathbf{F},e)\cdot e\,d\mu\Big| \leq \sum_{n=1}^{\infty}\Big|\int_{E(a_n,b_n)} (f - R(\mathbf{F},e)\cdot e)\,d\mu\Big|$$

$$\leq (b_n - a_n)\sum_{n=1}^{\infty}\int_{E(a_n,b_n)} e\,d\mu \leq \epsilon\int_X e\,d\mu.$$

Since $\epsilon > 0$ was arbitrary, this proves that (2) holds for $A = X = C$.

Step 2. Next suppose $\mathbf{F}$ is any nonnegative superadditive process in $L_1(\mu)$. As before, write

$$f^{\sim}(x) = \limsup_n\ F_{0,n}(x)\Big/\Big(\sum_{k=0}^{n-1} e\circ T^k(x)\Big)$$

and

$$f_{\sim}(x) = \liminf_n\ F_{0,n}(x)\Big/\Big(\sum_{k=0}^{n-1} e\circ T^k(x)\Big).$$

Since $F_{0,n+1}(x) \geq F_{0,1}(x) + F_{0,n}\circ T(x)$ and $\sum_{k=0}^{\infty} e(T^k x) = \infty$ a.e. on $X = C$, we have $f^{\sim}(Tx) \leq f^{\sim}(x)$ and $f_{\sim}(Tx) \leq f_{\sim}(x)$ a.e. on $X = C$. Thus for any $\alpha > 0$, $\{x : f^{\sim}(x) < \alpha\} \subset T^{-1}(\{x : f^{\sim}(x) < \alpha\})$ and if the set $E = T^{-1}(\{x : f^{\sim}(x) < \alpha\}) \setminus \{x : f^{\sim}(x) < \alpha\}$ is of positive measure, then there exists a nonnegative integrable function f with $\{x : f(x) > 0\} \subset E$ and $\int_X f\,d\mu > 0$. Since $T^n x \notin E$ for all $x \in E$ and $n \geq 1$, we then have

$$\lim_n \Big(\sum_{k=0}^{n-1} f(T^k x)\Big)\Big/\Big(\sum_{k=0}^{n-1} e(T^k x)\Big) = 0 \ \text{ for a.e. } \ x \in X.$$

But this is a contradiction by Step 1. It follows that $\{x : f^{\sim}(x) < \alpha\} \in \mathcal{I}$ and hence $f^{\sim}$ is an invariant function under T. Similarly $f_{\sim}$ is an invariant function under T. Since Theorem 1 can be applied to infer that $f^{\sim}(x) < \infty$ a.e. on X, it remains to prove that $f^{\sim}(x) = f_{\sim}(x)$ a.e. on X and (2) holds for any $A \in \mathcal{I}$. Here we may assume without loss of generality that $A = X = C$, and it suffices to prove that

$$\int_X f_{\sim} \cdot e \, d\mu = \int_X f^{\sim} \cdot e \, d\mu = \gamma(\mathbf{F}) < \infty. \tag{9}$$

To do so, let $F(a,b) = \{x : a < f^{\sim}(x) < b\}$, where $0 \leq a < b$. It follows that $F(a,b) \in \mathcal{I}$, and for $x \in F(a,b)$ we have

$$\sup_{n\geq 1} [F_{0,n}(x) - \sum_{k=0}^{n-1} a \cdot e(T^k x)] > 0.$$

Thus Theorem 1 implies

$$\int_{F(a,b)} a \cdot e \, d\mu \leq \lim_n \frac{1}{n} \int_{F(a,b)} F_{0,n} \, d\mu.$$

It follows that

$$\int_{F(a,b)} f^{\sim} \cdot e \, d\mu \leq \int_{F(a,b)} b \cdot e \, d\mu = \int_{F(a,b)} a \cdot e \, d\mu + (b-a) \int_{F(a,b)} e \, d\mu$$
$$\leq \lim_n \frac{1}{n} \int_{F(a,b)} F_{0,n} \, d\mu + (b-a) \int_{F(a,b)} e \, d\mu.$$

Now, given an $\epsilon > 0$, let us choose pairwise disjoint open interval $(a_n, b_n) \subset (0, \infty)$, $n \geq 1$, such that $b_n - a_n < \epsilon$ for each $n \geq 1$, and

$$X = \bigcup_{n=1}^{\infty} F(a_n, b_n) \qquad (\text{mod } \mu).$$

Then we have

$$\int_X f^{\sim} \cdot e \, d\mu = \sum_{n=1}^{\infty} \int_{F(a_n,b_n)} f^{\sim} \cdot e \, d\mu$$
$$\leq \sum_{n=1}^{\infty} [(\lim_{k\to\infty} \frac{1}{k} \int_{F(a_n,b_n)} F_{0,k} \, d\mu) + (b_n - a_n) \int_{F(a_n,b_n)} e \, d\mu]$$
$$\leq \gamma(\mathbf{F}) + \epsilon \int e \, d\mu.$$

Since $\epsilon > 0$ was arbitrary, we get $\int_X f^{\sim} \cdot e \, d\mu \leq \gamma(\mathbf{F})$.

Next for an integer $K > 1$, let $e_K = \sum_{i=0}^{k-1} e \circ T^i$. Applying Step 1 to T^K and e_K instead of T and e, we see that for almost all $x \in X$

$$\lim_n e_K(T^{nK}x) \Big/ \Big(\sum_{i=0}^{n-1} e_K)(T^{iK}x)\Big) = 0, \tag{10}$$

and thus the function

$$R(\mathbf{F}(K), e_K)(x) = \lim_n \frac{\sum_{i=0}^{n-1} F_{0,K}(T^{iK}x)}{\sum_{i=0}^{n-1} e_K(T^{iK}x)}$$

satisfies

$$R(\mathbf{F}(K), e_K)(x) \leq \liminf_n \frac{F_{0,nK}(x)}{\sum_{i=0}^{n-1} e_K(T^{iK}x)} = f_\sim(x) \quad \text{a.e. on } X,$$

where the last equality follows from (10). Since $f_\sim$ is invariant under T, we then have

$$\begin{aligned} \int_X f_\sim \cdot e \, d\mu &= \frac{1}{K} \int_X f_\sim \cdot \Big(\sum_{i=0}^{K-1} e \circ T^i\Big) \, d\mu \\ &\geq \frac{1}{K} \int_X R(\mathbf{F}(K), e_K) \cdot e_K \, d\mu = \frac{1}{K} \int_X F_{0,K} \, d\mu, \end{aligned}$$

where the last equality follows from Step 1 applied to T^K and e_K. Letting $K \uparrow \infty$ shows that $\int_X f_\sim \cdot e \, d\mu \geq \gamma(\mathbf{F})$, and hence (9) follows. This completes the proof.

Theorem 3 (cf. [2]). Let $\mathbf{F}' = \{F'_{i,k}\}$ be a nonnegative extended superadditive process and let $P = \{x : \lim_n F'_{0,n}(x) > 0\}$. If $\mathbf{F} = \{F_{i,k}\}$ is a nonnegative superadditive process in $L_1(\mu)$, then the limit

$$R(\mathbf{F}, \mathbf{F}')(x) = \lim_n \frac{F_{0,n}(x)}{F'_{0,n}(x)}$$

exists and is finite a.e. on the set P. In particular, if $X = C$ then $P \in \mathcal{I}$ and the limit function $R(\mathbf{F}, \mathbf{F}')(x)$ is invariant under T.

Proof. By Lemma we may restrict ourselves to the case $X = C$. Since

$$\frac{F_{0,n}(x)}{F'_{0,n}(x)} = \frac{F_{0,n}(x)/(\sum_{i=0}^{n-1} e \circ T^i(x))}{F'_{0,n}(x)/(\sum_{i=0}^{n-1} e \circ T^i(x))},$$

it then suffices by Theorem 2 to prove that the limit

$$R(\mathbf{F}', e)(x) = \lim_n \frac{F'_{0,n}(x)}{\sum_{i=0}^{n-1} e \circ T^i(x)}$$

exists (but may be infinite) a.e. on P and $R(\mathbf{F}', e)(x) > 0$ a.e. on P. To do so, let

$$\mathcal{I}(\mathbf{F}') = \{A \in \mathcal{I} : \sup_n \frac{1}{n} \int_A F'_{0,n}\, d\mu < \infty\}.$$

Since the measure μ is σ-finite, there exist sets A_k in $\mathcal{I}(\mathbf{F}'), k \geq 1$, such that if we set $Y = \cup_{k=1}^{\infty} A_k$ then $Y \in \mathcal{I}$ and for any $B \in \mathcal{I}$ with $\mu B > 0$ and $B \subset X \setminus Y$ we have

$$\sup_n \frac{1}{n} \int_B F'_{0,n}\, d\mu = \infty.$$

Since Theorem 2 implies that $A_k \cap P \in \mathcal{I}$ and for almost all x in $A_k \cap P$ the limit $R(\mathbf{F}', e)(x)$ exists and satisfies

$$0 < R(\mathbf{F}', e)(x) = R(\mathbf{F}', e)(Tx) < \infty,$$

it only remains to consider the part $Z = X \setminus Y$. It is now enough to show that the function

$$f_*(x) = \liminf_n \frac{F'_{0,n}(x)}{\sum_{i=0}^{n-1} e \circ T^i(x)}$$

satisfies $f_*(x) = \infty$ a.e. on Z. Since $\sum_{k=0}^{\infty} e(T^k x) = \infty$ a.e. on Z, we see, as before, that the function f_* is invariant under T. Thus, for any $a > 0$, the set $E(a) = \{x : f_*(x) < a\} \cap Z$ is in $\mathcal{I}$. Further, for the function $e_K = \sum_{i=0}^{K-1} e \circ T^i$, where K is an integer with $K \geq 1$, we see that if $x \in E(a)$ then

$$a > f_*(x) = \liminf_n \frac{F'_{0,nK}(x)}{\sum_{i=0}^{n-1} e_K(T^{iK}x)} \geq \liminf_n \frac{\sum_{i=0}^{n-1} F'_{0,K}(T^{iK}x)}{\sum_{i=0}^{n-1} e_K(T^{iK}x)},$$

where the equality follows from (10). Thus $x \in E(a)$ implies

$$\sup_n \Big[\sum_{i=0}^{n-1} a \cdot e_K(T^{iK}x) - \sum_{i=0}^{n-1} F'_{0,K}(T^{iK}x)\Big] > 0,$$

and hence by Theorem 1

$$\int_{E(a)} F'_{0,K}\, d\mu \leq \int_{E(a)} a \cdot e_K\, d\mu = aK \int_{E(a)} e\, d\mu.$$

It follows that

$$\sup_{K \geq 1} \frac{1}{K} \int_{E(a)} F'_{0,K}\, d\mu \leq a \int_{E(a)} e\, d\mu < \infty$$

and hence $E(a) \subset Y$. But, since $E(a) \subset Z$ by definition, we get $\mu E(a) = 0$. Consequently $f_*(x) = \infty$ a.e. on Z. The proof is complete.

Corollary 1. If $\mathbf{F} = \{F_{i,k}\}$ is a superadditive process in $L_1(\mu)$ then the limit $f(x) = \lim_n n^{-1}F_{0,n}(x)$ exists and is finite a.e. on X.

Proof. Immediate from Theorem 3.

Corollary 2. If $\mathbf{F}' = \{F'_{i,k}\}$ is a nonnegative extended superadditive process then the limit $f'(x) = \lim_n n^{-1}F'_{0,n}(x)$ exists (but it may be infinite) a.e. on the set $C_1 = \{x : \lim_n n^{-1}\sum_{i=0}^{n-1} e(T^i x) > 0\}$. Further there exists a nonnegative superadditive process $\mathbf{G}' = \{G'_{i,k}\}$ for which the averages $n^{-1}G'_{0,n}(x)$ fail to converge for almost all x in $X \setminus C_1$.

Proof. By the relation

$$n^{-1}F'_{0,n}(x) = \frac{F'_{0,n}(x)}{\sum_{i=0}^{n-1} e \circ T^i(x)} \cdot \frac{\sum_{i=0}^{n-1} e \circ T^i(x)}{n},$$

the first half of the corollary follows directly from the proof of Theorem 3.

To prove the second half, let $\{A_N\}$ be an increasing sequence of measurable sets such that $\mu A_N < \infty$ for all $N \geq 1$ and $X \setminus C_1 = \lim_N A_N$. Let

$${}_1g_n = \sum_{i=0}^{n-1} \chi_{A_1} \circ T^i$$

By Theorem 2, if f is a nonnegative integrable function with $\{x : f(x) > 0\} \subset X \setminus C_1$ then $\lim_n n^{-1}\sum_{i=0}^{n-1} f(T^i x) = 0$ a.e. on X. Using this, we see that there exists an integer k_1 , $k_1 \geq 1$, such that

$$\mu(A_1 \setminus \{x : \frac{1}{k_1} \cdot {}_1g_{k_1}(x) < 2^{-1}\}) < 2^{-1}.$$

Take a positive real number a_1 and a measurable set B_1 such that

$$a_1/(k_1 + 1) \geq 1, \quad \mu B_1 < \infty \quad \text{and} \quad \mu(A_1 \setminus T^{-k_1}(B_1)) < 2^{-1}.$$

Then define

$${}_2g_n = \begin{cases} 0 & \text{if } 1 \leq n \leq k_1 \\ \sum_{i=k_1}^{n-1} (a_1 \cdot \chi_{B_1}) \circ T^i & \text{if } n \geq k_1 + 1. \end{cases}$$

As above, there exists an integer k_2 , $k_2 > k_1$, such that

$$\mu(A_2 \setminus \{x : \frac{1}{k_2} \cdot [{}_1g_{k_2}(x) + {}_2g_{k_2}(x)] < 2^{-2}\}) < 2^{-2}.$$

Next take a positive real number a_2 and a measurable set B_2 such that

$$a_2/(k_2 + 1) \geq 1, \quad \mu B_2 < \infty \quad \text{and} \quad \mu(A_2 \setminus T^{-k_2}(B_2)) < 2^{-2}.$$

By repeating this process we can find sequences $\{k_j\}, \{a_j\}$ and $\{B_j\}$ such that $1 \leq k_1 < k_2 < \ldots$, $a_j/(k_j+1) \geq 1$, $\mu B_j < \infty$, $\mu(A_j \setminus T^{-k_j}(B_j)) < 2^{-j}$ and

$$\mu(A_j \setminus \{x : \frac{1}{k_j} \sum_{l=1}^{j} {}_l g_{k_j}(x) < 2^{-j}\}) < 2^{-j} \quad \text{for all } j \geq 1,$$

where we let for $l \geq 2$

$${}_l g_n = \begin{cases} 0 & \text{if } 1 \leq n \leq k_{l-1} \\ \sum_{i=k_{l-1}}^{n-1} (a_{l-1} \cdot \chi_{B_{l-1}}) \circ T^i & \text{if } n \geq k_{l-1}+1. \end{cases}$$

If we put $k_0 = 0$ then for each $n \geq 1$ there exists a unique integer j, $j \geq 1$, such that $k_{j-1} < n \leq k_j$. Then we define

$$G'_{i,i+n} = \sum_{l=1}^{j} {}_l g_n \circ T^i \quad \text{for} \quad i \geq 0.$$

It follows by construction that $\mathbf{G}' = \{G'_{i,k} : 0 \leq i < k\}$ becomes a nonnegative extended superadditive process, and for almost all x in $X \setminus C_1$ we have

$$\limsup_n n^{-1} G'_{0,n}(x) \geq 1 \qquad \text{and} \qquad \liminf_n n^{-1} G'_{0,n}(x) = 0,$$

whence the proof is complete.

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