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QUASILINEAR ABSTRACT PARABOLIC EVOLUTION EQUATIONS AND EXPONENTIAL ATTRACTORS

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Abstract

The Exponential attractor, one of notions of limit set in infinite-dimensional dynamical systems, is known to have strong robustness and is known to be constructed under a simple compact smoothing condition. In this paper, we study a dynamical system determined from the Cauchy problem for a quasilinear abstract parabolic evolution equation. We give a general strategy for constructing the exponential attractor and apply the abstract result to a chemotaxis-growth system in non smooth domain.

1. Introduction

Exponential attractor which has been introduced by Eden, Foias, Nicolaenko and Temam [4] is one of very important notions of limit sets in the theory of dynamical systems in infinite-dimensional spaces (see [2, 3, 19, 26]). The exponential attractor is, if it exists, a compact set with finite fractal dimension which contains a global attractor interiorly and attracts every trajectory in an exponential rate. In many mathematical models, exponential attractors are considered essential limit sets. In some pattern formation model, the formation is considered to perform in an exponential attractor rather than in a global attractor. And the fractal dimension of an exponential attractor is taken as a number of active modes and the attraction of every trajectory is taken as a reduction of the degrees of freedom in the process of pattern formation which is called the slaving principle.

Exponential attractors are also known to have very strong stability in approximation. Indeed the first and third authors [1] have shown under suitable conditions that an exponential attractor attracts even approximate solutions in its neighborhood exponentially and continues to trap them in the neighborhood forever. This then shows that we have global reliability of numerical computations which are practiced for investigating profiles of the solutions which evolve in the exponential attractor and for knowing a structure of the exponential attractor.

Eden et al. [4] presented also a very useful method for construction of exponential attractors in Hilbert spaces. They showed a method how to construct an exponential

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attractor from a property called the squeezing property of a nonlinear semigroup which defines the dynamical system in consideration. More recently, Miranville, Zelik and the second author [6] presented another more general method which is available even in Banach spaces. They presented a new condition of semigroup called the compact smoothing Lipschitz property, see (5.2), and showed a method of constructing an exponential attractor from this property. The latter method seems to have advantages in several view points. In [4], the authors consider a maximal set for the relation of cone property of semigroup which is closely related to the squeezing property. Such a maximal set is however obtained only by using Zorn's lemma. In [6], a compact smoothing Lipschitz condition of semigroup is only used, which gives us hope for numerical implementations. As we have no uniqueness of exponential attractors, these different methods may give different exponential attractors. But we have to remark that Miranville and the second author [5] have shown for reaction-diffusion systems that the two methods have the same sharpness in the estimate of fractal dimensions of attractors.

In this paper we are concerned with construction of an exponential attractor for a dynamical system determined from the quasilinear abstract parabolic evolution equation in a Banach space. As observed in [4, Chapter 3], the squeezing property seems to fit only to semigroups which are determined from semilinear evolution equations and does not necessarily seem to fit to semigroups determined from quasilinear equations. So we intend to verify the compact smoothing Lipschitz property of semigroup. To this end we shall utilize a representation formula of solutions for the quasilinear equation in terms of the evolution operators for the linear abstract equations. The theory of linear abstract parabolic evolution equations was originated by Tanabe [22, 23] on the basis of the theory of analytic semigroups. Then it was developed by many authors (see [7, 24, 25]). To verify the desired compact smoothing, we need however very refined properties of the evolution operator which may not be necessarily used in the linear theory itself.

We shall also consider an application of our abstract results to a chemotaxis-growth model presented by Mimura et al. [14] in mathematical biology. In the paper Osaki et al. [18], an exponential attractor was already constructed when the region Ω is a two-dimensional bounded domain of class C^3 (cf. also [17]). In [18] the authors established the squeezing property of semigroup to use the method of Eden et al., but this required us a shift property that $\Delta u \in H^1(\Omega)$ with $\partial u/\partial n = 0$ on $\partial\Omega$ implies $u \in H^3(\Omega)$. This is the reason why we needed C^3-regularity of Ω. But the compact smoothing requires us only a weaker shift property that $\Delta u \in L^2(\Omega)$ with $\partial u/\partial n = 0$ implies $u \in H^2(\Omega)$. Therefore, C^2-regularity of Ω is sufficient and more interestingly we can work even in a convex domain (see [9]). As well known, the spatially discretized approximate problems are usually formulated in polygonal domains (see [15, 16]).

Not only to the chemotaxis-growth model but our abstract results can be expected to apply to many other interaction diffusion systems (see [28]).

This paper is organized as follows. In Section 2 we review the evolution operators for linear parabolic evolution equations and list their properties used in the subsequent sections. Section 3 is devoted to constructing local solutions to a quasilinear abstract equation and to representing them by the evolution operators. First result on this subject was obtained by Sobolevskiĭ [20] (cf. also [8]), afterward his result was generalized by Lunardi [12] and Yagi [29, 30]. We shall present in this paper a very refined result with its proof. In Section 4 we establish Lipshitz continuity of local solutions with respect to initial values, which provides directly the compact smoothing Lipschitz condition. We shall present in Section 5 a general strategy for constructing an exponential attractor for a dynamical system determined from the quasilinear abstract equation. Along these lines we shall apply in Section 6 our abstract result to the chemotaxis-growth system.

NOTATION. Let X be a Banach space with norm $\|\cdot\|_X$. If there is no fear of confusion, $\|\cdot\|_X$ is denoted by $\|\cdot\|$. Let $\mathcal{X}$ be a subset of X, then $\mathcal{X}$ is a metric space with the induced distance $d(U,V) = \|U - V\|_X$ $(U, V \in \mathcal{X})$. For $U \in \mathcal{X}$ and a set $B \subset \mathcal{X}$, $d(U,B)$ is defined by $d(U,B) = \inf_{V\in B} d(U,V)$. For two sets $B_1, B_2 \subset \mathcal{X}$, their distance $d(B_1,B_2)$ is defined by $d(B_1,B_2) = \max\{h(B_1,B_2), h(B_2,B_1)\}$, where $h(B_1,B_2)$ denotes the Hausdorff pseudodistance given by

$$h(B_1,B_2) = \sup_{U\in B_1} d(U,B_2) = \sup_{U\in B_1} \inf_{V\in B_2} d(U,V). \tag{1.1}$$

For two Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X into Y with the uniform operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. For each $f \in X$, $p_f(A) = \|Af\|_Y$ is a seminorm of $\mathcal{L}(X,Y)$. The topology defined by all these seminorms is called the strong topology of $\mathcal{L}(X,Y)$. For example, a sequence $\{A_n\}_{n=1,2,3,\ldots}$ of linear operators in $\mathcal{L}(X,Y)$ is said to be strongly convergent to an operator $A \in \mathcal{L}(X,Y)$ on X if $textitY\text{-}\lim_{n\to\infty} A_n f = Af$ for all $f \in X$. When $X = Y$, $\mathcal{L}(X,X)$ is abbreviated as $\mathcal{L}(X)$.

Let X be a Banach space and let I be an interval. $C(I;X)$, $C^\theta(I;X)$ $(0<\theta<1)$ and $C^1(I;X)$ denote the space of X-valued continuous functions, Hölder continuous functions with exponent θ, and continuously differentiable functions equipped with the usual function norms, respectively. $\mathcal{B}(I;X)$ is the space of X-valued bounded functions (not necessarily measurable) equipped with the norm $\|f\|_{\mathcal{B}} = \sup_{t\in I}\|f(t)\|_X$.

2. Review of evolution operators

Let X be a Banach space with norm $\|\cdot\|$. We consider a family of densely defined closed linear operators $A(t)$, $0 \le t \le T$, acting in X. We assume that the spectral set $\sigma(A(t))$ is contained in a fixed open sectorial domain

$$\sigma(A(t)) \subset \Sigma_\phi = \{\lambda \in \mathbb{C}; |\arg \lambda| < \phi\}, \quad 0 < \phi < \frac{\pi}{2},$$

and the resolvent satisfies

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda| + 1}, \quad \lambda \notin \Sigma_\phi, \quad 0 \le t \le T \tag{2.1}$$

with some constant $M \ge 1$. In addition, $A(t)$ is assumed to have a constant domain $\mathcal{D}(A(t)) \equiv \mathcal{D}$ and to satisfy a Hölder condition of the form

$$\|A(t)\{A(t)^{-1} - A(s)^{-1}\}\|_{\mathcal{L}(X)} \le N|t - s|^\mu, \quad 0 \le s, t \le T \tag{2.2}$$

with some exponent $0 < \mu \le 1$ and some constant $N > 0$, $\mathcal{D}$ being a Banach space equipped with a graph norm $\|\cdot\|_{\mathcal{D}} = \|A(0)\cdot\|_X$.

The condition (2.1) yields that each $-A(t)$ is the generator of an analytic semigroup $e^{-\tau A(t)}$, $\tau \ge 0$, on X, and the semigroup satisfies

$$\|A(t)^\theta e^{-\tau A(t)}\|_{\mathcal{L}(X)} \le C_\theta \tau^{-\theta}, \quad \theta \ge 0, \ \tau > 0,$$

where $A(t)^\theta$ denote the fractional powers of $A(t)$. From this estimate the following estimate is easily obtained:

$$\|\{e^{-\tau A(t)} - 1\}A(t)^{-\theta}\|_{\mathcal{L}(X)} \le C\tau^\theta, \quad 0 < \theta \le 1, \ \tau \ge 0. \tag{2.3}$$

Under (2.1) and (2.2), Tanabe [22, 23] constructed a unique evolution operator $U(t, s)$ for the family $A(t)$, $0 \le t \le T$. That is, $U(t, s)$ is a family of bounded linear operators on X defined for $0 \le s \le t \le T$ with the following basic properties: a) $U(t, s)U(s, r) = U(t, r)$ for $0 \le r \le s \le t \le T$, $U(s, s) = 1$ for $0 \le s \le T$; b) $U(t, s)$ (resp. $A(t)U(t, s)$) is strongly continuous on X for $0 \le s \le t \le T$ (resp. $0 \le s < t \le T$) with the estimate $\|U(t, s)\|_{\mathcal{L}(X)} \le C$ (resp. $\|A(t)U(t, s)\|_{\mathcal{L}(X)} \le C(t - s)^{-1}$); $U(t, s)$ is strongly differentiable on X in t for $t > s$ with $\partial U(t, s)/\partial t = -A(t)U(t, s)$; and d) $U(t, s)$ is strongly differentiable in s for $s < t$ on the domain $\mathcal{D}$ with $\partial U(t, s)/\partial s = U(t, s)A(s)$. In this section, C denotes a universal constant which is determined in each occurrence by the exponent and initial constants appearing in (2.1) and (2.2).

By further investigations, we can establish the following estimates of the evolution operator; for the proofs, see [29, 30]. As for estimates of operator norms,

$$\|A(t)^\theta U(t, s)\|_{\mathcal{L}(X)} \le C(t - s)^{-\theta}, \quad 0 \le \theta < 1 + \mu, \ 0 \le s < t \le T, \tag{2.4}$$

$$\|U(t, s)A(s)^\theta\|_{\mathcal{L}(X)} \le C_\theta(t - s)^{-\theta}, \quad 0 \le \theta < \mu, \ 0 \le s < t \le T, \tag{2.5}$$

$$\|A(t)U(t, s)A(s)^{-\theta}\|_{\mathcal{L}(X)} \le C(t - s)^{\theta - 1}, \quad 0 \le \theta \le 1, \ 0 \le s < t \le T, \tag{2.6}$$

(2.7) $$\|A(t)^{\theta}U(t,s)A(s)^{-\theta}\|_{\mathcal{L}(X)} \le C, \quad 0 \le \theta \le 1, \ 0 \le s < t \le T.$$

As for difference of the evolution operator and the semigroup, we verify the following. For $0 \le \theta \le 1$ and $0 \le \varphi \le 1$,

(2.8) $$\|A(t)^{\theta}\{U(t,s) - e^{-(t-s)A(t)}\}A(s)^{-\varphi}\|_{\mathcal{L}(X)} \le C(t-s)^{\varphi-\theta+\mu}, \quad 0 \le s < t \le T.$$

For $0 \le \theta \le 1$ and $0 \le \varphi \le 1$,

(2.9) $$\|A(t)^{\theta}\{U(t,s) - e^{-(t-s)A(s)}\}A(s)^{-\varphi}\|_{\mathcal{L}(X)} \le C(t-s)^{\varphi-\theta+\mu}, \quad 0 \le s < t \le T.$$

For $k = 0, 1$,

(2.10) $$\|A(t)^{k}U(t,s)A(s)^{-k} - e^{-(t-s)A(s)}\|_{\mathcal{L}(X)} \le C(t-s)^{\mu}, \quad 0 \le s \le t \le T.$$

For $0 < \theta < 1$,

(2.11) $$\begin{aligned}\|A(t)^{\theta}U(t,s)A(s)^{-\theta} - e^{-(t-s)A(s)}\|_{\mathcal{L}(X)} &\le C\log((t-s)^{-1}+1)(t-s)^{\mu},\\ &\qquad 0 \le s \le t \le T.\end{aligned}$$

Let us now consider the Cauchy problem of a linear evolution equation

(2.12) $$\begin{cases} \dfrac{dU}{dt} + A(t)U = F(t), \quad 0 < t \le T, \\ U(0) = U_0 \end{cases}$$

in X. We assume that the initial value U_0 is from

(2.13) $$U_0 \in \mathcal{D}(A(0)^{\beta}), \quad 0 \le \beta \le 1.$$

In addition, F is an X-valued Hölder continuous function such that

(2.14) $$F \in C^{\sigma}([0,T];X), \quad 0 < \sigma \le 1.$$

Then it is well known that (2.12) possesses a unique solution in the function space:

$$U \in C([0,T];X) \cap C^1((0,T];X) \cap C((0,T];\mathcal{D}).$$

And the solution is represented by the formula

(2.15) $$U(t) = U(t,0)U_0 + \int_0^t U(t,s)F(s)\,ds, \quad 0 \le t \le T$$

with an estimate

$$\left\| A(t)\int_0^t U(t,s)F(s)\,ds \right\|_X \le C_\sigma \|F\|_{C^\sigma}, \quad 0 < t \le T \tag{2.16}$$

(see [22, 23]).

Furthermore it is possible to verify the following refined properties

$$A^\beta U \in C([0,T];X) \quad \text{and} \quad t^{1-\beta}U \in C([0,T];\mathcal{D}). \tag{2.17}$$

In fact, let us first prove these in the case when $0 \le \beta < 1$. From (2.15),

$$\begin{aligned} A(t)^\beta U(t) &= A(0)^\beta U_0 + \{e^{-tA(0)} - 1\}A(0)^\beta U_0 \\ &\quad + \{A(t)^\beta U(t,0)A(0)^{-\beta} - e^{-tA(0)}\}A(0)^\beta U_0 + \int_0^t A(t)^\beta U(t,s)F(s)\,ds. \end{aligned}$$

Then, by (2.4) and (2.11), we conclude that $A(t)^\beta U(t)$ is convergent to $A(0)^\beta U_0$ as $t \to 0$; that is, $A(t)^\beta U(t)$ is continuous at $t = 0$ in X-norm. For $0 < t \le T$, we write $A(t)^\beta U(t) = A(t)^{\beta-1}A(t)U(t)$. By some calculation it is seen from (2.2) that $A(t)^{\beta-1}$ is Hölder continuous in t in $\mathcal{L}(X)$-norm. Hence, $A(t)^\beta U(t)$ is continuous for $0 < t \le T$ also.

To see the second assertion of (2.17), we write

$$\begin{aligned} A(t)U(t) &= A(t)U(t,0)A(0)^{-\beta}A(0)^\beta U_0 + \int_0^t A(t)U(t,s)\{F(s) - F(t)\}\,ds \\ &\quad + \int_0^t \{A(t)U(t,s) - A(t)e^{-(t-s)A(t)}\}\,ds\,F(t) + \{1 - e^{-tA(t)}\}F(t). \end{aligned} \tag{2.18}$$

Here it follows from (2.4) that

$$t^{1-\beta}\|A(t)U(t,0)A(0)^{-\beta}\|_{\mathcal{L}(X)} \le C, \quad 0 < t \le T;$$

and it is clear that $t^{1-\beta}A(t)U(t,0)A(0)^{-\beta}u \to 0$ as $t \to 0$ for every $u \in \mathcal{D}$; then, since $\mathcal{D}$ is dense in X, it follows that $t^{1-\beta}A(t)U(t,0)A(0)^{-\beta}f \to 0$ as $t \to 0$ for every $f \in X$. It is easy to see that

$$\left\| \int_0^t A(t)U(t,s)\{F(s) - F(t)\}\,ds \right\| \le C_\sigma t^\sigma \|F\|_{C^\sigma},$$

$$\left\| \int_0^t \{A(t)U(t,s) - A(t)e^{-(t-s)A(t)}\}\,ds\,F(t) \right\| \le Ct^\mu \|F\|_C.$$

Hence we conclude that $t^{1-\beta}A(t)U(t)$ converges to 0 as $t \to 0$ in X-norm. It is the same for $t^{1-\beta}A(0)U(t) = A(0)A(t)^{-1}t^{1-\beta}A(t)U(t)$. This means that $t^{1-\beta}U(t)$ is continuous at $t = 0$ with respect to $\mathcal{D}$-norm. Therefore the second assertion of (2.17) is also verified.

Let us now consider the case when $\beta = 1$. We use again the formula (2.18) with $\beta = 1$. From (2.10) it is seen that, as $t \to 0$, $A(t)U(t,0)A(0)^{-1}$ converges strongly to 1 on X. It is also easily seen that, as $t \to 0$, $e^{-tA(t)}$ converges strongly to 1 on X. Therefore, $A(t)U(t)$ converges to $A(0)U_0$. As the continuity of $A(t)U(t)$ is known for $0 < t \le T$, the first assertion of (2.17) is proved when $\beta = 1$. From $A(0)U(t) = A(0)A(t)^{-1}A(t)U(t)$, the second assertion is also proved.

3. Quasilinear abstract parabolic evolution equations

We consider the Cauchy problem for an abstract evolution equation

$$\begin{cases} \dfrac{dU}{dt} + A(U)U = F(U), & 0 < t < \infty, \\ U(0) = U_0 \end{cases} \tag{3.1}$$

in a Banach space X. Let Z be a second Banach space which is continuously embedded in X, and let K be an open ball of Z] such that

$$K = \{U \in Z; \|U\|_Z < R\}, \quad 0 < R < \infty.$$

For each $U \in K$, $A(U)$ is a densely defined closed linear operator in X with the domain $\mathcal{D}(A(U))$ independent of $U \in K$. F is a nonlinear operator from K into X. U_0 is an initial value at least from K.

We make the following structural assumptions.

The spectral set $\sigma(A(U))$ is contained in a fixed open sectorial domain

$$\sigma(A(U)) \subset \Sigma_\phi = \{\lambda \in \mathbb{C}; |\arg\lambda| < \phi\}, \quad 0 < \phi < \frac{\pi}{2},$$

and the resolvent satisfies

$$\|(\lambda - A(U))^{-1}\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda| + 1}, \quad \lambda \notin \Sigma_\phi, U \in K. \tag{3.2}$$

The domain $\mathcal{D}(A(U)) \equiv \mathcal{D}$ is independent of $U \in K$, $\mathcal{D}$ being a Banach space with a graph norm $\|\cdot\|_{\mathcal{D}} = \|A(0)\cdot\|_X$. And $A(U)$ is assumed to satisfy a Lipschitz condition

$$\|A(U)\{A(U)^{-1} - A(V)^{-1}\}\|_{\mathcal{L}(X)} \le N\|U - V\|_Y, \quad U, V \in K, \tag{3.3}$$

where Y is a third Banach space such that $Z \subset Y \subset X$ with continuous embedding.

The nonlinear operator F also satisfies a usual Lipschitz condition

$$\|F(U) - F(V)\|_X \le L\|U - V\|_Y, \quad U, V \in K. \tag{3.4}$$

There are two exponents $0 \le \alpha < \beta < 1$ such that $\mathcal{D}(A(U)^\alpha) \subset Y$and $\mathcal{D}(A(U)^\beta) \subset Z$ for every $U \in K$ with the estimates

$$\begin{cases} \|\widetilde{U}\|_Y \le D_1 \|A(U)^\alpha \widetilde{U}\|_X & \widetilde{U} \in \mathcal{D}(A(U)^\alpha), \ U \in K, \\ \|\widetilde{U}\|_Z \le D_2 \|A(U)^\beta \widetilde{U}\|_X, & \widetilde{U} \in \mathcal{D}(A(U)^\beta), \ U \in K, \end{cases} \tag{3.5}$$

D_i $(i = 1, 2)$ being some constants independent of $U \in K$.

For the initial value $U_0 \in K$, we assume a compatibility condition

$$U_0 \in \mathcal{D}(A(U_0)^\beta) \quad \text{with the same } \beta \text{ as above.} \tag{3.6}$$

Then the following result on local existence is proved.

Theorem 1. *Under* (3.2)–(3.5), *let* $U_0 \in K$ *satisfy the condition* (3.6). *Then, there exists a unique local solution to* (3.1) *in the function space*:

$$\begin{cases} U \in C^1((0, T_{U_0}]; X) \cap C^{\beta-\alpha}([0, T_{U_0}]; Y) \cap C([0, T_{U_0}]; Z), \\ A(U)^\beta U \in C([0, T_{U_0}]; X), \quad t^{1-\beta} U \in C([0, T_{U_0}]; \mathcal{D}). \end{cases} \tag{3.7}$$

Here, $T_{U_0} > 0$ *is determined by the norm* $\|A(U_0)^\beta U_0\|_X$ *and the modulus of continuity*

$$\omega_{U_0}(t) = \sup_{0 \le s \le t} \|\{e^{-sA(U_0)} - 1\} U_0\|_Z \quad \text{as} \quad t \to 0. \tag{3.8}$$

Note that from (3.5) *and* (3.6) *it holds that* $Z\text{-}\lim_{t\to 0} e^{-tA(U_0)} U_0 = U_0$.

Proof. Similar results have already been obtained in [29, 30] in which the domains $\mathcal{D}(A(U)$ are allowed to vary with $U \in K$ but the space Z is assumed to be reflexive. To get rid of the reflexiveness we shall need in this proof more refined arguments than those in [29, 30].

Our proof consists of several steps. Throughout the proof C denotes a universal constant which is determined in each occurrence by the exponents and by the initial constants appearing in the structural assumptions in a specific way.

STEP 1. For S such that $0 < S < \infty$, we set a Banach space

$$\mathcal{Z}(S) = C^\mu_{\{0\}}([0, S]; Y) \cap \mathcal{B}([0, S]; Z)$$

with some fixed exponent μ such that $0 < \mu < \beta - \alpha$.

Here, $C^\mu_{\{0\}}([0, S]; Y)$ denotes the space of Y-valued continuous functions which are Hölder continuous at the initial time, namely

$$C^\mu_{\{0\}}([0, S]; Y) = \left\{ U \in \mathcal{C}([0, S]; Y); \ \sup_{0 < t \le S} \frac{\|U(t) - U(0)\|_Y}{t^\mu} < \infty \right\}$$

equipped with the norm

$$\|U\|_{C^{\mu}_{\{0\}}([0,S];Y)} = \|U\|_{C([0,S];Y)} + \sup_{0<t\le S} \frac{\|U(t)-U(0)\|_Y}{t^{\mu}}.$$

It is readily verified that $C^{\mu}_{\{0\}}([0,S];Y)$ becomes a Banach space.

We set also a subset of $\mathcal{Z}(S)$ in such a way that

$$\mathcal{K}(S) = \Big\{ U \in \mathcal{Z}(S);\, U(0) = U_0, \quad \sup_{0\le t\le S} \|U(t)\|_Z \le R_1 \text{ and } \sup_{0\le s<t\le S} \frac{\|U(t)-U(s)\|_Y}{|t-s|^{\mu}} \le 1 \Big\}.$$

Here, R_1 is a constant fixed as

(3.9) $$\|U_0\|_Z < R_1 < R.$$

The nonempty set $\mathcal{K}(S)$ is clearly closed in $\mathcal{Z}(S)$.

STEP 2. For each $V \in \mathcal{K}(S)$, $A_V(t)$ denotes a family of linear operators $A_V(t) = A(V(t))$, $0 \le t \le S$. And F_V is a Hölder continuous function $F_V(t) = F(V(t))$, $0 \le t \le S$. We consider the Cauchy problem of a linear evolution equation

(3.10) $$\begin{cases} \dfrac{dU}{dt} + A_V(t)U = F_V(t), & 0 < t \le S, \\ U(0) = U_0. \end{cases}$$

It is quite easy to observe that $A_V(t)$ satisfies (2.1) and (2.2) with the exponent μ fixed above. As U_0 and F_V satisfy (2.13) and (2.14), respectively, there exists a unique solution U to (3.10) in the space:

$$\begin{cases} U \in C([0,S];X) \cap C^1((0,S];X) \cap C((0,S];\mathcal{D}), \\ A_V^{\beta} U \in C([0,S];X), \quad t^{1-\beta} U \in C([0,S];\mathcal{D}). \end{cases}$$

The solution U is indeed given by

$$U(t) = U_V(t,0)U_0 + \int_0^t U_V(t,s)F_V(s)\,ds, \quad 0 \le t \le S,$$

where $U_V(t,s)$ denotes the evolution operator for the family $A_V(t)$.

We can then define a mapping Φ from $\mathcal{K}(S)$ into $\mathcal{Z}(S)$ by setting $\Phi(V)(t) = U(t)$, $0 \le t \le S$, for each $V \in \mathcal{K}(S)$.

STEP 3. If $S > 0$ is sufficiently small, then Φ maps the set $\mathcal{K}(S)$ into itself.

Indeed, for $U = \Phi(V)$, we write $U(t)$ as

$$U(t) - U_0 = \{e^{-tA(U_0)} - 1\}U_0 + \{U_V(t,0) - e^{-tA_V(0)}\}A_V(0)^{-\beta}A(U_0)^\beta U_0 + \int_0^t U_V(t,s)F_V(s)\,ds.$$

Then by the same calculations as in Step 3 of the proof of [30, Theorem 3.1] (note that (3.3) implies clearly [30, (A.ii)] with $\nu = 1$) of using (3.5), (3.8), (2.4), and (2.9), we can verify that

$$\|U(t) - U_0\|_Z \le C\{\omega_{U_0}(t) + t^\mu \|U_0\|_\beta + t^{1-\beta}\}, \quad 0 \le t \le S, \tag{3.11}$$

here and in what follows $\|U_0\|_\beta$ stands for the quantity $\|A(U_0)^\beta U_0\|_X$. Hence, if $S > 0$ is sufficiently small, then (3.9) implies that

$$\sup_{0\le t\le S} \|U(t)\|_Z \le R_1. \tag{3.12}$$

In a similar way, we can estimate $\|A_V(t)^\beta U(t)\|_X$ also. In fact, from

$$A_V(t)^\beta U(t) = A_V(t)^\beta U_V(t,0)A_V(0)^{-\beta}A(U_0)^\beta U_0 + \int_0^t A_V(t)^\beta U_V(t,s)F_V(s)\,ds,$$

we verify that

$$\|A_V(t)^\beta U(t)\|_X \le C(\|U_0\|_\beta + 1), \quad 0 \le t \le S. \tag{3.13}$$

In order to verify the Hölder condition of U, let us write

$$\begin{aligned} U(t) - U(s) &= \{U_V(t,s) - 1\}U(s) + \int_s^t U_V(t,\tau)F_V(\tau)\,d\tau \\ &= \left[\{U_V(t,s) - e^{-(t-s)A_V(s)}\} + \{e^{-(t-s)A_V(s)} - 1\}\right] A_V(s)^{-\beta}A_V(s)^\beta U(s) \\ &\quad + \int_s^t U_V(t,\tau)F_V(\tau)\,d\tau, \quad 0 \le s < t \le S. \end{aligned}$$

Then, by the same calculations as in Step 3 of the proof of [30, Theorem 3.1] of using (2.3), (2.4), (2.9), (3.5), and (3.13), we can verify that

$$\|U(t) - U(s)\|_Y \le C(\|U_0\|_\beta + 1)(t-s)^{\beta-\alpha}, \quad 0 \le s \le t \le S. \tag{3.14}$$

Hence, if $S > 0$ is sufficiently small, then

$$\sup_{0\le s<t\le S} \frac{\|U(t) - U(s)\|_Y}{(t-s)^\mu} \le 1.$$

STEP 4. If $S > 0$ is sufficiently small, then the mapping $\Phi\colon \mathcal{K}(S) \to \mathcal{K}(S)$ is a contraction with respect to $\|\cdot\|_{\mathcal{Z}(S)}$-norm.

Indeed, for $U_i = \Phi(V_i)$, $V_i \in \mathcal{K}(S)$, $i = 1, 2$, we have

$$U_1(t) - U_2(t) = \{U_{V_1}(t,0) - U_{V_2}(t,0)\}U_0 + \int_0^t \{U_{V_1}(t,s) - U_{V_2}(t,s)\}F_{V_1}(s)\,ds + \int_0^t U_{V_2}(t,s)\{F_{V_1}(s) - F_{V_2}(s)\}\,ds.$$

Here we establish the following lemma.

Lemma 1. *For* $0 \le \theta < 1$,

$$\|A_{V_1}(t)^\theta\{U_{V_1}(t,0) - U_{V_2}(t,0)\}U_0\|_X \le C_\theta t^{\beta+\mu-\theta}\|U_0\|_\beta\|V_1 - V_2\|_{C^\mu_{\{0\}}([0,S];Y)}, \quad 0 \le t \le S. \tag{3.15}$$

Let $F \in C^\sigma([0,T];X), \sigma > 0$. *Then, for* $0 \le \theta < 1$,

$$\left\|A_{V_1}(t)^\theta \int_0^t \{U_{V_1}(t,s) - U_{V_2}(t,s)\}F(s)\,ds\right\|_X \le C_{\theta,\sigma}t^{1+\mu-\theta}\|F\|_{C^\sigma}\|V_1 - V_2\|_{C^\mu_{\{0\}}([0,S];Y)}, \quad 0 \le t \le S. \tag{3.16}$$

Proof. In order to verify these fundamental results, we have to employ the evolution operators $U_{V_i,n}(t,s)$ $(i = 1,2)$ for the families of Yosida approximation $A_{V_i,n}(t)$ $(i = 1,2)$ of $A_{V_i}(t)$ (cf. [23, p.207]). Indeed we observe that

$$\begin{aligned}
&A_{V_1,n}(t)^\theta\{U_{V_1,n}(t,0) - U_{V_2,n}(t,0)\}A_{V_2,n}(0)^{-\beta}\\
&= \int_0^t A_{V_1,n}(t)^\theta U_{V_1,n}(t,s)\\
&\qquad \times A_{V_1,n}(s)\{A_{V_1,n}(s)^{-1} - A_{V_2,n}(s)^{-1}\}A_{V_2,n}(s)U_{V_2,n}(s,0)A_{V_2,n}(0)^{-\beta}\,ds.
\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\begin{aligned}
&A_{V_1}(t)^\theta\{U_{V_1}(t,0) - U_{V_2}(t,0)\}A_{V_2}(0)^{-\beta}\\
&= \int_0^t A_{V_1}(t)^\theta U_{V_1}(t,s)\\
&\qquad \times A_{V_1}(s)\{A_{V_1}(s)^{-1} - A_{V_2}(s)^{-1}\}A_{V_2}(s)U_{V_2}(s,0)A_{V_2}(0)^{-\beta}\,ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|A_{V_1}(t)^\theta\{U_{V_1}(t,0)-U_{V_2}(t,0)\}A_{V_2}(0)^{-\beta}\|_{\mathcal{L}(X)}\\
&\le C\int_0^t (t-s)^{-\theta}s^{\beta-1}\|V_1(s)-V_2(s)\|_Y\,ds\\
&\le C\int_0^t (t-s)^{-\theta}s^{\mu+\beta-1}\,ds\|V_1-V_2\|_{C^\mu_{\{0\}}([0,S];Y)}.
\end{aligned}$$

From this the first assertion is verified.

Next, we write

$$\begin{aligned}
&A_{V_1,n}(t)^\theta\int_0^t\{U_{V_1,n}(t,s)-U_{V_2,n}(t,s)\}F(s)\,ds\\
&=\int_0^t\int_s^t A_{V_1,n}(t)^\theta U_{V_1,n}(t,\tau)\\
&\qquad\times A_{V_1,n}(\tau)\{A_{V_1,n}(\tau)^{-1}-A_{V_2,n}(\tau)^{-1}\}A_{V_2,n}(\tau)U_{V_2,n}(\tau,s)F(s)\,d\tau\,ds\\
&=\int_0^t A_{V_1,n}(t)^\theta U_{V_1,n}(t,\tau)A_{V_1,n}(\tau)\{A_{V_1,n}(\tau)^{-1}-A_{V_2,n}(\tau)^{-1}\}\\
&\qquad\times A_{V_2,n}(\tau)\int_0^\tau U_{V_2,n}(\tau,s)F(s)\,ds\,d\tau.
\end{aligned}$$

From (2.16), $A_{V_2,n}(\tau)\int_0^\tau U_{V_2,n}(\tau,s)F(s)\,ds$ satisfies a uniform estimate

$$\left\|A_{V_2,n}(\tau)\int_0^\tau U_{V_2,n}(\tau,s)F(s)\,ds\right\|_X\le C_\sigma\|F\|_{C^\sigma([0,S];X)},$$

and as $n\to\infty$,

$$A_{V_2,n}(\tau)\int_0^\tau U_{V_2,n}(\tau,s)F(s)\,d\tau\to A_{V_2}(\tau)\int_0^\tau U_{V_2}(\tau,s)F(s)\,ds=g(\tau).$$

Then, letting $n\to\infty$, we obtain that

$$\begin{aligned}
&A_{V_1}(t)^\theta\int_0^t\{U_{V_1}(t,s)-U_{V_2}(t,s)\}F(s)\,ds\\
&=\int_0^t A_{V_1}(t)^\theta U_{V_1}(t,\tau)A_{V_1}(\tau)\{A_{V_1}(\tau)^{-1}-A_{V_2}(\tau)^{-1}\}g(\tau)\,d\tau.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\|A_{V_1}(t)^\theta\int_0^t\{U_{V_1}(t,s)-U_{V_2}(t,s)\}F(s)\,ds\right\|_X\\
&\le C\int_0^t(t-\tau)^{-\theta}\|V_1(\tau)-V_2(\tau)\|_Y\,d\tau\|F\|_{C^\sigma}
\end{aligned}$$

$$\leq C \int_0^t (t-\tau)^{-\theta} \tau^{\mu} \, d\tau \|V_1 - V_2\|_{C^{\mu}_{\{0\}}((0,S];Y)} \|F\|_{C^{\sigma}}.$$

We have thus proved the second assertion of the lemma. □

Using this lemma with $\theta = \beta$, we obtain that

$$\begin{aligned}&\|\{U_{V_1}(t,0) - U_{V_2}(t,0)\}U_0\|_Z + \left\| \int_0^t \{U_{V_1}(t,s) - U_{V_2}(t,s)\} F_{V_1}(s)\, ds \right\|_Z \\ &\leq Ct^{\mu}(\|U_0\|_{\beta} + 1)\|V_1 - V_2\|_{C^{\mu}_{\{0\}}([0,S];Y)}.\end{aligned}$$

In addition, by (3.4) and (3.5), it is easy to see that

$$\left\| \int_0^t U_{V_2}(t,s)\{F_{V_1}(s) - F_{V_2}(s)\}\, ds \right\|_Z \leq Ct^{1+\mu-\beta}\|V_1 - V_2\|_{C^{\mu}_{\{0\}}([0,S];Y)}.$$

Hence,

$$\|U_1 - U_2\|_{\mathcal{B}([0,S];Z)} \leq CS^{\mu}(\|U_0\|_{\beta} + 1)\|V_1 - V_2\|_{C^{\mu}_{\{0\}}([0,S];Y)}. \tag{3.17}$$

$C^{\mu}_{\{0\}}$-norm of $U_1 - U_2$ is also estimated in a quite similar way by applying the lemma with $\theta = \alpha$. Indeed,

$$\begin{aligned}&\|\{U_{V_1}(t,0) - U_{V_2}(t,0)\}U_0\|_Y + \left\| \int_0^t \{U_{V_1}(t,s) - U_{V_2}(t,s)\} F_{V_1}(s)\, ds \right\|_Y \\ &\leq Ct^{\beta-\alpha+\mu}(\|U_0\|_{\beta} + 1)\|V_1 - V_2\|_{C^{\mu}_{\{0\}}([0,S];Y)}, \quad 0 < t \leq S.\end{aligned}$$

In addition,

$$\left\| \int_0^t U_{V_2}(t,s)\{F_{V_1}(s) - F_{V_2}(s)\}\, ds \right\|_Y \leq Ct^{1+\mu-\alpha}\|V_1 - V_2\|_{C^{\mu}_{\{0\}}([0,S];Y)}.$$

Therefore,

$$\begin{aligned}&\sup_{0<t\leq S} t^{-\mu}\|\{U_1(t) - U_2(t)\} - \{U_1(0) - U_2(0)\}\|_Y \\ &\leq CS^{\beta-\alpha}(\|U_0\|_{\beta} + 1)\|V_1 - V_2\|_{C^{\mu}_{\{0\}}([0,S];Y)}, \quad 0 < S \leq T.\end{aligned} \tag{3.18}$$

This together with (3.17) then yields that

$$\|U_1 - U_2\|_{\mathcal{Z}(S)} \leq CS^{\mu}(\|U_0\|_{\beta} + 1)\|V_1 - V_2\|_{\mathcal{Z}(S)}, \quad V_1, V_2 \in \mathcal{K}(S).$$

Hence, Φ is a contraction from $\mathcal{K}(S)$ into itself, provided S is sufficiently small.

STEP 5. Take a $T_{U_0} = S > 0$ in such a way that the results of Steps 3 and 4 are valid. Then, there exists a unique fixed point $U \in \mathcal{K}(S)$ of Φ. Since U satisfies the formula

$$U(t) = U_U(t,0)U_0 + \int_0^t U_U(t,s)F_U(s)\,ds, \quad 0 \le t \le T_{U_0}, \tag{3.19}$$

U is shown to be a local solution to (3.1) on the interval $[0, T_{U_0}]$ which satisfies all the conditions of (3.7), except that $U \in C([0, T_{U_0}]; Z)$.

From (3.11), it is seen that $U(t)$ is continuous at $t = 0$ in the Z-norm. Meanwhile, $U(t)$ is already known that $U \in C((0, T_{U_0}]; \mathcal{D}) \subset C((0, T_{U_0}]; Z)$. Therefore, $U \in C([0, T_{U_0}]; Z)$.

STEP 6. Finally the uniqueness of local solution in the space (3.7) is verified by the same arguments as in Step 6 of the proof of [30, Theorem 3.1]. So we omit the proof.

We have thus accomplished the proof of the theorem. □

For more regular initial values such as $U_0 \in \mathcal{D}(A(U_0)^\gamma)$ with an exponent γ, $\beta < \gamma \le 1$, we can prove a stronger result.

Corollary 1. *Let an initial value $U_0 \in K$ satisfy a stronger compatibility condition*

$$U_0 \in \mathcal{D}(A(U_0)^\gamma), \quad \beta < \gamma \le 1. \tag{3.20}$$

Then, the local solution U obtained in Theorem 1 *satisfies*:

$$\begin{cases} U \in C^1((0, T_{U_0}]; X) \cap C^{\gamma-\alpha}([0, T_{U_0}]; Y) \cap C^{\gamma-\beta}([0, T_{U_0}]; Z), \\ A(U)^\gamma U \in C([0, T_{U_0}]; X), \quad t^{1-\gamma} U \in C([0, T_{U_0}]; \mathcal{D}). \end{cases} \tag{3.21}$$

Furthermore, $T_{U_0} > 0$ is determined by $\|A(U_0)^\gamma U_0\|_X$ alone.

Proof. By the same arguments as in Steps 3 and 5 of the proof of Theorem 1, we can verify from (3.20) that the solution U belongs to (3.21). Dependence of T_{U_0} on $\|A(U_0)^\gamma U_0\|_X$ alone is verifed as in [30, Remark 3.1]. □

We finally notice some global existence result. For an initial values U_0 satisfying (3.20), assume that every local solution to (3.1) satisfies a priori estimates

$$\begin{cases} \|U(t)\|_Z \le R_{U_0} < R, & 0 \le t \le T_U, \\ \|A(U(t))^\gamma U(t)\|_X \le C_{U_0}, & 0 \le t \le T_U \end{cases}$$

with some uniform constants R_{U_0} and C_{U_0} independent of T_U. Then (3.1) possesses a global solution on the whole interval $[0, \infty)$. In fact this is now clear, because Corollary 1 yields that any local solution on an interval $[0, T_U]$ can be extended over the interval $[0, T_U + \tau]$ with some fixed time length $\tau > 0$ independent of T_U.

4. Lipschitz continuity for initial values

We shall verify Lipschitz continuity of solutions to (3.1) with respect to the initial values. For this purpose we introduce a set of initial values

$$B = \{U_0 \in Z;\ \|U_0\|_Z \leq R_1 \text{ and } \|A(U_0)^\gamma U_0\|_X \leq C_1\}, \quad \beta < \gamma \leq 1$$

with some constants $0 < R_1 < R$ and $0 < C_1 < \infty$. Then, for each $U_0 \in B$, there exists a unique local solution. Moreover, by Corollary 1, we see that (3.1) possesses a local solution in the space (3.21) at least over a fixed interval $[0, T_B]$ for every initial value $U_0 \in B$, $T_B > 0$ being determined from the set B.

We then show the following theorem.

Theorem 2. *Let* (3.2)–(3.5) *be satisfied. Let U and V be the local solutions to* (3.1) *with initial values U_0 and V_0 in the set B, respectively. Then there exists some constant $C_B > 0$ depending on the set B alone such that*

$$\begin{aligned} &t^\beta \|U(t) - V(t)\|_Z + t^\alpha \|U(t) - V(t)\|_Y + \|U(t) - V(t)\|_X \\ &\leq C_B \|U_0 - V_0\|_X, \quad 0 \leq t \leq T_B. \end{aligned}$$

Proof. Let $U_U(t, s)$ (resp. $U_V(t, s)$) denote the evolution operator for a family of linear operators $A_U(t) = A(U(t))$ (resp. $A_V(t) = A(V(t))$).

From (3.19) we have

(4.1)
$$\begin{aligned} U(t) - V(t) &= U_U(t, 0)(U_0 - V_0) + \{U_U(t, 0) - U_V(t, 0)\}V_0 \\ &\quad + \int_0^t \{U_U(t, s) - U_V(t, s)\}F_V(s)\, ds + \int_0^t U_U(t, s)\{F_U(s) - F_V(s)\}\, ds. \end{aligned}$$

Let us first estimate Y-norm of $U(t) - V(t)$. By (2.4) and (3.5) we have

$$\|U_U(t, 0)(U_0 - V_0)\|_Y \leq D_1 \|A_U(t)^\alpha U_U(t, 0)(U_0 - V_0)\|_X \leq C t^{-\alpha} \|U_0 - V_0\|_X.$$

By (3.4) and (3.5),

$$\left\| \int_0^t U_U(t, s)\{F_U(s) - F_V(s)\}\, ds \right\|_Y \leq C \int_0^t (t - s)^{-\alpha} \|U(s) - V(s)\|_Y\, ds.$$

For estimating other terms in the right hand side of (4.1), we repeat the same argument as in the proof of Lemma 1. Indeed, arguing in the same way as for (3.15) with $\theta = \alpha$, we observe that

$$\|A_U(t)^\alpha\{U_U(t,0) - U_V(t,0)\}V_0\|_X \le C\|V_0\|_\beta \int_0^t (t-s)^{-\alpha} s^{\beta-1}\|U(s) - V(s)\|_Y\, ds.$$

Similarly, in the same way as for (3.16) with $\theta = \alpha$,

$$\left\|A_U(t)^\alpha \int_0^t \{U_U(t,s) - U_V(t,s)\}F_V(s)\,ds\right\|_X \le C\|F_V\|_{C^\mu} \int_0^t (t-s)^{-\alpha}\|U(s) - V(s)\|_Y\, ds.$$

Thus we obtain an integral inequality

$$\varphi(t) \le C\|U_0 - V_0\|_X + C_B t^\alpha \int_0^t (t-s)^{-\alpha} s^{\beta-\alpha-1}\varphi(s)\, ds$$

which is satisfied by $\varphi(t) = t^\alpha\|U(t) - V(t)\|_Y$.

For all s such that $0 \le s \le t$, we then see that

$$\begin{aligned}\varphi(s) &\le C\|U_0 - V_0\|_X + C_B s^\alpha \int_0^s (s-\sigma)^{-\alpha}\sigma^{\beta-\alpha-1} d\sigma \sup_{0\le\sigma\le s}\varphi(\sigma)\\ &\le C\|U_0 - V_0\|_X + C_B t^{\beta-\alpha} \sup_{0\le s\le t}\varphi(s).\end{aligned}$$

Therefore,

$$\{1 - C_B t^{\beta-\alpha}\} \sup_{0\le s\le t}\varphi(s) \le C\|U_0 - V_0\|_X.$$

This shows that, if t is sufficiently small, say $0 \le t \le \varepsilon_B$ with some fixed $\varepsilon_B > 0$, then

$$\varphi(t) \le \sup_{0\le s\le t}\varphi(s) \le C\|U_0 - V_0\|_X, \quad 0 \le t \le \varepsilon_B.$$

It now suffices to consider the case when $t \ge \varepsilon_B$. Then,

$$\begin{aligned}\varphi(t) &\le C\|U_0 - V_0\|_X + C_B t^\alpha\|U_0 - V_0\|_X \int_0^{\varepsilon_B} (t-s)^{-\alpha} s^{\beta-\alpha-1}\, ds\\ &\quad + C_B t^\alpha \varepsilon_B^{\beta-\alpha-1} \int_{\varepsilon_B}^t (t-s)^{-\alpha}\varphi(s)\, ds \le C_B\left(\|U_0 - V_0\|_X + \int_{\varepsilon_B}^t (t-s)^{-\alpha}\varphi(s) ds\right).\end{aligned}$$

Solving this integral inequality of Gronwall's type, we conclude that

$$\varphi(t) \le C_B\|U_0 - V_0\|_X, \quad \varepsilon_B \le t \le T_B.$$

Hence,

$$t^\alpha\|U(t) - V(t)\|_Y \le C_B\|U_0 - V_0\|_X, \quad 0 \le t \le T_B. \tag{4.2}$$

Estimation of Z and X-norms of $U(t) - V(t)$ is now immediate. By (2.4) and (3.5),

$$\|A_U(t)^\beta U_U(t,0)(U_0 - V_0)\|_X \le Ct^{-\beta}\|U_0 - V_0\|_X.$$

Similarly, by (2.4) and (4.2),

$$\begin{aligned}
&\left\|A_U(t)^\beta \int_0^t U_U(t,s)\{F_U(s) - F_V(s)\}\,ds\right\|_X \\
&\le C\int_0^t (t-s)^{-\beta}\|U(s) - V(s)\|_Y\,ds \\
&\le C_B\int_0^t (t-s)^{-\beta}s^{-\alpha}ds\|U_0 - V_0\|_X \le C_B t^{1-\alpha-\beta}\|U_0 - V_0\|_X.
\end{aligned}$$

In addition, by the same argument as for (3.15) and (3.16) with $\theta = \beta$, we verify that

$$\begin{aligned}
&\|A_U(t)^\beta\{U_U(t,0) - U_V(t,0)\}V_0\|_X + \left\|A_U(t)^\beta \int_0^t \{U_U(t,s) - U_V(t,s)\}F_V(s)\,ds\right\|_X \\
&\le C_B\int_0^t (t-s)^{-\beta}s^{\beta-1}\|U(s) - V(s)\|_Y\,ds \\
&\le C_B\int_0^t (t-s)^{-\beta}s^{\beta-\alpha-1}\,ds\|U_0 - V_0\|_X \le C_B t^{-\alpha}\|U_0 - V_0\|_X.
\end{aligned}$$

Summing up these estimates, we conclude that

$$t^\beta\|U(t) - V(t)\|_Z \le D_2 t^\beta\|A_U(t)^\beta\{U(t) - V(t)\}\|_X \le C_B\|U_0 - V_0\|_X, \quad 0 \le t \le T_B.$$

It is similar for the estimation of $\|U(t) - V(t)\|_X$. We may argue as for (3.15) and (3.16) with $\theta = 0$. □

5. Exponential attractors

Let X be a Banach space with norm $\|\cdot\|_X$. Let $\mathcal{X}$ be a subset of X, $\mathcal{X}$ being a metric space with the distance $d(\cdot\,,\,\cdot)$ induced from $\|\cdot\|_X$. A family of nonlinear operators $S(t)$, $0 \le t < \infty$, from $\mathcal{X}$ into itself is called a semigroup on $\mathcal{X}$ if $S(0) = 1$ (identity in $\mathcal{X}$) and $S(t+s) = S(t)S(s)$ for $0 \le t, s < \infty$. A semigroup is called a continuous semigroup on $\mathcal{X}$ if

(5.1) $\quad G(t, U_0) = S(t)U_0$ is a continuous mapping from $[0,\infty)\times\mathcal{X}$ into $\mathcal{X}$.

Let $S(t)$ be a continuous semigroup on $\mathcal{X}$. Then the set of all $\mathcal{X}$-valued continuous functions $S(\cdot)U_0$, $U_0 \in \mathcal{X}$, on $[0,\infty)$ is called a dynamical system determined by the semigroup $S(t)$ on the phase space $\mathcal{X}$ in the universal space X. The system is denoted by $(S(t), \mathcal{X}, X)$.

From now on we assume that the phase space is a compact set of X. From the compactness of $\mathcal{X}$, it is immediately seen that the set

$$\mathcal{A} = \bigcap_{0 \le t < \infty} S(t)\mathcal{X}$$

is a global attractor of $(S(t), \mathcal{X}, X)$. That is, $\mathcal{A}$ is a compact set of X, $\mathcal{A}$ is an invariant set of $S(t)$ (this means that $S(t)\mathcal{A} = \mathcal{A}$ for every $t \ge 0$), and $\mathcal{A}$ attracts $\mathcal{X}$ in the sense that $h(S(t)\mathcal{X}, \mathcal{A})$ converges to 0 as $t \to \infty$, where $h(\,\cdot\,,\,\cdot\,)$ is the Hausdorff pseudodistance defined by (1.1).

The exponential attractor is then defined as follows (see Eden et al. [4]). a subset $\mathcal{M}$ such that $\mathcal{A} \subset \mathcal{M} \subset \mathcal{X}$ is called an exponential attractor of $(S(t), \mathcal{X}, X)$ if

(1) $\mathcal{M}$ is a compact subset of X with finite fractal dimension;
(2) $\mathcal{M}$ is a positively invariant set of $S(t)$, namely $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t \ge 0$;
(3) $\mathcal{M}$ attracts the whole space $\mathcal{X}$ exponentially in the sense that

$$h(S(t)\mathcal{X}, \mathcal{M}) \le Ce^{-\delta t}, \quad 0 \le t < \infty$$

with some exponent $\delta > 0$ and a constant $C > 0$.

Concerning construction of exponential attractors we present a method of [6]. We assume the following two conditions. There exists another Banach space $Z \subset X$ with a compact embedding such that the operator $S(t^*)$ with some fixed $t^* > 0$ satisfies a Lipschitz condition of the form

$$\|S(t^*)U_0 - S(t^*)V_0\|_Z \le L_1\|U_0 - V_0\|_X, \quad U_0, V_0 \in \mathcal{X} \tag{5.2}$$

with a constant $L_1 > 0$. The mapping $G(t, U_0) = S(t)U_0$ from $[0, t^*] \times \mathcal{X}$ into $\mathcal{X}$ satisfies the usual Lipschitz condition

$$\|G(t, U_0) - G(s, V_0)\|_X \le L_2\{|t - s| + \|U_0 - V_0\|_X\}, \quad t, s \in [0, t^*],\ U_0, V_0 \in \mathcal{X}. \tag{5.3}$$

Theorem 3. *Let $S(t^*)$ satisfy* (5.2) *with some Banach space Z embedded compactly in X and let G satisfy* (5.3). *Then, an exponential attractor $\mathcal{M}$ is constructed for the dynamical system $(S(t), \mathcal{X}, X)$.*

Proof. It is known by [6, Proposition 1] that, under the Lipschitz condition (5.2), an exponential attractor $\mathcal{M}^*$ is constructed for a discrete dynamical system $(S(t^*)^n, \mathcal{X}, X)$ defined by $S(t^*)$. Then it is easy to construct an exponential attractor for the continuous dynamical system on the basis of $\mathcal{M}^*$ and (5.3), see [4, Theorem 3.1]. □

In the second half of this section we shall describe a general strategy for applying Theorem 3 to a dynamical system determined from the Cauchy problem of an abstract parabolic evolution equation.

Let X be a reflexive Banach space. We consider the Cauchy problem for an abstract parabolic evolution equation

$$\begin{cases} \dfrac{dU}{dt} + A(U)U = F(U), & 0 < t < \infty, \\ U(0) = U_0 \end{cases} \tag{5.4}$$

in X. For each $U \in Z$, $A(U)$ is a densely defined closed linear operator in X with a constant domain $\mathcal{D}(A(U)) \equiv \mathcal{D}$, where $Z \subset X$ is a second Banach space with a continuous embedding. The domain $\mathcal{D}$ is a Banach space with a graph norm $\|\cdot\|_{\mathcal{D}} = \|A(0)\cdot\|_X$. F is a nonlinear operator from Z into X.

For $0 < R < \infty$, let

$$K_R = \{U \in Z;\ \|U\|_Z < R\}.$$

We assume that, for each $R > 0$, the family of linear operators $A(U)$, $U \in K_R$, and the nonlinear operator $F\colon K_R \to X$ satisfy all the structural conditions (3.2)–(3.5) announced in Section 3 with a third Banach space Y such that $Z \subset Y \subset X$ which is independent of R. If necessary, we may replace $A(U)$ (resp. F) by $A(U) + k_R$ (resp. $F + k_R$) in the equation of (5.4), where k_R is some sufficiently large constant depending on R, for verifying (3.2) and (3.3). Since

$$F(U) - A(U)U = \{F(U) + k_R U\} - \{A(U) + k_R\}U, \quad U \in \mathcal{D},$$

such replacement does not cause any essential change of equations.

In addition to these conditions, we assume that

$$Z \text{ is compactly embedded in } X. \tag{5.5}$$

Let γ, where $\beta < \gamma \leq 1$, be an exponent such that the condition

$$\mathcal{D}_\gamma \equiv \mathcal{D}(A(U)^\gamma), \quad U \in Z \tag{5.6}$$

holds. Of course this condition is always true if we take $\gamma = 1$. $\mathcal{D}_\gamma$ is a Banach space with a graph norm $\|\cdot\|_{\mathcal{D}_\gamma} = \|A(0)^\gamma \cdot\|_X$. Since $\mathcal{D}_\gamma = \mathcal{D}(A(0)^\gamma) \subset \mathcal{D}(A(0)^\beta) \subset Z$, (5.5) implies naturally that $\mathcal{D}_\gamma$ is also compactly embedded in X. Meanwhile, the reflexivity of X implies that of $\mathcal{D}_\gamma$.

Let B be any bounded set of $\mathcal{D}_\gamma$, and take a semidiameter R of K_R sufficiently large in such a way that $B \subset K_R$. By Corollary 1, for every $U_0 \in B$, there exists a unique local solution to (5.4) on a fixed interval $[0, T_B]$, $T_B > 0$ is determined from B. If we can show a priori estimates for all local solutions starting from B, then the global solutions are constructed. In fact, assume that there exist constants R_B

and C_B such that the estimates

$$\begin{cases} \|U(t)\|_Z \le R_B < R, & 0 \le t \le T_U, \\ \|U(t)\|_{\mathcal{D}_\gamma} \le C_B, & 0 \le t \le T_U \end{cases} \tag{5.7}$$

hold for every local solution U on $[0, T_U]$ with $U(0) = U_0 \in B$. Then, (5.4) possesses a global solution on $[0, \infty)$ for every $U_0 \in B$.

Furthermore, if such a result is true for each bounded set $B \subset \mathcal{D}_\gamma$, then (5.4) possesses a global solution for every initial value $U_0 \in \mathcal{D}_\gamma$ in the space:

$$U \in C^{\gamma-\alpha}([0,\infty); Y) \cap C^{\gamma-\beta}([0,\infty); Z) \cap C([0,\infty); \mathcal{D}_\gamma), \quad t^{1-\gamma} U \in C([0,\infty); \mathcal{D}).$$

As a result, we can define a semigroup $S(t)$ which maps $\mathcal{D}_\gamma$ into itself and maps $\mathcal{D}_\gamma$ into $\mathcal{D}$ for $t > 0$ by setting $S(t)U_0 = U(t)$, where U is the global solution with $U(0) = U_0$. Set, for each bounded set B,

$$\mathcal{B} = \overline{\bigcup_{0 \le t < \infty} S(t)B} \quad \text{(the closure in the norm } \|\cdot\|_X). \tag{5.8}$$

The second estimate of (5.7) jointed with reflexivity of $\mathcal{D}_\gamma$ implies that $\mathcal{B}$ is a bounded set of $\mathcal{D}_\gamma$. Therefore, $\mathcal{B}$ is a compact set of X. Utilizing Theorem 2 finite times in view of (5.7), $S(t)$ is, for any t, a continuous mapping from $(\mathcal{B}, d)$ into X. Consequently,

$$S(t)\overline{\bigcup_{0 \le t < \infty} S(t)B} \subset \overline{S(t) \bigcup_{0 \le t < \infty} S(t)B} \subset \overline{\bigcup_{0 \le t < \infty} S(t)B},$$

this shows that $\mathcal{B}$ is a positively invariant set of $S(t)$. According to Theorem 2 again, $S(t)$ is Lipschitz continuous from $(\mathcal{B}, d)$ into X and the Lipschitz constant is uniform in any bounded interval $[0, T]$; this then yields that $G(t, U_0) = S(t)U_0$ is a continuous mapping from $[0, \infty) \times (\mathcal{B}, d)$ into X. In this way we have constructed a dynamical system $(S(t), \mathcal{B}, X)$ determined from the problem (5.4).

The crucial part is to establish an absorbing estimate. We show that there is an absolute constant C such that, for every bounded set B of $\mathcal{D}_\gamma$, there is a time $t_B > 0$ for which the following estimate holds:

$$\sup_{t \ge t_B} \sup_{U_0 \in B} \|S(t)U_0\|_{\mathcal{D}} \le C.$$

Using this constant C, we define a set

$$\mathcal{X}_1 = \{U \in \mathcal{D}; \|U\|_{\mathcal{D}} \le C\}.$$

In terms of the dynamical system, when $\mathcal{B} \supset \mathcal{X}_1$, $\mathcal{X}_1$ is always an absorbing set of $(S(t), \mathcal{B}, X)$, that is there is a time $t_{\mathcal{B}} > 0$ such that

$$S(t)\mathcal{B} \subset \mathcal{X}_1 \quad \text{for all } t \geq t_{\mathcal{B}}. \tag{5.9}$$

In addition we set

$$\mathcal{X} = \overline{\bigcup_{0 \leq t < \infty} S(t)S(t_{\mathcal{X}_1})\mathcal{X}_1} = \overline{\bigcup_{t_{\mathcal{X}_1} \leq t < \infty} S(t)\mathcal{X}_1} \subset \mathcal{X}_1, \tag{5.10}$$

where $t_{\mathcal{X}_1} > 0$ is a time such that $S(t)\mathcal{X}_1 \subset \mathcal{X}_1$ for all $t \geq t_{\mathcal{X}_1}$. Then, by the same argument as above, $\mathcal{X}$ is a compact set of X and is a positively invariant set of $S(t)$; in particular, $(S(t), \mathcal{X}, X)$ is also a dynamical system. Furthermore in the sense of (5.9), every dynamical system $(S(t), \mathcal{B}, X)$ with $\mathcal{B} \supset \mathcal{X}$ is reduced to the dynamical system $(S(t), \mathcal{X}, X)$ in finite time $(\geq t_{\mathcal{B}} + t_{\mathcal{X}_1})$.

We are now ready to apply Theorem 3 to the system $(S(t), \mathcal{X}, X)$. From Theorem 2, $S(t^*)$ satisfies with sufficiently small time $t^* > 0$ the condition (5.2). Similarly, we have

$$\|S(t)U_0 - S(t)V_0\|_X \leq C\|U_0 - V_0\|_X, \quad 0 \leq t \leq t^*, U_0, V_0 \in \mathcal{X}.$$

In addition, for $S(t)U_0 = U(t)$,

$$\begin{aligned} \|S(t)U_0 - S(s)U_0\|_X &= \left\| \int_s^t \frac{dU}{d\tau}(\tau)\, d\tau \right\|_X = \left\| \int_s^t \{F(U(\tau)) - A(U(\tau))U(\tau)\}\, d\tau \right\|_X \\ &\leq C(t-s) \sup_{0 \leq \tau \leq t^*} \|U(\tau)\|_{\mathcal{D}} \leq C(t-s), \quad 0 \leq s < t \leq t^*. \end{aligned}$$

Therefore, (5.3) is also fulfilled.

In this way we can construct an exponential attractor for $(S(t), \mathcal{X}, X)$.

6. Outline of application to chemotaxis-growth system

6.1. Chemotaxis-growth system. We are concerned with the Cauchy problem of the following chemotaxis-growth system

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - \nabla \cdot \{u \nabla \chi(\rho)\} + f(u) & \text{in} \quad \Omega \times (0, \infty), \\ \dfrac{\partial \rho}{\partial t} = b\Delta\rho - c\rho + du & \text{in} \quad \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x) & \text{in} \quad \Omega, \end{cases} \tag{6.1}$$

where Ω is a bounded convex domain in $\mathbb{R}^2$.

Here, $\chi(\rho)$ is a real smooth function of $\rho \in (-\infty, \infty)$ with uniformly bounded derivatives up to the third order

$$\sup_{-\infty<\rho<\infty} \left| \frac{d^i \chi}{d\rho^i}(\rho) \right| < \infty, \quad i = 1, 2, 3. \tag{6.2}$$

The function $f(u)$ is a real smooth function of $u \in (-\infty, \infty)$ such that $f(0) = 0$ and

$$f(u) = (-\mu u + \nu)u \quad \text{for sufficiently large} \quad |u| \tag{6.3}$$

with two constants $\mu > 0$ and $-\infty < \nu < \infty$.

As for derivation of this system, see [14] and [18].

In this section we use the following notations. Ω is a bounded convex domain in the plane. As well known, a convex domain is a Lipschitz domain (cf. [9, Corollary 1.2.2.3]). For $0 \le s < \infty$, $H^s(\Omega)$ denotes the Sobolev space, its norm being denoted by $\|\cdot\|_{H^s}$ (see [9, Chap. 1] and [27]). For $0 \le s_0 \le s \le s_1 \le 2$, $H^s(\Omega)$ coincides with the complex interpolation space $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$, where $s = (1-\theta)s_0 + \theta s_1$, and the estimate

$$\|\cdot\|_{H^s} \le C \|\cdot\|_{H^{s_0}}^{1-\theta} \|\cdot\|_{H^{s_1}}^{\theta} \tag{6.4}$$

holds. When $0 \le s < 1$, $H^s(\Omega) \subset L^p(\Omega)$, where $1/p = (1-s)/2$, with continuous embedding. When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ for any finite $1 \le q < \infty$ with the estimate

$$\|\cdot\|_{L^q} \le C_{p,q} \|\cdot\|_{H^1}^{1-p/q} \|\cdot\|_{L^p}^{p/q},$$

where $1 \le p < q < \infty$ (by virtue of Stein [21, Chap. VI,Theorem 5] this can be verified even in a Lipschitz domain). When $s > 1$, $H^s(\Omega) \subset C(\overline{\Omega})$ with continuous embedding.

We shall make use of the following known estimates (see [18]). For any $0 < \varepsilon \le 1$,

$$\|uv\|_{H^{1+\varepsilon}} \le C_\varepsilon \|u\|_{H^{1+\varepsilon}} \|v\|_{H^{1+\varepsilon}}, \quad u, v \in H^{1+\varepsilon}(\Omega). \tag{6.5}$$

Let $\chi_1(\rho)$ be a smooth function defined for $-\infty < \rho < \infty$. Then, for any $0 < \varepsilon \le 1$,

$$\|\chi_1(\mathrm{Re}\,\rho)\|_{H^{1+\varepsilon}} \le p_\varepsilon(\|\rho\|_{H^{1+\varepsilon}}), \quad \rho \in H^{1+\varepsilon}(\Omega), \tag{6.6}$$

$$\begin{aligned} \|\chi_1(\mathrm{Re}\,\rho) - \chi_1(\mathrm{Re}\,\eta)\|_{H^{1+\varepsilon}} \le p_\varepsilon(\|\rho\|_{H^{1+\varepsilon}} + \|\eta\|_{H^{1+\varepsilon}}) \|\rho - \eta\|_{H^{1+\varepsilon}}, \\ \rho, \eta \in H^{1+\varepsilon}(\Omega), \end{aligned} \tag{6.7}$$

where $p_\varepsilon(\,\cdot\,)$ denotes some continuous increasing function determined from $\chi_1(\,\cdot\,)$.

From these facts we immediately verify that

$$\begin{aligned} &\|\nabla \cdot \{u\chi_1 \nabla\rho\}\|_{L^2} \le C_\varepsilon \|u\|_{H^{1+\varepsilon}} \|\chi_1\|_{H^{1+\varepsilon}} \|\rho\|_{H^2}, \\ &u \in H^{1+\varepsilon}(\Omega),\ \chi_1 \in H^{1+\varepsilon}(\Omega),\ \rho \in H^2_N(\Omega) \end{aligned} \tag{6.8}$$

with an arbitrary ε, $0 < \varepsilon < 1$. For the definition of the space $H^2_N(\Omega)$, see (6.11). The $H^1(\Omega)'$-norm of $\nabla \cdot \{u\chi_1 \nabla\rho\}$ is estimated as follows. We have

$$\begin{aligned}
&|\langle \nabla \cdot \{u\chi_1\nabla\rho\}, w\rangle_{(H^1)'\times H^1}| \\
&= \left| \int_\Omega u\chi_1 \nabla\rho \cdot \nabla w dx \right| \\
&\le \|u\|_{L^{2/(1-\varepsilon)}} \|\chi_1\|_{L^\infty} \|\nabla\rho\|_{L^{2/(\varepsilon)}} \|\nabla w\|_{L^2} \le C_\varepsilon \|u\|_{H^\varepsilon} \|\chi_1\|_{H^{1+\varepsilon}} \|\rho\|_{H^{2-\varepsilon}} \|w\|_{H^1}
\end{aligned}$$

with $0 < \varepsilon \le 1$. Therefore,

$$\begin{aligned}
&\|\nabla \cdot \{u\chi_1\nabla\rho\}\|_{(H^1)'} \le C_\varepsilon \|u\|_{H^\varepsilon} \|\chi_1\|_{H^{1+\varepsilon}} \|\rho\|_{H^{2-\varepsilon}}, \\
&u \in H^\varepsilon(\Omega),\ \chi_1 \in H^{1+\varepsilon}(\Omega),\ \rho \in H^2_N(\Omega)
\end{aligned} \tag{6.9}$$

with an arbitrary $0 < \varepsilon < 1$. Let us consider $\nabla \cdot \{u\chi_1\nabla\rho\}$ as a linear operator with respect to u, then it is a bounded operator from $H^{1+\varepsilon}(\Omega)$ to $L^2(\Omega)$ and, at the same time, from $H^\varepsilon(\Omega)$ to $H^1(\Omega)'$. So by interpolation we obtain that

$$\begin{aligned}
&\|\nabla \cdot \{u\chi_1\nabla\rho\}\|_{(H^{1/2})'} \le C_\varepsilon \|u\|_{H^{1/2+\varepsilon}} \|\chi_1\|_{H^{1+\varepsilon}} \|\rho\|_{H^2}, \\
&u \in H^{1/2+\varepsilon}(\Omega),\ \chi_1 \in H^{1+\varepsilon}(\Omega),\ \rho \in H^2_N(\Omega)
\end{aligned} \tag{6.10}$$

with $0 < \varepsilon < 1$.

Consider a sesquilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla\overline{v}\, dx + \int_\Omega u\overline{v}\, dx, \quad u, v \in H^1(\Omega).$$

From this form we can define realization of the Laplace operator $-\Delta + 1$ in Ω under the Neumann boundary conditions on $\partial\Omega$ (see Lions and Magenes [11, Chap. 2, No 9]). Identifying $L^2(\Omega)$ and its dual $L^2(\Omega)'$, we consider a triplet of spaces $H^1(\Omega) \subset L^2(\Omega) \subset H^1(\Omega)'$. Then, $\mathcal{A} = -\Delta + 1$ becomes a densely defined closed linear operator of $H^1(\Omega)'$ with $\mathcal{D}(\mathcal{A}) = H^1(\Omega)$. Meanwhile, the part of $\mathcal{A}$ in $L^2(\Omega)$, which is defined by $Au = \mathcal{A}u$ for $u \in \mathcal{D}(A) = \{u \in H^1(\Omega); \mathcal{A}u \in L^2(\Omega)\}$, is a positive definite self-adjoint operator of $L^2(\Omega)$ with

$$\mathcal{D}(A) = H^2_N(\Omega) = \left\{ u \in H^2(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}. \tag{6.11}$$

By the convexity of Ω, one can conclude $u \in H^2(\Omega)$ from $\Delta u \in L^2(\Omega)$ (see Grisvard [9, Theorem 3.2.1.3]). Then, we have

$$\mathcal{D}(\mathcal{A}^\theta) = [H^1(\Omega)', H^1(\Omega)]_\theta = \begin{cases} H^{1-2\theta}(\Omega)', & \text{when } 0 \le \theta \le \dfrac{1}{2}, \\ H^{2\theta-1}(\Omega), & \text{when } \dfrac{1}{2} \le \theta \le 1 \end{cases} \tag{6.12}$$

(cf. [11, Chap. 1, Proposition 2.1]), and

$$\mathcal{D}(A^\theta) = [L^2(\Omega), H^2_N(\Omega)]_\theta = \begin{cases} H^{2\theta}(\Omega), & \text{when } 0 \le \theta < \dfrac{3}{4} \\ H^{2\theta}_N(\Omega), & \text{when } \dfrac{3}{4} < \theta \le 1 \end{cases} \tag{6.13}$$

(see [16, Sec. 2]).

6.2. Abstract formulation. In formulating the chemotaxis-growth system as an abstract equation, we set the underlying space as

$$X = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix} ; u \in H^1(\Omega)' \text{ and } \rho \in L^2(\Omega) \right\}. \tag{6.14}$$

In addition, Y and Z are set as

$$Y = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix} ; u \in H^{\varepsilon_1}(\Omega) \text{ and } \rho \in H^{1+\varepsilon_1}(\Omega) \right\}, \tag{6.15}$$

$$Z = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix} ; u \in H^{\varepsilon_2}(\Omega) \text{ and } \rho \in H^{1+\varepsilon_2}(\Omega) \right\} \tag{6.16}$$

with arbitrarily fixed two positive exponents $0 < \varepsilon_1 < \varepsilon_2 < 1/2$.

For each $U = \binom{u}{\rho} \in Z$, a linear operator $A(U)$ is defined by

$$A(U)\widetilde{U} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1(U) \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} \widetilde{u} \\ \widetilde{\rho} \end{pmatrix}, \quad \widetilde{U} = \begin{pmatrix} \widetilde{u} \\ \widetilde{\rho} \end{pmatrix} \in \mathcal{D}(A(U)). \tag{6.17}$$

Here, $\mathcal{A}_1 = -a\Delta + 1$ is a closed linear operator of $H^1(\Omega)'$ with $\mathcal{D}(\mathcal{A}_1) = H^1(\Omega)$, $A_2 = -b\Delta + c$ is a self-adjoint operator of $L^2(\Omega)$ with $\mathcal{D}(A_2) = H^2_N(\Omega)$. And $\mathcal{B}_1(U)$ is a linear operator of $H^1(\Omega)'$ defined by

$$\mathcal{B}_1(U)\widetilde{\rho} = \nabla \cdot \{u\chi'(\operatorname{Re}\rho)\nabla\widetilde{\rho}\}, \quad U \in Z, \widetilde{\rho} \in \mathcal{D}(\mathcal{B}_1(U)) = H^2_N(\Omega).$$

We notice by (6.6) and (6.9) that, for any $0 < \varepsilon \le \varepsilon_2$,

$$\|\mathcal{B}_1(U)\widetilde{\rho}\|_{(H^1)'} \le C_\varepsilon \|u\|_{H^\varepsilon} p_\varepsilon(\|\rho\|_{H^{1+\varepsilon}})\|\widetilde{\rho}\|_{H^{2-\varepsilon}}, \quad U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in Z, \widetilde{\rho} \in H^2_N(\Omega) \tag{6.18}$$

with some continuous increasing function $p_\varepsilon(\,\cdot\,)$ determined by $\chi'(\,\cdot\,)$. The domain of $A(U)$ is therefore given by

$$\mathcal{D}(A(U)) \equiv \mathcal{D} = \left\{ \begin{pmatrix} \widetilde{u} \\ \widetilde{\rho} \end{pmatrix} ; \widetilde{u} \in H^1(\Omega) \text{ and } \widetilde{\rho} \in H^2_N(\Omega) \right\}, \quad U \in Z.$$

The nonlinear operator F of (5.4) is defined by

$$F(U) = \begin{pmatrix} u + f(\operatorname{Re} u) \\ du \end{pmatrix}, \quad U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in Z. \tag{6.19}$$

We notice from (6.3) that

$$\begin{aligned} &|\langle f(\operatorname{Re} u), w\rangle_{(H^1)'\times H^1}| \le C \int_\Omega (|u|^2+1)|w|dx \\ &\le C(\|u\|^2_{L^{2/(1-\varepsilon_2)}}+1)\|w\|_{L^{1/(\varepsilon_2)}} \le C(\|u\|^2_{H^{\varepsilon_2}}+1)\|w\|_{H^1}, \quad u \in H^{\varepsilon_2}(\Omega), \quad w \in H^1(\Omega). \end{aligned}$$

Consequently,

$$\|f(\operatorname{Re} u)\|_{(H^1)'} \le C(\|u\|^2_{H^{\varepsilon_2}}+1), \quad u \in H^{\varepsilon_2}(\Omega).$$

In this way we have an abstract formulation of (6.1) as the Cauchy problem of the form (5.4) in the product space X.

6.3. Construction of local solutions. Let $0 < R < \infty$, and set

$$K_R = \{U \in Z;\ \|U\|_Z < R\}.$$

We have to verify that all the structural assumptions (3.2)–(3.5) are fulfilled by $A(U)$, $U \in K_R$, and F.

For $\lambda \in \mathbb{C} - (0,\infty)$,

$$(\lambda - A(U))\widetilde{U} = \widetilde{F}, \quad U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in K_R, \quad \widetilde{U} = \begin{pmatrix} \widetilde{u} \\ \widetilde{\rho} \end{pmatrix} \in \mathcal{D}, \quad \widetilde{F} = \begin{pmatrix} \widetilde{f} \\ \widetilde{\eta} \end{pmatrix} \in X$$

if and only if

$$\begin{cases} \widetilde{u} = (\lambda - \mathcal{A}_1)^{-1}\{\mathcal{B}_1(U)(\lambda - A_2)^{-1}\widetilde{\eta} + \widetilde{f}\}, \\ \widetilde{\rho} = (\lambda - A_2)^{-1}\widetilde{\eta}. \end{cases}$$

By (6.18) ($\varepsilon = \varepsilon_2$), we observe that

$$\begin{cases} \|\widetilde{u}\|_{(H^1)'} \le C_R\|(\lambda - \mathcal{A}_1)^{-1}\|_{\mathcal{L}((H^1)')}\{\|(\lambda - A_2)^{-1}\widetilde{\eta}\|_{H^2} + \|\widetilde{f}\|_{(H^1)'}\}, \\ \|\widetilde{\rho}\|_{L^2} \le \|(\lambda - A_2)^{-1}\|_{\mathcal{L}(L^2)}\|\widetilde{\eta}\|_{L^2}. \end{cases}$$

Then, for an arbitrarily fixed $0 < \phi < \pi/2$, the spectral set $\sigma(A(U))$ is contained in an open sectorial domain

$$\sigma(A(U)) \subset \Sigma_\phi = \{\lambda \in \mathbb{C};\ |\arg\lambda| < \phi\}, \quad U \in K_R$$

with the estimate

$$\|(\lambda - A(U))^{-1}\|_{\mathcal{L}(X)} \le \frac{C_R}{|\lambda| + 1}, \quad \lambda \notin \Sigma_\phi, U \in K_R.$$

Hence, (3.2) is verified.

Let $U, V \in K_R$ with $U = \binom{u}{\rho}$ and $V = \binom{v}{\zeta}$. Then, since

$$A(U)\{A(U)^{-1} - A(V)^{-1}\}\widetilde{F} = \{A(V) - A(U)\}A(V)^{-1}\widetilde{F}$$
$$= \begin{pmatrix} 0 & \mathcal{B}_1(V) - \mathcal{B}_1(U) \\ 0 & 0 \end{pmatrix} A(V)^{-1}\widetilde{F}, \quad \widetilde{F} \in X,$$

(3.3) is reduced to the condition

$$\|\{\mathcal{B}_1(U) - \mathcal{B}_1(V)\}\widetilde{\rho}\|_{(H^1)'} \le C_R(\|u - v\|_{H^{\varepsilon_1}} + \|\rho - \zeta\|_{H^{1+\varepsilon_1}})\|\widetilde{\rho}\|_{H^2}, \quad \widetilde{\rho} \in H^2_N(\Omega).$$

But this is verified by similar calculations as for (6.9), utilizing (6.6) and (6.7) with $\varepsilon = \varepsilon_1$.

From (6.3) we observe that

$$|\langle f(\mathrm{Re}\, u) - f(\mathrm{Re}\, v), w\rangle_{(H^1)' \times H^1}| \le C \int_\Omega |u - v|(|u| + |v| + 1)|w|\, dx$$
$$\le C\|u - v\|_{L^{2/(1-\varepsilon_1)}}(\|u\|_{L^{2/(1-\varepsilon_2)}} + \|v\|_{L^{2/(1-\varepsilon_2)}} + 1)\|w\|_{L^{2/(\varepsilon_1+\varepsilon_2)}}$$
$$\le C\|u - v\|_{H^{\varepsilon_1}}(\|u\|_{H^{\varepsilon_2}} + \|v\|_{H^{\varepsilon_2}} + 1)\|w\|_{H^1}, \quad u, v \in H^{\varepsilon_2}(\Omega), \ w \in H^1(\Omega).$$

Then, from (6.19), (3.4) is also verified.

To verify (3.5) we consider a decomposition $A(U) = A + B(U)$ with

$$A = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad B(U) = \begin{pmatrix} 0 & \mathcal{B}_1(U) \\ 0 & 0 \end{pmatrix}.$$

From (6.12) and (6.13) it is seen that, for $0 \le \theta \le 1$,

$$\mathcal{D}(A^\theta) = \left\{ \begin{pmatrix} \widetilde{u} \\ \widetilde{\rho} \end{pmatrix} ; \widetilde{u} \in [H^1(\Omega)', H^1(\Omega)]_\theta \text{ and } \widetilde{\rho} \in [L^2(\Omega), H^2_N(\Omega)]_\theta \right\}. \tag{6.20}$$

Meanwhile (6.18) ($\varepsilon = \varepsilon_2$) yields that

$$\|\mathcal{B}_1(U)\widetilde{\rho}\|_{(H^1)'} \le C_R\|\widetilde{\rho}\|_{H^{2-\varepsilon_2}}, \quad \widetilde{\rho} \in H^2_N(\Omega).$$

Therefore, on account of (6.13),

$$\|B(U)\widetilde{U}\|_X \le C_R\|A^{\theta_2}\widetilde{U}\|_X, \quad \widetilde{U} \in \mathcal{D}$$

with $\theta_2 = (2 - \varepsilon_2)/2 < 1$. This shows that $B(U)$ is dominated by A with an exponent θ_2. From the relation of resolvent

$$(\lambda - A(U))^{-1} - (\lambda - A)^{-1} = (\lambda - A(U))^{-1} B(U)(\lambda - A)^{-1},$$

it follows that

$$\|(\lambda - A(U))^{-1} - (\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq C_R(|\lambda| + 1)^{-1-(\varepsilon_2/2)}, \quad \lambda \notin \Sigma_\phi.$$

By a standard argument (cf. [10, Chap. 1,Theorem 7.6]), we can then verify that

$$\mathcal{D}(A(U)^\theta) = \mathcal{D}(A^\theta), \quad 0 \leq \theta \leq 1, U \in K_R. \tag{6.21}$$

with uniform norm equivalence. Hence, in view of (6.15) and (6.16), it is sufficient to take

$$\alpha = \frac{1 + \varepsilon_1}{2} \quad \text{and} \quad \beta = \frac{1 + \varepsilon_2}{2}$$

for the space condition (3.5).

Therefore, by virtue of Theorem 1, for any initial functions $u_0 \in H^{\varepsilon_2}(\Omega)$ and $\rho_0 \in H^{1+\varepsilon_2}(\Omega)$, there exists a unique local solution to (5.4). Moreover, Corollary 1 provides that, if

$$u_0 \in H^\varepsilon(\Omega) \quad \text{and} \quad \rho_0 \in H^{1+\varepsilon}(\Omega) \quad \text{with} \quad \varepsilon_2 < \varepsilon < \frac{1}{2}, \tag{6.22}$$

then the local solution belongs to the function space:

$$\begin{cases} u \in C^1((0, T_{U_0}]; H^1(\Omega)') \cap C^{(\varepsilon-\varepsilon_1)/2}([0, T_{U_0}]; H^{\varepsilon_1}(\Omega)) \cap C([0, T_{U_0}]; H^\varepsilon(\Omega)), \\ \qquad t^{(1-\varepsilon)/2} u \in C([0, T_{U_0}]; H^1(\Omega)), \\ \rho \in C^1((0, T_{U_0}]; L^2(\Omega)) \cap C^{(\varepsilon-\varepsilon_1)/2}([0, T_{U_0}]; H^{1+\varepsilon_1}(\Omega)) \cap C([0, T_{U_0}]; H^{1+\varepsilon}(\Omega)), \\ \qquad t^{(1-\varepsilon)/2} \rho \in C([0, T_{U_0}]; H^2_N(\Omega)). \end{cases} \tag{6.23}$$

Here, $T_{U_0} > 0$ is determined by the norm $\|A(U_0)^\gamma U_0\|_X$ alone, where $\gamma = (1 + \varepsilon)/2$; and, from (6.20) and (6.21), this norm is equivalent to $\|u_0\|_{H^\varepsilon} + \|\rho_0\|_{H^{1+\varepsilon}}$.

If we appeal to the maximal regularity of linear abstract equations, then we can obtain the optimal regularity for u also. In fact, u belongs to

$$u \in C((0, T_{U_0}]; H^2_N(\Omega)). \tag{6.24}$$

To prove this we consider u as a solution to the linear equation

$$\frac{du}{dt} + \mathcal{A}_1 u = F_1(t), \quad 0 < t \leq T_{U_0} \tag{6.25}$$

in the space $H^1(\Omega)'$ with $F_1(t) = -\mathcal{B}_1(U(t))\rho(t) + u(t) + f(\mathrm{Re}\, u(t))$. By (6.10) and (6.23), we observe that

$$\mathcal{B}_1(U)\rho \in C((0, T_{U_0}]; H^{1/2}(\Omega)').$$

From (6.3) it is clear that

$$f(\mathrm{Re}\, u) \in C((0, T_{U_0}]; L^2(\Omega)).$$

As a consequence, $F_1 \in C((0, T_{U_0}]; H^{1/2}(\Omega)')$.

We now notice the fact that u is written as

$$\mathcal{A}_1^{9/8} u(t) = \mathcal{A}_1^{9/8} e^{-(t-s)\mathcal{A}_1} u(s) + \int_s^t \mathcal{A}_1^{7/8} e^{-(t-\tau)\mathcal{A}_1} \mathcal{A}_1^{1/4} F_1(\tau)\, d\tau, \quad 0 < s \le t \le T_{U_0}.$$

Since $\mathcal{A}_1^{1/4}$ is a bounded operator from $H^{1/2}(\Omega)'$ to $H^1(\Omega)'$ (see (6.12)), we obtain that

$$u(t) \in \mathcal{D}(\mathcal{A}_1^{9/8}) \subset H^{5/4}(\Omega) \quad \text{and} \quad u \in C((0, T_{U_0}]; H^{5/4}(\Omega)).$$

Furthermore, since $u \in C^{1/2}((0, T_{U_0}]; L^2(\Omega))$, it follows by interpolation property (6.4) that

$$u \in C^{1/20}((0, T_{U_0}]; H^{9/8}(\Omega)).$$

We next notice that ρ is a solution to the evolution equation

$$\frac{d\rho}{dt} + A_2\rho = du(t), \quad 0 < t \le T_{U_0} \tag{6.26}$$

in $H^1(\Omega) = \mathcal{D}(A_2^{1/2})$. As $u \in C^{1/20}((0, T_{U_0}]; H^1(\Omega))$, the maximal regularity of this equation provides that $A_2\rho \in C^{1/20}((0, T_{U_0}]; H^1(\Omega))$, that is

$$\rho \in C^{1/20}((0, T_{U_0}]; \mathcal{D}(A_2^{3/2})) \subset C^{1/20}((0, T_{U_0}]; H^2(\Omega)).$$

Then we can use (6.8) with $\varepsilon = 1/8$ to obtain that $F_1 \in C^{1/20}((0, T_{U_0}]; L^2(\Omega))$. This means that the evolution equation (6.25) can be considered in $L^2(\Omega)$ substituting the part A_1 of $\mathcal{A}_1$ in $L^2(\Omega)$ for the coefficient operator $\mathcal{A}_1$. As a result we obtain the desired regularity (6.24).

6.4. Global solutions. We assume nonnegativity of initial functions u_0 and ρ_0 in addition to (6.22). Then, by the truncation method (see [18, Theorem 3.5]) nonnegativity of u and ρ is verified.

The goal of this subsection is to show a priori estimates of local solutions to (5.4) in the space:

$$\begin{cases} 0 \le u \in C^1((0, T_U]; L^2(\Omega)) \cap C([0, T_U]; H^\varepsilon(\Omega)) \cap C((0, T_U]; H^2_N(\Omega)), \\ 0 \le \rho \in C^1((0, T_U]; L^2(\Omega)) \cap C([0, T_U]; H^{1+\varepsilon}(\Omega)) \cap C((0, T_U]; \mathcal{D}(A_2^{3/2})) \end{cases} \tag{6.27}$$

and to obtain global solutions.

Let $0 < R < \infty$, and consider a set of initial functions

$$B_R^+ = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix};\ 0 \le u_0 \in H^\varepsilon(\Omega) \quad \text{and} \quad 0 \le \rho_0 \in H^{1+\varepsilon}(\Omega) \right. \\ \left. \text{with } \|u_0\|_{H^\varepsilon}^2 + \|\rho_0\|_{H^{1+\varepsilon}}^2 < R \right\}, \quad \varepsilon_2 < \varepsilon < \frac{1}{2}. \tag{6.28}$$

We first notice local estimates of the solutions starting from initial values in B_R^+. As a matter of fact, Theorem 1 and Corollary 1 provide not only (6.23) but also the estimates

$$\|u(t)\|_{H^\varepsilon} + \|\rho(t)\|_{H^{1+\varepsilon}} + t^{(1-\varepsilon)/2}(\|u(t)\|_{H^1} + \|\rho(t)\|_{H^2}) \le C_R, \quad 0 \le t \le T_R \tag{6.29}$$

for all local solutions on a fixed interval $[0, T_R]$ with initial functions from B_R^+, C_R and T_R being dependent only on R.

In view of this fact, we can assume that initial functions satisfy

$$0 \le u_0 \in H^1(\Omega) \quad \text{and} \quad 0 \le \rho_0 \in H_N^2(\Omega).$$

We next show global estimates of the solutions starting from initial functions like this. We can verify that the estimate

$$\|u(t)\|_{H^1} + \|\rho(t)\|_{H^2} \le p(\|u_0\|_{H^1} + \|\rho_0\|_{H^2}), \quad 0 \le t \le T_U \tag{6.30}$$

holds for every local solution in the space

$$\begin{cases} 0 \le u \in C^1((0, T_U]; L^2(\Omega)) \cap C([0, T_U]; H^1(\Omega)) \cap C((0, T_U]; H_N^2(\Omega)), \\ 0 \le \rho \in C^1((0, T_U]; L^2(\Omega)) \cap C([0, T_U]; H^2(\Omega)) \cap C((0, T_U]; \mathcal{D}(A_2^{3/2})) \end{cases}$$

with some continuous increasing function $p(\,\cdot\,)$ determined absolutely.

In fact, this result is proved by an analogous method to the proof of [18, Proposition 4.1]. The arguments in Steps 1–3 of [18] are available without any change, because the norms $\|u\|_{H^3}$ and $\|\rho\|_{H^3}$ are not used yet. The arguments in Step 4 can be recovered as follows, although $\|\rho\|_{H^3}$ was used. Indeed, we have

$$\begin{aligned} -\int_\Omega \chi'(\rho)u^2\Delta\rho\,dx &\le C\|u\|_{L^3}^2\|(A_2 - c)\rho\|_{L^3} \le C\|u\|_{L^3}^2\|(A_2 - c)\rho\|_{H^1}^{1/3}\|(A_2 - c)\rho\|_{L^2}^{2/3} \\ &\le C\|u\|_{L^3}^2\|A_2^{3/2}\rho\|_{L^2}^{1/3}\|A_2\rho\|_{L^2}^{2/3} \le C\|u\|_{L^3}^2\|A_2^{3/2}\rho\|_{L^2}^{2/3}\|\rho\|_{H^1}, \end{aligned}$$

utilizing the moment inequality

$$\|A_2\rho\|_{L^2} \le \|A_2^{3/2}\rho\|_{L^2}^{1/2}\|A_2^{1/2}\rho\|_{L^2}^{1/2}, \quad \rho \in \mathcal{D}(A_2^{3/2}).$$

Similarly,

$$-\int_\Omega \chi'' u^2 |\nabla\rho|^2\,dx$$
$$\le C\|u\|_{L^3}^2\|\nabla\rho\|_{L^6}^2 \le C\|u\|_{L^3}^2\|\rho\|_{H^2}^{4/3}\|\rho\|_{H^1}^{2/3} \le C\|u\|_{L^3}^2\|A_2^{3/2}\rho\|_{L^2}^{2/3}\|\rho\|_{H^1}^{4/3}.$$

In this way we can substitute $\|A_2^{3/2}\rho\|_{L^2}$ for $\|\rho\|_{H^3}$.

In the meantime we obtain from (6.26) that

$$\frac{1}{2}\frac{d}{dt}\|A_2\rho(t)\|_{L^2}^2 + \|A_2^{3/2}\rho(t)\|_{L^2}^2 = d(A_2^{1/2}u(t), A_2^{3/2}\rho(t))$$
$$\le \frac{1}{2}\|A_2^{3/2}\rho(t)\|_{L^2}^2 + \frac{d^2}{2}\|A_2^{1/2}u(t)\|_{L^2}^2.$$

Thus we have arrived at the same estimate as [18, (4.15)].

As the norms $\|u\|_{H^3}$ and $\|\rho\|_{H^3}$ are not used in the first half of Step 5, the same estimate as [18, (4.17)] is valid. Hence we have established (6.30).

As an immediate consequence of (6.29) and (6.30), we obtain the global existence of solutions. For any initial functions u_0 and ρ_0 in B_R^+, (5.4) possesses a global solution in the space (6.27) with an arbitrary $0 < T_U < \infty$.

6.5. Exponential attractor. We are ready to define a dynamical system from the Cauchy problem (5.4) and to construct an exponential attractor.

We fix an exponent $\gamma = (1+\varepsilon)/2$ with $\varepsilon_2 < \varepsilon < 1/2$. Then, from (6.20) and (6.21),

$$\mathcal{D}(A(U)^\gamma) \equiv \mathcal{D}_\gamma = \left\{U_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix};\, u_0 \in H^\varepsilon(\Omega) \text{ and } \rho_0 \in H^{1+\varepsilon}(\Omega)\right\}$$

is independent of $U \in Z$. We then set

$$\mathcal{D}_\gamma^+ = \left\{U_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in \mathcal{D}_\gamma;\, u_0 \ge 0 \text{ and } \rho_0 \ge 0\right\}.$$

We have already known, for each bounded set B_R^+ of $\mathcal{D}_\gamma^+$ given by (6.28), that (5.4) possesses a unique global solution. Since $0 < R < \infty$ is arbitrary, this means that a nonlinear semigroup $S(t)$ is defined on $\mathcal{D}_\gamma^+$. We can now argue along the lines announced in the preceding section. We define a set $\mathcal{B}_R^+$ from B_R^+ by the same way as (5.8), then $\mathcal{B}_R^+$ is a compact set of X and is a positively invariant set of $S(t)$. If $\mathcal{B}_R^+$ is equipped with the induced metric from $\|\cdot\|_X$, then $S(t)$ satisfies (5.1). Thus, $(S(t), \mathcal{B}_R^+, X)$ is shown to become a dynamical system.

Furthermore from the decaying estimate (6.3) of $f(u)$, we can establish the absorbing estimates for $S(t)$ as in [18]. In fact the same estimates as [18, (4.4), (4.9), (4.12), (4.15)] and the first half of the estimate [18, (4.18)], namely

$$\|u(t)\|_{H^1}^2 \le Ce^{-\delta t}\|u_0\|_{H^1}^2 + p(\|u_0\|_{L^2} + \|\rho_0\|_{H^2}), \quad 0 \le t < \infty,$$

are all verified by the quite similar techniques as in [18]. These estimates then imply existence of an absolute constant C such that, for any bounded set B_R^+ of $\mathcal{D}_\gamma^+$, it holds that

$$\sup_{t\geq t_R}\sup_{U_0\in B_R^+}\|S(t)U_0\|_{\mathcal{D}}\leq C$$

with a suitable time $t_R > 0$ depending on R. Furthermore, a set of the form $\mathcal{X}_1^+ = \{U \in \mathcal{D}_1^+;\ \|U\|_{\mathcal{D}} \leq C\}$ is an absorbing set.

We finally define a set $\mathcal{X}^+$ from $\mathcal{X}_1^+$ by the same way as (5.10). Then, $\mathcal{X}^+$ is a compact set of X and is an absorbing and positively invariant set of $S(t)$. Therefore a dynamical system $(S(t), \mathcal{X}^+, X)$ is defined which absorbs every larger system $(S(t), \mathcal{B}_R^+, X)$ in finite time.

As the Lipschitz conditions (5.2) and (5.3) of $S(t)$ and $G(t, U_0)$ are easily verified by Theorem 2, we conclude by Theorem 3 that the dynamical system $(S(t), \mathcal{X}^+, X)$ possesses an exponential attractor.

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