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AN IDENTITY THEOREM FOR LOGARITHMIC POTENTIALS

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Introduction

The main result of this paper is Theorem 5 below which gives an answer to a question put by R. Grothmann concerning a uniqueness criterion for representing measures of logarithmic potentials. The key to the proof are propositions 3 and 4. In terms of the “fine topology” one might restate Proposition 3 as follows: the fine closure and the natural closure of a connected subset of \mathbf{C} coincide. We remark that this result is true only for the fine topology associated with the logarithmic (2-dimensional) potential theory. Its proof is based on an elementary—fairly known—inequality. For the sake of completeness we prove it in Proposition 1. Proposition 4 is based on a regularity criterion for boundary points due to O. Frostman which will be remembered in Proposition 2.

Throughout this paper we shall use the following notations:

- 1) $D(0, r) := \{z \in \mathbf{C} : |z| < r\}$, $r \in \mathbf{R}_+^*$,
- 2) χ_A : the characteristic function of the set A .
- 3) H_f^G : the solution on an open set G of the Dirichlet problem with boundary value f .

1. Some auxiliary results

Proposition 1. *Let $F \subset \mathbf{C} \setminus \{0\}$ be closed, denote $F^* := \{x \in \mathbf{R} : x = |z|, z \in F\}$, $f := \chi_F$ and $f^* := \chi_{F^*}$. Then we have for any $R > 0$:*

$$H_f^{D(0,R) \setminus F}(0) \geq H_{f^*}^{D(0,R) \setminus F^*}(0).$$

Proof. Assume $R=1$ and denote by

$$g: (z, w) \rightarrow \log \frac{|1 - z\bar{w}|}{|z - w|}, \quad z, w \in D(0, 1),$$

the Green function of $D(0, 1)$. Take ν a (positive) measure on $D(0, 1)$ and denote by λ the measure defined by

$$\lambda(\phi) := \int_{D(0,1)} \phi(|w|)d\nu(w),$$

where ϕ is a continuous function with compact support on $D(0, 1)$. Further put

$$p_\nu: z \rightarrow \int_{D(0,1)} g(z, w)d\nu(w), \quad z \in D(0, 1),$$

the Green potential of ν . Analogously p_λ will be defined. By a straightforward calculation one sees that for any $z \in D(0, 1)$

$$g(|z|, |w|) \geq g(z, w).$$

From this inequality we get for any $z \in D(0, 1)$

$$p_\lambda(|z|) \geq p_\nu(z).$$

Indeed we have:

$$\begin{aligned} p_\lambda(|z|) &= \int g(|z|, w)d\lambda(w) = \int g(|z|, |w|)d\nu(w) \geq \int g(z, w)d\nu(w) \\ &= p_\nu(z). \end{aligned}$$

Using the obvious equalities

$$g(0, w) = \log \frac{1}{|w|} = g(0, |w|), \quad w \in D(0, 1)$$

we get in a similar way

$$p_\nu(0) = p_\lambda(0).$$

Assume now $F \subset D(0, 1) \setminus \{0\}$ and let $\varepsilon > 0$ be given. Then we may find a measure ν such that $p_\nu \geq 1$ on F and

$$p_\nu(0) \leq H_F^{D(0,1) \setminus F}(0) + \varepsilon.$$

Using the first part of the proof we have $p_\lambda \geq 1$ on F^* hence

$$p_\lambda \geq H_{F^*}^{D(0,1) \setminus F^*}.$$

Since $p_\nu(0) = p_\lambda(0)$ we get

$$H_F^{D(0,1) \setminus F}(0) + \varepsilon \geq p_\nu(0) \geq H_{F^*}^{D(0,1) \setminus F^*}(0).$$

The required inequality follows now making ε tend to 0. If F is arbitrary, denote by

$$F_n := \{z \in F: |z| \leq 1 - 1/n\} \quad n \in \mathbb{N},$$

and use the relations:

$$\begin{aligned} H_f^{D(0,1)\setminus F}(0) &= \lim_{n \rightarrow \infty} H_f^{D(0,1)\setminus F_n}(0), \\ H_{f_*}^{D(0,1)\setminus F^*}(0) &= \lim_{n \rightarrow \infty} H_{f_*}^{D(0,1)\setminus F_n^*}(0). \end{aligned}$$

Proposition 2. *Let G be a domain of \mathbf{C} possessing a Green function and denote by g_b the Green function of G with pole at $b \in G$. Then for any open set $U \subset G$ and any boundary point $b \in \partial U$ which is regular for the Dirichlet problem on U we have $g_b = H_{g_b}^U$ on U .*

Proof. Assume U is connected and denote for any $n \in \mathbf{N}$ by $U_n := U \cup D(b, 1/n)$. Fix $a \in U$ and put g_a^U (resp. $g_a^{U_n}$) the Green function of U (resp. U_n) with pole at a . We show first that $g_a^U = \lim_{n \rightarrow \infty} g_a^{U_n}$ on U . Indeed if we denote

$$f_n: \partial U \rightarrow \mathbf{R} \quad \begin{cases} f_n := g_a^{U_n} & \text{on } D(b, 1/n) \cap \partial U \\ f_n := 0 & \text{on } \partial U \setminus D(b, 1/n) \end{cases}$$

We have on U

$$g_a^{U_n} = g_a^U + H_{f_n}^U.$$

The equality $g_a^U = \lim_{n \rightarrow \infty} g_a^{U_n}$ on U , follows now from the fact that the harmonic measure on U of the sets $D(b, 1/n) \cap \partial U$ goes to 0 for $n \rightarrow \infty$ and that $(f_n)_{n \in \mathbf{N}}$ is a decreasing sequence of bounded functions.

We show that

$$\lim_{n \rightarrow \infty} g_a^{U_n}(b) = 0.$$

Let us denote

$$\begin{aligned} u: G \setminus \{a\} \rightarrow \mathbf{R} \quad & \begin{cases} u := g_a^U & \text{on } U \setminus \{a\} \\ u := 0 & \text{on } G \setminus U \end{cases}, \\ (\text{resp. } u_n: G \setminus \{a\} \rightarrow \mathbf{R} \quad & \begin{cases} u_n := g_a^{U_n} & \text{on } U_n \setminus \{a\} \\ u_n := 0 & \text{on } G \setminus U_n \end{cases}). \end{aligned}$$

For any disc

$$D := D(b, r) \subset \bar{D} \subset G \setminus \{a\}, \quad r > 0$$

we have $u_n \leq H_{u_n}^D$ on D . Using the fact that on $G \setminus \{a\} \setminus \{b\}$ we have $u = \lim_{n \rightarrow \infty} u_n$ we get

$$\lim_{n \rightarrow \infty} g_a^{U_n}(b) = \lim_{n \rightarrow \infty} u_n(b) \leq H_u^D(b).$$

From the fact that b was assumed regular we have

$$\lim_{z \rightarrow 0} u(z) = 0$$

and therefore

$$\lim_{r \rightarrow 0} H_u^{D(b,r)}(b) = 0$$

thus we get $\lim_{n \rightarrow \infty} g_a^{U_n}(b) = 0$.

The proposition follows now from

$$g_a(b) \geq H_{g_b}^U(a) \geq H_{g_b}^{U_n}(a) = g_a(b) - g_a^{U_n}(b).$$

Proposition 3. *Let s be a superharmonic function on an open set $U \subset \mathbf{C}$, A be a connected set in \mathbf{C} and $z \in U \cap \bar{A}$. Then we have*

$$s(z) = \liminf_{w \rightarrow z, w \in U \cap A} s(w).$$

Proof. We may assume that A contains more than one point and that $z = 0$. Replacing if necessary U by a smaller open set and s by $s + c$ for a suitable $c \in \mathbf{R}^*$ we may also assume that $s \geq 0$. Take $\alpha \in \mathbf{R}$, $\alpha < \liminf_{w \rightarrow 0, w \in U \cap A} s(w)$ and $R \in \mathbf{R}^*$ such that $\{z \in \mathbf{C}: |z| = R\} \cap A \neq \emptyset$, $D(0, 2R) \subset U$ and $s > \alpha$ on $D(0, 2R) \cap A$. Denote

$$G := \{z \in D(0, 2R): s(z) > \alpha\} \cup \{z \in \mathbf{C}: |z| > R\}.$$

The set G is open and contains A . Let B be the connected component of G containing A . We have $0 \in \bar{B}$ and $\{z \in \mathbf{C}: |z| = R\} \cap B \neq \emptyset$. Choose $(z_n)_{n \in \mathbf{N}}$ a sequence in $B \cap D(0, R)$ converging to 0 and construct for any $n \in \mathbf{N}$ a connected compact set $K_n \subset B$ such that $z_n \in K_n$ and $\{z \in \mathbf{C}: |z| = R\} \cap K_n \neq \emptyset$ (for instance a polygonal curve linking z_n with the boundary of $D(0, R)$). Since the superharmonic function $\frac{1}{\alpha}s$ is non-negative and ≥ 1 on K_n for any $n \in \mathbf{N}$, we have $s(0) \geq \alpha H_{\chi_{(K_n)}}^{D(0,R)}(0)$. Using now proposition 1 we have $\lim_{n \rightarrow \infty} H_{\chi_{(K_n)}}^{D(0,R)}(0) = 1$ hence $s(0) \geq \alpha$. Because α was arbitrary and s is lower semicontinuous we get

$$s(0) = \liminf_{w \rightarrow 0, w \in U \cap A} s(w).$$

Proposition 4. *Let U, G be open subsets of \mathbf{C} such that G has only regular boundary points and \bar{G} is compact and is contained in U . Then for any superharmonic function s on U which is harmonic on G we have $s = H_s^G$ on G .*

Proof. Replacing if necessary U by a smaller open set and s by $s + c$ for a suitable $c \in \mathbf{R}$ we may assume that $s \geq 0$. Using the Riesz representation theorem we may consider s of the form $s(z) = \int g(z, w) d\mu(w)$ where g is the Green function of U and μ a positive Radon-measure on U . Since s is harmonic on G we have $\mu(G) = 0$. Fix a point $z \in G$ and denote by μ_z the harmonic measure of G at z , i.e. the positive Radon-measure on the boundary of G for which

$$H_f^G(z) = \int f d\mu_z \quad f \text{ continuous on } \partial G.$$

Using proposition 2 we have for any $w \in \partial G$, $g(z, w) = \int g(\cdot, w) d\mu_z$. From the theorem of Fubini we have

$$H_s^G(z) = \int s d\mu_z = \iint g d\mu d\mu_z = \iint g d\mu_z d\mu = \int g(z, \cdot) d\mu = s(z).$$

2. The main theorem

Theorem 5. *Let s, t be superharmonic functions on C and $A \subset C$. The functions s and t are equal if following conditions are fulfilled:*

- 1) $s=t$ on A ,
- 2) both s and t are harmonic on the complement of \bar{A} ,
- 3) if A is not bounded then

$$\liminf_{z \rightarrow \infty} \frac{s(z)}{\log |z|} \neq -\infty, \quad \liminf_{z \rightarrow \infty} \frac{t(z)}{\log |z|} \neq -\infty,$$

- 4) if A is bounded then

$$\liminf_{z \rightarrow \infty} \frac{s(z)}{\log |z|} = \liminf_{z \rightarrow \infty} \frac{t(z)}{\log |z|} \neq -\infty,$$

- 5) the set A has finitely many bounded connected components each of which consisting of more than one point.

Proof. Assume that A is bounded and let $(A_j)_{j=1, \dots, n}$ be the connected components of A . From proposition 3 we have for any j , $1 \leq j \leq n$, $s=t$ on \bar{A}_j , and therefore $s=t$ on $\bar{A} = \cup_{j=1, \dots, n} \bar{A}_j$.

Put G the unbounded connected component of $C \setminus \bar{A}$. Also from proposition 3 we get that $C \setminus \bar{A}$ has only regular boundary points hence from proposition 4 $s=t$ on every bounded component of $C \setminus \bar{A}$ i.e. on the set $C \setminus \bar{A} \setminus G$. It remained only to show that $s=t$ on G . We may assume

$$\liminf_{z \rightarrow \infty} \frac{s(z)}{\log |z|} = -1 = \liminf_{z \rightarrow \infty} \frac{t(z)}{\log |z|}.$$

Then we have:

$$s(z) = u(z) - \log |z|, \quad t(z) = v(z) - \log |z|,$$

where u and v are harmonic on G and bounded in a neighborhood of ∞ . For $r \in R_+^*$ such that $\{z \in C: |z| \geq r\} \subset G$ denote

$$G_r := \{z \in G: |z| < r\} \quad \text{and} \quad f_r := \begin{cases} u-v & \text{on } \{z \in C: |z| = r\} \\ 0 & \text{on } \partial G \end{cases}.$$

Again from proposition 4 we have on G_r

$$H_s^{G_r} = s, \quad H_t^{G_r} = t,$$

and since $s=t$ on ∂G we get

$$s-t = H_{f_r}^{G_r}.$$

By a straightforward calculation one may show that $\lim_{r \rightarrow \infty} H_{f_r}^{G_r} = 0$, and thus $s=t$ on G . Let now A be unbounded, assume that $0 \notin \bar{A}$ and fix a negative real number

$$\alpha < \min \left(\liminf_{z \rightarrow \infty} \frac{s(z)}{\log |z|}, \liminf_{z \rightarrow \infty} \frac{t(z)}{\log |z|} \right).$$

Further denote:

$$\begin{aligned} A_* &:= \{z \in \mathbb{C} \setminus \{0\} : 1/z \in A\}, \\ s_*(z) &:= s(z^{-1}) + \alpha \log |z|, \quad z \in \mathbb{C} \setminus \{0\}, \\ t_*(z) &:= t(z^{-1}) + \alpha \log |z|, \quad z \in \mathbb{C} \setminus \{0\}, \end{aligned}$$

The functions s_*, t_* are superharmonic on $\mathbb{C} \setminus \{0\}$ and from the above condition 3) they are non-negative on a neighbourhood of 0, hence they may be extended to superharmonic functions on the whole of \mathbb{C} . Obviously they are equal on A_* , and applying proposition 3 we have $s_* = t_*$ on the closure of each connected component of A_* . Since the union of these closures is a set having only finitely many connected components and is a bounded set we get from the first part of the proof $s_* = t_*$, hence $s = t$.

DEFINITION. For a measure μ on \mathbb{C} with compact carrier, we shall denote by

$$p_\mu: z \rightarrow \int_{\mathbb{C}} \log \frac{1}{|z-w|} d\mu(w), \quad z \in \mathbb{C},$$

the **logarithmic potential** of μ .

Corollary 6. Let $K \subset \mathbb{C}$ be connected and compact and let μ, ν be two measures on K . Then we have:

$$\mu(K) = \nu(K) \text{ and } p_\mu = p_\nu \text{ on } K \Rightarrow p_\mu = p_\nu \text{ on } \mathbb{C}.$$

Proof. If $K = \{a\}$, $a \in \mathbb{C}$ we have $\mu = \mu(K)\delta_a = \nu$. Assume that K has more than one point. By a direct calculation we have

$$\lim_{z \rightarrow \infty} \frac{p_\mu(z)}{\log |z|} = -\mu(K).$$

Thus condition 4) of the theorem 5 is fulfilled because $\mu(K)=\nu(K)$.

REMARK. Let $A \subset C$ be given. For a point $z \in C$ denote

$$A_z := \{x \in R: \exists w \in A, |z-w| = x\}.$$

Put also

$$A_\infty := \{x \in R: \exists z \in A, |z|^{-1} = x\}.$$

We may generalize the above theorem 5 by replacing the condition 5) there with the following less restrictive one:

5*) for any $z \in \bar{A}$ the set A_z is "not thin at 0".

As an example consider the following condition:

5**) for any $z \in \bar{A}$ there exists $r(z) \in R_+^*$ such that

$$]0, r(z)[\subset A_z,$$

and if A is not bounded there exists $r \in R_+^*$ such that

$$[0, r] \subset A_\infty.$$

Indeed using arguments like in the proof of proposition 1 one may show first that if A_z is not thin at 0 then A is not thin at z . From this result we deduce:

$$s(z) = \liminf_{w \rightarrow z, w \in A} s(w),$$

for any $z \in \bar{A}$ and any superharmonic function s .

REMARK. Generalizations of theorem 5 to higher dimensions might be obtained by generalizing condition 5* which might be viewed as a thinness preserving property by certain projections. First we show that Lipschitz maps preserve thinness.

Proposition 7. Let $A \subset R^d$, $a \in R^d \setminus A$, $b \in R^d$, $M \in R_+^*$, and $T: A \rightarrow R^d$, be such that:

$$\begin{aligned} x, y \in A &\Rightarrow \|T(x) - T(y)\| \leq M \|x - y\|, \\ x \in A &\Rightarrow \|b - T(x)\| \geq M^{-1} \|a - x\|. \end{aligned}$$

If A is thin at a then $T(A)$ is thin at b .

Proof. For any Radon measure μ denote by p_μ the Newtonian potential generated by μ . If A is thin at a and $a \in \bar{A}$ then there exists a measure ν such that

$$p_\nu(a) < +\infty, \lim_{z \rightarrow a, z \in A} p_\nu(x) = +\infty.$$

Let us denote by λ the measure defined by

$$f \rightarrow \int_{\mathbf{R}^d} f \circ T d\nu, \quad f \in C^0(\mathbf{R}^d).$$

There exists $c \in \mathbf{R}_+^*$ such that:

$$\begin{aligned} p_\lambda \circ T &\geq c p_\nu \text{ on } A, \\ p_\lambda(b) &\leq c^{-1} p_\nu(a). \end{aligned}$$

Then we have

$$p_\lambda(b) < +\infty, \quad \lim_{y \rightarrow b, y \in T(A)} p_\lambda(y) = +\infty.$$

Proposition 8. Fix $v \in \mathbf{R}^d$ with $\|v\|=1$. For any $x \in \mathbf{R}^d$ put $T_v(x) := x - \langle x, v \rangle v$. If $A \subset \mathbf{R}^d$ is thin at 0 and $\sup_{x \in A} \frac{\langle x, v \rangle}{\|x\|} < 1$, then $T_v(A)$ is thin at 0.

CONJECTURE. A set $A \in \mathbf{R}^d$ is thin at 0 if there exist $v_1, v_2, v_3 \in \mathbf{R}^d$ with $\|v_j\|=1, j=1, 2, 3$ linearly independent and such that $T_{v_j}(A)$ is thin at 0, $j=1, 2, 3$.

REMARK. The above conjecture is true if the set A is contained in a set of the form $\cup_{j=0}^n G_j$ where G_j is a Lipschitz manifold (graph of a Lipschitz function).

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