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# AN IDENTITY THEOREM FOR LOGARITHMIC POTENTIALS

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## Introduction

The main result of this paper is Theorem 5 below which gives an answer to a question put by R. Grothmann concerning a uniqueness criterion for representing measures of logarithmic potentials. The key to the proof are propositions 3 and 4. In terms of the "fine topology" one might restate Proposition 3 as follows: the fine closure and the natural closure of a connected subset of $\boldsymbol{C}$ coinside. We remark that this result is true only for the fine topology associated with the logarithmic (2-dimensional) potential theory. Its proof is based on an elementary—fairly known—inequality. For the sake of completeness we prove it in Proposition 1. Proposition 4 is based on a regularity criterion for boundary points due to O. Frostman which will be remembered in Proposition 2.

Throughout this paper we shall use the following notations:

1) $D(0, r):=\{z\in\boldsymbol{C}: |z|<r\}$, $r\in\boldsymbol{R}_+^*$,
2) $\chi_A$: the characteristic function of the set $A$.
3) $H_f^G$: the solution on an open set $G$ of the Dirichlet problem with boundary value $f$.

## 1. Some auxiliary results

**Proposition 1.** *Let $F\subset\boldsymbol{C}\setminus\{0\}$ be closed, denote $F^*:=\{x\in\boldsymbol{R}: x=|z|, z\in F\}$, $f:=\chi_F$ and $f^*:=\chi_{F^*}$. Then we have for any $R>0$:*

$$H_f^{D(0,R)\setminus F}(0)\geq H_{f^*}^{D(0,R)\setminus F^*}(0).$$

Proof. Assume $R=1$ and denote by

$$g: (z, w)\to \log\frac{|1-z\overline{w}|}{|z-w|}, \qquad z, w\in D(0, 1),$$

the Green function of $D(0, 1)$. Take $\nu$ a (positive) measure on $D(0, 1)$ and denote by $\lambda$ the measure defined by

$$\lambda(\phi) := \int_{D(0,1)} \phi(|w|)d\nu(w),$$

where $\phi$ is a continuous function with compact support on $D(0, 1)$. Further put

$$p_\nu: z \to \int_{D(0,1)} g(z, w)d\nu(w), \qquad z \in D(0, 1),$$

the Green potential of $\nu$. Analogously $p_\lambda$ will be defined. By a straightforward calculation one sees that for any $z \in D(0, 1)$

$$g(|z|, |w|) \geq g(z, w).$$

From this inequality we get for any $z \in D(0, 1)$

$$p_\lambda(|z|) \geq p_\nu(z).$$

Indeed we have:

$$p_\lambda(|z|) = \int g(|z|, w)d\lambda(w) = \int g(|z|, |w|)d\nu(w) \geq \int g(z, w)d\nu(w)$$
$$= p_\nu(z).$$

Using the obvious equalities

$$g(0, w) = \log \frac{1}{|w|} = g(0, |w|), \qquad w \in D(0, 1)$$

we get in a similar way

$$p_\nu(0) = p_\lambda(0).$$

Assume now $F \subset D(0, 1) \setminus \{0\}$ and let $\varepsilon > 0$ be given. Then we may find a measure $\nu$ such that $p_\nu \geq 1$ on $F$ and

$$p_\nu(0) \leq H_f^{D(0,1)\setminus F}(0) + \varepsilon.$$

Using the first part of the proof we have $p_\lambda \geq 1$ on $F^*$ hence

$$p_\lambda \geq H_{f^*}^{D(0,1)\setminus F^*}.$$

Since $p_\nu(0) = p_\lambda(0)$ we get

$$H_f^{D(0,1)\setminus F}(0) + \varepsilon \geq p_\nu(0) \geq H_{f^*}^{D(0,1)\setminus F^*}(0).$$

The required inequality follows now making $\varepsilon$ tend to 0. If $F$ is arbitrary, denote by

$$F_n := \{z \in F: |z| \leq 1 - 1/n\} \qquad n \in \mathbf{N},$$

and use the relations:

$$\mathrm{H}_f^{D^{(0,1)\setminus F}}(0) = \lim_{n\to\infty} \mathrm{H}_f^{D^{(0,1)\setminus F_n}}(0)\,,$$
$$\mathrm{H}_{f^*}^{D^{(0,1)\setminus F^*}}(0) = \lim_{n\to\infty} \mathrm{H}_{f^*}^{D^{(0,1)\setminus F_n^*}}(0)\,.$$

**Proposition 2.** *Let $G$ be a domain of $\boldsymbol{C}$ possessing a Green function and denote by $g_b$ the Green function of $G$ with pole at $b\in G$. Then for any open set $U\subset G$ and any boundary point $b\in\partial U$ which is regular for the Dirichlet problem on $U$ we have $g_b=H_{g_b}^U$ on $U$.*

Proof. Assume $U$ is connected and denote for any $n\in\mathbf{N}$ by $U_n:=U\cup D(b,\,1/n)$. Fix $a\in U$ and put $g_a^U$ (resp. $g_a^{U_n}$) the Green function of $U$ (resp. $U_n$) with pole at $a$. We show first that $g_a^U=\lim_{n\to\infty} g_a^{U_n}$ on $U$. Indeed if we denote

$$f_n\colon \partial U\to \boldsymbol{R} \quad \begin{cases} f_n := g_a^{U_n} & \text{on } D(b,\,1/n)\cap\partial U \\ f_n := 0 & \text{on } \partial U\setminus D(b,\,1/n) \end{cases}$$

We have on $U$

$$g_a^{U_n} = g_a^U + \mathrm{H}_{f_n}^U\,.$$

The equality $g_a^U=\lim_{n\to\infty} g_a^{U_n}$ on $U$, follows now from the fact that the harmonic measure on $U$ of the sets $D(b,\,1/n)\cap\partial U$ goes to 0 for $n\to\infty$ and that $(f_n)_{n\in N}$ is a decreasing sequence of bounded functions.
We show that

$$\lim_{n\to\infty} g_a^{U_n}(b) = 0\,.$$

Let us denote

$$u\colon G\setminus\{a\}\to \boldsymbol{R} \quad \begin{cases} u := g_a^U & \text{on } U\setminus\{a\} \\ u := 0 & \text{on } G\setminus U \end{cases},$$

$$\left(\text{resp. } u_n\colon G\setminus\{a\}\to \boldsymbol{R} \quad \begin{cases} u_n := g_a^{U_n} & \text{on } U_n\setminus\{a\} \\ u_n := 0 & \text{on } G\setminus U_n \end{cases}\right).$$

For any disc

$$D := D(b,\,r)\subset\bar{D}\subset G\setminus\{a\}, \qquad r>0$$

we have $u_n\leq H_{u_n}^D$ on $D$. Using the fact that on $G\setminus\{a\}\setminus\{b\}$ we have $u=\lim_{n\to\infty} u_n$ we get

$$\lim_{n\to\infty} g_a^{U_n}(b) = \lim_{n\to\infty} u_n(b)\leq H_u^D(b)\,.$$

From the fact that $b$ was assumed regular we have

$$\lim_{z\to 0} u(z) = 0$$

and therefore

$$\lim_{r\to 0} H_u^{D(b,r)}(b) = 0$$

thus we get $\lim_{n\to\infty} g_a^{U_n}(b)=0$.
The proposition follows now from

$$g_a(b)\geq H_{g_b}^U(a)\geq H_{g_b}^{U_n}(a) = g_a(b)-g_a^{U_n}(b)\,.$$

**Proposition 3.** *Let $s$ be a superharmonic function on an open set $U\subset\boldsymbol{C}$, $A$ be a connected set in $\boldsymbol{C}$ and $z\in U\cap\bar{A}$. Then we have*

$$s(z) = \liminf_{w\to z, w\in U\cap A} s(w)\,.$$

Proof. We may assume that $A$ contains more than one point and that $z=0$. Replacing if necessary $U$ by a smaller open set and $s$ by $s+c$ for a suitable $c\in\boldsymbol{R}_+^*$ we may also assume that $s\geq 0$. Take $\alpha\in\boldsymbol{R}$, $\alpha<\liminf_{w\to 0, w\in U\cap A} s(w)$ and $R\in\boldsymbol{R}_+^*$ such that $\{z\in\boldsymbol{C}\colon |z|=R\}\cap A\neq\emptyset$, $D(0, 2R)\subset U$ and $s>\alpha$ on $D(0, 2R)\cap A$. Denote

$$G := \{z\in D(0, 2R)\colon s(z)>\alpha\}\cup\{z\in C\colon |z|>R\}\,.$$

The set $G$ is open and contains $A$. Let $B$ be the connected component of $G$ containing $A$. We have $0\in\bar{B}$ and $\{z\in\boldsymbol{C}\colon |z|=R\}\cap B\neq\emptyset$. Choose $(z_n)_{n\in N}$ a sequence in $B\cap D(0, R)$ converging to 0 and construct for any $n\in\boldsymbol{N}$ a connected compact set $K_n\subset B$ such that $z_n\in K_n$ and $\{z\in\boldsymbol{C}\colon |z|=R\}\cap K_n\neq\emptyset$ (for instance a polygonal curve linking $z_n$ with the boundary of $D(0, R)$). Since the superharmonic function $\frac{1}{\alpha}s$ is non-negative and $\geq 1$ on $K_n$ for any $n\in N$, we have $s(0)\geq\alpha H_{\chi(K_n)}^{D(0,R)\setminus K_n}(0)$. Using now proposition 1 we have $\lim_{n\to\infty} H_{\chi(K_n)}^{D(0,R)\setminus K_n}(0)=1$ hence $s(0)\geq\alpha$. Because $\alpha$ was arbitrary and $s$ is lower semicontinuous we get

$$s(0) = \liminf_{w\to 0, w\in U\cap A} s(w)\,.$$

**Proposition 4.** *Let $U$, $G$ be open subsets of $\boldsymbol{C}$ such that $G$ has only regular boundary points and $\bar{G}$ is compact and is contained in $U$. Then for any superharmonic function $s$ on $U$ which is harmonic on $G$ we have $s=H_s^G$ on $G$.*

Proof. Replacing if necessary $U$ by a smaller open set and $s$ by $s+c$ for a suitable $c\in\boldsymbol{R}$ we may assume that $s\geq 0$. Using the Riesz representation theorem we may consider $s$ of the form $s(z):=\int g(z, w)d\mu(w)$ where $g$ is the Green function of $U$ and $\mu$ a positive Radon-measure on $U$. Since $s$ is harmonic on $G$ we have $\mu(G)=0$. Fix a point $z\in G$ and denote by $\mu_z$ the harmonic measure of $G$ at $z$, i.e. the positive Radon-measure on the boundary of $G$ for which

$$H_f^G(z) = \int f d\mu_z \quad f \text{ continuous on } \partial G .$$

Using proposition 2 we have for any $w \in \partial G$, $g(z, w) = \int g(\cdot, w) d\mu_z$. From the theorem of Fubini we have

$$H_s^G(z) = \int s d\mu_z = \iint g d\mu d\mu_z = \iint g d\mu_z d\mu = \int g(z, \cdot) d\mu = s(z) .$$

## 2. The main theorem

**Theorem 5.** *Let $s, t$ be superharmonic functions on $\boldsymbol{C}$ and $A \subset \boldsymbol{C}$. The functions $s$ and $t$ are equal if following conditions are fulfilled:*

1) *$s = t$ on $A$,*
2) *both $s$ and $t$ are harmonic on the complement of $\bar{A}$,*
3) *if $A$ is not bounded then*

$$\liminf_{z \to \infty} \frac{s(z)}{\log|z|} \neq -\infty, \quad \liminf_{z \to \infty} \frac{t(z)}{\log|z|} \neq -\infty ,$$

4) *if $A$ is bounded then*

$$\liminf_{z \to \infty} \frac{s(z)}{\log|z|} = \liminf_{z \to \infty} \frac{t(z)}{\log|z|} \neq -\infty ,$$

5) *the set $A$ has finitely many bounded connected components each of which consisting of more than one point.*

Proof. Assume that $A$ is bounded and let $(A_j)_{j=1,\dots,n}$ be the connected components of $A$. From proposition 3 we have for any $j$, $1 \leq j \leq n$, $s = t$ on $\bar{A}_j$, and therefore $s = t$ on $\bar{A} = \cup_{j=1,\dots,n} \bar{A}_j$.
Put $G$ the unbounded connected component of $\boldsymbol{C} \setminus \bar{A}$. Also from proposition 3 we get that $\boldsymbol{C} \setminus \bar{A}$ has only regular boundary points hence from proposition 4 $s = t$ on every bounded component of $\boldsymbol{C} \setminus \bar{A}$ i.e. on the set $\boldsymbol{C} \setminus \bar{A} \setminus G$. It remained only to show that $s = t$ on $G$. We may assume

$$\liminf_{z \to \infty} \frac{s(z)}{\log|z|} = -1 = \liminf_{z \to \infty} \frac{t(z)}{\log|z|} .$$

Then we have:

$$s(z) = u(z) - \log|z|, \quad t(z) = v(z) - \log|z| ,$$

where $u$ and $v$ are harmonic on $G$ and bounded in a neighborhood of $\infty$. For $r \in R_+^*$ such that $\{z \in \boldsymbol{C}: |z| \geq r\} \subset G$ denote

$$G_r := \{z \in G: |z| < r\} \quad \text{and} \quad f_r := \begin{cases} u - v & \text{on } \{z \in \boldsymbol{C}: |z| = r\} \\ 0 & \text{on } \partial G \end{cases} .$$

Again from proposition 4 we have on $G_r$

$$H_s^{G_r} = s, \quad H_t^{G_r} = t,$$

and since $s=t$ on $\partial G$ we get

$$s-t = H_{f_r}^{G_r}.$$

By a straightforward calculation one may show that $\lim_{r\to\infty} H_{f_r}^{G_r}=0$, and thus $s=t$ on $G$. Let now $A$ be unbounded, assume that $0\notin\bar{A}$ and fix a negative real number

$$\alpha<\min\left(\liminf_{z\to\infty}\frac{s(z)}{\log|z|}, \liminf_{z\to\infty}\frac{t(z)}{\log|z|}\right).$$

Further denote:

$$\begin{aligned} A_* &:= \{z\in\boldsymbol{C}\setminus\{0\} : 1/z\in A\}, \\ s_*(z) &:= s(z^{-1})+\alpha\log|z|, \qquad z\in\boldsymbol{C}\setminus\{0\}, \\ t_*(z) &:= t(z^{-1})+\alpha\log|z|, \qquad z\in\boldsymbol{C}\setminus\{0\}, \end{aligned}$$

The functions $s_*$, $t_*$ are superharmonic on $\boldsymbol{C}\setminus\{0\}$ and from the above condition 3) they are non-negative on a nieghbourhood of 0, hence they may be extended to superharmonic functions on the whole of $\boldsymbol{C}$. Obviously they are equal on $A_*$, and applying proposition 3 we have $s_*=t_*$ on the closure of each connected component of $A_*$. Since the union of these closures is a set having only finitely many connected components and is a bounded set we get from the first part of the proof $s_*=t_*$, hence $s=t$.

DEFINITION. For a measure $\mu$ on $\boldsymbol{C}$ with compact carrier, we shall denote by

$$p_\mu: z \to \int_{\boldsymbol{C}} \log\frac{1}{|z-w|}\,d\mu(w), \qquad z\in\boldsymbol{C},$$

the **logarithmic potential** of $\mu$.

**Corollary 6.** *Let $K\subset\boldsymbol{C}$ be connected and compact and let $\mu$, $\nu$ be two measures on $K$. Then we have*:

$$\mu(K)=\nu(K) \text{ and } p_\mu=p_\nu \text{ on } K \Rightarrow p_\mu=p_\nu \text{ on } \boldsymbol{C}.$$

Proof. If $K=\{a\}$, $a\in\boldsymbol{C}$ we have $\mu=\mu(K)\delta_a=\nu$. Assume that $K$ has more than one point. By a direct calculation we have

$$\lim_{z\to\infty}\frac{p_\mu(z)}{\log|z|} = -\mu(K).$$

Thus condition 4) of the theorem 5 is fulfilled because $\mu(K)=\nu(K)$.

REMARK. Let $A\subset \boldsymbol{C}$ be given. For a point $z\in\boldsymbol{C}$ denote

$$A_z := \{x\in\boldsymbol{R}: \exists w\in A,\ |z-w| = x\}.$$

Put also

$$A_\infty := \{x\in\boldsymbol{R}: \exists z\in A,\ |z|^{-1} = x\}.$$

We may generalize the above theorem 5 by replacing the condition 5) there with the following less restrictive one:

5*) *for any $z\in\bar{A}$ the set $A_z$ is "not thin at 0".*

As an example consider the following condition:

5**) *for any $z\in\bar{A}$ there exists $r(z)\in\boldsymbol{R}_+^*$ such that*

$$]0, r(z)[\subset A_z,$$

*and if $A$ is not bounded there exists $r\in\boldsymbol{R}_+^*$ such that*

$$[0, r]\subset A_\infty.$$

Indeed using arguments like in the proof of proposition 1 one may show first that if $A_z$ is not thin at 0 then $A$ is not thin at $z$. From this result we deduce:

$$s(z) = \liminf_{w\to z,\ w\in A} s(w),$$

for any $z\in\bar{A}$ and any superharmonic function $s$.

REMARK. Generalizations of theorem 5 to higher dimensions might be obtained by generalizing condition 5* which might be viewed as a thinness preserving property by certain projections. First we show that Lipschitz maps preserve thinness.

**Proposition 7.** *Let $A\subset\boldsymbol{R}^d$, $a\in\boldsymbol{R}^d\backslash A$, $b\in\boldsymbol{R}^d$, $M\in\boldsymbol{R}_+^*$, and $T: A\to\boldsymbol{R}^d$, be such that:*

$$x, y\in A \Rightarrow \|T(x)-T(y)\|\leq M\|x-y\|,$$
$$x\in A \Rightarrow \|b-T(x)\|\geq M^{-1}\|a-x\|.$$

*If $A$ is thin at $a$ then $T(A)$ is thin at $b$.*

Proof. For any Radon measure $\mu$ denote by $p_\mu$ the Newtonian potential generated by $\mu$. If $A$ is thin at $a$ and $a\in\bar{A}$ then there exists a measure $\nu$ such that

$$p_\nu(a)<+\infty,\ \lim_{x\to a,\ x\in A} p_\nu(x) = +\infty.$$

Let us denote by $\lambda$ the measure defined by

$$f \to \int_{R^d} f \circ T d\nu, \qquad f \in C^0(R^d).$$

There exists $c \in R_+^*$ such that:

$$p_\lambda \circ T \geq c p_\nu \text{ on } A,$$
$$p_\lambda(b) \leq c^{-1} p_\nu(a).$$

Then we have

$$p_\lambda(b) < +\infty, \lim_{y \to b,\, y \in T(A)} p_\lambda(y) = +\infty.$$

**Proposition 8.** *Fix* $v \in R^d$ *with* $||v|| = 1$. *For any* $x \in R^d$ *put* $T_v(x) := x - \langle x, v \rangle v$. *If* $A \subset R^d$ *is thin at* 0 *and* $\sup_{x \in A} \frac{\langle x, v \rangle}{||x||} < 1$, *then* $T_v(A)$ *is thin at* 0.

CONJECTURE. A set $A \in R^d$ is thin at 0 if there exist $v_1, v_2, v_3 \in R^d$ with $||v_j|| = 1$, $j = 1, 2, 3$ linearly independent and such that $T_{v_j}(A)$ is thin at 0, $j = 1, 2, 3$.

REMARK. The above conjecture is true if the set $A$ is contained in a set of the form $\cup_{j=0}^{n} G_j$ where $G_j$ is a Lipschitz manifold (graph of a Lipschitz function).

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