Cornea, A.
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AN IDENTITY THEOREM FOR LOGARITHMIC POTENTIALS

AUREL CORNEA

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Introduction

The main result of this paper is Theorem 5 below which gives an answer to a question put by R. Grothmann concerning a uniqueness criterion for representing measures of logarithmic potentials. The key to the proof are propositions 3 and 4. In terms of the "fine topology" one might restate Proposition 3 as follows: the fine closure and the natural closure of a connected subset of $\boldsymbol{C}$ coinside. We remark that this result is true only for the fine topology associated with the logarithmic (2-dimensional) potential theory. Its proof is based on an elementary—fairly known—inequality. For the sake of completeness we prove it in Proposition 1. Proposition 4 is based on a regularity criterion for boundary points due to O. Frostman which will be remembered in Proposition 2.

Throughout this paper we shall use the following notations:
1) $D(0, r):=\{z\in\boldsymbol{C}:|z|<r\}$, $r\in\boldsymbol{R}_+^*$,
2) χ_A: the characteristic function of the set A.
3) H_f^G: the solution on an open set G of the Dirichlet problem with boundary value f.

1. Some auxiliary results

Proposition 1. *Let $F\subset\boldsymbol{C}\setminus\{0\}$ be closed, denote $F^*:=\{x\in\boldsymbol{R}: x=|z|, z\in F\}$, $f:=\chi_F$ and $f^*:=\chi_{F^*}$. Then we have for any $R>0$:*

$$\mathrm{H}_f^{D(0,R)\setminus F}(0)\geq\mathrm{H}_{f^*}^{D(0,R)\setminus F^*}(0).$$

Proof. Assume $R=1$ and denote by

$$g:(z, w)\rightarrow\log\frac{|1-z\overline{w}|}{|z-w|},\qquad z, w\in D(0, 1),$$

the Green function of $D(0, 1)$. Take ν a (positive) measure on $D(0, 1)$ and denote by λ the measure defined by

$$\lambda(\phi) := \int_{D(0,1)} \phi(|w|)d\nu(w),$$

where ϕ is a continuous function with compact support on $D(0, 1)$. Further put

$$p_\nu : z \to \int_{D(0,1)} g(z, w)\, d\nu(w), \qquad z \in D(0, 1),$$

the Green potential of ν. Analogously p_λ will be defined. By a straightforward calculation one sees that for any $z \in D(0, 1)$

$$g(|z|, |w|) \geq g(z, w).$$

From this inequality we get for any $z \in D(0, 1)$

$$p_\lambda(|z|) \geq p_\nu(z).$$

Indeed we have:

$$p_\lambda(|z|) = \int g(|z|, w)d\lambda(w) = \int g(|z|, |w|)d\nu(w) \geq \int g(z, w)\, d\nu(w)$$
$$= p_\nu(z).$$

Using the obvious equalities

$$g(0, w) = \log \frac{1}{|w|} = g(0, |w|), \qquad w \in D(0, 1)$$

we get in a similar way

$$p_\nu(0) = p_\lambda(0).$$

Assume now $F \subset D(0, 1) \setminus \{0\}$ and let $\varepsilon > 0$ be given. Then we may find a measure ν such that $p_\nu \geq 1$ on F and

$$p_\nu(0) \leq H_f^{D(0,1)\setminus F}(0) + \varepsilon.$$

Using the first part of the proof we have $p_\lambda \geq 1$ on F^* hence

$$p_\lambda \geq H_{f^*}^{D(0,1)\setminus F^*}.$$

Since $p_\nu(0) = p_\lambda(0)$ we get

$$H_f^{D(0,1)\setminus F}(0) + \varepsilon \geq p_\nu(0) \geq H_{f^*}^{D(0,1)\setminus F^*}(0).$$

The required inequality follows now making ε tend to 0. If F is arbitrary, denote by

$$F_n := \{z \in F : |z| \leq 1 - 1/n\} \qquad n \in \boldsymbol{N},$$

and use the relations:

$$\mathrm{H}_f^{D(0,1)\setminus F}(0) = \lim_{n\to\infty} \mathrm{H}_f^{D(0,1)\setminus F_n}(0)\,,$$
$$\mathrm{H}_{f^*}^{D(0,1)\setminus F^*}(0) = \lim_{n\to\infty} \mathrm{H}_{f^*}^{D(0,1)\setminus F_n^*}0)\,.$$

Proposition 2. *Let G be a domain of $\boldsymbol{C}$ possessing a Green function and denote by g_b the Green function of G with pole at $b\in G$. Then for any open set $U\subset G$ and any boundary point $b\in\partial U$ which is regular for the Dirichlet problem on U we have $g_b=H_{g_b}^U$ on U.*

Proof. Assume U is connected and denote for any $n\in\mathbf{N}$ by $U_n:=U\cup D(b,1/n)$. Fix $a\in U$ and put g_a^U (resp. $g_a^{U_n}$) the Green function of U (resp. U_n) with pole at a. We show first that $g_a^U=\lim\limits_{n\to\infty} g_a^{U_n}$ on U. Indeed if we denote

$$f_n\colon \partial U\to\boldsymbol{R}\ \begin{cases} f_n := g_a^{U_n} & \text{on } D(b,1/n)\cap\partial U \\ f_n := 0 & \text{on } \partial U\setminus D(b,1/n)\end{cases}$$

We have on U

$$g_a^{U_n} = g_a^U + \mathrm{H}_{f_n}^U\,.$$

The equality $g_a^U=\lim\limits_{n\to\infty} g_a^{U_n}$ on U, follows now from the fact that the harmonic measure on U of the sets $D(b,1/n)\cap\partial U$ goes to 0 for $n\to\infty$ and that $(f_n)_{n\in N}$ is a decreasing sequence of bounded functions.
We show that

$$\lim_{u\to\infty} g_a^{U_n}(b) = 0\,.$$

Let us denote

$$u\colon G\setminus\{a\}\to\boldsymbol{R}\ \begin{cases} u := g_a^U & \text{on } U\setminus\{a\} \\ u := 0 & \text{on } G\setminus U\end{cases},$$

$$\left(\text{resp. } u_n\colon G\setminus\{a\}\to\boldsymbol{R}\ \begin{cases} u_n := g_a^{U_n} & \text{on } U_n\setminus\{a\} \\ u_n := 0 & \text{on } G\setminus U_n\end{cases}\right).$$

For any disc

$$D := D(b,r)\subset\bar{D}\subset G\setminus\{a\}\,,\qquad r>0$$

we have $u_n\leq H_{u_n}^D$ on D. Using the fact that on $G\setminus\{a\}\setminus\{b\}$ we have $u=\lim\limits_{n\to\infty} u_n$ we get

$$\lim_{n\to\infty} g_a^{U_n}(b) = \lim_{n\to\infty} u_n(b)\leq H_u^D(b)\,.$$

From the fact that b was assumed regular we have

$$\lim_{z\to 0} u(z) = 0$$

and therefore

$$\lim_{r\to 0} \mathrm{H}_u^{D(b,r)}(b) = 0$$

thus we get $\lim_{n\to\infty} g_a^{U_n}(b)=0$.
The proposition follows now from

$$g_a(b) \geq \mathrm{H}_{g_b}^{U}(a) \geq \mathrm{H}_{g_b}^{U_n}(a) = g_a(b) - g_a^{U_n}(b)\,.$$

Proposition 3. *Let s be a superharmonic function on an open set $U\subset \boldsymbol{C}$, A be a connected set in $\boldsymbol{C}$ and $z\in U\cap \bar{A}$. Then we have*

$$s(z) = \liminf_{w\to z, w\in U\cap A} s(w)\,.$$

Proof. We may assume that A contains more than one point and that $z=0$. Replacing if necessary U by a smaller open set and s by $s+c$ for a suitable $c\in \boldsymbol{R}_+^*$ we may also assume that $s\geq 0$. Take $\alpha\in \boldsymbol{R}$, $\alpha < \liminf_{w\to 0, w\in U\cap A} s(w)$ and $R\in \boldsymbol{R}_+^*$ such that $\{z\in \boldsymbol{C}\colon |z|=R\}\cap A\neq\emptyset$, $D(0, 2R)\subset U$ and $s>\alpha$ on $D(0, 2R)\cap A$. Denote

$$G := \{z\in D(0, 2R)\colon s(z)>\alpha\} \cup \{z\in C\colon |z|>R\}\,.$$

The set G is open and contains A. Let B be the connected component of G containing A. We have $0\in\bar{B}$ and $\{z\in \boldsymbol{C}\colon |z|=R\}\cap B\neq\emptyset$. Choose $(z_n)_{n\in \boldsymbol{N}}$ a sequence in $B\cap D(0, R)$ converging to 0 and construct for any $n\in \boldsymbol{N}$ a connected compact set $K_n\subset B$ such that $z_n\in K_n$ and $\{z\in \boldsymbol{C}\colon |z|=R\}\cap K_n\neq\emptyset$ (for instance a polygonal curve linking z_n with the boundary of $D(0, R)$). Since the superharmonic function $\frac{1}{\alpha}s$ is non-negative and ≥ 1 on K_n for any $n\in N$, we have $s(0)\geq \alpha \mathrm{H}_{\chi(K_n)}^{D(0,R)\setminus K_n}(0)$. Using now proposition 1 we have $\lim_{n\to\infty} \mathrm{H}_{\chi(K_n)}^{D(0,R)\setminus K_n}(0) =1$ hence $s(0)\geq\alpha$. Because α was arbitrary and s is lower semicontinuous we get

$$s(0) = \liminf_{w\to 0, w\in U\cap A} s(w)\,.$$

Proposition 4. *Let U, G be open subsets of $\boldsymbol{C}$ such that G has only regular boundary points and $\bar{G}$ is compact and is contained in U. Then for any superharmonic function s on U which is harmonic on G we have $s=\mathrm{H}_s^G$ on G.*

Proof. Replacing if necessary U by a smaller open set and s by $s+c$ for a suitable $c\in \boldsymbol{R}$ we may assume that $s\geq 0$. Using the Riesz representation theorem we may consider s of the form $s(z):=\int g(z, w)d\mu(w)$ where g is the Green function of U and μ a positive Radon-measure on U. Since s is harmonic on G we have $\mu(G)=0$. Fix a point $z\in G$ and denote by μ_z the harmonic measure of G at z, i.e. the positive Radon-measure on the boundary of G for which

$$\mathrm{H}_f^G(z) = \int f d\mu_z \quad f \text{ continuous on } \partial G\,.$$

Using proposition 2 we have for any $w \in \partial G$, $g(z, w) = \int g(\cdot, w) d\mu_z$. From the theorem of Fubini we have

$$\mathrm{H}_s^G(z) = \int s d\mu_z = \iint g d\mu d\mu_z = \iint g d\mu_z d\mu = \int g(z, \cdot) d\mu = s(z)\,.$$

2. The main theorem

Theorem 5. *Let s, t be superharmonic functions on $\boldsymbol{C}$ and $A \subset \boldsymbol{C}$. The functions s and t are equal if following conditions are fulfilled:*

1) *$s = t$ on A,*
2) *both s and t are harmonic on the complement of $\bar{A}$,*
3) *if A is not bounded then*

$$\liminf_{z\to\infty} \frac{s(z)}{\log|z|} \neq -\infty, \quad \liminf_{z\to\infty} \frac{t(z)}{\log|z|} \neq -\infty\,,$$

4) *if A is bounded then*

$$\liminf_{z\to\infty} \frac{s(z)}{\log|z|} = \liminf_{z\to\infty} \frac{t(z)}{\log|z|} \neq -\infty\,,$$

5) *the set A has finitely many bounded connected components each of which consisting of more than one point.*

Proof. Assume that A is bounded and let $(A_j)_{j=1,\dots,n}$ be the connected components of A. From proposition 3 we have for any j, $1 \leq j \leq n$, $s = t$ on $\bar{A}_j$, and therefore $s = t$ on $\bar{A} = \bigcup_{j=1,\dots,n} \bar{A}_j$.
Put G the unbounded connected component of $\boldsymbol{C} \setminus \bar{A}$. Also from proposition 3 we get that $\boldsymbol{C} \setminus \bar{A}$ has only regular boundary points hence from proposition 4 $s = t$ on every bounded component of $\boldsymbol{C} \setminus \bar{A}$ i.e. on the set $\boldsymbol{C} \setminus \bar{A} \setminus G$. It remained only to show that $s = t$ on G. We may assume

$$\liminf_{z\to\infty} \frac{s(z)}{\log|z|} = -1 = \liminf_{z\to\infty} \frac{t(z)}{\log|z|}\,.$$

Then we have:

$$s(z) = u(z) - \log|z|\,, \quad t(z) = v(z) - \log|z|\,,$$

where u and v are harmonic on G and bounded in a neighborhood of ∞. For $r \in R_+^*$ such that $\{z \in \boldsymbol{C} \colon |z| \geq r\} \subset G$ denote

$$G_r := \{z \in G \colon |z| < r\} \quad \text{and} \quad f_r := \begin{cases} u - v & \text{on } \{z \in \boldsymbol{C} \colon |z| = r\} \\ 0 & \text{on } \partial G \end{cases}\,.$$

Again from proposition 4 we have on G_r

$$\mathrm{H}_s^{G_r} = s, \quad \mathrm{H}_t^{G_r} = t,$$

and since $s=t$ on ∂G we get

$$s-t = \mathrm{H}_{f_r}^{G_r}.$$

By a straightforward calculation one may show that $\lim_{r\to\infty} \mathrm{H}_{f_r}^{G_r}=0$, and thus $s=t$ on G. Let now A be unbounded, assume that $0\notin\bar{A}$ and fix a negative real number

$$\alpha<\min\left(\liminf_{z\to\infty}\frac{s(z)}{\log|z|}, \liminf_{z\to\infty}\frac{t(z)}{\log|z|}\right).$$

Further denote:

$$\begin{aligned} A_* &:= \{z\in \boldsymbol{C}\setminus\{0\} : 1/z\in A\} , \\ s_*(z) &:= s(z^{-1})+\alpha \log|z|, \qquad z\in\boldsymbol{C}\setminus\{0\} , \\ t_*(z) &:= t(z^{-1})+\alpha\log|z|, \qquad z\in\boldsymbol{C}\setminus\{0\} , \end{aligned}$$

The functions s_*, t_* are superharmonic on $\boldsymbol{C}\setminus\{0\}$ and from the above condition 3) they are non-negative on a nieghbourhood of 0, hence they may be extended to superharmonic functions on the whole of $\boldsymbol{C}$. Obviously they are equal on A_*, and applying proposition 3 we have $s_*=t_*$ on the closure of each connected component of A_*. Since the union of these closures is a set having only finitely many connected components and is a bounded set we get from the first part of the proof $s_*=t_*$, hence $s=t$.

DEFINITION. For a measure μ on $\boldsymbol{C}$ with compact carrier, we shall denote by

$$p_\mu : z \to \int_{\boldsymbol{C}} \log\frac{1}{|z-w|}\,d\mu(w), \qquad z\in\boldsymbol{C},$$

the **logarithmic potential** of μ.

Corollary 6. *Let $K\subset\boldsymbol{C}$ be connected and compact and let μ, ν be two measures on K. Then we have:*

$$\mu(K)=\nu(K) \text{ and } p_\mu=p_\nu \text{ on } K \Rightarrow p_\mu=p_\nu \text{ on } \boldsymbol{C}.$$

Proof. If $K=\{a\}$, $a\in\boldsymbol{C}$ we have $\mu=\mu(K)\delta_a=\nu$. Assume that K has more than one point. By a direct calculation we have

$$\lim_{z\to\infty}\frac{p_\mu(z)}{\log|z|} = -\mu(K).$$

Thus condition 4) of the theorem 5 is fulfilled because $\mu(K)=\nu(K)$.

REMARK. Let $A\subset \boldsymbol{C}$ be given. For a point $z\in \boldsymbol{C}$ denote

$$A_z := \{x\in \boldsymbol{R}\colon \exists w\in A,\ |z-w| = x\}.$$

Put also

$$A_\infty := \{x\in \boldsymbol{R}\colon \exists z\in A,\ |z|^{-1} = x\}.$$

We may generalize the above theorem 5 by replacing the condition 5) there with the following less restrictive one:

5*) *for any $z\in \bar{A}$ the set A_z is "not thin at* 0".

As an example consider the following condition:

5**) *for any $z\in \bar{A}$ there exists $r(z)\in \boldsymbol{R}_+^*$ such that*

$$]0, r(z)[\subset A_z,$$

and if A is not bounded there exists $r\in \boldsymbol{R}_+^*$ such that

$$[0, r]\subset A_\infty .$$

Indeed using arguments like in the proof of proposition 1 one may show first that if A_z is not thin at 0 then A is not thin at z. From this result we deduce:

$$s(z) = \liminf_{w\to z,\ w\in A} s(w)\,,$$

for any $z\in \bar{A}$ and any superharmonic function s.

REMARK. Generalizations of theorem 5 to higher dimensions might be obtained by generalizing condition 5* which might be viewed as a thinness preserving property by certain projections. First we show that Lipschitz maps preserve thinness.

Proposition 7. *Let $A\subset \boldsymbol{R}^d$, $a\in \boldsymbol{R}^d\setminus A$, $b\in \boldsymbol{R}^d$, $M\in \boldsymbol{R}_+^*$, and $T\colon A\to \boldsymbol{R}^d$, be such that:*

$$x, y\in A \Rightarrow ||T(x)-T(y)||\le M||x-y||\,,$$
$$x\in A \Rightarrow ||b-T(x)||\ge M^{-1}||a-x||\,.$$

If A is thin at a then $T(A)$ is thin at b.

Proof. For any Radon measure μ denote by p_μ the Newtonian potential generated by μ. If A is thin at a and $a\in \bar{A}$ then there exists a measure ν such that

$$p_\nu(a)<+\infty,\ \lim_{x\to a,\ x\in A} p_\nu(x) = +\infty.$$

Let us denote by λ the measure defined by

$$f \to \int_{\boldsymbol{R}^d} f \circ T d\nu, \qquad f \in C^0(\boldsymbol{R}^d).$$

There exists $c \in \boldsymbol{R}_+^*$ such that:

$$p_\lambda \circ T \geq c p_\nu \text{ on } A,$$
$$p_\lambda(b) \leq c^{-1} p_\nu(a).$$

Then we have

$$p_\lambda(b) < +\infty, \lim_{y \to b,\, y \in T(A)} p_\lambda(y) = +\infty.$$

Proposition 8. *Fix $v \in \boldsymbol{R}^d$ with $\|v\| = 1$. For any $x \in \boldsymbol{R}^d$ put $T_v(x) := x - \langle x, v\rangle v$. If $A \subset \boldsymbol{R}^d$ is thin at 0 and $\sup_{x \in A} \frac{\langle x, v\rangle}{\|x\|} < 1$, then $T_v(A)$ is thin at 0.*

CONJECTURE. A set $A \in \boldsymbol{R}^d$ is thin at 0 if there exist $v_1, v_2, v_3 \in \boldsymbol{R}^d$ with $\|v_j\| = 1$, $j = 1, 2, 3$ linearly independent and such that $T_{v_j}(A)$ is thin at 0, $j = 1, 2, 3$.

REMARK. The above conjecture is true if the set A is contained in a set of the form $\cup_{j=0}^n G_j$ where G_j is a Lipschitz manifold (graph of a Lipschitz function).

Katholische Universität Eichstätt
Mathematisch-Geographsche Fakultat
Ostenstrasse 26–28
W-8078 Eichstät
FRG.