

Title	Divisibility conditions on signatures of fixed point sets
Author(s)	Kim, Sung Sook
Citation	Osaka Journal of Mathematics. 1992, 29(2), p. 387-392
Version Type	VoR
URL	https://doi.org/10.18910/11266
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

DIVISIBILITY CONDITIONS ON SIGNATURES OF FIXED POINT SETS

SUNG SOOK KIM

(Received June 12, 1991)

Let G denote the cyclic group of order p , where p is an odd prime. In [6], we constructed a smooth G -action on some \mathbf{Z}_q -homology sphere such that the fixed point set is a closed connected $4r$ -dimensional manifold with nonzero Pontryagin numbers, where q is an odd prime distinct from p .

In this paper we take some preliminary steps towards studying the divisibility conditions on the characteristic numbers of the fixed point set of a G -action on a \mathbf{Z}_q -homology sphere. One reason for interest in this topic is that the image of the fixed point set of a G -action on a \mathbf{Z}_q -homology sphere in $\Omega_*^{SO}/\text{torsion}$ is completely determined by these divisibility conditions. For some time it has been known that nontrivial conditions appear (compare [5]; [2]). Perhaps the simplest divisibility condition involves the signature of the fixed point set. If G acts smoothly and preserving orientation on a closed oriented even dimensional \mathbf{Z}_q -homology sphere, then the signature of the fixed point set must be even because the Euler characteristic number is 2 by the Lefschetz fixed point theorem and the signature and Euler characteristic number of a closed oriented manifold are always congruent modulo 2.

Our first theorem is the following, which is proved by using the G -signature theorem.

Theorem 1. *Let X be a smooth closed oriented manifold of even dimension such that $H^{(\dim X)/2}(X; \mathbf{Q})=0$. If the fixed point set F of a smooth G -action on X is 4-dimensional, then*

$$4 \mid \text{Sign } F, \text{ when } p > 3 \text{ and}$$

$$16 \mid \text{Sign } F, \text{ when } p = 3.$$

Following Kawakubo [5] we say that a smooth G -action is regular if the normal G vector bundle of the fixed point set is decomposed by only one eigenbundle; i.e. it is of the form $\xi_m \otimes t^m$ for some m ($1 \leq m \leq \frac{p-1}{2}$), where ξ_m is a complex vector bundle with trivial G -action and t^m is the complex 1-dimensional

G -module on which a fixed generator of G acts as multiplication by ζ^m ($\zeta = e^{2\pi i/p}$). Note that any G -action is regular in case $p=3$.

Kawakubo [5] showed by using G -bordism theory that for a regular G -action on a closed smooth orientable manifold X , $\text{Sign } F \equiv \text{Sign } X \pmod{p}$ provided $2(p-1) > \dim X$, where F denotes the G -fixed point set. If X is, in particular, a \mathbf{Z}_q -homology sphere, then $\text{Sign } X = 0$; so this gives a divisibility condition on $\text{Sign } F$ provided $2(p-1) > \dim X$. In this paper we obtain divisibility conditions even for $2(p-1) < \dim X$. So it would be interesting to compare Kawakubo's result for the following theorem, which is proved by using the the Atiyah-Singer index theorem with a Dirac operator (i.e. the $\hat{\mathfrak{A}}$ -genus).

Theorem 2. *Let X be a smooth closed Spin manifold of even dimension such that the rational first Pontryagin class vanishes and $H^1(X; \mathbf{Z}) = 0$. If F is the 4-dimensional fixed point set of a smooth regular G -action on X , then*

$$4 \cdot p^{\lfloor (\dim X)/2(p-1) \rfloor} \mid \text{Sign } F.$$

In [2] tom Dieck obtains formal equivariant integrality theorems that can, in principle, be translated into divisibility conditions for characteristic numbers of unitary G -manifolds. A precise understanding of the relationship between these results and Theorem 2 would be enlightening.

ACKNOWLEDGMENTS. I would like to thank Professor Reinhard Schultz for his invaluable advice. I am indebted to Professor Mikiya Masuda for his encouragement and helpful conversations.

1. Divisibility by using G -signature formula

In this section, we prove Theorem 1 by using the Atiyah-Singer G -signature formula. Decompose the normal G -vector bundle ξ of F into eigenbundles as follows; $\xi = \bigoplus_{k=1}^{(p-1)/2} \xi_k \oplus_{\mathcal{C}} t^k$.

By the G -signature theorem, we have the following formula.

$$(1.1) \quad \text{Sign}(g, X) = \text{Constant} \left\langle \mathcal{L}(F) \prod_{k=1}^{(p-1)/2} \prod_j \frac{e^{x_{kj}} \zeta^k + 1}{e^{x_{kj}} \zeta^k - 1}, [F] \right\rangle$$

where $\mathcal{L}(F)$ denotes the Atiyah-Singer L -class (page 577 of [1]) of the bundle tangent to F , $[F]$ denotes the fundamental class of F , $\langle \cdot, \cdot \rangle$ is the natural Kronecker pairing between cohomology and homology, and the symbols x_{kj} have the usual interpretation as roots of the total Chern class of ξ_k such that the total Chern class of ξ_k is $c(\xi_k) = \prod_j (1 + x_{kj})$. Since we assume $H^{(\dim X)/2}(X; \mathbf{Q}) = 0$, $\text{Sign}(g, X) = 0$. This yields the following Lemma:

Lemma 1.1. *Let F^4 be as in Theorem 1 and let $\xi = \bigoplus_{k=1}^{(p-1)/2} \xi_k \otimes t^k$ be the decomposition as above. Then (1.1) reduces to the following equation.*

$$(1.2) \quad -\frac{1}{4} \text{Sign } F = \sum_{k=1}^{(p-1)/2} \frac{\zeta^k + \zeta^{-k}}{(\zeta^k - \zeta^{-k})^2} p_1(\xi_k) [F] + 2 \left(\sum_{k=1}^{(p-1)/2} \frac{1}{\zeta^k - \zeta^{-k}} c_1(\xi_k) \right)^2 [F].$$

Proof. Since $\dim F=4$, $c_i(\xi_k)$ vanishes for $i>2$. Hence we can write $c(\xi_k)=(1+x_{k_1})(1+x_{k_2})$. Then it follows from (1.1) that

$$\begin{aligned} \text{Sign}(g, X) &= \text{Constant} \times \mathcal{L}(F) \prod_{k=1}^{(p-1)/2} \left[\left(\frac{\zeta^k e^{x_{k_1}} + 1}{\zeta^k e^{x_{k_1}} - 1} \right) \left(\frac{\zeta^k e^{x_{k_2}} + 1}{\zeta^k e^{x_{k_2}} - 1} \right) \right] [F] \\ &= \text{Constant} \times \mathcal{L}(F) \prod_{k=1}^{(p-1)/2} \left[1 - \frac{2\zeta^k}{\zeta^{2k} - 1} c_1(\xi_k) \right. \\ &\quad \left. + \frac{\zeta^k}{(\zeta^k - 1)^2} (c_1(\xi_k)^2 - 2c_2(\xi_k)) + \frac{(2\zeta^k)^2}{(\zeta^{2k} - 1)^2} c_2(\xi_k) \right] [F], \end{aligned}$$

where $\mathcal{L}(F) = 1 + \frac{p_1(F)}{12}$. By Hirzebruch signature theorem we have $\mathcal{L}(F) [F] = \frac{1}{4} \text{Sign } F$. Since $\text{Sign}(g, X) = 0$, the above equation reduces to

$$\begin{aligned} 0 &= \frac{1}{4} \text{Sign}(F) + \prod_{k=1}^{(p-1)/2} \left[1 - \frac{2\zeta^k}{\zeta^{2k} - 1} c_1(\xi_k) \right. \\ &\quad \left. + \frac{\zeta^k}{(\zeta^k - 1)^2} (c_1(\xi_k)^2 - 2c_2(\xi_k)) + \frac{(2\zeta^k)^2}{(\zeta^{2k} - 1)^2} c_2(\xi_k) \right] [F] \\ &= \frac{1}{4} \text{Sign}(F) + \left[\sum_{k=1}^{(p-1)/2} \frac{\zeta^k}{(\zeta^k - 1)^2} p_1(\xi_k) + \sum_{k=1}^{(p-1)/2} \frac{(2\zeta^k)^2}{(\zeta^{2k} - 1)^2} c_2(\xi_k) \right. \\ &\quad \left. + \sum_{\substack{k=1 \\ k < j}}^{(p-1)/2} \left(\frac{2\zeta^k}{\zeta^{2k} - 1} \right) \left(\frac{2\zeta^j}{\zeta^{2j} - 1} \right) c_1(\xi_k) c_1(\xi_j) \right] [F]. \end{aligned}$$

Since $p_1(\xi_k) = c_1(\xi_k)^2 - 2c_2(\xi_k)$ by an elementary calculation we have the formula (1.2). Q.E.D.

Multiply both sides of the above equation (1.2) by

$$\prod_{k=1}^{(p-1)/2} (\zeta^k - \zeta^{-k})^2 = (-1)^{(p-1)/2} p \quad (\text{see page 72 of [7]}).$$

Then the right hand side becomes a linear combination of $\zeta^k + \zeta^{-k}$ ($1 \leq k \leq (p-1)/2$) over \mathbf{Z} because it is invariant under the complex conjugation $\zeta \rightarrow \zeta^{-1}$, and $p_1(\xi_k) [F]$ and $c_1(\xi_j) c_1(\xi_k) [F]$ are both integers. This means that $p/4 \text{Sign } F \in \mathbf{Z}[\zeta + \zeta^{-1}] \cap \mathbf{Q}$. However it is well known that $\mathbf{Z}[\zeta + \zeta^{-1}] \cap \mathbf{Q} = \mathbf{Z}$, and therefore $p/4 \text{Sign } F \in \mathbf{Z}$. Since p is an odd prime it follows that $4 \mid \text{Sign } F$.

Consider the equation (1.2) when $p=3$. Then $k=1$, and an elementary calculation shows that (1.2) becomes

$$\text{Sign } F = \frac{16}{15} c_2(\xi) [F],$$

and consequently $16 \mid \text{Sign } F$.

2. Divisibility by using the $\hat{\mathfrak{A}}$ -genus

In this section, we prove Theorem 2 by using the index theorem for the Dirac operator. Consider the \mathbf{Spin}^c -structure P of X determined by a Spin structure (i.e. the \mathbf{Spin}^c -structure with trivial first Chern class). Since $H^1(X; \mathbf{Z}) = 0$ by assumption, the G -action on X lifts to a G action on P [4]. Therefore we can apply the Atiyah-Singer formula for index D , where D is the Dirac operator associated with the \mathbf{Spin}^c -structure P . In our case $c_1(P) = 0$ and ξ is of the form $\xi_m \otimes t^m$ by assumption that the action is regular, so the formula reduces to the following equation:

$$(2.1) \quad (\text{Index}(D))(\zeta) = (-1)^{(\dim X)/2} \hat{\mathfrak{A}}(F) \zeta^\lambda \prod_k \frac{1}{e^{x_{m_k}/2} - e^{-x_{m_k}/2} \zeta^{-m}} [F].$$

(see [3]) where $\hat{\mathfrak{A}}(F) = 1 - 1/24 p_1(F)$, the symbols x_{m_k} have the usual interpretation as formal two-dimensional cohomology classes such that $c(\xi_m) = \prod_k (1 + x_{m_k})$, and $[F]$ denotes the fundamental class of F as before.

Let $d = \dim_C \xi$. Then

$$\begin{aligned} & \prod_k \frac{1}{e^{x_{m_k}/2} - e^{-x_{m_k}/2} \zeta^{-m}} \\ &= \frac{1}{(1 - \zeta^{-m})^d} \prod_k \left(1 - \frac{1}{2} \left(\frac{1 + \zeta^{-m}}{1 - \zeta^{-m}} \right) x_{m_k} + \frac{1}{4} \left(\left(\frac{1 + \zeta^{-m}}{1 - \zeta^{-m}} \right)^2 - \frac{1}{2} \right) x_{m_k}^2 \right) \\ &= \frac{1}{(1 - \zeta^{-m})^d} \left(1 - \frac{1}{2} \left(\frac{1 + \zeta^{-m}}{1 - \zeta^{-m}} \right) c_1(\xi) + \frac{1}{4} \left(\left(\frac{1 + \zeta^{-m}}{1 - \zeta^{-m}} \right)^2 - \frac{1}{2} \right) p_1(\xi) \right. \\ & \quad \left. + \frac{1}{4} \left(\frac{1 + \zeta^{-m}}{1 - \zeta^{-m}} \right)^2 c_2(\xi) \right) \end{aligned}$$

Multiply the above equation by $\hat{\mathfrak{A}}(F) = 1 - 1/24 p_1(F) = 1 + 1/24 p_1(\xi)$ and evaluate it on $[F]$ (note that $p_1(F) + p_1(\xi) = i^* p_1(X) = 0$ by assumption). Then we obtain

$$(2.2) \quad \frac{1}{(1 - \zeta^{-m})^{d+2}} \left(\frac{1}{4} \left((1 + \zeta^{-m})^2 - \frac{1}{3} \right) p_1(\xi) + \frac{1}{4} (1 + \zeta^{-m})^2 c_2(\xi) \right) [F].$$

We note that the left hand side of (2.1) is an element of the ring $\mathbf{Z}[\zeta]$. Let $z = \zeta^{-m}$. Then (2.1) becomes

$$\begin{aligned} & 12(b_1 + b_1 z + b_2 z^2 + \dots + b_{p-1} z^{p-1}) (1 - z)^{d+2} \\ &= (3(1 + z)^2 - 1) p_1(\xi) [F] + 3(1 + z)^2 c_2(\xi) [F], \end{aligned}$$

with integers $b_i (1 \leq i \leq p-1)$.

Since $p_1(\xi) [F] = -p_1(F) [F] = -3 \text{ Sign } F$ we have

$$4(b_0 + b_1 z + b_2 z^2 + \dots + b_{p-1} z^{p-1}) (1 - z)^{d+2}$$

$$\begin{aligned}
 (2.3) \quad &= (1-3(1+z)^2) \text{Sign } F + (1+z)^2 c_2(\xi) [F] \\
 &= (-2 \text{Sign } F + c_2(\xi) [F]) + (2c_2(\xi) [F] - 6 \text{Sign } F) z \\
 &\quad + (c_2(\xi) [F] - 3 \text{Sign } F) z^2.
 \end{aligned}$$

Write $d+2=r(p-1)+s$, where $0 \leq s < p-1$. Since $(1-z)^{p-1} \equiv 0 \pmod{p}$ we have

$$(2.4) \quad p^r \mid (1-z)^{d+2} = ((1-z)^{p-1})^r (1-z)^s.$$

On the other hand

$$d+2 = \frac{\dim X - 4}{2} + 2 = \frac{\dim X}{2}.$$

So

$$(2.5) \quad r = \frac{\dim X - 2s}{2(p-1)} = \left[\frac{\dim X}{2(p-1)} \right].$$

It follows from (2.3) and (2.4) that

$$4p^r \mid (c_2(\xi) [F] - 2 \text{Sign } F)$$

and

$$4p^r \mid (c_2(\xi) [F] - 3 \text{Sign } F).$$

Therefore

$$4p^r \mid \text{Sign } F.$$

This together with (2.5) proves Theorem 2.

References

- [1] M.F. Atiyah and I.M. Singer: *The index of elliptic operators*, III, Ann. of Math. **87** (1968), 546-604.
- [2] T. tom Dieck: *Bordism of G-manifolds and integrality theorems*, Topology **9** (1970), 345-358.
- [3] A. Hattori: *Spin^c-structures and S¹-actions*, Inventiones Math. **48** (1978), 7-31.
- [4] A. Hattori and T. Yoshida: *Lifting compact group actions in fiber bundles*, Japan J. Math. **2**, No. 1 (1976), 13-25.
- [5] K. Kawakubo: *The index and the Todd genus of Z_p-actions*, Amer. J. Math. **97** (1975), 182-204.
- [6] S.S. Kim: *Characteristic numbers and group actions on homology spheres*, Ph. D. Thesis, Purdue University, 1988.
- [7] I.N. Stewart and D.O. Tall: *Algebraic Number Theory (second edition)*, Chapman and Hall, New York, 1987.

Department of Mathematics
Paichai University
Taejon, (302-735),
Korea