



Title	Three fold irregular branched coverings of some spatial graphs
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Citation	Osaka Journal of Mathematics. 1991, 28(3), p. 639-648
Version Type	VoR
URL	https://doi.org/10.18910/11279
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THREE-FOLD IRREGULAR BRANCHED COVERINGS OF SOME SPATIAL GRAPHS

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(Received May 22, 1990)

1. Introduction

A *spatial graph* is a graph embedded in a 3-sphere S^3 . In this paper, we consider three-fold irregular branched coverings of some spatial graphs. In particular, we investigate those of some of θ -curves and handcuff graphs in S^3 and prove that there exists at least one three-fold irregular branched covering of these graphs. Further, we identify these branched coverings. Hilden [4] and Montesinos [6] independently showed that every orientable closed 3-manifold is a three-fold irregular covering of S^3 , branched along a link.

Let L be a spatial graph and $G=\pi_1(S^3-L)$. Then there is a one-to-one correspondence between n -fold unbranched coverings of S^3-L and conjugacy classes of transitive representations of G into S_n , the symmetric group with n letters $\{0, 1, \dots, n-1\}$. Let μ be such a representation, called a *monodromy map*, and $T=\mu(G)$. Define T_0 as the subgroup of T that fixes letter 0. Then $\mu^{-1}(T_0)$ is the fundamental group of the unbranched covering associated with μ . To each unbranched covering of S^3-L there exists the unique completion $\tilde{M}_\mu(L)$ called the associated branched covering (see Fox [1]).

In this paper we investigate a monodromy map $\mu: G \rightarrow S_3$ which is surjective, i.e. the covering is irregular. We call μ an *S_3 -representation* of L . Further we only consider the case that the branched covering associated with μ is an orientable 3-manifold.

The author of the paper would like to express his sincere gratitude to Professor S. Kinoshita and Dr. K. Yoshikawa for their valuable advice.

2. Three-fold branched coverings of spatial θ -curves

In this section, let L denote a spatial θ -curve that consists of three edges e_1, e_2 and e_3 , each of which has distinct endpoints A and B . Suppose that each of e_1, e_2 and e_3 is oriented from A to B . Then $G=\pi_1(S^3-L)$ is generated by $x_1, \dots, x_l; y_1, \dots, y_m; z_1, \dots, z_n$, where each of x_i, y_j and z_k corresponds to a meridian of each of e_1, e_2 and e_3 , respectively. Note that every element of S_3 can be expressed as $a^\delta b^\epsilon$, where $a=(01), b=(012); \delta=0, 1, \epsilon=0, 1, 2$. We assume that

$\mu(x_i) = a^{\alpha_{1i}}b^{\alpha_{2i}}$, $\mu(y_j) = a^{\beta_{1j}}b^{\beta_{2j}}$, $\mu(z_k) = a^{\gamma_{1k}}b^{\gamma_{2k}}$. Let $r_1 = x_1y_1z_1 = 1$ be the relation corresponding to A . By applying $ba = ab^{-1}$ to $r_1 = 1$, we have $\alpha_{11} + \beta_{11} + \gamma_{11} \equiv 0 \pmod{2}$. We put $\alpha_{11} = \beta_{11} = 1$ and $\gamma_{11} = 0$ without loss of generality. Since $\mu(x_i)$ is a conjugation of $\mu(x_{i-1})$ with a^8b^8 , we have $\alpha_{1i} = 1$. Similarly we have $\beta_{1i} = 1$ and $\gamma_{1k} = 0$. Hence we have

$$(1) \quad \begin{cases} \mu(x_i) = ab^{\alpha_i}, & i = 1, \dots, l, \\ \mu(y_j) = ab^{\beta_j}, & j = 1, \dots, m, \quad \text{and} \\ \mu(z_k) = b^{\gamma_k}, & k = 1, \dots, n. \end{cases}$$

Let F be the free group generated by $x_1, \dots, x_l; y_1, \dots, y_m; z_1, \dots, z_n$ and ϕ the canonical projection from F to G . Further let $\psi: G \rightarrow H = \langle t \rangle$, where $\psi(x_i) = t$, $\psi(y_j) = t^{-1}$ and $\psi(z_k) = 1$. Then the Jacobian matrix $A(G, \psi)$ of G at ψ is defined as follows (see Kinoshita [5]): Let r be the p -th relation of G . Then the p -th row of $A(G, \psi)(t)$ can be expressed as

$$\left(\left(\frac{\partial r}{\partial x_i} \right)^{\psi\phi} \left(\frac{\partial r}{\partial y_j} \right)^{\psi\phi} \left(\frac{\partial r}{\partial z_k} \right)^{\psi\phi} \right),$$

where $\partial/\partial x_i$, $\partial/\partial y_j$, and $\partial/\partial z_k$ are the Fox's free derivatives. Let ν be the nullity of $A(G, \psi)(-1)$ in Z_3 -coefficients. Note that $\nu \geq 1$. Then we have

Theorem 2.1. *The number of conjugacy classes of S_3 -representations of L , each of which satisfies (1), is equal to $(3^\nu - 3)/3!$.*

Since one of the relations of G is a consequence of the others, the deficiency of G is equal to two. Hence $\nu \geq 2$. Therefore we have

Corollary 2.2. *There exists at least one S_3 -representation of L which satisfies (1).*

Proof of Theorem 2.1. We may deform a diagram of any spatial θ -curve so that there is no crossing on e_3 (see Figure 2.1). In Figure 2.1 let T be a 2-string tangle. Then G has generators $x_1, \dots, x_l; y_1, \dots, y_m; z$ and relations,

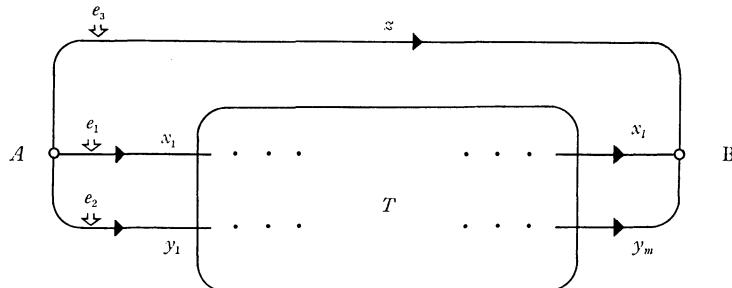


Fig. 2.1

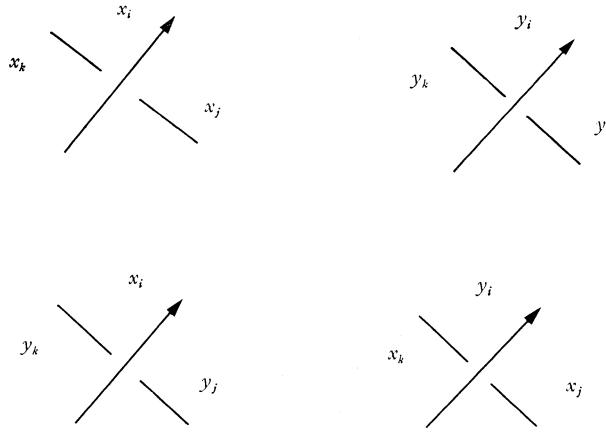


Fig. 2.2

each of which can be expressed as one of the following six types: $r_1=x_1y_1z$, $r_2=x_1y_my_mz$, $r_3=x_ix_jx_i^{-1}x_k^{-1}$, $r_4=y_iy_jy_i^{-1}y_k^{-1}$, $r_5=x_iy_jx_i^{-1}y_k^{-1}$ and $r_6=y_ix_jy_i^{-1}x_k^{-1}$, where r_1 and r_2 correspond to vertices A and B , and r_3 , r_4 , r_5 and r_6 correspond to four types of crossings as shown in Figure 2.2, respectively. Since $\mu(r_i)=1$, $i=1, \dots, 6$, we have the following equations which correspond to r_i , $i=1, \dots, 6$, respectively:

$$(2.1) \quad \alpha_1 - \beta_1 - \gamma \equiv 0 \pmod{3},$$

$$(2.2) \quad \alpha_i - \beta_m - \gamma \equiv 0 \pmod{3},$$

$$(2.3) \quad 2\alpha_i - \alpha_j - \alpha_k \equiv 0 \pmod{3},$$

$$(2.4) \quad 2\beta_i - \beta_j - \beta_k \equiv 0 \pmod{3},$$

$$(2.5) \quad 2\alpha_i - \beta_j - \beta_k \equiv 0 \pmod{3},$$

$$(2.6) \quad 2\beta_i - \alpha_j - \alpha_k \equiv 0 \pmod{3}.$$

On the other hand, for six types of relations of G we have

$$(3.1) \quad \left(\frac{\partial r_1}{\partial x_i} \right)^{\psi\phi} = 1, \quad \left(\frac{\partial r_1}{\partial y_i} \right)^{\psi\phi} = t, \quad \left(\frac{\partial r_1}{\partial z} \right)^{\psi\phi} = 1;$$

$$(3.2) \quad \left(\frac{\partial r_2}{\partial x_i} \right)^{\psi\phi} = 1, \quad \left(\frac{\partial r_2}{\partial y_m} \right)^{\psi\phi} = t, \quad \left(\frac{\partial r_2}{\partial z} \right)^{\psi\phi} = 1;$$

$$(3.3) \quad \left(\frac{\partial r_3}{\partial x_i} \right)^{\psi\phi} = 1-t, \quad \left(\frac{\partial r_3}{\partial x_j} \right)^{\psi\phi} = t, \quad \left(\frac{\partial r_3}{\partial x_k} \right)^{\psi\phi} = -1;$$

$$(3.4) \quad \left(\frac{\partial r_4}{\partial y_i} \right)^{\psi\phi} = 1-t^{-1}, \quad \left(\frac{\partial r_4}{\partial y_j} \right)^{\psi\phi} = t^{-1}, \quad \left(\frac{\partial r_4}{\partial y_k} \right)^{\psi\phi} = -1;$$

$$(3.5) \quad \left(\frac{\partial r_5}{\partial x_i} \right)^{\psi\phi} = 1 - t^{-1}, \quad \left(\frac{\partial r_5}{\partial y_j} \right)^{\psi\phi} = t, \quad \left(\frac{\partial r_5}{\partial y_k} \right)^{\psi\phi} = -1;$$

$$(3.6) \quad \left(\frac{\partial r_6}{\partial y_i} \right)^{\psi\phi} = 1 - t, \quad \left(\frac{\partial r_6}{\partial x_j} \right)^{\psi\phi} = t^{-1}, \quad \left(\frac{\partial r_6}{\partial x_k} \right)^{\psi\phi} = -1.$$

Therefore we have the following equation:

$$(4) \quad A(G, \psi)(-1) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_l \\ \beta_1 \\ \vdots \\ \beta_m \\ -\gamma \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{3}.$$

Since the nullity of $A(G, \psi)(-1)$ is ν , there are 3^ν solutions for (4). In order to count the number of S_3 -representations, we must omit three solutions $\alpha_i = \beta_j = 0, \gamma = 0$; $\alpha_i = \beta_j = 1, \gamma = 0$; $\alpha_i = \beta_j = 2, \gamma = 0$, since each of the corresponding monodromy maps is not surjective. The monodromy map corresponding to any other solution is surjective. Hence, by taking into account the six inner automorphisms of S_3 , the number of solutions corresponding to S_3 -representations (up to conjugation) is $(3^\nu - 3)/3!$.

EXAMPLES. (1) Let L be a θ -curve illustrated in Figure 2.3, where T is a 1-string tangle. Let K be a constituent knot $e_1 \cup e_2$ of L .

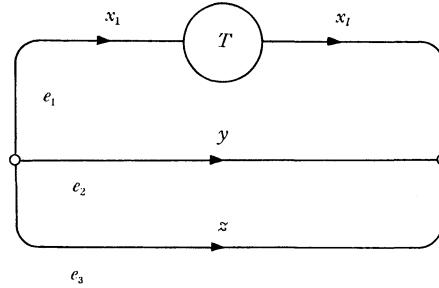


Fig. 2.3

Case 1. Suppose that $\mu(z) = b^\nu$, where ν is equal to 0, 1 or 2. Let $\tilde{M}_2(K)$ be the two-fold branched covering of K and $\tilde{M}_3(K)$ the three-fold irregular branched covering of K . If we denote the Betti number of $H_1(\tilde{M}_2(K); \mathbb{Z}_3)$ by λ , then $\nu = \lambda + 2$. Note that the number of conjugacy classes of S_3 -representations of K is equal to $(3^{\lambda+1} - 3)/3!$. By Theorem 2.1, the number of conjugacy classes of μ is equal to $(3^{\lambda+2} - 3)/3!$. Actually, the set of $\tilde{M}_\mu(L)$ consists of one

$\tilde{M}_2(K)$, $(3^{\lambda+1}-3)/3!$ $\tilde{M}_3(K)$'s and $2(3^{\lambda+1}-3)/3!$ $\tilde{M}_3(K) \# (S^2 \times S^1)$'s.

Case 2. Suppose that $\mu(x_i) = b^{\alpha_i}$, where α_i is equal to 1 or 2, $i=1, \dots, l$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, $\tilde{M}_\mu(L)$ is the three-fold cyclic branched covering of K .

(2) Let L be a rational θ -curve $\theta(p, q)$ illustrated in Figure 2.4, where

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{2n}}}}$$

(see Harikae [2]). Note that L has the symmetry for e_1 and e_2 .

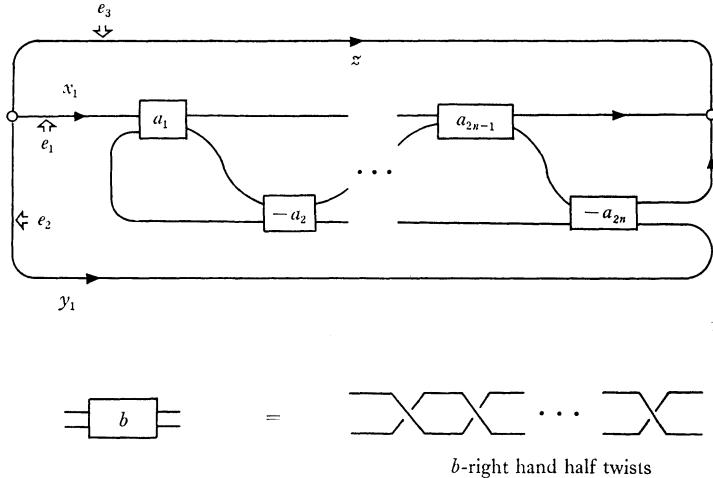


Fig. 2.4

Case 1. Suppose that $\mu(z) = b^\gamma$, where γ is equal to 0, 1 or 2. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, $\tilde{M}_\mu(L)$ is an S^3 .

Case 2. Suppose that $\mu(x_i) = b^{\alpha_i}$, where α_i is equal to 1 or 2, $i=1, \dots, l$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Further, we can see that $\tilde{M}_\mu(L)$ is a lens space.

(3) Let L be a pseudo-rational θ -curve $\theta(p_1, q_1; p_2, q_2)$ illustrated in Figure 2.5, where

$$\frac{p_1}{q_1} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n + \frac{1}{2}}}} \quad \text{and} \quad \frac{p_2}{q_2} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{2}}}}$$

(see [2]).

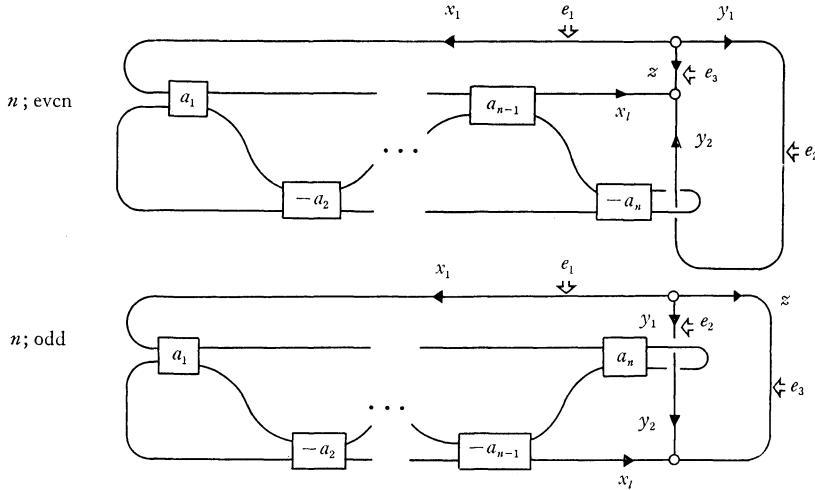


Fig. 2.5

Case 1. Suppose that $\mu(z) = b^\gamma$, where γ is equal to 0, 1 or 2. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, if $p_2 \equiv 0 \pmod{3}$, then $\tilde{M}_\mu(L)$ is an S^3 . If $p_2 \not\equiv 0 \pmod{3}$, then $\tilde{M}_\mu(L)$ is a real projective 3-space P^3 .

Case 2. Suppose that $\mu(y_i) = b^{\beta_i}$, where β_i is equal to 0, 1 or 2, $i=1, 2$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, if $p_1 \equiv 0 \pmod{3}$, then $\tilde{M}_\mu(L)$ is an S^3 . If $p_1 \not\equiv 0 \pmod{3}$, then $\tilde{M}_\mu(L)$ is a P^3 .

Case 3. Suppose that $\mu(x_i) = b^{\alpha_i}$, where α_i is equal to 1 or 2, $i=1, \dots, l$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one.

(4) Let L be the Kinoshita's θ -curve illustrated in Figure 2.6 (see [5]). Note that L has the symmetry for three edges. We assume that $\mu(z_k) = b^{\gamma_k}$, where γ_k is equal to 1 or 2 for $k=1, 2, 3$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, $\tilde{M}_\mu(L)$ is a lens space $L(5, 2)$.

(5) Let L be a θ -curve illustrated in Figure 2.7. Note that L has the symmetry for e_2 and e_3 .

Case 1. Suppose that $\mu(z_k) = b^{\gamma_k}$, where γ_k is equal to 1 or 2 for $k=1, 2, 3, 4$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, $\tilde{M}_\mu(L)$ is $L(4, 1)$.

Case 2. Suppose that $\mu(x_i) = b^{\alpha_i}$, where α_i is equal to 0, 1 or 2 for $i=1, 2$.

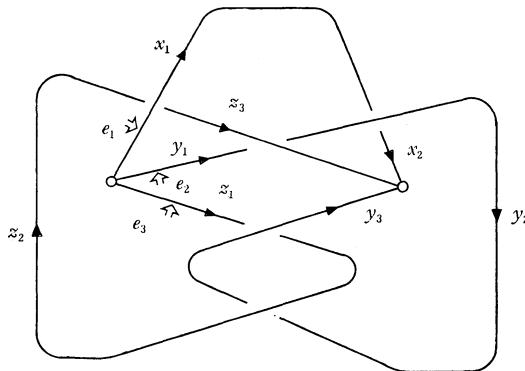


Fig. 2.6

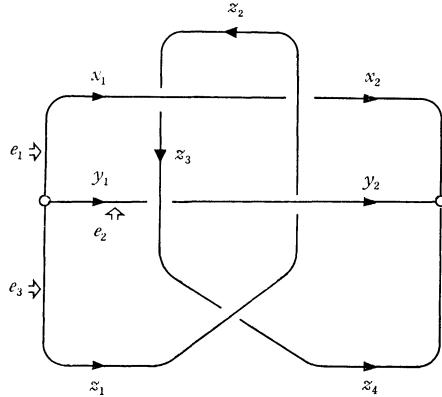


Fig. 2.7

Then we have $\nu=3$. Hence, the number of conjugacy classes of μ is equal to four. Actually, the set of $\tilde{M}_\mu(L)$ consists of S^3 , $S^2 \times S^1$, $L(3, 1)$ and $L(3, 1)$.

3. Three-fold branched coverings of spatial handcuff graphs

In this section, let L denote a spatial handcuff graph which consists of three edges e_1 , e_2 and e_3 , where e_3 has distinct endpoints A and B , and e_1 and e_2 are loops based at A and B , respectively. Suppose that e_3 is oriented from A to B . We shall use the same notations as Section 2. Then $G=\pi_1(S^3-L)$ is generated by $x_1, \dots, x_l; y_1, \dots, y_m; z_1, \dots, z_n$, where each of x_i, y_j and z_k corresponds to a meridian of each of e_1, e_2 and e_3 , respectively. Let $r_1=x_1x_1^{-1}z_1=1$ be the relation corresponding to A . By applying $ba=ab^{-1}$ to $r_1=1$, we have $\alpha_{11}-\alpha_{11}+\gamma_{11}\equiv 0 \pmod{2}$. Further we obtain $\alpha_{11}=\alpha_{11}$ by using the argument in Section 2. Hence we have $\gamma_{11}=0$, which leads $\gamma_{1k}=0$. Suppose that $\alpha_{1i}=\beta_{1j}=1$, then $\tilde{M}_\mu(L)$ is an orientable 3-manifold. Thus we have equations (1)

in Section 2. If we define ν as similar to Section 2, then we have

Theorem 3.1. *The number of conjugacy classes of S_3 -representations of L , each of which satisfies (1), is equal to $(3^\nu - 3)/3!$.*

Proof. Using the similar argument to the proof of Theorem 2.1, we can prove the statement of the theorem.

Since one of the relations of G is a consequence of the others, the deficiency of G is equal to two. Hence $\nu \geq 2$. Therefore we have

Corollary 3.2. *There exists at least one S_3 -representation of L which satisfies (1).*

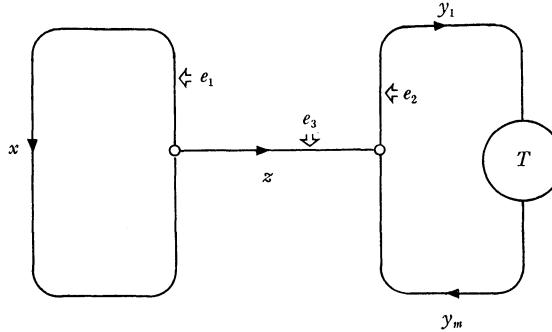


Fig. 3.1

EXAMPLES. (1) Let L be a handcuff graph illustrated in Figure 3.1, where T is a 1-string tangle. Let K be a constituent knot e_2 of L . Let $\tilde{M}_2(K)$ be the two-fold branched covering of K and $\tilde{M}_3(K)$ the three-fold irregular branched covering of K . If we denote the Betti number of $H_1(\tilde{M}_2(K); \mathbb{Z}_3)$ by λ , then $\nu = \lambda + 2$. Note that the number of conjugacy classes of S_3 -representations of K is equal to $(3^{\lambda+1} - 3)/3!$. Suppose that μ satisfies (1). Then by Theorem 3.1, the number of conjugacy classes of μ is equal to $(3^{\lambda+2} - 3)/3!$. Actually, the set of $\tilde{M}_\mu(L)$ consists of one $\tilde{M}_2(K)$ and $3(3^{\lambda+1} - 3)/3! \tilde{M}_3(K) \# (S^2 \times S^1)$'s.

(2) Let L be a rational handcuff graph $\phi(p, q)$ illustrated in Figure 3.2, where

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{2n+1}}}}$$

(see Harikae [3]). Suppose that μ satisfies (1). Then we have $\nu = 2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, $\tilde{M}_\mu(L)$ is an S^3 .

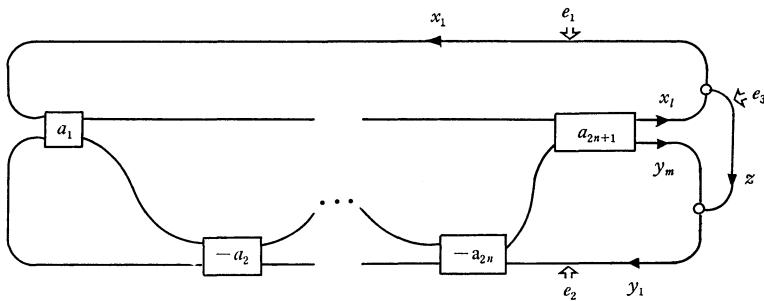


Fig. 3.2

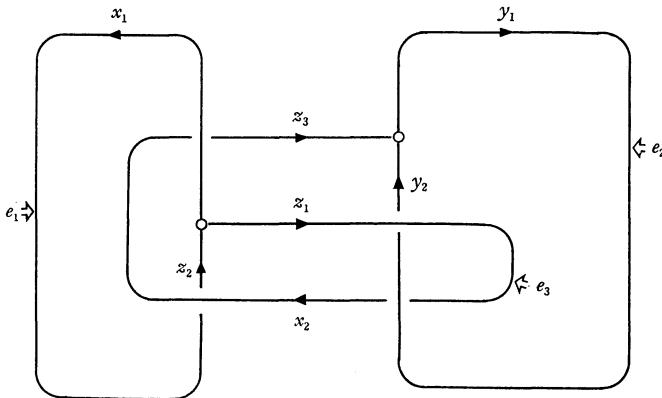


Fig. 3.3

(3) Let L be a handcuff graph illustrated in Figure 3.2 (see [5]). Suppose that μ satisfies (1). Then we have $\nu=3$. Hence, the number of conjugacy classes of μ is equal to four. Actually, the set of $\tilde{M}_\mu(L)$ consists of S^3 , $S^2 \times S^1$, $L(3, 1)$ and $L(3, 2)$.

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