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VANISHING THEOREMS FOR TYPE $(0, q)$ COHOMOLOGY OF LOCALLY SYMMETRIC SPACES

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1. Introduction

Let G be a connected non-compact semisimple Lie group which admits a finite-dimensional faithful representation. Let K be a maximal compact subgroup of G. We assume that the quotient G/K admits a G-invariant complex structure. We also assume that the complexification G^C of G is simply-connected. Choose a Cartan subgroup T of G such that $T \subset K$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$ denote the complexifications of the (real) Lie algebras $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{h}_0$ of G, K, T, respectively. Let

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \tag{1.1}$$

be a Cartan decomposition of $\mathfrak{g}_0$, let $\mathfrak{p}$ be the complexification of $\mathfrak{p}_0$, let Δ be the set of non-zero roots of $(\mathfrak{g}, \mathfrak{h})$, and let Δ_n, Δ_k denote the set of non-compact, compact roots, respectively. That is $\alpha \in \Delta$ is in Δ_n(or Δ_k) if and only if the corresponding (one-dimensional) root space $\mathfrak{g}_\alpha \subset \mathfrak{p}$ (or $\mathfrak{g}_\alpha \subset \mathfrak{k}$). Choose a system Δ^+ of positive roots compatible with the complex structure on G/K. That is if

$$\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^- \tag{1.2}$$

is a splitting of the complex tangent space at the origin in G/K into holomorphic and anti-holomorphic tangent vector $\mathfrak{p}^+$, $\mathfrak{p}^-$ respectively, then

$$\mathfrak{p}^\pm = \sum_{\alpha \in \Delta^+ \cap \Delta_n} \mathfrak{g}_{\pm\alpha} \tag{1.3}$$

We now fix a discrete subgroup Γ of G such that Γ acts freely on G/K and such that the quotient $X = \Gamma | G/K$ is compact. Thus X is a compact locally symmetric Hermitian domain. Given any finite-dimensional irreducible representation τ of K on a complex vector space, there is associated to τ a sheaf $\theta_\tau \to X$ over X in the following way. Let $E_\tau \to G/K$ be the induced homogeneous C^∞ vector bundle over G/K associated to the principal C^∞ fibration $K \to G \to G/K$. Then, as is well-known, E_τ has a holomorphic structure. We obtain a presheaf by assigning to each open set U in X the abelian group of Γ-invariant holomorphic sections of E_τ on the inverse image $\tilde{U}$ of U in G/K. θ_τ is the sheaf generated by this presheaf. Let $H^q(X, \theta_\tau)$ be the q^{th} cohomology space of X with coefficients in θ_τ. *The main result of this paper is Theorem* 2.3, a general result governing the vanishing of the spaces $H^q(X, \theta_\tau)$ for τ whose highest weight Λ relative to $\Delta_k^+=\Delta_k\cap\Delta^+$ belongs to the set

$$\begin{aligned}(1.4)\qquad \mathcal{F}_0' = \{&\Lambda\in\mathfrak{h}^* \mid \Lambda \text{ is integral},\\ &(\Lambda+\delta, \alpha)\neq 0 \text{ for all } \alpha\in\Delta,\ (\Lambda+\delta, \alpha)>0 \text{ for all } \alpha\in\Delta_k^+\}\end{aligned}$$

and for which the system of positive roots defined by the regular element $\Lambda+\delta$, $\delta=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$, is also compatible with a G-invariant complex structure on G/K. Here (,) denotes the Killing form and the integrality of Λ means that $\frac{2(\Lambda, \alpha)}{(\alpha, \alpha)}$ in an integer for every α in Δ. Other results and applications are given in section 3.

The key point in the proof of Theorem 2.3 is the application of Parthasarathy's new unitarizability criteria for highest weight modules [15]. Theorem 2.3 extends, and implies in particular, results of [3], [4], [5], [6], [7], [9], [10], [11], [15].

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2. Statement and proof of the main result

Let $\Lambda\in\mathcal{F}_0'$ in (1.4) and let τ_Λ be the finite-dimensional irreducible representation of K on a complex vector space V with highest weight Λ, relative to Δ_k^+. Since $\Lambda+\delta$ is regular

$$(2.1)\qquad P^{(\Lambda)} = \{\alpha\in\Lambda \mid (\Lambda+\delta, \alpha)>0\}$$

is a system of positive roots. We shall assume that *every non-compact root* α *in* $P^{(\Lambda)}$ *is totally positive*; i.e. $\alpha+\beta\in P_n^{(\Lambda)}=P^{(\Lambda)}\cap\Delta_n$ for every $\beta\in\Delta_k$ such that $\alpha+\beta\in\Delta$. It then follows, as is well-known, that there exists a G-invariant complex structure on G/K such that the space of holomorphic tangent vectors at the origin is given by $\sum_{\alpha\in P_n^{(\Lambda)}}\mathfrak{g}_\alpha$; cf. (1.3). In general, if $Q\subset\Delta$ we shall write $\langle Q\rangle=\sum_{\alpha\in Q}\alpha$. Put $P_k^{(\Lambda)}=P^{(\Lambda)}\cap\Delta_k$. Let

$$\begin{aligned}(2.2)\qquad &Q_\Lambda = \{\alpha\in\Delta_n^+ = \Delta^+\cap\Delta_n \mid (\Lambda+\delta, \alpha)>0\}\,,\\ &Q_\Lambda' = \Delta_n^+-Q_\Lambda\,,\qquad 2\delta^{(\Lambda)} = \langle P^{(\Lambda)}\rangle\,,\\ &2\delta_n^{(\Lambda)} = \langle P_n^{(\Lambda)}\rangle\,,\qquad 2\delta_k^{(\Lambda)} = \langle P_k^{(\Lambda)}\rangle\,.\end{aligned}$$

Let $|S|$ denote the cardinality of a set S. Our main theorem is

Theorem 2.3. *Let* $\Lambda\in\mathcal{F}_0'$ *as above and assume that every non-compact root in* $P^{(\Lambda)}$ *is totally positive. Suppose* $H^q(\Gamma\backslash G/K, \theta_{\tau_\Lambda})\neq 0$. *Then there exists a* *parabolic subalgebra* $\theta = \mathfrak{u} + \mathfrak{m}$ *of* $\mathfrak{g}$ *containing the Borel subalgebra* $\mathfrak{h} + \sum_{\alpha \in P^{(\Lambda)}} \mathfrak{g}_\alpha$ *with* $\mathfrak{u} =$ *the unipotent radical of* θ *and* $\mathfrak{m} =$ *the reductive part of* θ *such that*

(i) *if* $\theta_{u,n}$ *is the set of non-compact roots in* $\mathfrak{u}$ *then* $q = 2|\theta_{u,n} \cap Q_\Lambda| + |Q'_\Lambda| - |\theta_{u,n}|$

(ii) $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ *for every root* α *in* $\mathfrak{m}$. *In particular, if* $A_\Lambda = \{\alpha \in Q_\Lambda \cup -Q'_\Lambda | (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) > 0\}$, *then* $|A_\Lambda| \leqslant |\theta_{u,n}| = 2|\theta_{u,n} \cap Q_\Lambda| + |Q'_\Lambda| - q$.

REMARK: The conditions imposed on Λ in Theorem 2.3 are the same as those formulated by Parthasarathy in Theorem 1 of [14].

We begin the proof of Theorem 2.3 by first proving

Lemma 2.4. *Let* Λ *be as in the statement of Theorem* 2.3 *and suppose* $H^q(\Gamma | G/K, \theta_{\tau_\Lambda}) \neq 0$. *Then there exists an irreducible unitarizable highest weight* $\mathfrak{g}$*-module* H_π *with respect to the system of positive roots* $\bar{P}^{(\Lambda)} = P_k^{(\Lambda)} \cup -P_n^{(\Lambda)} = \Delta_k^+ \cup -Q_\Lambda \cup Q'_\Lambda$ *with* Δ_k^+ *highest weight* $\mu = \Lambda + \langle Q \rangle$ *where* $Q \subset \Delta_n^+$ *satisfies*

(i) $|Q| = q$,

(ii) $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ *for every* $\alpha \in (Q_\Lambda \cap Q') \cup (Q \cap Q'_\Lambda)$, $Q' = \Delta_n^+ - Q$,

(iii) $|\delta_k - \delta_n| = |\delta_k - \delta_n + \langle Q \rangle|$ *where* $2\delta_k = \langle \Delta_k^+ \rangle$, $2\delta_n = \langle \Delta_n^+ \rangle$. *The representation* π *satisfies* $\pi(\Omega) = (\Lambda, \Lambda + 2\delta)1$ *where* Ω *is the Casimir element of* $\mathfrak{g}$.

REMARK: In the special case when Λ is actually Δ^+ dominant, $P^{(\Lambda)} = \Delta^+$, $\bar{P}^{(\Lambda)} = \Delta_k^+ \cup -\Delta_n^+$, $Q_\Lambda = \Delta_n^+$, $Q'_\Lambda = \phi$, $\delta = \delta^{(\Lambda)}$, and Lemma 2.4 reduces to Lemma 2 of Hotta-Murakami [4]. We now prove Lemma 2.4 by abstracting parts of Parthasarathy's argument in his proof of Theorem 1 of [14]. Let W_G be the subgroup of the Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ generated by reflections with respect to compact roots. One has

$$P^{(\Lambda)} = \Delta_k^+ \cup Q_\Lambda \cup -Q'_\Lambda \tag{2.5}$$

so that

$$\begin{aligned} P_k^{(\Lambda)} &= \Delta_k^+ , & P_n^{(\Lambda)} &= Q_\Lambda \cup -Q'_\Lambda \\ Q_\Lambda &= \Delta_n^+ \cap P_n^{(\Lambda)} , & \delta_n + \delta^{(\Lambda)} &= \delta_k + \langle Q_\Lambda \rangle . \end{aligned} \tag{2.6}$$

Note

$$\delta - \delta^{(\Lambda)} = \delta_n + \delta_k - (\delta_k^{(\Lambda)} + \delta_n^{(\Lambda)}) = \delta_n - \delta_n^{(\Lambda)} . \tag{2.7}$$

Since every non-compact root in $P^{(\Lambda)}$ is totally positive, by assumption, $\sigma P_n^{(\Lambda)} = P_n^{(\Lambda)}$ for every σ in W_G. Already $\sigma \Delta_n^+ = \Delta_n^+$ for σ in W_G. In particular, if κ is the unique element of W_G such that $\kappa \Delta_k^+ = -\Delta_k^+$, then (2.5), (2.6) imply

$$-\kappa P^{(\Lambda)} = \Delta_k^+ \cup -Q_\Lambda \cup Q'_\Lambda \overset{\text{def.}}{=} \bar{P}^{(\Lambda)} \tag{2.8}$$

Now suppose $H^q(\Gamma | G/K, \theta_{\tau_\Lambda}) \neq 0$. Then by Theorem 3 of [12] or more explicitly formula (4.2) of [4], there exist an irreducible unitary representation π of

G on a Hilbert space H_π such that $\pi(\Omega)=(\Lambda, \Lambda+2\delta)1$ and an irreducible K module V_μ with Δ_k^+ highest weight $\mu\in h^*$ such that V_μ is contained in both $\pi|_K$ and $V_\Lambda\otimes\bigwedge^q \mathfrak{p}^+$. μ has the form

$$\mu=\Lambda+\langle Q\rangle, \quad Q\subset\Delta_n^+, \quad |Q|=q. \tag{2.9}$$

Parthasarathy's Theorem 1 of [14] is a statement about the vanishing of *square-integrable* cohomology and in his proof the role of H_π is played by a so-called *discrete series* representation of G. However since we have $\pi(\Omega)=(\Lambda, \Lambda+2\delta)1$ his arguments on pages 608–682 are applicable in our present context and one may conclude that for a certain unit vector $\psi_\mu\in H_\pi$ and basis $\{E_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$ (see pages 681, 682 of [14])

$$\begin{aligned} -2\sum_{\alpha\in\bar{P}_n^{(\Lambda)}}\|\pi(E_\alpha)\psi_\mu\|^2 = {} & -2(\Lambda+\delta-\delta^{(\Lambda)}, \gamma-\langle Q_\Lambda\rangle) \\ & -(\gamma-\langle Q_\Lambda\rangle+2\delta^{(\Lambda)}, \gamma-\langle Q_\Lambda\rangle) \end{aligned} \tag{2.10}$$

where $\gamma=\langle Q\rangle$, and both terms on the r.h.s. of (2.10) are non-negative. Here one needs that $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant. ψ_μ is chosen moreover to satisfy $\pi(E_\alpha)\psi_\mu=0$ for any $\alpha\in\Delta_k^+$ and $\pi(H)\psi_\mu=\mu(H)\psi_\mu$ for every H in $\mathfrak{h}$. Thus one has $\pi(E_\alpha)\psi_\mu=0$ for any $\alpha\in\bar{P}^{(\Lambda)}$so that H_π is a highest weight module relative to the system of positive roots $\bar{P}^{(\Lambda)}$with Δ_k^+ highest weight μ. Also by (2.10)

$$\begin{aligned} (\Lambda+\delta-\delta^{(\Lambda)}, \gamma-\langle Q_\Lambda\rangle) &= 0, \\ (\gamma-\langle Q_\Lambda\rangle+2\delta^{(\Lambda)}, \gamma-\langle Q_\Lambda\rangle) &= 0, \end{aligned} \tag{2.11}$$

with $\gamma-\langle Q_\Lambda\rangle = \langle Q\cap Q'_\Lambda\rangle-\langle Q_\Lambda\cap Q'\rangle$ (since $Q\cup(Q_\Lambda\cap Q')=(Q\cap Q'_\Lambda)\cup Q_\Lambda$).

Therefore, $-(\Lambda+\delta-\delta^{(\Lambda)}, \gamma-\langle Q\rangle)=0$ implies

$$0=\sum_{\alpha\in-(Q\cap Q'_\Lambda)}(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)+\sum_{\beta\in Q_\Lambda\cap Q'}(\Lambda+\delta-\delta^{(\Lambda)}, \beta)$$

and since $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant, (2.5) implies

$$(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=0 \quad \text{for} \quad \alpha\in-(Q\cap Q'_\Lambda)\cup(Q_\Lambda\cap Q') \tag{2.12}$$

which proves statement (ii) of Lemma 2.4. By page 682 of [14]

$$\begin{aligned} & -(\gamma-\langle Q_\Lambda\rangle+\delta^{(\Lambda)}, \gamma-\langle Q_\Lambda\rangle+\delta^{(\Lambda)})+(\delta^{(\Lambda)}, \delta^{(\Lambda)}) \\ & = -(\gamma-\langle Q_\Lambda\rangle+2\delta^{(\Lambda)}, \gamma-\langle Q_\Lambda\rangle). \end{aligned} \tag{2.13}$$

Hence by (2.11)

$$(\gamma-\langle Q_\Lambda\rangle+\delta^{(\Lambda)}, \gamma-\langle Q_\Lambda\rangle+\delta^{(\Lambda)})=(\delta^{(\Lambda)}, \delta^{(\Lambda)}). \tag{2.14}$$

Now let

$$\Delta'_+=\Delta_k^+\cup-\Delta_n^+, \quad 2\delta'=\langle\Delta'_+\rangle \quad \text{so that} \quad \delta'=\delta_k-\delta_n. \tag{2.15}$$

One knows that Δ'_+ is a system of positive roots. Hence $(\delta^{(\Lambda)}, \delta^{(\Lambda)})=(\delta', \delta')$. Since $\gamma=\langle Q\rangle$ and since $-\langle Q_\Lambda\rangle+\delta^{(\Lambda)}=\delta_k-\delta_n$ by (2.6), equation (2.14) is statement (iii) of Lemma 2.4. Thus the proof of Lemma 2.4 is completed.

By (iii) of Lemma 2.4, $|\delta'|=|\delta'-\langle -Q\rangle|$ (see (2.15)) so by a lemma of Kostant [8] there exists $\sigma\in W$ such that

$$\sigma\delta'=\delta'-\langle -Q\rangle\,,\quad l(\sigma)=q\,,\quad \sigma(-\Delta'_+)\cap\Delta'_+=-Q\,. \tag{2.16}$$

One knows in fact that

$$\Delta_k^+\subset\sigma\Delta'_+\,. \tag{2.17}$$

In (2.16), $l(\sigma)$ denotes the *length* of σ. It is easy to check that (2.16) implies

Proposition 2.18. $Q=\{\alpha\in\Delta_n^+\,|\,(\sigma\delta', \alpha)>0\}$.

Lemma 2.19. $(\mu+\delta_k^{(\Lambda)}-\delta_n^{(\Lambda)}, \alpha)\neq 0$ *for every* α *in* Δ *and for* μ *in Lemma* 2.4; (*see* 2.2).

Proof. $\delta_k^{(\Lambda)}=\delta_k$ and $\delta'=\delta_k-\delta_n$ so

$$\begin{aligned}\mu+\delta_k^{(\Lambda)}-\delta_n^{(\Lambda)} &= \Lambda+\langle Q\rangle+\delta_k-\delta_n^{(\Lambda)}\\ &= \Lambda+\sigma\delta'+\delta_n-\delta_n^{(\Lambda)} = \Lambda+\sigma\delta'+\delta-\delta^{(\Lambda)}\end{aligned} \tag{2.20}$$

by (2.16) and (2.7). It suffices to check Lemma 2.19 for $\alpha\in P^{(\Lambda)}=\Delta_k^+\cup Q_\Lambda\cup -Q'_\Lambda$ (see (2.5)). Again we use that $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant. For $\alpha\in\Delta_k^+\subset P^{(\Lambda)}$ in particular, $\alpha=\sigma\alpha_1$, $\alpha_1\in\Delta'_+$ by (2.17) so that $(\sigma\delta', \alpha)=(\delta', \sigma^{-1}\alpha)=(\delta', \alpha_1)>0$. Hence $(\Delta+\delta-\delta^{(\Lambda)}+\sigma\delta', \alpha)>0$. Also $Q_\Lambda\cap Q\subset Q_\Lambda\subset P^{(\Lambda)}$ and $(\sigma\delta', \alpha)>0$ for $\alpha\in Q$ by Proposition 2.18 so by the same argument

$$(\Lambda+\delta-\delta^{(\Lambda)}+\sigma\delta', \alpha)>0 \quad\text{for}\quad \alpha\in Q_\Lambda\cap Q\,. \tag{2.21}$$

Suppose $\alpha\in(Q'_\Lambda\cap Q)\cup(Q_\Lambda\cap Q')$. Then by (ii) of Lemma 2.4 $(\Lambda+\delta-\delta^{(\Lambda)}, \alpha)=0$ and $(\Lambda+\delta-\delta^{(\Lambda)}+\sigma\delta', \alpha)=(\sigma\delta', \alpha)\neq 0$ with $(\sigma\delta', \alpha)>0$ for $\alpha\in Q'_\Lambda\cap Q$ by Proposition 2.18. Then by (2.21)

$$(\Lambda+\delta-\delta^{(\Lambda)}+\sigma\delta', \alpha)>0 \quad\text{for}\quad \alpha\in Q=(Q_\Lambda\cap Q)\cup(Q'_\Lambda\cap Q)\,. \tag{2.22}$$

Since $Q'_\Lambda=(Q'_\Lambda\cap Q)\cup(Q'_\Lambda\cap Q')$ the final case to check is $\alpha\in Q'_\Lambda\cap Q'$. By Proposition 2.18, $(\sigma\delta', \alpha)<0$ for $\alpha\in Q'$; i.e. $(\sigma\delta', -\alpha)>0$. Also $-\alpha\in -Q'_\Lambda$ for $\alpha\in Q'_\Lambda$ and $(\Lambda+\delta-\delta^{(\Lambda)}, -\alpha)\geqslant 0$ since $-Q'_\Lambda\subset P^{(\Lambda)}$ and since $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant. That is

$$(\Delta+\delta-\delta^{(\Lambda)}+\sigma\delta', -\alpha)>0 \quad\text{for}\quad \alpha\in Q'_\Lambda\cap Q'\,. \tag{2.23}$$

This completes the proof of Lemma 2.19.

REMARK: One can observe directly that $(\mu+\delta_k^{(\Lambda)}-\delta_n^{(\Lambda)}, \alpha)\neq 0$ for $\alpha\in\Delta_k^+$.

Hence equation (2.17) is not needed for the proof of Lemma 2.19, nor for the proof of Theorem 2.3.

We now state Parthasarathy's necessary conditions for the unitarizability of highest weight modules. This is Theorem A of [15]. For sufficient conditions see Theorem B of [15].

Theorem 2.24 (Parthasarathy). *Suppose that P is a system of positive roots compatible with a G-invariant complex structure on G/K (here G is linear, as we have assumed in section* 1). *Let $2\delta_{P,n}=\langle\Delta_n\cap P\rangle$, $2\delta_{P,k}=\langle\Delta_k\cap P\rangle$. Suppose $\mu\in h^*$ is integral and $\Delta_k\cap P$ dominant and suppose that H_μ is an irreducible highest weight $\mathfrak{g}$-module (or G-module) with respect to the positive system $(\Delta_k\cap P)\cup-(\Delta_n\cap P)$, with $\Delta_k\cap P$ highest weight μ. Suppose that*

$$(2.25)\qquad (\mu+\delta_{P,k}-\delta_{P,n},\ \alpha)\neq 0 \quad \textit{for every} \quad \alpha\in\Delta\,.$$

Then if H_μ is unitarizable there exists a parabolic subalgebra θ of $\mathfrak{g}$, θ containing the Borel subalgebra $\mathfrak{h}+\sum_{\alpha\in P}\mathfrak{g}_\alpha$ of $\mathfrak{g}$, such that $\mu=\Lambda_0+2\delta_{\theta,n}$ where

(i) $2\delta_{\theta,n}$ *= the sum of non-compact roots in the unipotent radical of θ.*
(ii) Λ_0 *is P dominant integral, and*
(iii) $(\Lambda_0,\alpha)=0$ *for every root α in the reductive part of θ.*

As indicated in the Introduction the proof of Theorem 2.3 will be based upon Theorem 2.24. Suppose $\Lambda\in\mathcal{F}_0'$ satisfies the hypothesis of Theorem 2.3 and suppose $H^q(\Gamma\,|\,G/K,\ \theta_{\tau_\Lambda})\neq 0$. By the remarks following (2.1) we may take P of Theorem 2.24 to be $P^{(\Lambda)}$. Then $\Delta_k\cap P=P_k^{(\Lambda)}=\Delta_k^+$and $-(\Delta_n\cap P)=-P_n^{(\Lambda)}$. By Lemma 2.19 condition (2.25) is satisfied for μ in Lemma 2.4. Thus by Lemma 2.4 and Theorem 2.24 we can conclude the following: There exists a parabolic subalgebra $\theta=\mathfrak{u}+\mathfrak{m}$ of $\mathfrak{g}$ with unipotent radical $\mathfrak{u}$ and reductive part $\mathfrak{m}$ such that $\theta\supset\mathfrak{h}+\sum_{\alpha\in P^{(\Lambda)}}\mathfrak{g}_\alpha$, $\mu=\Lambda_0+2\delta_{\theta,n}$ where if $\theta_{u,n}$ is the set of non-compact roots in $\mathfrak{u}$, $2\delta_{\theta,n}=\langle\theta_{u,n}\rangle$. Also Λ_0 is $P^{(\Lambda)}$ dominant integral and $(\Lambda_0,\alpha)=0$ for every root α of $\mathfrak{m}$. Now we also know that $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant integral. We will show that in fact $\Lambda+\delta-\delta^{(\Lambda)}=\Lambda_0$. For this we need the following remark: The subalgebra $\theta=\mathfrak{u}+\mathfrak{m}$ in Theorem 2.24 can be chosen so that if $P'=\{\alpha\in\Delta\,|\,(\mu+\delta_{P,k}-\delta_{P,n},\ \alpha)>0\}$ is the positive system defined by the regular element $\mu+\delta_{P,k}-\delta_{P,n}$ (see (2.25)) then $\theta_{u,n}=(\Delta_n\cap P)\cap(\Delta_n\cap P')$. This follows by (3.49) of [15]. Thus we have by (2.20)

$$(2.26)\qquad \theta_{u,n}=P'\cap P_n^{(\Lambda)}$$

where $$P'=\{\alpha\in\Delta\,|\,(\Lambda+\delta-\delta^{(\Lambda)}+\sigma\delta,\ \alpha)>0\}\,.$$

If $\Delta(\mathfrak{m})$ denotes the set of roots of $\mathfrak{m}$, then

$$\mathfrak{m} = \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{m})} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \sum_{\alpha \in P^{(\Delta)} - \Delta(\mathfrak{m})} \mathfrak{g}_\alpha \tag{2.27}$$

and hence $\theta_{u,n} = P_n^{(\Lambda)} - \Delta(\mathfrak{m})$.

Lemma 2.28. $\theta_{u\ n} = (Q \cap Q_\Lambda) \cup (-Q'_\Lambda \cap -Q')$ *and* $P_n^{(\Lambda)} \cap \Delta(\mathfrak{m}) = P_n^{(\Lambda)} - P'$ $= (Q' \cap Q_\Lambda) \cup (-Q'_\Lambda \cap -Q)$.

Proof. By (2.6) and (2.26), $\theta_{u,n} = P' \cap P_n^{(\Lambda)} = P' \cap (Q_\Lambda \cup -Q'_\Lambda)$. By (2.22) and (2.23) Q, $-Q'_\Lambda \cap -Q' \subset P'$ so that $Q \cap Q_\Lambda$, $-Q'_\Lambda \cap -Q' \subset P' \cap (Q_\Lambda \cup -Q'_\Lambda)$. Conversely let $\alpha \in P' \cap (Q_\Lambda \cap -Q'_\Lambda)$. We consider two cases:

(i) $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ and (ii) $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) \neq 0$.

If (i) holds then $\alpha \in P'$ implies $(\sigma\delta', \alpha) > 0$. Hence if $\alpha \in Q_\Lambda \subset \Delta_n^+$ then $\alpha \in Q_\Lambda \cap Q$ by Proposition 2.18. If $\alpha \in -Q'_\Lambda$ then $-\alpha \in Q'_\Lambda \subset \Delta_n^+$ such that $(\sigma\delta', -\alpha) < 0$. By Proposition 2.18, $-\alpha \in Q'$ so $\alpha \in -Q'$ implies $\alpha \in -Q'_\Lambda \cap -Q'$. Suppose (ii) holds. Then by (ii) of Lemma 2.4 we have $\alpha \notin Q_\Lambda \cap Q'$, $-\alpha \notin Q \cap Q'_\Lambda$. Since $\alpha \in Q_\Lambda \cup -Q'_\Lambda$ we have $\alpha \in Q_\Lambda \cap Q$ if $\alpha \in Q_\Lambda$, and if $\alpha \in Q'_\Lambda$ then $-\alpha \in Q'$ so $\alpha \in -Q'_\Lambda \cap -Q'$. Since $P_n = Q_\Lambda \cup (-Q'_\Lambda)$, it follows that $P_n^{(\Lambda)} - P' = (Q' \cap Q_\Lambda) \cup (-Q'_\Lambda \cap -Q)$. Then $P_n^{(\Lambda)} = (P_n^{(\Lambda)} \cap \Delta(\mathfrak{m})) \cup (P_n^{(\Lambda)} - \Delta(\mathfrak{m}))$ and $P_n^{(\Lambda)} - \Delta(\mathfrak{m}) = P' \cap P_n^{)\Lambda)}$ implies that $P_n^{(\Lambda)} \cap \Delta(\mathfrak{m}) = P_n^{(\Lambda)} - P'$, which completes the proof of Lemma 2.28.

By Lemma 2.28 we have

$$Q \cap Q_\Lambda = \theta_{u,n} \cap Q_\Lambda \tag{2.29}$$

and $|\theta_{u,n}| = |Q \cap Q_\Lambda| + |Q'_\Lambda \cap Q'| = |Q \cap Q_\Lambda| + |Q'_\Lambda| - |Q'_\Lambda \cap Q| = 2|Q \cap Q_\Lambda|$ $- |Q| + |Q'_\Lambda|$ so that by (2.29) and (i) of Lemma 2.4, $q = 2|\theta_{u,n} \cap Q_\Lambda| - |\theta_{u,n}| +$ $|Q'_\Lambda|$, which proves statement (i) of Theorem 2.3. Recalling that $2\delta_{\theta,n} \overset{\text{def.}}{=} \langle \theta_{u,n} \rangle$ we also have by Lemma 2.28 that $2\delta_{\theta,n} = \langle Q \cap Q_\Lambda \rangle - \langle Q'_\Lambda \cap Q' \rangle = \langle Q \cap Q_\Lambda \rangle +$ $\langle Q'_\Lambda \cap Q \rangle - \langle Q'_\Lambda \rangle = \langle Q \rangle + \langle Q_\Lambda \rangle - 2\delta_n = \langle Q \rangle - \delta_n + \delta^{(\Lambda)} - \delta_k$ (by (2.6)) $= \langle Q \rangle +$ $\delta^{(\Lambda)} - \delta$. That is we have $\Lambda + \langle Q \rangle = \mu = \Lambda_0 + 2\delta_{\theta,n} = \Lambda_0 + \langle Q \rangle + \delta^{(\Lambda)} - \delta$ implies $\Lambda + \delta - \delta^{(\Lambda)} = \Lambda_0$. Hence $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ for every $\alpha \in \Delta(\mathfrak{m})$, which proves the first statement in (ii) of Theorem 2.3. Since $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ for every α in $\Delta(\mathfrak{m}) \cap P_n^{(\Lambda)}$ in particular, we must have the set

$$A_\Lambda = \{\alpha \in P_n^{(\Lambda)} \mid (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) > 0\} \subset P_n^{(\Lambda)} - \Delta(\mathfrak{m}) = \theta_{u,n}.$$

Hence $|A_\Lambda| \leqslant |\theta_{u,n}|$ and we have completed the proof of Theorem 2.3.

REMARK: One may check that statement (i) of Theorem 2.3 is equivalent to the statement

$$(2.30) \qquad n-q = 2|(P_n^{(\Lambda)} \cap \Delta(\mathfrak{m})) \cap Q_\Lambda| - |P_n^{(\Lambda)} \cap \Delta(\mathfrak{m})| + |Q'_\Lambda|$$

where $n = |\Delta_n^+| = \dim_C G/K$.

For later computational purposes it is convenient to consider parabolic subalgebras of $\mathfrak{g}$ which contain the Borel subglgebra $\mathfrak{h} + \sum_{\phi \in \bar{P}^{(\Lambda)}} \mathfrak{g}_\alpha$. Thus we give the following equivalent formulation of Theorem 2.3.

Theorem 2.3′. *Let $\Lambda \in \mathscr{F}'_0$ such that every non-compact root in $P^{(\Lambda)}$ is totally positive. Suppose that $H^q(\Gamma | G/K, \theta_{\tau_\Lambda}) \neq 0$. Then there exists a parabolic subalgebra $\theta = \mathfrak{u} + \mathfrak{m}$ of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{h} + \sum_{\alpha \in \bar{P}^{(\Lambda)}} \mathfrak{g}_\alpha$ with $\mathfrak{u}$ = the unipotent radical of θ and $\mathfrak{m}$ = the reductive part of θ such that*

(i) $n-q = 2|\theta_{u,m} \cap Q'_\Lambda| + |Q_\Lambda| - |\theta_{u,n}|$

(ii) $(\Lambda + \delta - \delta^{(\Lambda)}, \kappa\alpha) = 0$ *for every root α of $\mathfrak{m}$.*

Here $n = |\Delta_n^+| = \dim_C G/K$, $\theta_{u,n}$ is the set of non-compact roots in $\mathfrak{u}$, and κ denotes the unique element of W_G such that $\kappa\Delta_\kappa^+ = -\Delta_\kappa^+$.

Proof: One has $\kappa\Delta_n^+ = \Delta_n^+$ and $\kappa^2 = 1$. Given $\Delta \in \mathscr{F}'_0$ we have $-\kappa\Lambda - 2\delta_n \in \mathscr{F}'_0$ since $-\kappa\Lambda - 2\delta_n + \delta = -\kappa\Lambda - \kappa\delta = -\kappa(\Lambda + \delta)$. The latter equation also shows that $P^{(-\kappa\Lambda - 2\delta_n)} = -\kappa P^{(\Lambda)} = \bar{P}^{(\Lambda)}$; see (2.8). Moreover one may check that every non-compact root in the positive system $\bar{P}^{\Lambda)}$ is totally positive. By Serre duality, $H^q(\Gamma | G/K, \theta_{\tau_\Lambda}) \cong H^{n-q}(\Gamma | G/K, \theta_{\tau_{-\kappa\Lambda - 2\delta_n}})$. Hence if $H^q(\Gamma | G/K, \theta_{\tau_\Lambda}) \neq 0$. Theorem 2.3 says there exists a parabolic subalgebra $\theta = \mathfrak{u} + \mathfrak{m}$ of $\mathfrak{g}$ containing $\mathfrak{h} + \sum_{\alpha \in \bar{P}^{(\Lambda)}} \mathfrak{g}_\alpha$ such that (i) $n - q = 2|\theta_{u,n} \cap Q_{-\kappa\Lambda - 2\delta_n}| + Q'_{-\kappa\Lambda - 2\delta_n}| - |\theta_{u,n}|$, (ii) $(-\kappa\Lambda - 2\delta_n + \delta - \bar{\delta}^{(\Lambda)}, \alpha) = 0$ for every α in $\Delta(\mathfrak{m})$. Now $Q_{-\kappa\Lambda - 2\delta_n} = \kappa Q'_\Lambda$ so that $Q'_{-\kappa\Lambda - 2\delta_n} = \kappa Q_\Lambda$. Also $-\kappa\delta^{(\Lambda)} = \bar{\delta}^{(\Lambda)}$ by (2.8) so that

$$-\kappa\Lambda - 2\delta_n + \delta - \bar{\delta}^{(\Lambda)} = -\kappa(\Lambda + \delta) + \kappa\delta^{(\Lambda)} = -\kappa(\Lambda + \delta - \delta^{(\Lambda)})$$

implies $(\Lambda + \delta - \delta^{(\Lambda)}, \kappa\alpha) = 0$ for every α in $\Delta(\mathfrak{m})$. Thus Theorem 2.3′ follows.

3. Some applications

In the present discussion we shall see how Theorem 2.3 incorporates and extends some of the classical results on the vanishing of $H^q(\Gamma | G/K, \theta_{\tau_\Lambda})$. Here we consider the two extreme cases of $\Delta \in \mathscr{F}'_0$:

(i) $(\Lambda + \delta, \alpha) > 0$ for every in Δ_n^+ (i.e. Λ is Δ^+ dominant) and

(ii) $(\Lambda + \delta, \alpha) < 0$ for every α in Δ_n^+.

In case (i), $Q_\Lambda = \Delta_n^+$ so that $\delta^{(\Lambda)} = \delta$ and $P^{(\Lambda)} = \Delta^+$ by (2.5). In case (ii) $Q_\Lambda = \phi$, $P^{(\Lambda)} = \Delta_k^+ \cup -\Delta_n^+$ (by (2.5)) and $\delta^{(\Lambda)} = \delta' = \delta_k - \delta_n$. Thus in both cases every non-compact root in $P^{(\Lambda)}$ is totally positive and therefore Theorem 2.3 is applicable. For case (i) we get the following:

Theorem 3.1. *Suppose* Λ *is* Δ^+ *dominant integral. Suppose* $H^q(\Gamma | G/K, \theta_{\tau_\Lambda}) \neq 0$. *Then there exists a parabolic subalgebra* $\theta = \mathfrak{u} + \mathfrak{m}$ *of* $\mathfrak{g}$ *containing the Borel subalgebra* $\mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ *such that*

(i) $q = |\theta_{u,n}| =$ *the no. of non-compact roots in the unipotent radical* $\mathfrak{u}$ *of* θ

(ii) $(\Lambda, \alpha) = 0$ *for every root in the reductive part* $\mathfrak{m}$ *of* θ.

In particular, if $n_\Lambda = | \{\alpha \in \Delta_n^+ | (\Lambda, \alpha) > 0\}$ *then we have* $H^q(\Gamma | G/K, \theta_{\tau_\Lambda}) = 0$ *for* $q < n_\Lambda$.

The last statement follows since the set A_Λ in Theorem 2.3 is the set $\{\alpha \in \Delta_n^+ | (\Lambda, \alpha) > 0\}$ and $|\theta_{u,n}| = q$. This statement moreover was first proved by Y. Matsushima and S. Murakami; see [10], [11] [12],. Let r be the real rank of G. Suppose G is simple so that the Hermitian symmetric space G/K is irreducible. Then in [15] it is shown that there exists no parabolic subalgebra $\theta = \mathfrak{u} + \mathfrak{m}$ of $\mathfrak{g}$ such that $q = |\theta_{u,n}|$ for $1 \leqslant q < r$, $\theta \supset \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Hence, in particular, Theorem 3.1 implies

Corollary 3.2. $H^q(\Gamma | G/K, \theta_{\tau_\Lambda}) = 0$ *for* $1 \leqslant q < r$, *G simple and* Λ Δ^+ *dominant integral.*

Corollary 3.2 was also proved by R. Hotta and S. Murakami in [4]. However we shall see that statement (i) of Theorem 3.1 is generally sharper than the statement of Corollay 3.2. If we taken $\Lambda = 0$ in Theorem 3.1, then $\dim H^q(\Gamma | G/K, \theta_{\tau_\Lambda})$ is just the $(0, q)$ Betti number of the locally symmetric space $\Gamma | G/K$. Then Corollary 3.2 is a result of Y. Matsushima [18] and R. Hotta - N. Wallach [6] (also see A. Borel - N. Wallach [2]). Moreover Theorem 3.1 for $\Lambda = 0$ coincides with the sharper results of [15] obtained by R. Parthasarathy for the vanishing of $(0, q)$ Betti numbers.

Suppose now again that the Hermitian symmetric space G/K is irreducible and $r = r(G)$ is the real rank of G. Then G/K is one of the following spaces on E. Cartan's list:

$$(3.3) \qquad \begin{array}{lll}
\mathrm{I} & SU(n, m)/S(U(n) \times U(m)), & r = \min(n, m) \\
\mathrm{II} & Sp(n, R)/U(n), & r = n \\
\mathrm{III} & SO_0(n, 2)/SO(n) \times SO(2), & n > 2,\ r = 2 \\
\mathrm{IV} & SO^*(2n)/U(n), & n > 3,\ r = \left[\frac{n}{2}\right] \\
\mathrm{V} & G/K, & r = 2,\ G^C = E_6 \\
\mathrm{VI} & G/K, & r = 3,\ G^C = E_7 .
\end{array}$$

Parthasarathy [15] has computed all of the numbers $|\theta_{u,n}|$ as $\theta = \mathfrak{u} + \mathfrak{m}$ varies over the parabolics containing $\mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. We present his list in the form of the following

Table 3- 4

G	$\{\lvert\theta_{u,m}\rvert \mid \theta\supset\mathfrak{h}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha\}$
$SU(n, m)$, $n\geqslant m$	$\{nm-n'm' \mid 0\leqslant n'\leqslant n,\ 0\leqslant m'\leqslant m\}$
$SP(n, R)$	$\{0\}\cup\{n+(n-1)+\cdots+(n-j) \mid j=0,1,\cdots,n-1\}$
$SO_0(n, 2)$, $n>2$	$\{0\}\cup\left\{\left[\frac{n+1}{2}\right]\right\},\cdots,n\}$
$SO^*(2n)$, $n>3$	$\left\{\frac{n(n-1)}{2}-\frac{j(j-1)}{2} \mid j=3,\cdots,n\right\}$ $\cup\left\{\frac{n(n-1)}{2}-i \mid i=0,1,\cdots,n-1\right\}$
real form of E_6	$\{0, 8, 11, 12, 13, 14, 15, 16\}$
real from of E_7	$\{0 ,17, 21, 22, 23, 24, 25, 26, 27\}$

Theorem 3.1 now implies

Theorem 3.5. *Suppose G is simple as in Table 3.4 and suppose Λ is Δ^+ dominant integral. Then $H^q(\Gamma\mid G/K, \theta_{\tau_\Lambda})$ vanishes unless q belongs to the set $\{\lvert\theta_{u,n}\rvert \mid \theta\supset\mathfrak{h}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha\}$ corresponding to G in the Table 3.4.*

Consider the exceptional cases V, VI of (3.3) for example. For $G^C=E_6$, $r(G)=2$ and the classical result Corollary 3.2 predicts vanishing of H^q for $q=1$. However by Theorem 3.5 we get $H^q=0$ for $1\leqslant q\leqslant 7$, $q=9$, 10, which shows that Theorem 3.4 (or Theorem 3.1) is sharper than the main Theorem of [4] as we asserted earlier. For $G^C=E_7$ Corollary 3.2 gives $H^q=0$ for $q=1,2$ whereas Theorem 3.5 implies $H^q=0$ for $1\leq q\leq 16$ and $q=18$, 19, 20. We remark that $\dim_C G/K=16$, 27 respectively in cases V, VI. In case IV of (3.3) our result gives $H^q=0$ for $1\leqslant q\leqslant n-2$ (and for certain other values of q) even though $r(G)=[n/2]$. In case III of (3.3), $H^q=0$ for $1\leqslant q<[(n+1)/2]$, even though the real rank is only 2. Similarly in the other cases Theorem 3.5 improves known results.

Next let γ_1 be the unique non-compact simple root of Δ_n^+ and let $\beta_0\in\Delta^+$ be the largest root. Then β_0 is the highest Δ^+ weight of the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ and β_0 is the highest Δ_k^+ weight of the adjoint representation ad_+ of $\mathfrak{k}$ on $\mathfrak{p}^+$; $\beta_0\in\Delta_n^+$. For the special representation $\tau=\mathrm{ad}_+$ of K one has

$$(3.6)\qquad H^q(\Gamma\mid G/K, \theta_{\tau_{\beta_0}})\cong H^q(\Gamma\mid G/K, \Theta)$$

where Θ is the sheaf of germs of holomorphic vector fields on $\Gamma|G/K$. The number $n_\Delta = n_{\beta_0}$ in the statement of Theorem 3.1 (for G simple) is given by

$$n_{\beta_0} = \frac{1}{(\gamma_1, \gamma_1)} - 1 \tag{3.7}$$

; see [11] or [13]. Thus, following Matsushima and Murakami [11], we obtain from Theorem 3.1 the following classical theorem of E. Calabi and E. Vesentini [3]:

Theorem 3.8. *$H_q(\Gamma|G/K, \Theta)=0$ for $q<\frac{1}{(\gamma_1, \gamma_1)}-1$, G simple.*

One knows also that

$$n_{\beta_0} = m(G/K)+1 \tag{3.9}$$

where $\quad m(G/K) = |\{\alpha\in\Delta_n^+ - \{\gamma_1\} \,|\, \alpha-\gamma_1\in\Delta\}|$.

In [1] A. Borel shows that for E_6, E_7 respectively $m(G/K)=10, 16$. Thus by (3.7), (3.9) the Calabi-Vesentini theorem gives $H^q=0$ for $q<11$ for G=the real form of E_6, and $H^q=0$ for $q<17$ for G=the real form of E_7. However we have already observed that $H^q=0$ for $1\leqslant q\leqslant 16$ and $q=18, 19, 20$ for G=the real form of E_7. Thus we have the following slight improvement of the Calabi-Vesentini theorem:

Theorem 3.10. *Let G be the unique real form of E_7 such that G/K is Hermitian symmetric. Then $H^q(\Gamma|G/K, \Theta)=0$ for $0\leqslant q<17$ and for $q=18, 19, 20$.*

For $G^C=E_6$ and for the cases III, IV in (3.3) our results give no improvement of the Calabi-Vesentini result. However in cases I, II we do obtain further improvements (even more so than in case VI of Theorem 3.10).

Indeed, for the irreducible Hermitian symmetric spaces G/K in (3.3) the corresponding complex dimensions n and the values $n_{\beta_0}=\frac{1}{(\gamma_1, \gamma_1)}-1$ in Theorem 3.8 are given as follows:

G/K	n	n_{β_0}
I	nm	$n+m-1$
II	$\frac{n(n+1)}{2}$	n
III	n	$n-1$
IV	$\frac{n(n-1)}{2}$	$2n-3$
V	16	11
VI	27	17

We turn now to the consideration of (ii) above: $(\Lambda+\delta, \alpha)<0$ for every α in Δ_n^+. Here, as we have seen, $P^{(\Lambda)}=\Delta'_+=\Delta_k^+ \cup -\Delta_n^+$, $\delta^{(\Lambda)}=\delta'=\delta_k-\delta_n$, $Q_\Lambda=\phi$; we assume that Λ is integral and Δ_k^+ dominant. Let

$$B_\Lambda = \{\alpha \in \Delta_n^+ \mid (\Lambda+2\delta_n, \alpha)<0\} . \tag{3.11}$$

Then the set A_Λ in the statement of Theorem 2.3 is given by $A=\{\alpha\in -\Lambda_n^+ \mid (\Lambda+2\delta_n, \alpha)>0\}=-B_\Lambda$. Hence by Theorem 2.3 and Theorem 2.3′ we get

Theorem 3.12. *Suppose* Λ *is integral,* Δ_k^+ *dominant, and satisfies* $(\Lambda+\delta, \alpha)<0$ *for every* $\alpha\in\Delta_n^+$. *Then*

(i) $H^q(\Gamma|G/K, \theta_{\tau_\Lambda})=0$ *for* $q>n-|B_\Lambda|$ (*see* 3.11) *where* $n=|\Delta_n^+|=\dim_C G/K$.

(ii) *If* $H^q(\Gamma|G/K, \theta_{\tau_\Lambda})\neq 0$ *then a parabolic subalgebra* $\theta=\mathfrak{u}+\mathfrak{m}$ *of* $\mathfrak{g}$ *containing* $\mathfrak{h}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ *with* $\mathfrak{u}$=*the unipotent radical of* θ *and* $\mathfrak{m}$=*the reductive component of* θ *such that* (*a*) $n-q=|\theta_{u,n}|$, $\theta_{u,n}$=*set of non-compact roots in u, and* (*b*) $(\Lambda+2\delta_n, \kappa\alpha)=0$ *for every root* α *in* $\mathfrak{m}$; κ=*unique element of* W_G *such that* $\kappa\Delta_k^+=-\Delta_k^+$.

REMARK. In Theorem 3.12 G of course is *not* assumed to be simple.

For Λ integral and Δ_k^+ dominant, consider the following three assumptions:

(X) $(\Lambda+2\delta_n, \alpha)\leqslant 0$ for every α in Δ_n^+
(Y) $(\Lambda+2\delta_n, \alpha)<0$ for every α in Δ_n^+
(Z) $(\Lambda+2\delta, \alpha)<0$ for every α in Δ_n^+.

One has that (Z)⇒(Y)⇒(X)⇒$(\Lambda+\delta, \alpha)<0$ for every α in Δ_n^+ (using that $\delta=2\delta_n+\delta'$). In cases (Z) and (Y), $B_\Lambda=\Delta_n^+$ in (3.11). Hence by (i) of Theorem 3.12 we obtain the following result of Hotta-Parthasarathy [5] and M. Ise [7].

Corollary 3.13. *Suppose* Λ *is integral and* Δ_k^+ *dominant. If* Λ *satisfies either (Y) or (Z), then* $H^q(\Gamma|G/K, \theta_{\tau_\Lambda})=0$ *for* $q>0$; *(Z)⇒(Y).*

REMARK: In Corollary 2 page 231 of [5], Hotta and Parthasarathy assume that (Y) holds and that $\Lambda\in\mathcal{F}'_0$ such that $(\Lambda+\delta, \alpha)<0$ for every α in Δ_n^+. However as we have just observed, (Y)⇒$\Lambda\in\mathcal{F}'_0$ and that $(\Lambda+\delta, \alpha)<0$ for every α in Δ_n^+. Thus the latter two assumptions are superfluous. In particular the Hotta-Parthasarathy multiplicity formula of Corollary 2 for holomorphic discrete series representations is valid under assumption (Y) only.

In case (X), (i) of Theorem 3.12 implies that $H^q(\Gamma|G/K, \theta_{\tau_\Lambda})=0$ for $q>|\{\alpha\in\Delta_n^+ \mid (\Lambda+2\delta_n, \alpha)=0\}|$. From (ii) of Theorem 3.12 we obtain

Theorem 3.14. *Suppose* G *is simple as in Table* 3.4 *and* Λ *is integral,* Δ_k^+ *dominant, and satisfies* $(\Lambda+\delta, \alpha)<0$ *for every* α *in* Δ_n^+. *Then* $H^q(\Gamma|G/K, \theta_{\tau_\Lambda})=0$

unless $n-q$ *belongs to the set* $\{|\theta_{u,n}|\,|\,\theta\supset\mathfrak{h}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha\}$ *corresponding to* G *in the Table* 3.4. *Again* $n=|\Delta_n^+|=\dim_C G/K$.

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