

Title	Elliptic curves of prime power conductor with $\mathbb{Q}$ -rational points of finite order
Author(s)	Miyawaki, Isao
Citation	Osaka Journal of Mathematics. 10(2) p.309-p.323
Issue Date	1973
oaire:version	VoR
URL	<a href="https://doi.org/10.18910/11265">https://doi.org/10.18910/11265</a>
rights	
Note	

*Osaka University Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

Osaka University

## ELLIPTIC CURVES OF PRIME POWER CONDUCTOR WITH $\mathbf{Q}$ -RATIONAL POINTS OF FINITE ORDER

ISAO MIYAWAKI

(Received August 4, 1972)

### Introduction

Let  $E$  be an elliptic curve (an abelian variety of dimension one) defined over the rational number field  $\mathbf{Q}$ . After Weil [9], we can define the conductor of  $E$ . But, in general, it would be difficult to find all the curves of given conductor. But it seems to be easier to find all the curves of given conductor having  $\mathbf{Q}$ -rational points of finite order  $>2$ .

In this paper we determine all the curves of prime power conductor which have at least three rational points of finite order. There are only finitely many such curves up to  $\mathbf{Q}$ -isomorphism. They are listed in the table at the end of the paper.

Since each of them has  $\mathbf{Q}$ -rational points of finite order, we can take a special cubic equation as a global minimal model for it. Further, since that curve has a prime power conductor, the coefficients of that equation must be a solution of a certain diophantine equation. Therefore, the determination of such curves is reduced to elementary diophantine problems.

Some of them have no complex multiplication and non-integral invariants. Let  $E$  be one of them and  $\mathbf{Q}(E_n)$  be the field generated by the coordinates of the  $n$ -division points of  $E$  over  $\mathbf{Q}$ . Then we can determine the Galois group of  $\mathbf{Q}(E_l)$  over  $\mathbf{Q}$  for all prime  $l$ .

### 1. Integrality of rational points of finite order

Let  $E$  be an elliptic curve defined over  $\mathbf{Q}$ . A Weierstrass model for  $E$  over  $\mathbf{Q}$  is plane cubic equation of the form

$$(1) \quad y^2 + a_1xy + a_3y + x^3 + a_2x^2 + a_4x + a_6 = 0$$

with  $a_j \in \mathbf{Q}$ , the zero of  $E$  being the point at infinity. We define auxiliary quantities by

$$\begin{aligned} \beta_2 &= a_1^2 - 4a_2 \\ \beta_4 &= 2a_4 - a_1a_3 \end{aligned}$$

$$\begin{aligned}\beta_6 &= a_3^2 - 4a_6 \\ \beta_8 &= a_4^2 - a_1a_3a_4 + a_1^2a_6 + a_2a_3^2 - 4a_2a_6.\end{aligned}$$

The discriminant  $\Delta$  is defined by

$$\Delta = \beta_2^2\beta_8 - 8\beta_4^3 - 27\beta_6^2 + 9\beta_2\beta_4\beta_6$$

and the invariant  $J$  is defined by

$$J = (\beta_2^2 - 24\beta_4)^3 / \Delta.$$

This model is minimal at a prime  $p$  if each  $a_j$  is integral at  $p$ , and  $\text{ord}_p(\Delta)$  is as small as possible. Since the ring  $\mathbf{Z}$  of rational integers is a principal ideal domain, it is easy to see that we can find a global minimal Weierstrass model for  $\mathbf{E}$  over  $\mathbf{Q}$ , i.e., a cubic equation as above with each  $a_j \in \mathbf{Z}$ , which is simultaneously minimal at all  $p$ . For a detailed discussion of minimal models of elliptic curves, see Néron [2]. It should be noted that the conductor  $N$  of  $\mathbf{E}$  and the discriminant  $\Delta$  of the global minimal model for  $\mathbf{E}$  have same prime divisors.

**Lemma 1.** *Suppose that each  $a_j$  is a rational integer in (1). If  $t_0 = (x_0, y_0)$  is a rational point on (1) of finite order  $m > 2$ , then  $x_0$  and  $y_0$  are integers.*

*Proof.* After the transformation

$$(2) \quad \begin{aligned}6^2x &= X + 3a_1^2 - 12a_2 \\ 6^3y &= Y - 3a_1(X + 3a_1^2 - 12a_2) - 4 \cdot 3^3a_3,\end{aligned}$$

we get

$$(3) \quad Y^2 + X^3 + AX + B = 0$$

where

$$\begin{aligned}A &= 3(3a_1^2 - 12a_2)^2 + 2(6^2a_2 - 3^2a_1^2)(3a_1^2 - 12a_2) + 6^4a_4 - 2^3 \cdot 3^4a_1a_3 \\ B &= (3a_1^2 - 12a_2)^3 + (6^2a_2 - 3^2a_1^2)(3a_1^2 - 12a_2)^2 + 6^6a_6 - 2^4 \cdot 3^6a_3^2 \\ &\quad + (6^4a_4 - 2^3 \cdot 3^4a_1a_3)(3a_1^2 - 12a_2).\end{aligned}$$

Hence  $3^3 | A$ ,  $3^3 | B$  and  $4A^3 + 27B^2 = -2^8 \cdot 3^{12} \Delta$ .

Let  $(X_0, Y_0)$  be the transform of  $(x_0, y_0)$  by (2). Then by Theorem 22.1 of Cassels [1]  $X_0$  and  $Y_0$  are integers. Hence  $6^2x_0$  and  $6^3y_0$  are integers. By the assumption we have  $Y_0 \neq 0$ . So the tangent of (3) at  $(X_0, Y_0)$  is

$$Y - Y_0 = -(3X_0^2 + A)(2Y_0)^{-1}(X - X_0).$$

Put  $(X_1, Y_1) = -2(X_0, Y_1)$ , then  $X_1$  and  $Y_1$  are also integers, and we have

$$(4) \quad (3X_0^2 + A)^2 / (2Y_0)^2 = -(2X_0 + X_1).$$

Hence

$$(5) \quad (3X_0^2 + A)/(2Y_0) \in \mathbf{Z}.$$

First, we shall show that  $\text{ord}_2(x_0)$  and  $\text{ord}_2(y_0)$  are non-negative. If  $X_0$  is odd, then  $A$  is odd by (5), so is  $a_1$ . Hence  $6^2x_0$  is even by (2). If  $X_0$  is even, then  $A$  is even by (5), so is  $a_1$ . Hence  $6^2x_0$  is even by (2). In either case, we have  $\text{ord}_2(x_0) \geq -1$ . Multiplying (1) by  $2^4$ , we see that  $2^2y_0$  is integral at 2, hence  $\text{ord}_2(y_0) \geq -2$ . Hence  $\text{ord}_2(y_0^3) \geq -3$  by (1), i.e.,  $\text{ord}_2(y_0) \geq -1$ . This implies  $\text{ord}_2(x_0^3) \geq -2$ , i.e.,  $\text{ord}_2(x_0) \geq 0$  and  $\text{ord}_2(y_0) \geq 0$ .

Next, we shall show that  $\text{ord}_3(x_0)$  and  $\text{ord}_3(y_0)$  are non-negative. If we suppose  $X_0 \not\equiv 0 \pmod 3$ , then  $Y_0 \not\equiv 0 \pmod 3$  by (3), hence  $X_0 \equiv -1 \pmod 3$ . Hence we have  $X_1 \equiv -1 \pmod 3$  by (4). Further we can easily see  $X_0 \not\equiv X_1$  by (4). The points  $-(X_0, Y_0) - (X_1, Y_1)$  and  $-(X_0, Y_0) - (X_1, -Y_1)$  on (3) are also integral. Hence, by the same reasoning as above, we have

$$\begin{aligned} (Y_0 - Y_1)/(X_0 - X_1) &\in \mathbf{Z} \\ (Y_0 + Y_1)/(X_0 - X_1) &\in \mathbf{Z}. \end{aligned}$$

Since  $X_0 - X_1 \equiv 0 \pmod 3$ , we have

$$Y_0 - Y_1 \equiv Y_0 + Y_1 \equiv 0 \pmod 3 \text{ i.e. } Y_0 \equiv 0 \pmod 3,$$

which is a contradiction. Hence  $X_0 \equiv 0 \pmod 3$  i.e.,  $\text{ord}_3(x_0) \geq -1$  by (2). By the same method as above we have  $\text{ord}_3(x_0) \geq 0$  and  $\text{ord}_3(y_0) \geq 0$ . This completes the proof of the lemma.

REMARK. Assumption being as in Lemma 1, we obtain the  $x$ -coordinates of 2-division points on (1) as the roots of the equation

$$(a_1x + a_3)^2 - 4(x^3 + a_2x^2 + a_4x + a_6) = 0$$

Hence, if  $(x_0, y_0)$  is a rational point on (1) of order 2, then we have  $2^2x_0 \in \mathbf{Z}$  and  $2^3y_0 \in \mathbf{Z}$ . Moreover, if the other 2-division points, say  $(x_1, y_1)$  and  $(x_2, y_2)$ , are also rational points, then we have

$$x_0x_1x_2 = -(a_6 - a_3^2/4).$$

Hence, at least one of them is an integral point.

**Lemma 2.** *Let  $E$  be an elliptic curve defined over a finite field  $k$  with  $q$ -elements and  $N$  be the number of  $k$ -rational points of  $E$ .*

*Then we have*

$$N = 1 - a + q,$$

*where*

$$|a| \leq 2\sqrt{q}.$$

This is the ‘‘Riemann hypothesis’’ for elliptic curves.

## 2. Determination of curves of prime power conductor with $Q$ -rational points of finite order

In this section, we shall determine all the elliptic curves of prime power conductor having at least three rational points of finite order. Since all the curves of 2-power conductor have been determined in Ogg [3], we shall deal with the curves of odd prime power conductor. So we shall always mean by  $p$  an odd prime integer. Let  $E$  be such a curve and  $F$  be the group of rational points of finite order on  $E$ . By Mordell-Weil theorem,  $F$  is a finite group. Let  $E(l)$  be the reduction of  $E$  at a prime  $l$ . Since  $E$  has a good reduction at 2,  $E(2)$  is also an elliptic curve defined over the field  $GF(2)$ . Hence  $E(2)$  has at most five  $GF(2)$ -rational points by Lemma 2. It is well-known that the reduction map of  $F$  into the group of  $GF(2)$ -rational points on  $E(2)$  is a homomorphism and the order of its kernel is 2-power. Therefore the order of  $F$  is  $2^\lambda$ ,  $2^\lambda \cdot 3$  or  $2^\lambda \cdot 5$ .

As was shown in Ogg [3], we note that there is no curve with  $\Delta = \pm 1$ .

**Theorem 1.** *There are nine elliptic curves having rational points of order 3 and they have no rational point of order 2.*

*Proof.* Let  $E$  be such a curve and (1) be a global minimal model of  $E$ . By Lemma 1, we may assume that  $a_6 = 0$  and  $t = (0, 0)$  is a point of order 3, hence  $a_3$  is not zero. Since  $2(0,0) + (0,0) = 0$ , we have

$$(-a_4/a_3)^2 + a_1(-a_4/a_3) + a_2 = 0.$$

Hence  $a_4/a_3$  is an integer. Put  $a_4 = a_3 a_4'$ , then

$$\Delta = a_3^3 \{-8(2a_4' - a_1)^3 - 27a_3 + 9(a_1^2 - 4a_2)(2a_4' - a_1)\}.$$

By  $\Delta = \pm p^m$ , we see that  $a_3$  is odd. Hence  $a_1$  is even. Hence by the transformation  $y \rightarrow y - (a_1/2)x$ , we can take  $a_1 = 0$ . Hence we get the global minimal model of  $E$  of the form

$$(6) \quad y^2 + a_3 y + x^3 + a_2 x^2 + a_3 a_4 x = 0$$

with

$$a_j \in \mathbf{Z} \\ a_4^2 + a_2 = 0, \quad \Delta = a_3^3(8a_4^3 - 27a_3) = \pm p^m, \quad a_3 > 0.$$

(I) The case  $a_3 = 1$

In this case, we have

$$8a_4^3 - 27 = (2a_4 - 3)(4a_4^2 + 6a_4 + 9) = \pm p^m$$

If  $2a_4 - 3 = \pm 1$ , then we get  $a_4 = 1$  or  $2$ , curves 1, 2. Next we shall consider the cases  $2a_4 - 3 = \pm p^n$  ( $n > 0$ ). If  $p \neq 3$  we have  $a_4 \not\equiv 0 \pmod p$ , hence  $4a_4^2 + 6a_4 + 9 = (2a_4 - 3)^2 + 18a_4 = \pm 1$ . This contradicts with  $a_4 \in \mathbf{Z}$ . Hence  $p = 3$ . If  $n = 1$ , then we get

$$2a_4 = 3 \pm 3, \quad (2a_4 - 3)^2 + 18a_4 = 9(1 + 3 \pm 3) = \pm 3^{m'}$$

Hence we get  $a_4 = 0$ , the curve 3. If  $n = 2$ , then we have

$$2a_4 = 3 \pm 9, \quad (2a_4 - 3)^2 + 18a_4 = 27(3 + 1 \pm 3) = \pm 3^{m'}$$

Hence we get  $a_4 = -3$ , the curve 4. If  $n \geq 3$ , then we have

$$2a_4 = 3 \pm 3^n, \quad (2a_4 - 3)^2 + 18a_4 = 3^{2n} + 9(3 \pm 3^n) = \pm 3^{m'}$$

But this is impossible.

(II) The case  $a_3 = p$

First, we shall consider the case  $p = 3$ . Then we have

$$8a_4^3 - 3^4 = \pm 3^{m'}$$

If  $m' = 0$ , then  $8a_4^3 - 3^4 = \pm 1$ . But this is impossible. Hence  $a_4$  is divisible by 3. Put  $a_4 = 3a_4'$ , then we have

$$8a_4'^3 - 3 = \pm 3^{m''}$$

If  $m'' = 0$ , then  $8a_4'^3 = 3 \pm 1$ . This is impossible. Hence  $a_4'$  is divisible by 3. Put  $a_4' = 3a_4''$ , then we have

$$8 \cdot 3^2 a_4''^3 - 1 = \pm 1$$

Hence we get  $a_4 = 0$ , the curve 5.

Next, we shall consider the case  $p \neq 3$ . If  $a_4$  were divisible by  $p$ , we would have

$$8(a_4/p)^3 p^2 - 27 = \pm 1$$

But this is impossible. Hence  $8a_4^3 - 27p = \pm 1$ , and we have

$$27p = 8a_4^3 \pm 1 = (2a_4 \pm 1) \{ (2a_4 \pm 1)^2 \mp 6a_4 \}$$

If  $2a_4 \pm 1$  is divisible by  $p$ , then we get  $(2a_4 \pm 1)^2 \mp 6a_4 = \pm 3, \pm 9$ . This implies  $2a_4 \pm 1 = \pm 3$ . But this contradicts with  $2a_4 \pm 1 \equiv 0 \pmod p$ . Hence  $2a_4 \pm 1$  is not divisible by  $p$ . This implies  $2a_4 \pm 1 = \pm 3, \pm 9$ . We can easily see  $2a_4 \pm 1 = 9$ , then we get curves 6, 7.

(III) The case  $a_3=p^2$

First, we shall consider the case  $p \neq 3$ . If  $a_4$  were divisible by  $p$ , we would have

$$8(a_4/p)^3 p - 27 = \pm 1.$$

But this is impossible. Hence  $a_4$  is not divisible by  $p$ , i.e.,  $8a_4^3 - 27p^2 = \pm 1$ . Hence we get  $27p^2 = 8a_4^3 \pm 1 = (2a_4 \pm 1)\{(2a_4 \pm 1)^2 \mp 6a_4\}$ . But we can easily see by the same method as in (II) that this is impossible. So we suppose  $p=3$ . Then we have

$$8a_4^3 - 3^5 = \pm 3^{m'}.$$

It can be easily seen that  $m'$  is not zero. Put  $a_4 = 3a_4'$ . Then we have

$$8a_4'^3 - 3^2 = \pm 3^{m''}.$$

If  $m''=0$ , then we get  $a_4'=1$ , the curve 8. If  $m''>0$ , put  $a_4'=3a_4''$ , then we have

$$3 \cdot 8a_4''^3 - 1 = \pm 1.$$

Hence we get  $a_4''=0$ , the curve 9.

(IV) The case  $a_3=p^n$  ( $n \geq 2$ )

If  $a_4$  is divisible by  $p$ , then  $a_2$  is divisible by  $p^2$ . But this implies that (6) is not a minimal model. Hence we have  $8a_4^3 - 27p^n = \pm 1$ , i.e.,  $27p^n = (2a_4 \pm 1)\{(2a_4 \pm 1)^2 \mp 6a_4\}$ . If we suppose  $p=3$ , then we have

$$3^{3+n} = (2a_4 \pm 1)\{(2a_4 \pm 1)^2 \mp 6a_4\}.$$

Hence we get  $(2a_4 \pm 1)^2 \mp 6a_4 = \pm 3$ . But this contradicts with  $n \geq 3$ . Hence we may assume  $p \neq 3$ . Then we have  $2a_4 \pm 1 = \pm 3, \pm 9$  or  $(2a_4 \pm 1)^2 \mp 6a_4 = \pm 3, \pm 9$ . But we can easily see that this also contradicts with  $n \geq 3$ .

Next, we shall show that  $E$  has no rational point of order 2. Suppose  $(x_0, y_0)$  be a rational point on (6) of order 2. Then  $x_0$  is a root of the equation  $a_3^2 - 4(x_0^3 + a_2x_0^2 + a_3a_4x_0) = 0$ . Put  $2x_0 = X_0$ , then  $X_0^3 + 2a_2X_0^2 + 4a_3a_4X_0 - 2a_3^2 = 0$ . Hence  $X_0$  is an even integer. This implies that  $a_3$  is even. But this is a contradiction. This completes the proof of Theorem 1.

**Theorem 2.** *There are only two elliptic curves which have three rational points of order 2, and they have no other rational point of finite order except zero.*

Proof. Let (1) be a global minimal model of such a curve, then by the Remark we may assume  $a_3 = a_6 = 0$ , i.e.,

$$(7) \quad y^2 + a_1xy + x^3 + a_2x^2 + a_4x = 0,$$

then we have

$$(8) \quad \Delta = a_4^2 \{ (a_1^2 - 4a_2)^2 - 2^6 a_4 \} = \pm p^m.$$

By the assumption, the equation  $(a_1 x)^2 - 4(x^3 + a_p x^2 + a_4 x) = 0$  has three rational roots. Hence we get

$$(a_1^2 - 4a_2)^2 - 2^6 a_4 = r^2 \quad \text{with some } r \in \mathbb{Q}.$$

By (8) we get  $r = p^n$ . Hence we have

$$(9) \quad 2^6 a_4 = (a_1^2 - 4a_2 - p^n)(a_1^2 - 4a_2 + p^n).$$

If we replace  $y$  by  $y + ax$ , then  $a_1$  is replaced by  $a_1 + 2a$ . By (8) we see that  $a_1$  is odd, so we can take  $a_1$  to be an arbitrary odd integer.

(I) The case  $n = 0$

We get  $\Delta = a_4^2$  and  $2^6 a_4 = (a_1^2 - 4a_2 - 1)(a_1^2 - 4a_2 + 1)$ , hence we have

$$\begin{aligned} a_1^2 - 4a_2 - 1 &= \pm 32p^\alpha \\ a_1^2 - 4a_2 + 1 &= \pm 32p^\beta + 2 = \pm 2p^\beta \end{aligned}$$

If we suppose  $\alpha > 0$ , then we get  $\pm 16p + 1 = \pm 1$ . But this is impossible. Hence, by  $\pm 16 + 1 = \pm p^\beta$ , we get  $p = 17$ . Then we get the curve 10 by taking  $a_1 = 1$ .

(II) The case  $n = 1$

First, we shall consider the case  $p \equiv 1 \pmod{4}$ , then by (9) we have

$$\begin{aligned} a_1^2 - 4a_2 - p &= \pm 32p^\alpha \\ a_1^2 - 4a_2 + p &= \pm 32p^\beta + 2p = \pm 2p^\beta \end{aligned}$$

If  $\alpha = 0$ , then  $\pm 16 + p = \pm 1$ . Hence we get  $p = 17$  and  $a_4 = -1$  by taking  $a_1 = 1$ . But we can easily see that this curve is isomorphic to the curve 10 over  $\mathbb{Q}$ . If  $\alpha = 1$ , then  $\pm 16 + 1 = p^{\beta'}$ . Hence we get  $p = 17$ , the curve 11 by taking  $a_1 = 1$ . If  $\alpha \geq 2$ , then  $\pm 16p^{\alpha-1} + 1 = \pm 1$ . But this is impossible.

Next, we shall consider the case  $p \equiv 3 \pmod{4}$ , then by (9) we have

$$\begin{aligned} a_1^2 - 4a_2 - p &= \pm 2p^\alpha \\ a_1^2 - 4a_2 + p &= \pm 2p^\beta + 2p = \pm 32p^\beta. \end{aligned}$$

By the same method as above, we obtain  $\alpha = 2$  and  $p = 17$ . But this contradicts with  $p \equiv 3 \pmod{4}$ .

(III) The case  $n \geq 2$

Put  $a_4 = \pm p^\alpha$ . If  $\alpha \geq 3$ , then by (9)  $a_1^2 - 4a_2 \equiv 0 \pmod{p^2}$ . We may assume  $a_1 \equiv 0$



mod  $p$ , hence  $a_2 \equiv 0 \pmod{p^2}$ . Therefore by (9)  $a_4 \equiv 0 \pmod{p^4}$ . But these imply that (7) is not a minimal model. Hence we get  $0 \leq \alpha \leq 2$ . If we suppose  $\alpha = 0$ , then by (9) we get

$$\pm 2^s = (a_1^2 - 4c_2 - p^n)(a_1^2 - 4a_2 + p^n).$$

If  $p \equiv 3 \pmod{4}$  and  $n$  is odd, then we get

$$\begin{aligned} a_1^2 - 4a_2 - p^n &= \pm 2 \\ a_1^2 - 4a_2 + p^n &= \pm 2 + 2p^n = \pm 32. \end{aligned}$$

Hence we get  $2p^n = \pm 32 \pm 2$ . But this contradicts with  $n \geq 2$ . If  $p \equiv 1 \pmod{4}$  or  $n$  is even, then we get

$$\begin{aligned} a_1^2 - 4a_2 - p^n &= \pm 32 \\ a_1^2 - 4a_2 + p^n &= \pm 32 + 2p^n = \pm 2 \end{aligned}$$

Hence we get  $2p^n = \pm 32 \pm 2$ . But this also contradicts with  $n \geq 2$ . If we suppose  $\alpha \geq 1$ , then by (9) we get  $\pm 2^s p^\alpha = (a_1^2 - 4a_2 - p^n)(a_1^2 - 4a_2 + p^n)$ . Hence we have  $a_1^2 - 4a_2 \equiv 0 \pmod{p}$ , and we may assume  $\alpha = 2$ . If  $p \equiv 1 \pmod{4}$  or  $n$  is even, then we get

$$\begin{aligned} a_1^2 - 4a_2 - p^n &= \pm 32p \\ a_1^2 - 4a_2 + p^n &= \pm 32p + 2p^n = \pm 2p. \end{aligned}$$

Hence we get  $n=2$ ,  $p=17$ ,  $a_2=64$ , and  $a_4=-17^2$  by taking  $a_1=1$ . But we can easily see that this curve is isomorphic to the the curve 11 over  $\mathbb{Q}$ . If  $p \equiv 3 \pmod{4}$  and  $n$  is odd, then we get

$$\begin{aligned} a_1^2 - 4a_2 - p^n &= \pm 2p \\ a_1^2 - 4a_2 + p^n &= \pm 32p. \end{aligned}$$

This implies  $p=17$ . But this is a contradiction.

Since these curves have a good reduction at  $p$  for  $p=3$  and  $p=5$ , the latter half of Theorem is obvious by Lemma 2.

**Theorem 3.** *There are three elliptic curves which have rational points of order 4. Each of them has only four rational points of finite order.*

*Proof.* Let  $E$  be such a curve and (1) be a global minimal model of  $E$ , then by Lemma 1 we may assume that  $a_6=0$  and  $t=(0, 0)$  is a point of order 4. Hence  $a_3 \neq 0$ . Moreover, after the transformation  $y \rightarrow -y$  (if necessary), we assume  $a_3 > 0$ . Put  $-2t=(x_0, y_0)$ . Then we have

$$\begin{aligned} (-a_4/a_3)^2 + a_1(-a_4/a_3) + a_2 &= -x_0 \\ y_0 &= (-a_4/a_3)x_0. \end{aligned}$$

Since  $(x_0, y_0)$  is a point of order 2, we get

$$2y_0 + a_1x_0 + a_3 = 0.$$

Hence

$$-2(-a_4/a_3)\{(-a_4/a_3)^2 + a_1(-a_4/a_3) + a_2\} - a_1\{(-a_4/a_3)^2 + a_1(-a_4/a_3) + a_2\} + a_3 = 0.$$

This shows that  $2a_4/a_3$  is an integer. Put

$$(10) \quad 2a_4 = a_3a_4',$$

then

$$(11) \quad (a_4' - a_1)(a_4'^2 - 2a_1a_4' + 4a_2) + 4a_3 = 0.$$

Put

$$(12) \quad \begin{aligned} a_4' - a_1 &= \alpha \\ a_4'^2 - 2a_1a_4' + 4a_2 &= \beta, \end{aligned}$$

then

$$(13) \quad \alpha\beta + 4a_3 = 0$$

$$(14) \quad a_1^2 - 4a_2 = \alpha^2 - \beta.$$

Hence we get

$$\begin{aligned} \Delta &= a_3^2\{(\alpha^2 - \beta)^2\beta/4 + 2\alpha^4\beta - 27(-\alpha\beta/4)^2 + 9(-\alpha\beta/4)\alpha(\alpha^2 - \beta)\} \\ &= a_3^2\beta^2(\alpha^2 + 4\beta)/16. \end{aligned}$$

(I) The case  $a_4'$  is even

By (12),  $\beta$  is divisible by 4. Put  $\beta = 4\beta'$ , then we have

$$\Delta = a_3^2\beta'^2(\alpha^2 + 16\beta') = \pm p^m.$$

First, we shall consider the cases  $\beta' = \pm 1$ , then by (13) we get  $a_3 = \pm \alpha$ . Hence

$$\Delta = a_3^2(a_3^2 \pm 16) = \pm p^m.$$

If  $a_3$  is divisible by  $p$ , then  $a_3^2 \pm 16 = \pm 1$ . But this is impossible. Hence  $a_3 = \pm 1$ . Therefore we may assume  $a_3 = 1$  and  $a_1 = 1$ . Hence we get the curve 12. Next, we shall consider the cases  $\beta' = \pm p^n$  with  $n > 0$ . If  $\alpha = \pm 1$ , then by (13)  $\beta' = \mp a_3$ . Hence we get  $\Delta = a_3^2(1 \mp 16a_3)$ , so  $1 \mp 16a_3 = \pm 1$ . But this is impossible. Hence we may assume  $\alpha \equiv 0 \pmod p$  and  $a_1 \equiv 0 \pmod p$ , then by (12) we have  $a_4' \equiv 0 \pmod p$ . If  $\beta'$  is divisible by  $p^2$ , then by (12) we see that  $a_2$  is divisible by  $p^2$ . Hence we get  $a_3 \equiv 0 \pmod{p^3}$  by (13) and  $a_4 \equiv 0 \pmod{p^4}$  by (10). But this contradicts with the fact that (1) is a minimal model of  $E$ . Hence we get

$\beta' = \pm p$  and  $\alpha \equiv 0 \pmod{p}$ , so

$$\Delta = a_3^2 p^2 (\alpha^2 \pm 16p) = \pm p^m.$$

By (13) we may put  $\alpha = \pm p^{m'}$  with  $m' > 0$ , then we get  $p^{2m'-1} \pm 16 = \pm 1$ , so  $p = 17$ , the curve 13 by taking  $a_1 = 17$ .

(II) The case  $a_4'$  is odd

By (10), (12), (13), we see that  $a_3$  is even and  $\alpha$  is divisible by 8. Put  $a_3 = 2a_3'$  and  $\alpha = 8\alpha'$ , then we have

$$\Delta = a_3'^2 \beta^2 (16\alpha'^2 + \beta) = \pm p^m.$$

First, we shall consider the cases  $\beta = \pm 1$ . By looking at (14) modulo 4 we have  $\beta = -1$ , then by (13)  $\alpha' = a_3'$ . Hence we have

$$\Delta = a_3'^2 (16a_3'^2 - 1) = \pm p^m.$$

Therefore  $a_3'$  is not divisible by  $p$ , so we assume  $a_3' = 1$ . But this is impossible. Next, we shall consider the case  $\beta \equiv 0 \pmod{p}$ . If  $\alpha' = \pm 1$ , then  $\beta = \mp a_3'$ . Hence we have

$$\Delta = a_3'^4 (16 \mp a_3') = \pm p^m.$$

Hence we get  $a_3' = p = 17$ , the curve 14. If  $\alpha'$  is divisible by  $p$ , then we may assume that  $\beta$  is not divisible by  $p^2$ . For if  $\beta$  is divisible by  $p^2$ , then we get  $a_4' \equiv 0 \pmod{p}$  by (12),  $a_2 \equiv 0 \pmod{p^2}$  by (12),  $a_3 \equiv 0 \pmod{p^3}$  by (13) and  $a_4 \equiv 0 \pmod{p^4}$  by (10) by taking  $a_1 = p$ . But this contradicts with the fact that (1) is a minimal model of  $E$ . Hence we have  $\beta = \pm p$  and  $\Delta = p^2 a_3'^2 (16p^{m'} \pm p)$  with  $m' \geq 2$ . Hence we have

$$16p^{m'-1} \pm 1 = 1.$$

But this is impossible.

The latter half of Theorem 3 is immediate by the same reasoning as in the proof of Theorem 2.

**Theorem 4.** *There are only two elliptic curves which have rational points of order 5, and then they have no rational point of order 2.*

*Proof.* Let  $E$  be such a curve and (1) be a global minimal model of  $E$ . By the lemma 1 we may assume  $a_6 = 0$  and  $t = (0, 0)$  be a point on (1) of order 5. Hence  $a_3 \neq 0$  and  $-t = (0, -a_3)$ . Put  $-2t = (x_0, y_0)$ , then  $x_0$  and  $y_0$  are integers, and we have

$$(-a_4/a_3)^2 + a_1(-a_4/a_3) + a_2 = -x_0.$$

Hence  $a_4/a_3$  is an integer. Put  $a_4 = a_3 a_4'$ , then we have

$$\Delta = a_3^2 \{ (a_1^2 - 4a_2)^2 (a_4'^2 - a_1 a_4' + a_2) - 8a_3 (2a_4' - a_1)^3 - 27a_3^2 + 9a_3 (a_1^2 - 4a_2) (2a_4' - a_1) \} = \pm p^m.$$

Put  $2t = (x_0, y_0')$ , then  $a_1 x_0 + a_3 = -(y_0 + y_0')$ . Let  $E(2)$  be the reduction of  $E$  at 2, then  $E(2)$  has just five  $GF(2)$ -rational points and they are the reductions of  $\{nt \mid n=1, 2, 3, 4, 5\}$ . Since  $t \pmod 2 = (0, 0)$  and  $-t \pmod 2 = (0, 1)$ , we have either  $2t \pmod 2 = (1, 0)$ ,  $-2t \pmod 2 = (1, 1)$  or  $2t \pmod 2 = (1, 1)$ ,  $-2t \pmod 2 = (1, 0)$ . In either case, we get  $y_0 + y_0' \equiv 1 \pmod 2$ , i.e.,  $a_1 x_0 + a_3 \equiv 1 \pmod 2$ . Since  $a_3$  and  $x_0$  are odd,  $a_1$  must be even. Hence we may assume  $a_1 = 0$ . Hence we can assume that

$$(15) \quad y^2 + a_3 y + x^3 + a_2 x^2 + a_3 a_4 x = 0$$

is a global minimal model of  $E$ ; and we have

$$\Delta = a_3^2 \{ 16a_2^2 (a_2 + a_4^2) - 64a_3 a_4^3 - 27a_3^2 - 72a_2 a_3 a_4 \},$$

and  $t = (0, 0)$ ,  $-t = (0, -a_3)$ ,  $-2t = -(a_2 + a_4^2, a_4(a_2 + a_4^2))$ .

After a suitable translation, if necessary, we may assume that  $a_3$  and  $a_2 + a_4^2$  are positive. By the assumption, the tangent of  $E$  at the point  $-2t$  intersects  $E$  at the point  $-t$ . Hence we get

$$(16) \quad \frac{a_4(a_2 + a_4^2) + a_3}{-(a_2 + a_4^2)} = -\frac{3x_0^2 + 2a_2 x_0 + a_3 a_4}{2y_0 + a_3} = -\frac{3(a_2 + a_4^2)^2 - 2a_2(a_2 + a_4^2) + a_3 a_4}{2a_4(a_2 + a_4^2) + a_3}.$$

By the same reasoning as before this is an integer. Hence  $a_2 + a_4^2$  divides  $a_3$ . Put  $a_3 = a_3'(a_2 + a_4^2)$ , then we have by (16)

$$a_4 + a_3' = \frac{3(a_2 + a_4^2) - 2a_2 + a_3' a_4}{2a_4 + a_3'}.$$

Hence

$$(17) \quad a_2 + a_4^2 = a_3'(2a_4 + a_3').$$

First we shall consider the case  $a_3' = 1$ . By (17),  $a_2 + a_4^2 = 2a_4 + 1 = a_3$ , hence we get

$$\begin{aligned} 2a_4 &= a_3 - 1 \\ 4a_2 &= 4a_3 - (2a_4)^2 = -a_3^2 + 6a_3 - 1, \end{aligned}$$

and consequently

$$\begin{aligned} \Delta &= a_3^2 \{ (-a_3^2 + 6a_3 - 1)^2 a_3 - 8a_3 (a_3 - 1)^3 - 27a_3^2 - 9a_3 (a_3 - 1) (-a_3^2 + 6a_3 - 1) \} \\ &= a_3^5 (a_3^2 - 11a_3 - 1) = \pm p^m. \end{aligned}$$

Therefore we have  $a_3 = 1$  or  $a_3^2 - 11a_3 - 1 = \pm 1$ . Hence we get  $a_3 = 1, 11$ , curves 15, 16. Next, we shall consider the case  $a_3' = p^{n'}$  with  $n' > 0$ . Then by (17) we

have  $a_3 = (a_2 + a_4^2)a_3' = a_3'^2(2a_4 + a_3')$ . If we suppose  $2a_4 + a_3' \equiv 0 \pmod{p}$ , then  $a_4 \equiv 0 \pmod{p}$ . Hence by (17) we get  $a_3 \equiv 0 \pmod{p^3}$  and  $a_2 \equiv 0 \pmod{p^2}$ . But this contradicts with the fact that (15) is a minimal model of  $E$ . Hence we get  $2a_4 + a_3' = 1$ , i.e.,  $a_3'^2 = a_3$  and  $a_2 + a_4^2 = a_3'$ , so

$$\begin{aligned} 2a_4 &= -a_3' + 1 \\ 4a_2 &= -a_3'^2 + 6a_3' - 1. \end{aligned}$$

Hence we have

$$\begin{aligned} \Delta &= a_3'^4 \{(-a_3'^2 + 6a_3' - 1)^2 a_3' - 8a_3'^2(-a_3' + 1)^3 - 27a_3'^4 - 9a_3'^2(-a_3'^2 + 6a_3' - 1) \\ &\quad (-a_3' + 1)\} \\ &= a_3'^5(-a_3'^2 - 11a_3' + 1) = \pm p^m. \end{aligned}$$

Therefore we have  $-a_3'^2 - 11a_3' + 1 = \pm 1$ . So  $a_3' = -11$ . But this contradicts with  $a_3' > 0$ .

The latter half of Theorem 4 is immediate by the same reasoning as in the proof of Theorem 1.

### 3. Determination of the Galois group

Let  $E$  be an elliptic curve defined over  $\mathbf{Q}$ . For a prime number  $l$ , put

$$E_l = \{t \in E \mid lt = 0\}.$$

Let  $\mathbf{Q}(E_l)$  be the field generated by the coordinates of the points of  $E_l$ , then the field  $\mathbf{Q}(E_l)$  is a normal extension of  $\mathbf{Q}$  of finite degree. Put

$$G_l = \text{Gal}(\mathbf{Q}(E_l)/\mathbf{Q}).$$

Taking a basis  $\{t_1, t_2\}$  of  $E_l$ , we get a faithful representation  $R_l$  of the group  $G_l$  in  $GL(2, \mathbf{Z}/l\mathbf{Z})$ ;

$$\begin{pmatrix} t_1^\sigma \\ t_2^\sigma \end{pmatrix} = R_l(\sigma) \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \quad \text{for any } \sigma \in G_l.$$

If the invariant of  $E$  is not an integer, then it is known that  $G_l$  is isomorphic to  $GL(2, \mathbf{Z}/l\mathbf{Z})$  for almost all  $l$  (see the theorem of Serre [4] IV-20). Moreover in some cases, following Serre [4] IV-20, we can determine the (finite) set of  $l$ 's with  $R_l(G_l) \cong GL(2, \mathbf{Z}/l\mathbf{Z})$ .

**Lemma 3.** *If  $R_l(G_l)$  satisfies the next three conditions a), b), c), then  $R_l(G_l)$  is equal to  $GL(2, \mathbf{Z}/l\mathbf{Z})$ .*

a)  $\det R_l(G_l) = (\mathbf{Z}/l\mathbf{Z})^\times$ .

b)  $R_l(G_l)$  contains the element  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  with respect to a suitable basis of  $E_l$ .

c)  $E_l$  is irreducible as  $G_l$ -module.

Proof. See the lemma 2 of Serre [4] IV-20.

**Theorem 5.** *Let  $E$  be one of the curves 1, 2, 6, 7, 10, 11, 12, 13, 14, 15, 16, in the table at the end of the paper. If  $E$  has a  $\mathbf{Q}$ -rational point of order  $l_0$  ( $l_0=2, 3,$  or  $5$ ), then  $R_l(G_l)=GL(2, \mathbf{Z}/l\mathbf{Z})$  for every prime  $l \neq l_0$ .*

Proof. We shall consider the case where  $E$  is the curve 16. This is the case dealt with in Shimura [6]. In this case we have

$$E: y^2 + 11y + x^3 - 14x^2 + 55x = 0$$

$$J = -2^{12} \cdot 31^3 \cdot 11^{-5}.$$

Let  $l$  be a prime number not equal to 5, then  $R_l(G_l)$  satisfies the condition  $b)$  of Lemma 3 by the lemma 1 of Serre [4] IV-20. The condition  $a)$  is equivalent to the fact that  $\mathbf{Q}(E_l)$  contains a primitive  $l$ -th root of unity, which is well-known. We shall show that the condition  $c)$  is satisfied for such  $l$ . Assume this is not the case. Then taking a suitable basis  $\{t_1, t_2\}$ , we have

$$R_l(G_l) \subset \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL(2, \mathbf{Z}/l\mathbf{Z}) \right\}$$

Put

$$A = \langle t_2 \rangle$$

$$E' = E/A,$$

then  $E'$  is an elliptic curve defined over  $\mathbf{Q}$  and has rational points of order 5. Let  $\lambda$  be the canonical isogeny of degree  $l$  from  $E$  to  $E'$ , then  $E'$  is not isomorphic to  $E$ , because  $E$  has no complex multiplication. By Serre-Tate [5],  $E'$  has a conductor of 11-power. Hence  $E'$  must be the curve 15. By Vélú [7], we see that there is an isogeny  $\lambda'$  of degree 5 from the curve 15 to  $E$ , i.e., we get

$$E \xrightarrow{\lambda} E' \xrightarrow{\lambda'} E.$$

Then  $\lambda' \circ \lambda$  is an endomorphism of  $E$  and its degree is  $5l$ . But this is impossible since  $E$  has no complex multiplication.

By Vélú [7],  $E_5$  is completely reducible as  $G_5$ -module. This shows that  $R_5(G_5)$  is isomorphic to  $(\mathbf{Z}/5\mathbf{Z})^\times$  and  $\mathbf{Q}(E_5) = \mathbf{Q}(e^{2\pi i/5})$ .

For the other curves we can prove the theorem by the same method as above,

TABLE  $y^2+a_1xy+a_3y+x^3+a_2x^2+a_4x=0$

	$a_1$	$a_3$	$a_2$	$a_4$	$d$	$N$	$J$	type of $F$
1	0	1	-1	1	-19	19	$2^{15} \cdot 19^{-1}$	a cyclic group of order 3
2	0	1	-4	2	37	37	$2^{15} \cdot 5^3 \cdot 37^{-1}$	"
3	0	1	0	0	$-3^3$	$3^3$	0	"
4	0	1	-9	-3	$-3^5$	$3^3$	$-2^{15} \cdot 3 \cdot 5^3$	"
5	0	3	0	0	$-3^7$	$3^5$	0	"
6	0	19	-16	76	$-19^3$	19	$-2^{18} \cdot 7^3 \cdot 19^{-3}$	"
7	0	37	-25	185	$37^3$	37	$2^{15} \cdot 5^3 \cdot 7^3 \cdot 37^{-3}$	"
8	0	9	-9	27	$-3^9$	$3^3$	0	"
9	0	9	0	0	$-3^{11}$	$3^5$	0	"
10	1	0	-8	17	$17^2$	17	$3^3 \cdot 7^3 \cdot 13^3 \cdot 17^{-2}$	(2, 2)-type
11	1	0	-140	$17^3$	$17^8$	$17^2$	$3^3 \cdot 7^3 \cdot 13^3 \cdot 17^{-2}$	"
12	1	1	1	0	17	17	$3^3 \cdot 11^3 \cdot 17^{-1}$	a cyclic group of order 4
13	17	$17^3$	-17	$17^3$	$17^7$	$17^2$	$3^3 \cdot 11^3 \cdot 17^{-1}$	"
14	1	34	-20	153	$-17^4$	17	$-3^3 \cdot 11^3 \cdot 17^{-4}$	"
15	0	1	1	0	-11	11	$-2^{12} \cdot 11^{-1}$	a cyclic group of order 5
16	0	11	-14	55	-11 <sup>5</sup>	11	$-2^{12} \cdot 31^3 \cdot 11^{-5}$	"

There are some isogenies connecting these curves, which are easily computed by the method of Vélú [8]. Let  $E^q$  be the curve  $q$  on the table, then we obtain the isogenies of degree 2

$$E^{12} \sim E^{10} = E^{12} / \langle (-1, 0) \rangle, E^{14} \sim E^{10} = E^{14} / \langle \left(\frac{17}{4}, \frac{153}{8}\right) \rangle$$

$$E^{13} \sim E^{11} = E^{13} / \langle (17, -17) \rangle,$$

the isogenies of degree 3

$$E^1 \sim E^6 = E^1 / \langle (0, 0) \rangle, E^2 \sim E^7 = E^2 / \langle (0, 0) \rangle,$$

the isogeny of degree 4

$$E^{12} \sim E^{14} = E^{12} / \langle (0, 0) \rangle,$$

and the isogeny of degree 5

$$E^{15} \sim E^{16} = E^{15} / \langle (0, 0) \rangle.$$

REMARK. There are some other isogenies of degree 3 among the curves  $E^3, E^4, E^5, E^8, E^9$ . But we exclude them, since we have no need for them to complete the proof of the theorem 5.

## References

- [1] J.W.S. Cassels: *Diophantine equations with special references to elliptic curves*, J. London Math. Soc. **41** (1966), 193–291.
- [2] A. Néron: *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, Inst. Hautes Études Sci. Publ. Math. **21** (1964) 5–125.
- [3] A.P. Ogg: *Abelian curves of 2-power conductor*, Proc. Cambridge Philos. Soc. **62** (1966), 143–148.
- [4] J.-P. Serre: *Abelian  $l$ -adic Representations and Elliptic Curves*, Lecture notes, New York, 1968.
- [5] J.-P. Serre and J. Tate: *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492–517.
- [6] G. Shimura: *A reciprocity law in non-solvable extensions*, J. Reine Angew. Math. **221** (1966), 209–220.
- [7] J. Vélú: *Courbes elliptiques sur  $\mathbf{Q}$  ayant bonne réduction en dehors de  $\{11\}$* , C.R. Acad. Sci. Paris **273** (1971), 73–75.
- [8] J. Vélú: *Isogénies entre courbes elliptiques*. C.R. Acad. Sci. Paris **273** (1971), 238–241.
- [9] A. Weil: *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **168** (1967), 149–156.



