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DIVISIBILITY CONDITIONS ON SIGNATURES OF FIXED POINT SETS

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Let G denote the cyclic group of order p, where p is an odd prime. In [6], we constructed a smooth G-action on some $\boldsymbol{Z}_q$-homology sphere such that the fixed point set is a closed connected $4r$-dimensional manifold with nonzero Pontryagin numbers, where q is an odd prime distinct from p.

In this paper we take some preliminary steps towards studying the divisibility conditions on the characteristic numbers of the fixed point set of a G-action on a $\boldsymbol{Z}_q$-homology sphere. One reason for interest in this topic is that the image of the fixed point set of a G-action on a $\boldsymbol{Z}_q$-homology sphere in Ω_*^{SO}/torsion is completely determined by these divisibility conditions. For some time it has been known that nontrivial conditions appear (compare [5]; [2]). Perhaps the simplest divisibility condition involves the signature of the fixed point set. If G acts smoothly and preserving orientation on a closed oriented even dimensional $\boldsymbol{Z}_q$-homology sphere, then the signature of the fixed point set must be even because the Euler characteristic number is 2 by the Lefschetz fixed point theorem and the signature and Euler characteristic number of a closed oriented manifold are always congruent modulo 2.

Our first theorem is the following, which is proved by using the G-signature theorem.

Theorem 1. *Let X be a smooth closed oriented manifold of even dimension such that $H^{(\dim X)/2}(X; \boldsymbol{Q})=0$. If the fixed point set F of a smooth G-action on X is 4-dimensional, then*

$$4 \mid \operatorname{Sign} F\,, \text{ when } p>3 \text{ and}$$
$$16 \mid \operatorname{Sign} F\,, \text{ when } p=3\,.$$

Following Kawakubo [5] we say that a smooth G-action is regular if the normal G vector bundle of the fixed point set is decomposed by only one eigenbundle; i.e. it is of the form $\xi_m \otimes t^m$ for some m $(1 \leq m \leq \frac{p-1}{2})$, where ξ_m is a complex vector bundle with trivial G-action and t^m is the complex 1-dimensional

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G-module on which a fixed generator of G acts as multiplication by ζ^m ($\zeta=e^{2\pi i/p}$). Note that any G-action is regular in case $p=3$.

Kawakubo [5] showed by using G-bordism theory that for a regular G-action on a closed smooth orientable manifold X, Sign $F\equiv$ Sign X (mod p) provided $2(p-1)>\dim X$, where F denotes the G-fixed point set. If X is, in particular, a $\boldsymbol{Z}_q$-homology sphere, then Sign $X=0$; so this gives a divisibility condition on Sign F provided $2(p-1)>\dim X$. In this paper we obtain divisibility conditions even for $2(p-1)<\dim X$. So it would be interesting to compare Kawakubo's result for the following theorem, which is proved by using the the Atiyah-Singer index theorem with a Dirac operator (i.e. the $\hat{\mathfrak{A}}$-genus).

Theorem 2. *Let X be a smooth closed Spin manifold of even dimension such that the rational first Pontryagin class vanishes and $H^1(X;\boldsymbol{Z})=0$. If F is the 4-dimensional fixed point set of a smooth regular G-action on X, then*

$$4\cdot p^{[(\dim X)/2(p-1)]}\mid \text{Sign } F\,.$$

In [2] tom Dieck obtains formal equivariant integrality theorems that can, in principle, be translated into divisibility conditions for characteristic numbers of unitary G-manifolds. A precise understanding of the relationship between these results and Theorem 2 would be enlightening.

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1. Divisibility by using G-signature formula

In this section, we prove Theorem 1 by using the Atiyah-Singer G-signature formula. Decompose the normal G-vector bundle ξ of F into eigenbundles as follows; $\xi=\oplus_{k=1}^{(p-1)/2}\xi_k\oplus_C t^k$.

By the G-signature theorem, we have the following formula.

$$\text{Sign}\,(g, X) = \text{Constant}\left\langle \mathcal{L}(F)\prod_{k=1}^{(p-1)/2}\prod_j \frac{e^{x_{kj}}\zeta^k+1}{e^{x_{kj}}\zeta^k-1}\,,\,[F]\right\rangle \tag{1.1}$$

where $\mathcal{L}(F)$ denotes the Atiyah-Singer L-class (page 577 of [1]) of the bundle tangent to F, $[F]$ denotes the fundamental class of F, $\langle\cdot,\cdot\rangle$ is the natural Kronecker pairing between cohomology and homology, and the symbols x_{kj} have the usual interpretation as roots of the total Chern class of ξ_k such that the total Chern class of ξ_k is $c(\xi_k)=\prod_j(1+x_{kj})$. Since we assume $H^{(\dim X)/2}(X;\boldsymbol{Q})=0$, Sign$(g,X)=0$. This yields the following Lemma:

Lemma 1.1. *Let F^4 be as in Theorem 1 and let $\xi=\oplus_{k=1}^{(p-1)/2}\xi_k\otimes t^k$ be the decomposition as above. Then* (1.1) *reduces to the following equation.*

$$(1.2)\quad -\frac{1}{4}\,\mathrm{Sign}\,F=\sum_{k=1}^{(p-1)/2}\frac{\zeta^k+\zeta^{-k}}{(\zeta^k-\zeta^{-k})^2}\,p_1(\xi_k)\,[F]+2\Big(\sum_{k=1}^{(p-1)/2}\frac{1}{\zeta^k-\zeta^{-k}}\,c_1(\xi_k)\Big)^2[F]\,.$$

Proof. Since dim $F=4$, $c_i(\xi_k)$ vanishes for $i>2$. Hence we can write $c(\xi_k)=(1+x_{k_1})(1+x_{k_2})$. Then it follows from (1.1) that

$$\begin{aligned}\mathrm{Sign}\,(g,X)&=\mathrm{Constant}\times\mathcal{L}(F)\prod_{k=1}^{(p-1)/2}\Big[\Big(\frac{\zeta^k e^{x_{k_1}}+1}{\zeta^k e^{x_{k_1}}-1}\Big)\Big(\frac{\zeta^k e^{x_{k_2}}+1}{\zeta^k e^{x_{k_2}}-1}\Big)\Big][F]\\&=\mathrm{Constant}\times\mathcal{L}(F)\prod_{k=1}^{(p-1)/2}\,[1-\frac{2\zeta^k}{\zeta^{2k}-1}\,c_1(\xi_k)\\&\quad+\frac{\zeta^k}{(\zeta^k-1)^2}\,(c_1(\xi_k)^2-2c_2(\xi_k))+\frac{(2\zeta^k)^2}{(\zeta^{2k}-1)^2}\,c_2(\xi_k)]\,[F]\,,\end{aligned}$$

where $\mathcal{L}(F)=1+\frac{p_1(F)}{12}$. By Hirezebruch signature theorem we have $\mathcal{L}(F)\,[F]=\frac{1}{4}$ Sign F. Since Sign $(g,X)=0$, the above equation reduces to

$$\begin{aligned}0&=\frac{1}{4}\,\mathrm{Sign}\,(F)+\prod_{k=1}^{(p-1)/2}\,[1-\frac{2\zeta^k}{\zeta^{2k}-1}\,c_1(\xi_k)\\&\quad+\frac{\zeta^k}{(\zeta^k-1)^2}\,(c_1(\xi_k)^2-2c_2(\xi_k))+\frac{(2\zeta^k)^2}{(\zeta^{2k}-1)^2}\,c_2(\xi_k)]\,[F]\\&=\frac{1}{4}\,\mathrm{Sign}\,(F)+[\sum_{k=1}^{(p-1)/2}\frac{\zeta^k}{(\zeta^k-1)^2}\,p_1(\xi_k)+\sum_{k=1}^{(p-1)/2}\frac{(2\zeta^k)^2}{(\zeta^{2k}-1)^2}\,c_2(\xi_k)\\&\quad+\sum_{\substack{k=1\\k<j}}^{(p-1/2)}\Big(\frac{2\zeta^k}{\zeta^{2k}-1}\Big)\Big(\frac{2\zeta^j}{\zeta^{2j}-1}\Big)\,c_1(\xi_k)\,c_1(\xi_j)]\,[F]\,.\end{aligned}$$

Since $p_1(\xi_k)=c_1(\xi_k)^2-2c_2(\xi_k)$ by an elementary calculation we have the formula (1.2). Q.E.D.

Mulitply both sides of the above equation (1.2) by

$$\prod_{k=1}^{(p-1)/2}(\zeta^k-\zeta^{-k})^2=(-1)^{(p-1)/2}p\quad\text{(see page 72 of [7])}\,.$$

Then the right hand side becomes a linear combination of $\zeta^k+\zeta^{-k}(1\leq k\leq (p-1)/2)$ over $\boldsymbol{Z}$ because it is invariant under the complex conjugation $\zeta\to\zeta^{-1}$, and $p_1(\xi_k)\,[F]$ and $c_1(\xi_j)\,c_1(\xi_k)\,[F]$ are both integers. This means that $p/4$ Sign $F\in\boldsymbol{Z}[\zeta+\zeta^{-1}]\cap\boldsymbol{Q}$. However it is well known that $\boldsymbol{Z}[\zeta+\zeta^{-1}]\cap\boldsymbol{Q}=\boldsymbol{Z}$, and therefore $p/4$ Sign $F\in\boldsymbol{Z}$. Since p is an odd prime it follows that $4\,|\,$Sign F.

Consider the equation (1.2) when $p=3$. Then $k=1$, and an elementary calculation shows that (1.2) becomes

$$\mathrm{Sign}\,F=\frac{16}{15}\,c_2(\xi)\,[F]\,,$$

and consequently 16/Sign F.

2. Divisibility by using the $\hat{\mathfrak{A}}$-genus

In this section, we prove Theorem 2 by using the index theorem for the Dirac operator. Consider the **Spin**^{c}-structure P of X determined by a Spin structure (i.e. the **Spin**^{c}-structure with trivial first Chern class). Since $H^1(X; \boldsymbol{Z})=0$ by assumption, the G-action on X lifts to a G action on P [4]. Therefore we can apply the Atiyah-Singer formula for index D, where D is the Dirac operator associated with the **Spin**^{c}-structure P. In our case $c_1(P)=0$ and ξ is of the form $\xi_m \otimes t^m$ by assumption that the action is regular, so the formula reduces to the following equation:

$$(2.1)\qquad (\mathrm{Index}\,(D))\,(\zeta) = (-1)^{(\dim X)/2}\,\hat{\mathfrak{A}}(F)\,\zeta^\lambda \prod_k \frac{1}{e^{x_{m_k}/2}-e^{x_{m_k}/2}\,\zeta^{-m}}\,[F]\,.$$

(see [3]) where $\hat{\mathfrak{A}}(F)=1-1/24\,p_1(F)$, the symbols x_{m_k} have the usual interpretation as formal two-dimensional cohomology classes such that $c(\xi_m)=\prod_k(1+x_{m_k})$, and $[F]$ denotes the fundamental class of F as before.

Let $d=\dim_C \xi$. Then

$$\begin{aligned}
&\prod_k \frac{1}{e^{x_{m_k}/2}-e^{x_{m_k}/2}\,\zeta^{-m}}\\
&= \frac{1}{(1-\zeta^{-m})^d}\prod_k\left(1-\frac{1}{2}\left(\frac{1+\zeta^{-m}}{1-\zeta^{-m}}\right)x_{m_k}+\frac{1}{4}\left(\left(\frac{1+\zeta^{-m}}{1-\zeta^{-m}}\right)^2-\frac{1}{2}\right)x_{m_k}^2\right)\\
&= \frac{1}{(1-\zeta^{-m})^d}\left(1-\frac{1}{2}\left(\frac{1+\zeta^{-m}}{1-\zeta^{-m}}\right)c_1(\xi)+\frac{1}{4}\left(\left(\frac{1+\zeta^{-m}}{1-\zeta^{-m}}\right)^2-\frac{1}{2}\right)p_1(\xi)\right.\\
&\qquad\left.+\frac{1}{4}\left(\frac{1+\zeta^{-m}}{1-\zeta^{-m}}\right)^2 c_2(\xi)\right)
\end{aligned}$$

Multiply the above equation by $\hat{\mathfrak{A}}(F)=1-1/24\,p_1(F)=1+1/24\,p_1(\xi)$ and evaluate it on $[F]$ (note that $p_1(F)+p_1(\xi)=i^*p_1(X)=0$ by assumption). Then we obtain

$$(2.2)\qquad \frac{1}{(1-\zeta^{-m})^{d+2}}\left(\frac{1}{4}\left((1+\zeta^{-m})^2-\frac{1}{3}\right)p_1(\xi)+\frac{1}{4}\,(1+\xi^{-m})^2\,c_2(\xi)\right)[F]\,.$$

We note that the left hand side of (2.1) is an element of the ring $\boldsymbol{Z}[\zeta]$. Let $z=\zeta^{-m}$. Then (2.1) becomes

$$\begin{aligned}
&12(b_1+b_1\,z+b_2\,z^2+\cdots+b_{p-1}\,z^{p-1})\,(1-z)^{d+2}\\
&\quad= (3(1+z)^2-1)\,p_1(\xi)\,[F]+3(1+z)^2\,c_2(\xi)\,[F]\,,
\end{aligned}$$

with integers $b_i (1\leq i\leq p-1)$.

Since $p_1(\xi)\,[F]=-p_1(F)\,[F]=-3$ Sign F we have

$$4(b_0+b_1\,z+b_2\,z^2+\cdots+b_{p-1}\,z^{p-1})\,(1-z)^{d+2}$$

$$(2.3) \qquad = (1-3(1+z)^2)\,\mathrm{Sign}\,F+(1+z)^2 c_2(\xi)\,[F]$$
$$= (-2\,\mathrm{Sign}\,F+c_2(\xi)\,[F])+(2c_2(\xi)\,[F]-6\,\mathrm{Sign}\,F)\,z$$
$$+(c_2(\xi)\,[F]-3\,\mathrm{Sign}\,F)\,z^2.$$

Write $d+2=r(p-1)+s$, where $0\leq s<p-1$. Since $(1-z)^{p-1}\equiv 0 \pmod p$ we have

$$(2.4) \qquad p^r\,|\,(1-z)^{d+2} = ((1-z)^{p-1})^r\,(1-z)^s\,.$$

On the other hand

$$d+2 = \frac{\dim X-4}{2}+2 = \frac{\dim X}{2}\,.$$

So

$$(2.5) \qquad r = \frac{\dim X-2s}{2(p-1)} = \left[\frac{\dim X}{2(p-1)}\right].$$

It follows from (2.3) and (2.4) that

$$4p^r\,|\,(c_2(\xi)\,[F]-2\,\mathrm{Sign}\,F)$$

and

$$4p^r\,|\,(c_2(\xi)\,[F]-3\,\mathrm{Sign}\,F)\,.$$

Therefore

$$4p^r\,|\,\mathrm{Sign}\,F\,.$$

This together with (2.5) proves Theorem 2.

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