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COMPLETIONS OF HEREDITARY NOETHERIAN PRIME RINGS

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Let R be a hereditary noetherian prime ring with quotient ring Q and let $A=M_1\cap\cdots\cap M_p$ be a maximal invertible ideal of R, where $M_1, \cdots, M_p$ is a cycle (cf. [2] for the definition of cycles). The main purpose of this paper is to prove the following theorem:

Theorem 1.1. (1) *The completion $\hat{R}$ of R with respect to A is a bounded hereditary noetherian prime ring with quotient ring $Q\otimes\hat{R}$. The Jacobson radical $\hat{A}$ of $\hat{R}$ is $A\hat{R}=\hat{R}A$ and $\hat{A}^p$ is a principal right and left ideal of $\hat{R}$.*

(2) *$\hat{R}$ has the following decomposition;*

$$\hat{R}=(\overbrace{e_1\hat{R}\oplus\cdots\oplus e_1\hat{R}}^{k_1})\oplus(\overbrace{e_2\hat{R}\oplus\cdots\oplus e_2\hat{R}}^{k_2})\oplus\cdots\oplus(\overbrace{e_p\hat{R}\oplus\cdots\oplus e_p\hat{R}}^{k_p})$$

such that each $e_i\hat{R}$ is a uniform right ideal of $\hat{R}$, e_i is an idempotent in $\hat{R}$ and $e_i\hat{R}/e_i\hat{A}$ is a simple right R-module which is annihilated by M_i, where k_i is the Goldie dimension of R/M_i.

In case R is a Dedekind prime ring and A is a maximal ideal of R, Gwynne and Robson proved that $\hat{R}$ is also a Dedekind prime ring [5] (in fact, it is a principal ideal ring). We can not use their techniques to prove the theorem. The theorem is proved by using properties of cotosion R-modules.

Applying the theorem to module theory, we prove, in section 2, the following theorems:

Theorem 2.1. *Any module over $\hat{R}$ has a basic submodule.*

Theorem 2.2. *Under the same notations as in Theorem 1.1, any indecomposable right $\hat{R}$-module is isomorphic to one of the following $\hat{R}$-modules;*

$$e_i\hat{R}/e_i\hat{A}^n\ (n=1,2,\cdots),\quad e_i\hat{R},\quad e_i(Q\otimes\hat{R}),\quad E(e_i\hat{R}/e_i\hat{A})\ (i=1,\cdots,p)$$

where $E(e_i\hat{R}/e_i\hat{A})$ is the R-injective hull of $e_i\hat{R}/e_i\hat{A}$.

In [18], Singh determined the structure of those bounded hereditary noetherian prime rings over which every module admits a basic submodule. If R is a commutative complete discrete valuation ring, then Theorem 2.2 was proved by Kaplansky [7, p. 53]. The author generalized the result to modules over g-discrete valuation rings [11, Corollary 4.4]

In an appendix we present some properties on cotorsion R-modules which are obtained by modifying the methods used in the corresponding ones in modules over Dedekind prime rings.

This paper was written while the author was a visitor at Guru Nanak Dev university, India. As I was doing my research, I got several hints from Prof. Singh's lectures and from discussions with him. I would like to express my tahnks to him for his kind invitation to G.N.D. univ. and foı his hospitality.

1. The proof of Theorem 1.1

Throughout this paper, R denotes a hereditary neotherian prime ring (for short: hnp-ring) with quotient ring Q and $K=Q/R\neq 0$. In place of $\otimes_R$, Hom_R, Ext_R and Tor^R, we just write $\otimes$, Hom, Ext and Tor, respectively. Since R is hereditary, $\mathrm{Tor}_n=0=\mathrm{Ext}^n$ for all $n>1$ and so we use Ext for Ext^1 and Tor for Tor_1. Let M be a right R-module. An element m of M is said to be *torsion* ıf $O(m)=\{r\in R\,|\,mr=0\}$ is an essential right ideal of R. We say that M is a *torsion module* if every element of M is torsion. If M has no nonzero torsion elements, then it is called *torsion-free.* M is called *divisible* if $MJ=M$ for every essentail left ideal J of R. Since R is an hnp-ring, the divisibility is equivalent to the injectivity by [10]. We denote the Jacobson radical of a ring S by $J(S)$. Let I be an essential right ideal of R. Define I^* by $I^*=\{q\in Q\,|\,qI\subseteq R\}$ Similarly $^*J=\{q\in\,|\,Jq\subseteq R\}$ for essential left ideal J of R. An ideal B of R is called *invertible* if $(B^*)B=B(^*B)=R$. In this case we have $B^*={}^*B$, denote it by B^{-1}. Let A be a maximal invertible ideal of R. The cancellation set of A, $C(A)$, is defined to be $\{c\in R\,|\,cx\in A\Rightarrow x\in A\}=\{c\in R\,|\,xc\in A\Rightarrow x\in A\}\}$. By [9], each element of $C(A)$ is regular. We denote the subring of Q generated by $\{a, c^{-1}\,|\,a\in R, c\in C(A)\}$ by R_A. The following lemma was proved by Kuzmanovich [9, §3].

Lemma 1.1. (1) *R satisfies the Ore condition with respect to $C(A)$, i.e.,* $R_A=\{ac^{-1}\,|\,a\in R, c\in C(A)\}=\{d^{-1}b\,|\,b\in R, d\in C(A)\}$.

(2) $J(R_A)=AR_A=R_AA$ *and* $R/A^n\cong R_A/J(R_A)^n$ *for all* n.

(3) *If A is a maximal ideal, then R_A is a principal ideal ring with a unique maximal ideal $J(R_A)$. So it is a Dedekind prime ring and every ideal of R_A is a power of $J(R_A)$.*

(4) *If A is an intersection of a cycle, say, $A=M_1\cap\cdots\cap M_p$, where $M_1, \cdots, M_p$ is a cycle, then $J(R_A)=M_1R_A\cap\cdots\cap M_pR_A$ and $M_1R_A, \cdots, M_pR_A$ is a cycle.*

M_iR_A's are only maximal ideals of R_A, all are idempotents and $M_iR_A = R_AM_i$.

(5) *$R/M_i \cong R_A/M_iR_A$ for all i.*

We denote the inverse limit of the rings R/A^n $(n=1, 2, \cdots)$ by $\hat{R}$. If A is a maximal ideal of R, then $\hat{R}$ is a principal ideal ring by Theorem 2.3 of [5] and Lemma 1.1. So, to prove Theorem 1.1, we may assume that A is not a maximal ideal of R. Further, since $\hat{R} \cong \hat{R}_A$, we may assume that R satisfies the following two conditions;

(a) $J(R) = A$ is a maximal invertible ideal of R, and

(b) $A = M_1 \cap \cdots \cap M_p$, where M_i are idempotent maximal ideals of R and $M_1, \cdots, M_p$ is a cycle.

From now on, R denotes an hnp-ring which satisfies the above conditions (a) and (b) unless otherwise stated. Then, by [2], we have

(i) *Every invertible ideal of R is a power of A.*

(ii) *R is bounded and any essential one-sided ideal of R contains a power of A. Especially $Q = \cup_n A^{-n}$.*

Let F be the family of all essential right ideals of R and let F_l be the family of all essential left ideals of R. We write $\hat{R}_F = \varprojlim R/I (I \in F)$ and $\hat{R}_{F_l} = \varprojlim R/J (J \in F_l)$. They are both rings (cf. [21] for more detailed results). The ring homomorphisms $\varphi: \hat{R}_F \to \hat{R}$ and $\psi: \hat{R}_{F_l} \to \hat{R}$, given by $\varphi(\hat{r}) = ([r_{A^n} + A^n])$ and $\psi(\hat{s}) = ([s_{A^n} + A^n])$, where $\hat{r} = ([r_I + I]) \in \hat{R}_F$ and $\hat{s} = ([s_J + J]) \in \hat{R}_{F_l}$ are both isomorphisms by the above (ii). Thus we have

Lemma 1.2. *There is a commutative diagram;*

$$\begin{array}{ccccc} R & = & R & = & R \\ \downarrow & & \downarrow & & \downarrow \\ \hat{R}_F & \xrightarrow[\cong]{\varphi} & \hat{R} & \xrightarrow[\cong]{\psi} & \hat{R}_{F_l}, \end{array}$$

where the vertical maps are all natural inclusions. All maps are (R, R)-bihomomorphisms.

Lemma 1.3. (1) *$\hat{R}/R$ is torsion-free and injective as right and left R-modules.*

(2) *$\hat{R}$ is torsion-free as right and left R-modules. Especially, $\hat{R}$ and $\hat{R}/R$ are both flat as right and left R-modules.*

Proof. (1) In view of Proposition A.3 in the appendix, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & \hat{R}_{F_l} & \longrightarrow & \hat{R}_{F_l}/R & \longrightarrow & 0 \\ & & \| & & \wr\| & & \|\wr & & \\ 0 & \longrightarrow & R & \longrightarrow & \mathrm{Ext}(K, R) & \to & \mathrm{Ext}(Q, R) & \longrightarrow & 0 . \end{array}$$

$\mathrm{Ext}(Q, R)$ is a right Q-module. So it is R-injective and R-torsion free. By

Lemma 1.2, so is $\hat{R}/R$. By symmetry, $\hat{R}/R$ is torsion-free and injective as left R-modules. The second assertion is obvious, because R is hereditary.

Lemma 1.4. *Let M be a right $\hat{R}$-module. If M is $\hat{R}$-injective, then it is R-injective.*

Proof. By Lemma 1.3, $\mathrm{Tor}_n(N, \hat{R})=0$ for any right R-module N and any $n\geqq 1$. Thus the lemma follows from Proposition 4.1.3 of [1, Chap. VI].

From the exact sequence $0\to R\to Q\to K\to 0$, we get an exact sequence $0\to \hat{R}\to Q\otimes\hat{R}\to K\otimes\hat{R}\to 0$.

Lemma 1.5. (1) *$M\otimes\hat{R}\cong M$ for any torsion right R-module M. So M is a right $\hat{R}$-module. Especially, $K\cong K\otimes\hat{R}\cong(Q\otimes\hat{R})/\hat{R}$.*

(2) *$Q\otimes\hat{R}$ is injective and torsion-free as right and left R-modules, and $Q\otimes\hat{R}$ is the injective hull of $\hat{R}$ as left and right R-modules.*

Proof. From the exact sequence $0\to R\to\hat{R}\to\hat{R}/R\to 0$, we get the exact sequence $\mathrm{Tor}(M, \hat{R}/R)\to M\otimes R\to M\otimes\hat{R}\to M\otimes\hat{R}/R$. By Lemma 1.3, $\mathrm{Tor}(M, \hat{R}/R)=0=M\otimes R/\hat{R}$. Thus $M\cong M\otimes\hat{R}$.

(2) By Proposition A.9 and Lemma 1.2, $J(\hat{R})=A\hat{R}=\hat{R}A$. Thus we have $(Q\otimes\hat{R})A=(Q\otimes\hat{R}A)=Q\otimes A\hat{R}=Q\otimes\hat{R}$. This means $Q\otimes\hat{R}$ is divisible as right R-modules and so it is R-injective. To prove that $Q\otimes\hat{R}$ is torsion-free as right R-modules, let $x=c^{-1}\otimes\hat{r}$ be any element in $Q\otimes\hat{R}$, where c is a regular element in R and $\hat{r}=([r_n+A^n])$. If $xA^m=0$ for some m. Then $(1\otimes\hat{r})A^m=0$ and $\hat{r}A^m=0$. This means $r_lA^m\subseteqq A^l$ for every l and $r_l\in A^{l-m}$ $(l=m+1, m+2, \cdots)$. Write $\hat{s}=([r_l+A^{l-m}])$ $(l=m+1, m+2, \cdots)$ is zero in $\hat{R}$. Clearly $\hat{r}=\hat{s}$. Thus $Q\otimes\hat{R}$ is torsion-free as right R-modules. It is clear that $Q\otimes\hat{R}$ is torsion-free and injective as left R-modules. To prove that $Q\otimes\hat{R}$ is the R-injective hull of $\hat{R}$, we consider the exact sequence $0\to\hat{R}\to Q\otimes\hat{R}\to K\to 0$. Since K is torsion and $Q\otimes\hat{R}$ is torsion-free, $Q\otimes\hat{R}$ is an essential extension of $\hat{R}$ as right and left R-modules. Hence $Q\otimes\hat{R}$ is the injective hull of $\hat{R}$ as right and left R-modules.

Lemma 1.6. *Let M be a right $\hat{R}$-module such that it is torsion-free and injective as R-modules. Then M is $\hat{R}$-injective.*

Proof. We let E be the $\hat{R}$-injective hull of M. Then we have $E=M\oplus N$ for some R-submodule N of E. By Lemma 1.4, E is R-injective. So N is also R-injective. Write $N=\Sigma\oplus N_\alpha$, where N_α are uniform and injective right R-modules. If N_α is torsion for some α, then it is an $\hat{R}$-module by Lemma 1.5. Thus we have $M\subseteqq M\oplus N_\alpha\subseteqq E$. This is a contradiction. So N_α are all torsion-free as R-modules and hence $\bar{E}=E/M$ is torsion-free. $\bar{E}$ is R-injective, because E is R-injective. It follows that $\bar{E}$ is embeddable in a direct sum of Q. From the exact sequence $0\to R\to\hat{R}\to\hat{R}/R\to 0$, we have the following diagram with exact rows and colums:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] \\ 0=\mathrm{Tor}(\Sigma\oplus Q/\bar{E}, R) \arrow[r] & \bar{E}\otimes R \arrow[r] \arrow[d] & (\Sigma\oplus Q)\otimes R \arrow[d] \\ 0=\mathrm{Tor}(\Sigma\oplus Q/\bar{E}, \hat{R}) \arrow[r] & \bar{E}\otimes\hat{R} \arrow[r] & (\Sigma\oplus Q)\otimes\hat{R}. \end{tikzcd}$$

By Proposition A.10, the right singular ideal $Z_{\hat{R}}(\hat{R})$ of $\hat{R}$ is zero and so $Z_{\hat{R}}(Q\otimes\hat{R})=0$. It follows that $Z_{\hat{R}}((\Sigma\oplus Q)\otimes\hat{R})=0$. Thus $Z_{\hat{R}}(\bar{E})=\bar{E}\cap Z_{\hat{R}}((\Sigma\oplus Q)\otimes\hat{R})=0$. On the other hand, $Z_{\hat{R}}(\bar{E})=\bar{E}$. This means $\bar{E}=0$, from which we have M is $\hat{R}$-injective.

We know from Lemmas 1.5 and 1.6 that $Q\otimes\hat{R}$ is the injective hull of $\hat{R}$ as right and left $\hat{R}$-modules. Thus $Q\otimes\hat{R}$ is the maximal right and left quotient ring of $\hat{R}$ by 1. +2. Theorem of [3, p. 69]. We denote the ring $Q\otimes\hat{R}$ by $\hat{Q}$. From the exact sequence $0\to R\to\hat{R}\to\hat{R}/R\to 0$, we get the exact sequence $0\to Q\otimes R\to Q\otimes\hat{R}$. Thus we may identify $q\otimes 1$ with q in $Q\otimes\hat{R}$, where $q\in Q$. The exact sequence $0\to R\to Q\to K\to 0$ induces the following exact sequence $\mathrm{Hom}(Q, M)\to M\to\mathrm{Ext}(K, M)\to\mathrm{Ext}(Q, M)$ for any right R-module M. Any indecomposable, injective right R-module is a homomorphic image of Q and any injective right R-module is a direct sum of indecomposable, injective right R-modules. So M is reduced, i.e., it has no nonzero injective submodules, if and only if $\mathrm{Hom}(Q, M)=0$. M is called *cotorsion* if $\mathrm{Ext}(Q, M)=0$.

Lemma 1.7. *$Tor_1^{\hat{R}}(M, \hat{Q})=0$ for any right $\hat{R}$-module M.*

Proof. It is enough to prove that any finitely generated left $\hat{R}$-submodule of $\hat{Q}$ is $\hat{R}$-projective. To prove this let $\hat{R}x_1+\cdots+\hat{R}x_n$ be any finitely generated $\hat{R}$-submodule of $\hat{Q}$. Write $x_i=c^{-1}\otimes\hat{r}_i$, where c is a regular element in R and $\hat{r}_i\in\hat{R}$. $c^{-1}A^n\subseteq R$ for some n. Thus we have $x_iA^n=(c^{-1}\otimes\hat{r}_i)A^n\subseteq(c^{-1}\otimes\hat{R})A^n=(c^{-1}A^n\otimes\hat{R})\subseteq\hat{R}$ and so $x_id\in\hat{R}$ for any regular element d in A^n. Thus we have $\sum_{i=1}^n\hat{R}x_i\cong\sum_{i=1}^n\hat{R}x_id$ which is contained in $\hat{R}$. Hence $\sum_{i=1}^n\hat{R}x_i$ is $\hat{R}$-projective by Proposition A.10.

Since A is invertible, $\dim A^{-1}/R=\dim R/A$ (dim denotes the (right) Goldie dimension). Clearly socle $K=A^{-1}/R$. Thus we have $k=\dim R/A=\dim K$. Write $K=\sum_{i=1}^k\oplus D_i$, where D_i are uniform, injective, torsion right R-modules. By periodicity theorem ([16] and also [4]), there exists a homomorphism f: $D_i\to D_j$ such that $\mathrm{Ker}\, f$ is zero or finite length.

Lemma 1.8. *$\hat{Q}$ is a simple arintian ring and $\dim_{\hat{R}}\hat{R}=\dim R/A$.*

Proof. Firstly we shall prove that $\hat{Q}$ is a semi-simple artinian ring. To prove this let I be any right ideal of $\hat{Q}$. It is a right Q-module. So it is torsion-free and injective as right R-modules. Since I is a right $\hat{R}$-module, it is $\hat{R}$-injective by Lemma 1.6 and so we have $I\oplus L=Q$ for some right $\hat{R}$-submodule L of $\hat{Q}$. It follows that $I\oplus L\hat{Q}=\hat{Q}$. This means $\hat{Q}$ is a semi-simple artinian ring. Next we shall prove that $k=\dim R/A=\dim_{\hat{R}} \hat{R}$. Let $K=\sum_{i=1}^{k}\oplus D_i$. By Proposition A.3, $\hat{R}=\mathrm{Hom}(K, K)=\sum_{i=1}^{k}\oplus \mathrm{Hom}(K, D_i)=\sum_{i=1}^{k}\oplus e_i\hat{R}$, where $e_i\in\hat{R}$ and $e_i=e_i^2$. Suppose that $e_i\hat{R}$ is not uniform for some i. Then $e\hat{Q}=X\oplus Y$ for some nonzero right ideals X, Y of $\hat{Q}$, where $e=e_i$. Since X is a direct summand of $\hat{Q}$, we have $X=O_{\hat{Q}}(g)=\{x\in\hat{Q}\,|\,gx=0\}$ for some idempotent g in $\hat{Q}$. There is a regular element c in R such that $cg\in\hat{R}$. Thus we have $X=O_{\hat{Q}}(cg)$. On the other hand, $e\hat{R}=e\hat{Q}\cap\hat{R}\supseteqq X\cap\hat{R}=O_{\hat{Q}}(cg)\cap\hat{R}=O_{\hat{R}}(cg)$. Thus, by Proposition A.10, $O_{\hat{R}}(cg)$ is a direct summand of $\hat{R}$. Write $O_{\hat{R}}(cg)=f\hat{R}$ for some idempotent f in $\hat{R}$. It follows that $e\hat{R}=f\hat{R}\oplus((1-f)\hat{R}\cap e\hat{R})$ and that $e\hat{R}=f\hat{R}$, because $e\hat{R}$ is indecomposable by Proposition A.6. So $e\hat{Q}=f\hat{Q}=X$, which is a contradiction. Therefore each $e_i\hat{R}$ is a uniform right ideal of $\hat{R}$ and thus $\dim_{\hat{R}} \hat{R}=\dim R/A$. Finally we shall prove that $\hat{Q}$ is a simple artinian ring. To prove this let D_i, D_j be any indecomposable, injective, torsion direct summands of K. As was shown in before the lemma, there exists an exact sequence $0\to\mathrm{Ker}f\to D_i\to D_j\to 0$ and $\mathrm{Ker}f$ is zero or finite length. Applying $\mathrm{Hom}(K,\ )$ to the exact sequence, we get the exact sequence $\mathrm{Hom}(K, \mathrm{Ker}f)\to\mathrm{Hom}(K, D_i)\to\mathrm{Hom}(K, D_j)$. The first term is zero, since $\mathrm{Ker}f$ is reduced and K is injective. Thus we have the exact sequence $0\to e_i\hat{R}\to{}_j\hat{R}$. Applying $\otimes_{\hat{R}}\hat{Q}$ to the sequence we get, by Lemma 1.7, the exact sequence $0\to e_i\hat{R}\otimes_{\hat{R}}\hat{Q}\to e_j\hat{R}\otimes_{\hat{R}}\hat{Q}$. But $e_j\hat{R}\otimes_{\hat{R}}\hat{Q}$ is a simple right $\hat{Q}$-module and so $e_i\hat{R}\otimes_{\hat{R}}\hat{Q}\cong e_j\hat{R}\otimes_{\hat{R}}\hat{Q}$. Now, since $\hat{R}=\sum_{i=1}^{k}\oplus e_i\hat{R}$, we have $\hat{Q}=\sum_{i=1}^{k}\oplus e_i\hat{Q}$ and $e_i\hat{Q}\cong e_j\hat{Q}$ for any pair i, j. This means Q is a simple artinian ring.

By Proposition A.9 and Lemma 1.2, $J(\hat{R})=A\hat{R}=\hat{R}A$. We denote it by $\hat{A}$. Clearly $\hat{A}^n=A^n\hat{R}=\hat{R}A^n$ for every n.

Lemma 1.9. (1) *Any ideal B of $\hat{R}$ contains a power of $\hat{A}$.*
(2) *$\hat{R}$ is a bounded hnp-ring with quotient ring $\hat{Q}$.*

Proof. (1) Since $\hat{Q}$ is a simple artinian ring, we have $\hat{Q}=\hat{Q}B\hat{Q}$. Write $1=\Sigma q_ib_ip_i$, where $q_i\in\hat{Q}$, $b_i\in B$ and $p_i\in\hat{Q}$. There exists a natural number l such that $A^lq_i\subseteq\hat{R}$. Write $p_i=\Sigma x_{ij}\otimes\hat{r}_{ij}$, where $x_{ij}\in Q$ and $\hat{r}_{ij}\in\hat{R}$. Again $x_{ij}A^m\subseteqq R$ for some m, and so $p_iA^m=(\Sigma x_{ij}\otimes\hat{r}_{ij})A\subseteqq(\Sigma x_{ij}\otimes\hat{R})A^m=\Sigma x_{ij}A^m\otimes\hat{R}\subseteqq\hat{R}$. Thus $B\supseteqq A^l(\Sigma q_ib_ip_i)A^m=A^{l+m}$ and so $B\supseteqq\hat{A}^{l+m}$.

(2) Since $A^n\neq 0$ for every n, $\hat{R}$ is a prime ring by (1). Let I be an essential right ideal of $\hat{R}$. Then $IQ=\hat{Q}$. By the same way as in (1), I contains a power of $\hat{A}$. So $\hat{R}$ is right bounded and I is a finitely generated right ideal of

$\hat{R}$, because $\hat{R}/\hat{A}^n \simeq R/A^n$, by Proposition A.9, which is an artinian ring for every n and A^n is finitely generated. Since $\dim_{\hat{R}} \hat{R}=k$, $\hat{R}$ is right noetherian. By symmetry, $\hat{R}$ is left bounded and left noetherian. Thus it follows that $\hat{R}$ is hereditary by Proposition A.10. Clearly $\hat{Q}$ is the classical quotient ring of $\hat{R}$.

Let $k=\dim R/A$, let $A=M_1 \cap \cdots \cap M_p$, where $M_1, \cdots, M_p$ is a cycle. We denote the $\dim R/M_i$ by k_i. Then $k=k_1+\cdots+k_p$, because $R/A \simeq R/M_1 \oplus \cdots \oplus R/M_p$. Let S_i be a simple right R-module such that $S_i M_i=0$.

Lemma 1.10. *$\hat{R}$ has the following decomposition*:

$$\hat{R} = (\overbrace{e_1\hat{R} \oplus \cdots \oplus e_1\hat{R}}^{k_1}) \oplus (\overbrace{e_2\hat{R} \oplus \cdots \oplus e_2\hat{R}}^{k_2}) \oplus \cdots \oplus (\overbrace{e_p\hat{R} \oplus \cdots \oplus e_p\hat{R}}^{k_p})$$

such that each $e_i\hat{R}$ is a uniform right ideal of $\hat{R}$, $e_i^2=e_i$ and $e_i\hat{R}/e_i\hat{A} \simeq S_i$ $(1 \leqq i \leqq p)$.

Proof. By Lemma 6, Theorems 7 and 8 of [4], we have

$$O_l(M_2)/R = O_r(M_1)/R = \sum^{k_1} \oplus S_2\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$O_l(M_p)/R = O_r(M_{p-1})/R = \sum^{k_{p-1}} \oplus S_p\,,$$
$$O_l(M_1)/R = O_r(M_p)/R = \sum^{k_p} \oplus S_1\,.$$

It is clear that socle $K=A^{-1}/R=O_l(M_2)/R \oplus \cdots \oplus O_l(M_p)/R \oplus O_l(M_1)/R$ (cf. Lemma 4.8 of [8]). Thus we get the following decomposition:

$$K = \overbrace{D_1 \oplus \cdots \oplus D_1}^{k_1} \oplus \overbrace{D_2 \oplus \cdots \oplus D_2}^{k_2} \oplus \cdots \oplus \overbrace{D_p \oplus \cdots \oplus D_p}^{k_p}\,,$$

where D_i are injective, uniform and torsion right R-modules such that socle $D_i \simeq S_{i+1}$ $(1 \leqq i \leqq p-1)$ and socle $D_p \simeq S_1$. By proposition A.3, we get $\hat{R}= \mathrm{Hom}(K, K) = \sum_{i=1}^{p} \oplus \sum^{k_i} \oplus \mathrm{Hom}(K, D_i) = \sum_{i=1}^{p} \oplus \sum^{k_i} \oplus e_i\hat{R}$, where $e_i\hat{R}$ are uniform right ideals of $\hat{R}$ and e_i are idempotents in $\hat{R}$. If $i \neq j$, then $e_i\hat{R}$ is non-isomorphic to $e_j\hat{R}$ by Proposition A.6. We consider the factor ring;

$$\hat{R}/\hat{A} = (e_1\hat{R}/e_1\hat{A} \oplus \cdots \oplus e_1\hat{R}/e_1\hat{A}) \oplus \cdots \oplus (e_p\hat{R}/e_p\hat{A} \oplus \cdots \oplus e_p\hat{R}/e_p\hat{A})\,.$$

$\hat{R}/\hat{A}$ is a right R/A-module. So it is completely reducible. Further $k= \dim R/A = \dim \hat{R}/\hat{A} = \dim_{\hat{R}} \hat{R}$. Thus each $e_i\hat{R}/e_i\hat{A}$ is a simple right R-module. For each i, we consider the exact sequence

$$(*) \qquad 0 \to e_i\hat{A} \to e_i\hat{R} \to e_i\hat{R}/e_i\hat{A} \to 0\,.$$

Applying Tor(, K) to (*), we have $\mathrm{Tor}(e_i/\hat{R}, K) \to \mathrm{Tor}(e_i\hat{R}/e_i\hat{A}, K) \to e_i\hat{A} \otimes K \to$

$e_i\hat{R}\otimes K\to e_i\hat{R}/e_i\hat{A}\otimes K$. The first and last terms are zero, because $e_i\hat{R}$ is R-flat by Lemma 1.3, $e_i\hat{R}/e_i\hat{A}$ is torsion and K is divisible. Further, $\mathrm{Tor}(e_i\hat{R}/e_i\hat{A}, K)\simeq e_i\hat{R}/e_i\hat{A}$ by Exersise 2 of [22, p. 81]. Thus we have the exact sequence

$$(**)\qquad 0\to e_i\hat{R}/e_i\hat{A}\to e_i\hat{A}\otimes K\to e_i\hat{R}\otimes K(\simeq D_i)\to 0\,.$$

Again, applying Hom(Q,) to (*), we get $0=\mathrm{Hom}(Q, e_i\hat{R}/e_i\hat{A})\to\mathrm{Ext}(Q, e_i\hat{A})\to\mathrm{Ext}(Q, e_i\hat{R})=0$, because $e_i\hat{R}$ is cotorsion. Hence $\mathrm{Ext}(Q, e_i\hat{A})=0$, from which we have $e_i\hat{A}$ is a reduced, cotorsion and uniform right ideal of $\hat{R}$. It follows that $e_i\hat{A}\otimes K\simeq D_j$ for some j by Proposition A.6. But, by periodicity theorem, if $i\neq 1$, then $j=i-1$, and if $i=1$, then $j=p$. Hence $e_i\hat{R}/e_i\hat{A}\simeq S_i$ for any i.

Lemma 1.11. *Under the same notations as in Lemma* 1.10, $\hat{A}^p=\hat{a}\hat{R}=\hat{R}\hat{a}$ *for some* $\hat{a}\in\hat{A}^p$.

Proof. We consider the decomposition;

$$\hat{R}/\hat{A}^{p+1}=(e_1\hat{R}/e_1\hat{A}^{p+1}\oplus\cdots\oplus e_1\hat{R}/e_1\hat{A}^{p+1})\oplus\cdots\oplus(e_p\hat{R}/e_p\hat{A}^{p+1}\oplus\cdots\oplus e_p\hat{R}/e_p\hat{A}^{p+1})\,.$$

Since $\hat{A}$ is invertible, $\dim_{\hat{R}}\hat{R}=\dim R/A=\dim\hat{R}/\hat{A}^{p+1}$. Thus each $e_i\hat{R}/e_i\hat{A}^{p+1}$ is a uniform R-module and so it is a uniserial R-module by Lemma 2 of [16]. Clearly the members of chain $e_i\hat{R}>e_i\hat{A}>\cdots>e_i\hat{A}^{p+1}$ are only $\hat{R}$-submodules of $e_i\hat{R}$ containing $e_i\hat{A}^{p+1}$. Especially, socle $e_i\hat{R}/e_i\hat{A}^{p+1}=e_i\hat{A}^p/e_i\hat{A}^{p+1}$ for each i. Periodicity theorem says that $e_i\hat{R}/e_i\hat{A}\simeq e_i\hat{A}^p/e_i\hat{A}^{p+1}$. Thus $\hat{R}/\hat{A}\simeq\hat{A}^p/\hat{A}^{p+1}$ and $\hat{A}^p/\hat{A}^{p+1}=[\hat{a}+\hat{A}^{p+1}]\hat{R}$ for some $\hat{a}\in\hat{A}^p$. It follows that $\hat{A}^p=\hat{a}\hat{R}+\hat{A}^{p+1}$. By Nakayama's Lemma, $\hat{A}^p=\hat{a}\hat{R}$ and, by symmetry, $\hat{A}^p=\hat{R}\hat{b}$ for some $\hat{b}\in\hat{A}^p$. But, by the same way as in [6, p. 37], we have $\hat{A}^p=\hat{R}\hat{a}$.

From Lemmas 1.2, 1.9, 1.10, 1.11 and Proposition A.9 we have the first theorem mentioned in the introduction.

Theorem 1.1. *Let R be an hnp-ring with quotient ring Q and let $A=M_1\cap\cdots\cap M_p$ be a maximal invertible ideal of R, where M_i are idempotent maximal ideals of R and $M_1,\cdots,M_p$ is a cycle. Then*

(1) *$\hat{R}$ is a bounded hnp-ring with quotient ring $Q\otimes\hat{R}$. $J(\hat{R})=A\hat{R}=\hat{R}A$ and $\hat{A}^p$ is a principal right and left ideal of $\hat{R}$.*

(2) *$\hat{R}$ has the following decomposition*:

$$\hat{R}=(\overbrace{e_1\hat{R}\oplus\cdots\oplus e_1\hat{R}}^{k_1})\oplus(\overbrace{e_2\hat{R}\oplus\cdots\oplus e_2\hat{R}}^{k_2})\oplus\cdots\oplus(\overbrace{e_p\hat{R}\oplus\cdots\oplus e_p\hat{R}}^{k_p})$$

such that each $e_i\hat{R}$ is a uniform right ideal of $\hat{R}$, e_i is an idempotent in $\hat{R}$ and $e_i\hat{R}/e_i\hat{A}$ is a simple right R-module which is annihilated by M_i, where $k_i=\dim R/M_i$.

2. Applications

In this section, we shall prove, by using Theorem 1.1, that any $\hat{R}$-module has a basic submodule, and shall characterize the structure of indecomposable $\hat{R}$-modules. By Theorem 1.1, $\hat{R}=(e_1\hat{R}\oplus\cdots\oplus e_1\hat{R})\oplus\cdots\oplus(e_p\hat{R}\oplus\cdots\oplus e_p\hat{R})$, where e_i are uniform idempotents in $\hat{R}$. Then $\hat{Q}=(e_1\hat{Q}\oplus\cdots\oplus e_1\hat{Q})\oplus\cdots\oplus(e_p\hat{Q}\oplus\cdots\oplus e_p\hat{Q})$. So $\hat{Q}/\hat{R}=\sum_{i=1}^{p}\oplus\sum^{k_i}\oplus e_i\hat{Q}/e_i\hat{R}$. Since $K\cong\hat{Q}/\hat{R}$ and $\dim K=\dim_{\hat{R}}\hat{R}$, each $e_i\hat{Q}/e_i\hat{R}$ is a uniform, injective and torsion right R-module. By Theorem 4 of [15], the set of right R-submodules of $e_i\hat{Q}/e_i\hat{R}$ is linearly ordered by inclusion. In this case, the set of right R-submodules of $e_i\hat{Q}/e_i\hat{R}$ is $\{e_i\hat{A}^{-n}/e_i\hat{R}\,|\,n=0,1,2,\cdots\}$. Thus $e_i\hat{R}<e_i\hat{A}^{-1}<\cdots<e_i\hat{A}^{-n}<\cdots$ are only proper right $\hat{R}$-submodules of $e_i\hat{Q}$ containing $e_i\hat{R}$.

Lemma 2.1. *Under the same notations as in Theorem* 1.1, *any torsion-free and uniform right $\hat{R}$-module is isomorphic to $e_i\hat{Q}$ or $e_i\hat{R}$ for some i.*

Proof. Let M be a torsion-free and uniform right $\hat{R}$-module. If M is $\hat{R}$-injective, then it is isomorphic to $e_i\hat{Q}$ for any i. If M is not injective, then it is reduced. Since $M=M\hat{R}=M(\sum_{i=1}^{p}\oplus\sum^{k_i}\oplus e_i\hat{R})$, we have $0\neq Me_j\hat{R}$ for some j and $0\neq xe_j\hat{R}$ for some $x\in M$. There exists an epimorphism $f\colon e_j\hat{R}\to xe_j\hat{R}$. If $\operatorname{Ker} f$ is non zero, then $e_j\hat{R}/\operatorname{Ker} f$ is torsion. But $xe_j\hat{R}$ is torsion-free. This is a contradiction. Thus f is an isomorphism. Consider the diagram

$$\begin{array}{ccc} 0 \to & xe_j\hat{R} & \to M \\ & \downarrow f^{-1} & \\ & e_j\hat{R} & \\ & \downarrow & \\ & e_j\hat{Q} & \end{array}$$

Since $e_j\hat{Q}$ is injective, f^{-1} is extended to $g\colon M\to e_j\hat{Q}$. It is clear that g is a monomorphism and $g(M)$ is a proper $\hat{R}$-submodule of $e_j\hat{Q}$ containing $e_j\hat{R}$, because M is reduced. Thus $g(M)=e_j\hat{A}^{-n}$ for some n. Since $e_j\hat{A}^{-n}/e_j\hat{A}^{-n+1}$ is a simple right R-module, $e_j\hat{A}^{-n}/e_j\hat{A}^{-n+1}=[\hat{a}+e_j\hat{A}^{-n+1}]\hat{R}$ and $e_j\hat{A}^{-n}=\hat{a}\hat{R}+(e_j\hat{A}^{-n})\hat{A}$ for some $\hat{a}\in e_j\hat{A}^{-n}$. By Nakayama's Lemma, $e_j\hat{A}^{-n}=\hat{a}\hat{R}$. Since $\hat{a}\hat{R}$ is $\hat{R}$-projective, it is isomorphic to a direct summand of $\hat{R}$ and so it is reduced, torsion-free, uniform and cotorsion $\hat{R}$-module. Thus, by Proposition A.6, $\hat{a}\hat{R}\cong\operatorname{Hom}(K,D_i)\cong e_i\hat{R}$ for some uniform, torsion, injective right R-module D_i. Hence $M\cong e_i\hat{R}$, as desired.

An $\hat{R}$-submodule N of a right $\hat{R}$-module M is called *pure* if any finite system of linear equations $\sum_j x_j\hat{r}_{ij}=s_i\in N$ is solvable in M, where $\hat{r}_{ij}\in\hat{R}$, then it possesses a solution in N. By the remark to Theorem 3.6 of [20], N is pure in M if and only if $Mc\cap N=Nc$ for every regular element c in $\hat{R}$. By using the above result, Theorem 10 of [16], Lemma 2 of [17] and Lemma 2.1, the proof of the following two lemmas proceeds just like that of Lemmas 3.4 and 3.5 of [11], respectively.

Lemma 2.2. *Any non injective right $\hat{R}$-module contains a non zero pure, uniform and cyclic right $\hat{R}$-submodule.*

Lemma 2.3. *Let M be a right $\hat{R}$-module and let N be a pure $\hat{R}$-submodule such that M/N is not injective. Then there exists an element $y \in M$ such that $N \cap y\hat{R}=0$ and $N \oplus y\hat{R}$ is pure in M.*

An $\hat{R}$-submodule B of a right $\hat{R}$-module M is said to be *basic* if it satisfies the following conditions:

(i) B is a direct sum of uniform, cyclic right $\hat{R}$-modules,
(ii) B is pure in M, and
(iii) M/B is an injective $\hat{R}$-module.

From Lemmas 2.2 and 2.3, we have

Theorem 2.1. *Any right $\hat{R}$-module possesses a basic $\hat{R}$-submodule.*

REMARK. Any two basic submodules of a right $\hat{R}$-module are isomorphic (cf. the remark to Theorem 3 of [18])

Corollary 2.1. *$\hat{R}$ is a block lower triangular matrix ring over D/M, where D is a discrete valuation ring with maximal ideal M* (cf. Theorem 2 of [18]).

Let R be an hnp-ring and let A be a maximal invertible ideal of R. A right R-module M is *A-primary* if any element in M is annihilated by a power of A.

Lemma 2.4. *Let R be an hnp-ring, let A be a maximal invertible ideal of R and let M be a right R-module. Then*

(1) *M is A-primary if and only if it is a right $\hat{R}$-module and is torsion as right $\hat{R}$-modules.*

(2) *If M is A-primary, then M is R-injective if and only if it is $\hat{R}$-injective.*

Proof. If M is A-primary, then $M \cong M \otimes R_A$ by the same way as in Lemma 1.5 and it is torsion as right R_A-modules. Thus it follows that M is R-injective if and only if it is R_A-injective by Proposition 3.11 of [23, p. 232]. So we may assume that $R=R_A$ and $J(R)=A$.

(1) is obvious, since $\hat{A}^n=A^nR=\hat{R}A^n$ for every n.

(2) Sufficiency follows from Lemma 1.4. To prove necessity, suppose that M is torsion and R-injective. Let E be any essential extension of M as right $\hat{R}$-modules. Any essential right ideal of $\hat{R}$ contains a power of $\hat{A}$. This means E/M is torsion as right R-modules and so E is a torsion right R-module.

By assumption we have a decomposition $\hat{R}=M\oplus N$, where N is a right R-module. But N is a right $\hat{R}$-module by (1). Thus $N=0$ and $M=E$. Hence M is $\hat{R}$-injective.

Lemma 2.5. *Under the same notations as in Theorem* 1.1, *any reduced, uniform and torsion right* $\hat{R}$*-module is isomorphic to* $e_i\hat{R}/e_i\hat{A}^n$ *for some* i *and some* n.

Proof. By the same way as in Lemma 1.11, $e_i\hat{R}/e_i\hat{A}^n$ is a uniserial, torsion right R-module of length n and socle $e_i\hat{R}/e_i\hat{A}^n=e_i\hat{A}^{n-1}/e_i\hat{A}^n$ for each i. So, by the periodicity theorem, we have $\{e_1\hat{A}^{n-1}/e_1\hat{A}^n, \cdots, e_p\hat{A}^{n-1}/e_p\hat{A}^n\}=\{S_1, \cdots, S_p\}$. Now let M be any reduced, uniform and torsion right $\hat{R}$-module and let socle $M\cong S_j$. Then, by Lemma 2 of [16], M is uniserial. Suppose that the length of M is n, then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & S_j \arrow[r] \arrow[d, phantom, "\wr\|"] & M \\ & e_i\hat{A}^{n-1}/e_i\hat{A}^n \arrow[d, hook] & \\ & E & \end{tikzcd}$$

for some i, where E is the injective hull of $e_i\hat{A}^{n-1}/e_i\hat{A}^n$. The monomorphism is extended $f\colon M\to E$. Clearly f is also a monomorphism. Hence $M\cong f(M)=e_i\hat{R}/e_i\hat{A}^n$, because $e_i\hat{R}/e_i\hat{A}^n$ is the only $\hat{R}$-submodule of E which is of length n.

Under the same notations as in §1, we obtained the exact sequence (cf. Lemma 1.10) $0\to S_i\to e_i\hat{A}\otimes K\to D_i\to 0$ and $D_{i-1}\cong e_i\hat{A}\otimes K\ (2\leqq i\leqq p)$, $D_p\cong e_1\hat{A}\otimes K$. By Proposition A.6, we have $f_i\colon e_i\hat{R}\cong e_{i+1}\hat{A}\ (1\leqq i\leqq p-1)$ and $f_p\colon e_p\hat{R}\cong e_1\hat{A}$. These f_j's induce the isomorphisms

$$f_i^{(n)}\colon\ e_i\hat{R}/e_i\hat{A}^n\cong e_{i+1}\hat{A}/e_{i+1}\hat{A}^{n+1}\,,\quad f_p^{(n)}\colon\ e_p\hat{R}/e_p\hat{A}^n\cong e_1\hat{A}/e_1\hat{A}^{n+1}$$

for every n. Thus we have the following ascending chains:

$$e_1\hat{R}/e_1\hat{A}\overset{f_1^{(1)}}{\cong} e_2\hat{A}/e_2\hat{A}^2\subseteqq e_2\hat{R}/e_2\hat{A}^2\subseteqq\cdots\subseteqq e_p\hat{R}/e_p\hat{A}^p\overset{f_p^{(p)}}{\cong} e_1\hat{A}/e_1\hat{A}^{p+1}\subseteqq e_1\hat{R}/e_1\hat{A}^{p+1}\subseteqq$$
$$\cdots\subseteqq e_i\hat{R}/e_i\hat{A}^{pn+i}\overset{f_i^{(pn+i)}}{\cong} e_{i+1}\hat{A}/e_{i+1}\hat{A}^{pn+i+1}\subseteqq e_{i+1}\hat{R}/e_{i+1}\hat{A}^{pn+i+1}\subseteqq\cdots.$$

We denote the inductive limit of $e_i\hat{R}/e_i\hat{A}^{pn+i}$ by $R(M_1^\infty)$. It is clear that $R(M_1^\infty)$ is a uniform, A-primary right R-module and that the length of it is infinite. Hence $R(M_1^\infty)=E(e_1\hat{R}/e_1\hat{A})$, the injecitve hull of $e_1\hat{R}/e_1\hat{A}$, by Theorem 19 of [4]. Similarly we can define $R(M_j^\infty)$ $(2\leqq j\leqq n)$. Thus we have

Proposition 2.1. *Let* R *be an hnp-ring and let* $A=M_1\cap\cdots\cap M_p$ *be a maximal invertible ideal of* R, *where* $M_1, \cdots, M_p$ *is a cycle. Then* $R(M_1^\infty), \cdots, R(M_p^\infty)$

are only non-isomorphic indecomposable, injective and A-primary R-modules.

REMARK. $R(M_i^\infty)$ are a natural generalization of the typical, divisible, indecomposable and torsion abelian group $Z(p^\infty)$.

Theorem 2.2. *Under the same notations as in Theorem* 1.1, *any indecomposable right $\hat{R}$-module is isomorphic to one of the following modules*:

$$e_i\hat{R}/e_i\hat{A}^n\ (n=1, 2, \cdots)\,,\quad e_i\hat{R}\,,\quad e_i(Q\otimes\hat{R})\,,\quad R(M_i^\infty)\quad (1\leqq i\leqq p)\,.$$

Proof. Let M be an indecomposable right $\hat{R}$-module. Suppose that M is $\hat{R}$-injective. Then it can not be mixed, i.e., it is torsion or torsion-free. If M is torsion, then $M\cong R(M_i^\infty)$ for some i by Lemma 2.4 and Proposition 2.1. If M is torsion-free, then it is isomorphic to $e_i(Q\otimes\hat{R})$. If M is not injective, then it is reduced. Assume that M is torsion-free. Then we have a following pure exact sequence $0\to e_i\hat{R}\to M\to M/e_i\hat{R}\to 0$ for some i by Lemmas 2.1 and 2.2. $M/e_i\hat{R}$ is torsion-free by Lemma 1.5 of [20]. Thus $e_i\hat{R}$ is a direct summand of M by Proposition A.8. Hence $M\cong e_i\hat{R}$. Finally if M is not torsion-free, then it has a uniserial torsion summand by Proposition 2.1 of [19]. Thus M is a uniserial torsion $\hat{R}$-module. By Lemma 2.5, we have $M\cong e_i\hat{R}/e_i\hat{A}^n$ for some i and some n.

Appendix

We shall present, in this section, some results on cotorsion modules over hnp-rings which are obtained by modifying the methods used in the corresponding ones in modules over Dedekind prime rings (cf. [12] and [13]). So we shall omit the proof of these except Proposition A.10. Since Proposition A.10 is a new result, we shall give the proof of it. Let R be an hnp-ring with quotient ring Q and let F be any right additive topology on R. An element m of a right R-module M is said to be *F-torsion* if $O(m)=\{r\in R\,|\,mr=0\}\in F$, and we denote the submodule of F-torsion elements of M by $t_F(M)$ (for short: $t(M)$). If $t(M)=0$, then we say that M is *F-torsion-free*. A right additive topology F on R is called *trivial* if all modules are F-torsion or F-torsion-free. By the same way as in [12, p. 548], F is non-trivial if and only if it consists of essential right ideals of R (This result is true if R is a prime Goldie ring (cf. [14])).

From now on, F denotes a non-trivial right additive topology on R. We put $R_F=\cup I^*(I\in F)$, a ring of quotients of R with respect to F. The family F_l of left ideals J of R such that $R_FJ=R_F$ is a left additive topology on R. We call it *the left additive topology corresponding to F*. F_l is also non-trivial by Proposition 1.1 of [12]. We write $R_{F_l}=\cup{}^*J\,(J\in F_l)$. Clearly $R_F=R_{F_l}$. It is well-known that R_F is R-flat and the inclusion map $R\to R_F$ is an epimorphism. A right R-module M is said to be *F_l-divisible* if $MJ=M$ for every

$J \in F_l$. We can define the concepts of *F_l-torsion* and *F-divisible* for any left R-module.

Proposition A.1. (1) *$t(K)=R_F/R=t_{F_l}(K)$, where $K=Q/R$. Thus $t(K)$ is (F, F_l)-divisible.*

(2) *Let I be an essential right ideal of R. Then $I \in F$ if and only if I^*/R is F_l-torsion* (cf. Proposition 1.4 of [12]).

Following [22], a right R-module D is *F-injective* if $\mathrm{Ext}(R/I, D)=0$ for every $I \in F$.

Proposition A.2. *A right R-module is F-injective if and only if it is F_l-divisible. In particular, $M \otimes R_F$ and $M \otimes t(K)$ are both F-injective for any right R-module M* (cf. Lemma 2.5 of [12]).

For a right R-module M, we define $\hat{M}_{F_l}=\varprojlim M/MJ\,(J \in F_l)$. Then it is a right R_{F_l}-module, where $R_{F_l}=\varprojlim R/J$, which is a ring (cf. [21, §4]).

Proposition A.3. *Let M be an F-torsion-free right R-module. Then there is a commutative diagram*:

$$\begin{tikzcd}
\hat{M}_{F_l} \arrow[r, phantom, "\overset{\alpha}{\cong}"] & \mathrm{Hom}\,(t(K), M \otimes t(K)) \arrow[r, phantom, "\cong"] & \mathrm{Ext}\,(t(K), M) \\
M \arrow[u] \arrow[r, equal] & M \arrow[u] \arrow[r, equal] & M \arrow[u, "\beta"']
\end{tikzcd}$$

Here $\alpha(\hat{m})(q)=m_L \otimes q$, where $\hat{m}=([m_J+MJ]) \in \hat{M}_{F_l}$ and $q \in t(K)$ such that $Lq=0$ and $L \in F_l$. β is the connecting homomorphism induced by the exact sequence $0 \to R \to R_F \to R_F/R \to 0$ (cf. Lemma 2.7 of [12]).

A right R-module G is said to be *F-cotorsion* if $\mathrm{Ext}(R_F, G)=0$. The union of all F_l-divisible sumbodules of a right R-module M is itself F_l-divisible and is denoted by MF^∞;if $MF^\infty=0$, then M is said to be *F-reduced*. From the exact sequence $0 \to R \xrightarrow{i} R_F \to t(K) \to 0$ we derive an exact sequence $\mathrm{Hom}\,(R_F, M) \xrightarrow{i^*} M \to \mathrm{Ext}\,(t(K), M)$ for any right R-module M.

Proposition A.4. (1) *M/MF^∞ is F-reduced.*

(2) $\mathrm{Im}\, i^*=MF^\infty$ (cf. Lemma 1.1 of [13]).

Proposition A.5. *Let G be an F-reduced right R-module. Then G is F-cotorsion if and only if it is F^∞-pure injective in the sense of* [13] (cf. Proposition 1.4 of [13]).

Proposition A.6 (Harrison duality for modules over hnp-rings). *The cor-*

respondence

$$(A^*) \qquad D \to G = \mathrm{Hom}\,(t(K), D)$$

is one-to-one between all F-torsion, F-injective right R-modules D and all F-reduced, F-torsion-free, F-cotorsion right R-modules G. The inverse of (A^*) *is given by the correspondence* $G \to G \otimes t(K)$. *The isomorphism* f: $\mathrm{Hom}\,(t(K), D) \otimes t(K) \to D$ *is given by* $f(x \otimes q) = x(q)$, *where* $x \in \mathrm{Hom}\,(t(K), D)$ *and* $q \in t(K)$ (cf. Theorem 2.2 of [13]).

Proposition A.7. (1) $\mathrm{Ext}\,(t(K), M)$ *is F-reduced and F-cotorsion for every right R-module M.*

(2) *Let G be F-reduced. Then G is F-cotorsion if and only if* $G \cong \mathrm{Ext}\,(t(K), G)$ (cf. Proposition 5.2 of [21] and Lemma 1.2 of [13]).

Proposition A.8. *Let G be F-reduced and F-cotorsion. Then* $\mathrm{Ext}(X, G) = 0$ *for every F-torsion-free right R-module X* (cf. Lemma 1.2 of [13]).

Let M be an F-torsion right R-module. Then M is a right $\hat{R}_F$-module as follows: For any $m \in M$, $\hat{r} = ([r_I + I]) \in \hat{R}_F$, we define $m\hat{r} = mr_J$, where $J = O(m)$. Similarly an F_l-torsion left R-module is a left $\hat{R}_{F_l}$-module. Let $S(t(K))$ be the right socle of $t(K)$. Then it is a left R-module and is F_l-torsion. Thus it is a left $\hat{R}_{F_l}$-module. Let $G = \mathrm{Hom}(t(K), D)$, where D is an F-torsion and F-injective right R-module. From the exact sequence $0 \to S(t(K)) \xrightarrow{j} t(K)$, we have an exact sequence $0 \to \mathrm{Ker}\, j^* \to G \xrightarrow{j^*} \mathrm{Hom}(S(t(K)), D) \to 0$ as right $\hat{R}_{F_l}$-modules.

Proposition A.9. (1) $\mathrm{Ker}\, j^* = \cap GJ$, *where J ranges over all maximal left ideals in* F_l. *Especially* $J(\hat{R}_{F_l}) = \cap \hat{R}_{F_l} J$ (cf. Lemma 2.6 and Corollary 2.7 of [13]).

(2) $R/J \cong \hat{R}_{F_l}/\hat{R}_{F_l} J$ *for every* $J \in F_l$ (cf. Corollary 2.8 of [12]).

By Proposition A.3, $\hat{R}_{F_l} \cong \mathrm{Hom}(t(K), t(K))$ and $t(K)$ is F-torsion and F-injective. So $\hat{R}_{F_l}$ is F-reduced, F-torsion-free and F-cotorsion by Proposition A.6. Let I be any finitely generated right ideal of $\hat{R}_{F_l}$. Then there exists an exact sequence:

$$(\mathrm{A}^{**}) \qquad 0 \to \mathrm{Ker}\, f \to \textstyle\sum_{i=1}^{n} \oplus \hat{R}_{F_l} \xrightarrow{f} I \to 0$$

for some n. Since $\hat{R}_{F_l}$ is F-reduced, $\mathrm{Ker}\, f$ and I are both F-reduced. Applying $\mathrm{Hom}(R_F, \ )$ to (A^{**}), we get the exact sequence $\mathrm{Hom}\,(R_F, I) \to \mathrm{Ext}(R_F, \mathrm{Ker}\, f) \to \mathrm{Ext}(R_F, \sum_{i=1}^{n} \oplus \hat{R}_{F_l}) \to \mathrm{Ext}(R_F, I) \to 0$. But $\mathrm{Hom}(R_F, I) = 0 = \mathrm{Ext}(R_F, \sum_{i=1}^{n} \oplus \hat{R}_{F_l})$, because R_F is F_l-divisible, I is F-reduced and $\hat{R}_{F_l}$ is F-cotorsion. Thus we have $\mathrm{Ext}(R_F, \mathrm{Ker}\, f) = 0$. So $\mathrm{Ker}\, f$ is F-cotorsion. By the same way as in Lemma 1.3, $\hat{R}_{F_l}$ is an F-torsion-free right R-module and so I is also F-torsion-free. It follows from Proposition A.8 that the sequence (A**) splits. Hence I is $\hat{R}_{F_l}$-projective. Thus we have

Proposition A.10. *$\hat{R}_{F_l}$ is a right semi-hereditary ring and so the right singular ideal of $\hat{R}_{F_l}$ is zero.*

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