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<td>Author(s)</td>
<td>Kobayashi, Tsuyoshi</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 26(4) P.699-P.742</td>
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<tr>
<td>Issue Date</td>
<td>1989</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
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<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/11295">https://doi.org/10.18910/11295</a></td>
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https://ir.library.osaka-u.ac.jp/repo/ouka/all/
1. Introduction

Let $L$ be an oriented link in a 3-manifold $M$. A Seifert surface $S$ for $L$ is a compact oriented surface, without closed components, such that $\partial S=L$. $\chi(L)$ denotes the maximal Euler characteristic of all Seifert surfaces for $L$. $L$ is a fibered link if the exterior $E(L)$ of $L$ is a surface bundle over $S^1$ such that a Seifert surface represents a fiber. An oriented surface $F$ in $M$ is a fiber surface if $\partial F$ is a fibered link, and $F \cap E(\partial F)$ is a fiber. Let $D$ be a disk in $M$, which intersects $L$ in two points of opposite orientations, $L'$ the image of $L$ after $\pm 1$ surgery along $\partial D$. We say that $L'$ is obtained from $L$ by a crossing change, and $D$ ($\partial D$ resp.) is called the crossing disk (crossing link resp.). For the links in the 3-sphere $S^3$, Scharlemann-Thompson [14] proved that if $L'$ is obtained from $L$ by a single crossing change along a crossing disk $D$, and $\chi(L')>\chi(L)$, then there is a minimal genus Seifert surface $S$ for $L$ such that $S$ is a plumbing of a surface $F$ and a Hopf band $A$ with $F \cap D=\phi$, and $A \cap D$ an essential arc in $A$. See Figure 1.1.

![Diagram](image)

Fig. 1.1

This work was supported by Grant-in Aid for Scientific Research, The Ministry of Education, Science and Culture.
In this paper, firstly, we show that a similar result holds for links in rational homology 3-spheres if \( L \) is a fibered link.

**Theorem 1.** Let \( L \) be a fibered link in a rational homology 3-sphere \( M \). Suppose that \( L' \) is obtained from \( L \) by a single crossing change along a crossing disk \( D \), and that \( \chi(L') > \chi(L) \). Then there is a minimal genus Seifert surface \( S \) for \( L \) such that \( S \) is a plumbing of a surface \( F \) in \( M \) and a Hopf band \( A \) with \( F \cap D = \phi \), and \( A \cap D \) an essential arc in \( A \).

**Remark.** We note that \( S \) and \( F \) are fiber surfaces (Lemma 2.2, [6, Theorem 7.4]).

Let \( S_0 \) be the image of \( S \) in Theorem 1 after the \( \pm 1 \) surgery along \( \partial D \), and \( S_1 = \text{cl}(S - A) \). Then \( S_0, S_1 \) are Seifert surfaces for \( L' \) (Figure 1.2). In section 4, we study the surfaces \( S_0, S_1 \).

**Theorem 2.** Let \( S_0, S_1 \) be as above. Then
1. \( S_0 \) is a pre-fiber surface,
2. if \( \chi(L') > \chi(L) + 2 \) (i.e. \( S_1 \) is not a minimal genus Seifert surface), then \( S_1 \) is also a pre-fiber surface.

For the definition of pre-fiber surface, see section 4. We prove Theorem 2 in sections 4, 5, and 6. In section 7, we give a characterization of a class of pre-fiber surfaces in case when they bound fibered links. For the statement of the result, we prepare some notations. Let \( \Sigma_n \) be the genus \( n(\geq 1) \) Seifert surface for a trivial knot in \( S^3 \) as in Figure 1.3. For the precise definition of \( \Sigma_n \), see section 7. Then we have;

**Theorem 3.** Suppose that a surface \( S_1 \) in a rational homology 3-sphere \( M \) is a pre-fiber surface of type 1 with \( L = \partial S_1 \) a fibered link. Then \( S_1 \) is a connected sum of a fiber surface for \( L \) and \( \Sigma_n \), where \( n = (\chi(L) - \chi(S_1))/2 \). Moreover a pair of canonical compressing disks for \( S_1 \) corresponds to that of \( \Sigma_n \).
Theorem 2 shows that we can get a pre-fiber surface from a fiber surface $S$ by adding a twist along a properly embedded arc in $S$, or by removing a band from $S$ (Figure 1.2). In section 8, we study the converse to this. Namely, we give a characterization of the arcs in a pre-fiber surface $S_\#$ the twists along which produce fiber surfaces, and a characterization of the bands for $S_\#$ to produce fiber surfaces in case when the ambient manifold is a rational homology 3-sphere. See the remarks of section 8.

We say that a knot in a 3-manifold $M$ is trivial if it bounds a non-singular disk in $M$. Suppose that a knot $K$ is contractible in $M$. Then it is easy to see that $K$ is tranformed into a trivial knot by a finite number of crossing changes. The unknotting number $u(K)$ is the minimal number of crossing changes that are necessary to transform $K$ into a trivial knot. Let $\Sigma_n$, $l_+$, $l_-$ ($\subset \Sigma_n$) be as in Figure 1.3. Then, as consequences of the above results, we have:

**Corollary 1.** A genus $g$ ($\geq 1$) surface $S$ in $S^3$ is a fiber surface with $\partial S$ an unknotting number 1 knot if and only if $S$ is obtained from $\Sigma_g$ by adding a twist along an arc $a$ ($\subset \Sigma_g$) such that $a$ intersects $l_+$ and $l_-$ transversely in one points.

**Corollary 2.** A genus $g$ ($>1$) surface $S$ in $S^3$ is a fiber surface with $\partial S$ an unknotting number 1 knot if and only if $S$ is obtained from $\Sigma_{g-1}$ by adding a band satisfying the properties (1), (2) of Proposition 8.2, and then plumbing a Hopf band along $b$. 

![Figure 1.3](image-url)
REMARK. Quach [9] proved that if \( A(t)(\pm 1) \) is an Alexander polynomial with leading coefficient \( \pm 1 \), then there exists an unknotting number 1, fibered knot \( K \) in \( S^3 \) with \( \Delta_K(t)=A(t) \), where \( \Delta_K(t) \) denotes the Alexander polynomial of \( K \). The result implies that, for each \( g(>1) \), there are infinitely many unknotting number 1, fibered knots of genus \( g \).

In section 9, by using Theorem 2, we study the rational homology 3-spheres containing unknotting number 1 fibered knots. We say that a 3-manifold is a lens space if it admits a Heegaard splitting of genus 1 [6]. Then we have;

**Theorem 4.** If a rational homology 3-sphere \( M \) contains an unknotting number 1 fibered knot, then \( M \) is a lens space.

**Remark.** Moreover we will show that, for each \( g(>1) \), every lens space contains an unknotting number 1 fibered knot of genus \( g \), and we will give the list of lens spaces containing genus 1, unknotting number 1, fibered knots. We note that there exist lens spaces which do not contain genus 1 fibered knots [7].

As an immediate consequence of Theorem 4, we have;

**Corollary 3.** If an integral homology 3-sphere \( \Sigma^3 \) contains an unknotting number 1 fibered knot, then \( \Sigma^3 \) is homeomorphic to \( S^3 \).

2. Preliminaries

Throughout this paper, we work in the piecewise linear category, all manifolds, including knots, links, and Seifert surfaces are oriented, and all submanifolds are in general position unless otherwise specified. For the definitions of standard terms of 3-dimensional topology, knot and link theory, see [6], and [10]. For a topological space \( B, \#B \) denotes the number of the components of \( B \). Let \( H \) be a subcomplex of a complex \( K \). Then \( N(H; K) \) denotes a regular neighborhood of \( H \) in \( K \). Let \( N \) be a manifold embedded in a manifold \( M \) with \( \dim N=\dim M \). Then \( \text{fr}_M N \) denotes the frontier of \( N \) in \( M \).

An arc \( a \) properly embedded in a surface \( S \) is *inessential* if it is rel \( \partial \) isotopic to an arc in \( \partial S \). If \( a \) is not inessential, then it is *essential*.

Let \( S \) be a surface properly embedded in a 3-manifold \( M \). A disk \( D \) in \( M \) is a *compressing disk* for \( S \) if \( D \cap S=\partial D \), and \( \partial D \) is not contractible in \( S \). If there does not exist a compressing disk for \( S \), then \( S \) is *incompressible*.

Let \( S_i \) be a surface with boundary in a 3-manifold \( M_i \) (\( i=1, 2 \)). Let \( B_i \) be a 3-ball in \( M_i \) such that \( B_i \cap \partial S_i \) is an arc, and \( B_i \cap S_i \) is a disk (Figure 2.1). Let \( h: \partial B_1 \to \partial B_2 \) be an orientation reversing homeomorphism such that \( h(\partial B_1 \cap S_i)=\partial B_2 \cap S_i \). Then \( (M_1-\text{Int } B_1) \cup_h (M_2-\text{Int } B_2) \) is a connected sum of \( M_i \) and \( M_2 \), and is denoted by \( M_1\#M_2 \). The image of \( S_1 \cup S_2 \) in \( M_1\#M_2 \) is called a connected sum of \( S_i \) and \( S_2 \).

A sutured manifold \((M, \gamma)\) is a compact 3-manifold \( M \) together with a set
\( \gamma ( \subset \partial M ) \) of mutually disjoint annuli \( A(\gamma) \) and tori \( T(\gamma) \) \[2\]. In this paper, we mainly treat the case of \( T(\gamma) = \phi \). The core curves of \( A(\gamma) \), \( s(\gamma) \), are the sutures. Every component of \( R(\gamma) = M - \text{Int } \gamma \) is oriented, and \( R_+ (\gamma) (R_- (\gamma) \text{ resp.)} \) denotes the union of the components whose normal vector point to (into resp.) \( M \). Moreover, the orientation of \( R(\gamma) \) must be coherent with respect to \( s(\gamma) \). We say that a sutured manifold \((M, \gamma)\) is a product sutured manifold if \( (M, \gamma) \) is homeomorphic to \((F \times I, \partial F \times I)\) with \( R_+(\gamma) = F \times \{1\} \), where \( F \) is a surface, and \( I \) is the unit interval \([0, 1]\).

Let \((M, \gamma)\) be a sutured manifold. A properly embedded annulus \( A \) in \( M \) is a product annulus if one boundary component of \( A \) is contained in \( R_+ (\gamma) \), and the other is contained in \( R_-(\gamma) \). A properly embedded disk \( D \) in \( M \) is a product disk if \( \partial D \cap \gamma \) consists of two essential arcs in \( A(\gamma) \). A product decomposition \((M, \gamma) \rightarrow (M', \gamma')\) is a sutured manifold decomposition \[2\] along a product disk. See Figure 2.2.

Let \( L \) be a link in a 3-manifold \( M \). The exterior \( E(L) \) of \( L \) is the closure of the complement of \( N(L; M) \). A meridian loop for \( L \) is a non-contractible simple loop in \( \partial E(L) \), which bounds a disk in \( N(L; M) \). Let \( S \) be a Seifert surface for \( L \). Then we often abbreviate \( S \cap E(L) \) to \( S \). \( S \) is a minimal genus Seifert surface if \( \chi(S) = \chi(L) \).

Let \( S \) be a Seifert surface for \( L \). Then \((N, \delta) = (N(S; E(L)), N(\partial S; \partial E(L)))\) has a product sutured manifold structure \((S \times I, \partial S \times I)\). \((N, \delta)\) is called the sutured manifold obtained from \( S \). Then the sutured manifold \((N', \delta') = \)}
\[ (\text{cl}(E(L)-N), \text{cl}(\partial E(L)-\delta)) \] with \( R_+ (\delta) = R_- (\delta) \) is the \textit{complementary sutured manifold} for \( S \). We say that a surface \( S \) in a 3-manifold is a \textit{fiber surface}, if \( \partial S \) is a fibered link with \( S \) a fiber. It is easy to see that \( S \) is a fiber surface if and only if the complementary sutured manifold for \( S \) is a product sutured manifold.

Then we easily see;

\textbf{Lemma 2.1.} \textit{Every fiber surface in a connected 3-manifold is connected.}

Let \( L \) be a link with a Seifert surface in a rational homology 3-sphere. It is easy to see that Seifert surfaces for \( L \) determine a unique non trivial element of \( H_2 (E(L), \partial E(L)) \), so that the cyclic covering space for \( L \) is well defined. Then the next lemma follows from the fact that the infinite cyclic covering space of a fibered link is homeomorphic to \((\text{surface}) \times R\), and details of the proof are left to the reader.

\textbf{Lemma 2.2.} \textit{For a surface \( S \) in a rational homology 3-sphere, with \( L = \partial S \) a fibered link, the following three conditions are equivalent.}

1. \( S \) is a fiber surface.
2. \( S \) is a minimal genus Seifert surface for \( L \).
3. \( S \) is incompressible.

Let \( S \) be a fiber surface. Then there is an orientation preserving homeomorphism \( \varphi \) of \( S \) such that \( \varphi |_{\partial S} = \text{id}_{\partial S} \), and \( E(L) \) is homeomorphic to \( S \times I / \sim \), where \((x, 1) \sim (\varphi(x), 0) \ (x \in S) \). \( \varphi \) is called a \textit{monodromy map}. \( \partial S \times I \) has an \( I \)-bundle structure such that each fiber projects to a meridian loop of \( \partial E(L) \). Let \( p: S \times I \rightarrow E(L) \) be a natural map, \( D (\subset S \times I) \) a product disk for the product sutured manifold \((S \times I, \partial S \times I)\) such that each component of \( \partial D \cap (\partial S \times I) \) is a fiber. Then the 2-complex \( \square = p(D) \) is called a \textit{projected product disk} (or \textit{pp disk} for short). For the pp disk \( \square \), \( \partial_- \square \), \( \partial_+ \square \) denotes \( p(D \cap (S \times \{0\})) \), \( p(D \cap (S \times \{1\})) \) respectively. Suppose that there is an ambient
isotopy $f_t$ for $S \times I$ such that $f_0 = \text{id}$, $f_t(D)$ is a product disk such that $\partial f_t(D) \cap (\partial S \times I)$ consists of fibers of $\partial S \times I$. Then we say that the pp disk $\square' = p(f_t(D))$ is isotopic to $\square$ by an isotopy as a pp disk.

**Example 2.3.** A Hopf band $A$ is a $\pm 1$ twisted unknotted annulus in $S^3$ (Figure 2.3). $A$ is a fiber surface, and a monodromy map for $A$ is a right or left hand Dehn twist along the core curve of $A$.

**Example 2.4.** The genus 0 surface $A^*$ of Figure 2.4 is a connected sum of two Hopf bands, and hence, by [3] or [13], is a fiber surface.

3. **Theorem 1**

In this section, we prove Theorem 1 stated in section 1. We assume that the reader is familiar with [5], and [14].

Let $L$, $L'$, and $D$ be as in Theorem 1. Let $S$ be a minimal genus Seifert surface for $L$ in $M$. Let $L_1$ be the link obtained from $L$ by splitting it as in Figure 3.1, $D_1$ the disk as in Figure 3.1, and $R_1$ a minimal genus Seifert surface for $L_1$ in $E(\partial D_1)$. By the arguments of the proof of [14, 1.4 Theorem], we may suppose that $R_1$ intersects $D_1$ in an arc $a_1$ (Figure 3.2 (i)). Let $R$ be the Seifert surface for $L$ obtained from $R_1$ by plumbing a Hopf band as in Figure 3.2 (ii).

**Claim 3.0.** *If $E(\partial D_1 \cup L_1)$ is not irreducible, then the conclusion of Theorem 1 holds.*
Proof. Let $P = D_1 \cap E(\partial D_1 \cup L)$. Then $P$ is a disk with two holes, with two boundary components $l_1, l_2$ are meridian loops of $L$, and the rest boundary component $l_3$ is parallel to $\partial D_1$ in $D_1$. Let $S_1$ be an essential 2-sphere in $E(L_1 \cup \partial D_1)$.

**Subclaim 1.** $S_1 \cap P \neq \emptyset$.

Proof. Assume that $S_1 \cap P = \emptyset$. Then, by Figure 3.2, we may suppose that $S_1$ is embedded in $E(\partial D_1 \cup L)$, and $\partial D_1 \cup L$ is contained in a component of $M - S_1$. Since $E(L)$ is irreducible, $S_1$ bounds a 3-ball in $E(\partial D_1 \cup L)$, so that $S_1$ bounds a 3-ball in $E(\partial D_1 \cup L)$, a contradiction.

Then we suppose that $\#(S_1 \cap P)$ is minimal among all essential 2-spheres in $E(\partial D_1 \cup L)$. Let $V(\subset S_1)$ be an innermost disk, i.e. $V \cap P = \partial V$. By the minimality of $\#(S_1 \cap P)$, we see that $\partial V$ is not contractible in $P$.

**Subclaim 2.** $\partial V$ is parallel to $l_3$ in $P$.

Proof. Assume not. Then $\partial V$ is parallel to $l_1$ or $l_2$. Let $D^*$ be the disk in $D_1$ such that $\partial D^* = \partial V$, and $S_1 = V \cup D^*$. $S_1$ is a 2-sphere, and intersects $L$ in one point. Then, by plumbing a Hopf band to $R_i$ in the right or left side of $D_1$ in Figure 3.2, we may suppose that $S_2 \cap L$ consists of one point. This shows that a meridian loop for $L$ is contractible in $E(L)$, contradicting the fact that $L$ is a fibered link.

**Subclaim 3.** $R_i$ is of minimal genus in $M$.

Proof. Let $D^*$ be the disk in $D_1$ such that $\partial D^* = \partial V$, and $S_2 = D^* \cup V$. By Subclaim 2, $S_2$ is a 2-sphere in $M$ which intersects in $L_1$ in two points. Let $R_i^\dagger$ be a minimal genus Seifert surface for $L_i$ in $M$. Since $S_2 \cap L_1$ consists of two points, by applying cut and paste arguments on $S_2$, we may suppose that $S_2 \cap R_i^\dagger = D_1 \cap R_i^\dagger$ consists of an arc whose endpoints are $S_2 \cap L_1$. This shows that $\chi(R_i) \geq \chi(R_i^\dagger)$. Clearly $\chi(R_i^\dagger) \geq \chi(R_i)$. Hence $\chi(R_i) = \chi(R_i^\dagger)$, so that $R_i$ is of minimal genus in $M$. 

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**Fig. 3.2**
**Subclaim 4.** \( E(L) \) is irreducible.

Proof. Assume not. Let \( S_3 \) be an essential 2-sphere in \( E(L) \). Since \( R_1 \) is incompressible (Subclaim 3), by using standard innermost disk arguments, we may suppose that \( S_3 \cap R_1 = \emptyset \). Hence we may suppose that \( S_3 \cap L = \emptyset \). It is easy to see that \( S_3 \) is an essential 2-sphere in \( E(L) \), contradicting the irreducibility of \( E(L) \).

By Subclaims 3 and 4, we see that \( R_1 \) is taut in terms of [2]. Hence, by [2, Theorem 5.5] and the argument of the proof of [3, Theorem 1.1], we see that \( E(L) \) possesses a taut foliation such that \( R \) is a leaf of the foliation. Hence \( R \) is a minimal genus Seifert surface for \( L \) in \( M \), and this completes the proof of Claim 3.0.

By Claim 3.0, hereafter, we suppose that \( E(\partial D_1 \cup L) \) is irreducible. Then, by the argument in the last paragraph of the proof of Claim 3.0, we see that \( E(\partial D_1 \cup L) \) possesses a taut foliation such that \( R \) is a leaf of the foliation, so that \( E(\partial D_1 \cup L) \) is irreducible, and \( R \) is a minimal genus Seifert surface for \( L \) in \( E(\partial D_1) \). Then we have the following two cases.

**Case 1.** \( E(L) \) is \( R_{\partial D_1} \)-atoroidal.

If \( R \) is a minimal genus Seifert surface for \( L \) in \( M \), then we have the conclusion of Theorem 1. Suppose that \( R \) is not of minimal genus in \( M \). Then by [5, Theorem 1.8] or [12, 5.1 Theorem], and by the arguments of the proof of [14, 1.14 Theorem], we see that the surface \( R^* \) obtained from \( R \) by cutting along \( a_1 \) is of minimal genus in \( M \) (Figure 3.3 (i)). Hence we see that the Seifert surface \( S' \) for \( L' \) obtained from \( R^* \) by removing the Hopf band is of minimal genus in \( M \) (Figure 3.3 (ii)). We note that \( \chi(S') = \chi(R) + 2 \). Since \( \chi(L') = \chi(L) + 2 \), this shows that \( R \) is a minimal genus Seifert surface for \( L \) in \( M \), a contradiction.

**Case 2.** \( E(L) \) is not \( R_{\partial D_1} \)-atoroidal.

Since \( E(L) \) is not \( R_{\partial D_1} \)-atoroidal, there is an incompressible, non-boundary parallel torus \( T \) in \( E(\partial D_1 \cup L) \) with the following properties.
(3.1) $T$ separates $E(\partial D_1)$ into $V_1$ and $V_2$ with $\partial E(\partial D_1) \subset V_1$, and $R \subset V_2$, and
(3.2) $i_*: H_1(T) \rightarrow H_1(V_1)$ is injective.

Let $T_1, T_2$ be incompressible, non-boundary parallel tori satisfying the above conditions (3.1), (3.2). We say that $T_1 \leq T_2$ if $T_1$ is isotopic to $T_2$ such that $T_1 \cap T_2 = \phi$, and $V_1 \subset V_2$, where $V_1$ ($V_2$ resp.) denotes the closure of the component of $E(L) - T_1$ ($E(L) - T_2$ resp.) which contains $\partial D_1$. Clearly $\leq$ is an order on the tori with the above properties (3.1), (3.2). Then we suppose that $T$ is maximal with respect to the order.

**Claim 3.1.** If $T$ is incompressible in $E(L)$, then $R$ is a minimal genus Seifert surface for $L$ in $M$.

Proof. Since $E(L)$ is irreducible, and $S$ is incompressible, by using standard innermost disk arguments, we may suppose that $T$ intersects $S$ in essential loops, so that each component of $T \cap N^c$ is an annulus, where $(N^c, \gamma^c)$ is the complementary sutured manifold for $S$ in $M$. Since $(N^c, \gamma^c)$ is a product sutured manifold, by [15, Corollary 3.2], we may suppose, by moving $T$ by an ambient isotopy, that each component of $T \cap N^c$ is a product annulus.

Since $T$ is incompressible, and $T \cap R = \phi$, we may suppose that $T$ intersects $D_1$ in essential loops in the annulus $cl(D_1 - N(a_1 \cup D_1))$. Suppose that some component of $T \cap D_1$ is contractible in $T$. Then, by using cut and paste arguments, we see that $\partial D_1$ bounds a disk in $E(L)$, contradicting the fact that $E(\partial D_1 \cup L)$ is irreducible. Hence we see that $\partial D_1$ is ambient isotopic to an essential loop $l$ on $T$. Then, by the above, we may suppose that either $l$ is ambient isotopic to a component of $T \cap S$ or each component of $l \cap N^c$ runs from $R_-(\gamma^c)$ to $R_+(\gamma^c)$. Then since $lk(l, L) = lk(\partial D_1, L) = 0$, we see that $l$ is ambient isotopic to a component of $T \cap S$. Hence we may suppose that $\partial D_1 \cap S = \phi$. This shows that $\chi(S) \leq \chi(R)$. Clearly $\chi(S) \geq \chi(R)$. Hence $\chi(S) = \chi(R)$, and $R$ is a minimal genus Seifert surface for $L$ in $M$.

**Claim 3.2.** If $T$ is compressible in $E(L)$, then $T$ bounds a solid torus in $E(L)$.

Proof. Since $E(L)$ is irreducible and $T$ separates $E(L)$, we see that $T$ bounds either a solid torus or a 3-manifold homeomorphic to the exterior of a non-trivial knot in $S^3$ such that the boundary of the compressing disk is a meridian loop. Assume that $T$ bounds the exterior $E$ of a non-trivial knot with a compressing disk $C$ for $T$ such that $\partial C$ is a meridian loop for $E$. Then $\partial D_1 \subset E$. Then $B = E \cup N(C; E(L))$ is a 3-ball such that $\partial D_1 \subset B$, contradicting the irreducibility of $E(\partial D_1 \cup L)$.

**Claim 3.3.** If $T$ is compressible in $E(L)$, then $R$ is a minimal genus Seifert surface for $L$ in $M$.

Proof. Assume that $R$ is not a minimal genus Seifert surface for $L$ in $M$.  

By Claim 3.2, $T$ bounds a solid torus $\tau$ such that $\partial D_1 \subset \tau$. Since $E(\partial D_1 \cup L)$ is irreducible, and $T$ is incompressible in $E(\partial D_1 \cup L)$, we may suppose that $T$ intersects $D_1$ in essential loops in the annulus $D_1 - N(a_1; D_1)$. By the argument of the second paragraph of the proof of Claim 3.1, we see that every component of $T \cap D_1$ is an essential loop in $T$. Then $\partial D_1$ is ambient isotopic to an essential loop $l$ on $T$.

Let $m$ be an essential simple loop on $T$. Then $M(m)$ denotes the manifold obtained from $D^2 \times S^1$ and $M - \text{Int} \tau$ by identifying their boundaries by a homeomorphism which takes $\partial(D^2 \times pt.)$ to $m$. Clearly $M(m)$ is obtained from $N$ by doing a Dehn surgery along the core curve $c$ of $\tau$. Then $R(m)$ denotes the image of $R$ in $M(m)$. Let $m_0$ be a simple loop on $T$ such that $M(m_0) = M$, and $R(m_0) = R$.

**Subclaim 1.** The absolute value of the intersection number of $m_0$ and $l$ in $T$ is greater than one.

Proof. Assume that $m_0$ does not intersect $l$, i.e. $m_0$ and $l$ are parallel. Then $l$ bounds a disk in $\tau$, contradicting the fact that $E(\partial D_1 \cup L)$ is irreducible. Assume that $m_0$ intersects $l$ in one point. Then $l$ is isotopic to $c$ in $\tau$, contradicting the fact that $T$ is not boundary parallel in $E(D_1 \cup L)$.

Let $l^*$ be a simple loop in $T$ intersecting $l$ in one point. By Subclaim 1, we see that $M$ is homeomorphic to the connected sum of $M(l^*)$ and a non-trivial lens space $L_a$ (Figure 3.4).

Since $T$ is incompressible, and $E(\partial D_1 \cup L)$ is irreducible, $E(c \cup L)$ ($\simeq E(L) - \text{Int} \tau$) is irreducible. By the maximality of $T$, it is easy to see that $E(L)$ is $R_c$-actoroidal. By Subclaim 1, $l$ is not ambient isotopic to $m_0$. Since $R(m_0) = R(l)$ is not of minimal genus, by [5, Theorem 1.8] or [12, 5.1 Theorem], we see that $R(l)$ is taut, so that of minimal genus.

Let $\bar{R}^*$ be the image of $R^*$ (Figure 3.3 (ii)) in $M(l^*)$. Then;

**Subclaim 2.** $\bar{R}^*$ is a minimal genus Seifert surface in $M(l^*)$.

Proof. The idea of the following proof can be found in [14]. Let $(N^0, \delta^0)$, $(N^1, \delta^1)$, $(N^*, \delta^*)$ be the complementary sutured manifolds for $R(=R(m_0))$, $R(l)$, $R(l^*)$ respectively. Let $S^2$ be a 2-sphere in $M(l)$ such that $S^2 \cap (M - \text{Int} \tau)$ is a disk whose boundary is $l$, and intersecting $R(l)$ in an essential arc (Figure 3.4 (i)). Then the image of $S^2$ in $N^1$ is a product disk $\mathcal{D}$ in $(N^1, \delta^1)$, and, by doing the product decomposition along $\mathcal{D}$, we get a sutured manifold $(\bar{N}, \bar{\delta})$, which is homeomorphic to the complementary sutured manifold for $\bar{R}^*$. Since $R(l)$ is taut, $(N^1, \delta^1)$ is taut. Hence, by [2, Lemma 3.12] or [12, 4.2 Lemma], $(\bar{N}, \bar{\delta})$ is taut, so that $\bar{R}^*$ is of minimal genus.

Since $M = M(l^*) \# L_a$ (Figure 3.4 (ii)), Subclaim 2 shows that $R^*$ of Figure
3.3 (i) is of minimal genus. Hence $S'$ of Figure 3.3 (ii) is of minimal genus. We note that $\chi(L')=\chi(S')=\chi(R)+2$, and $\chi(L)<\chi(L')$, i.e. $\chi(L)+2\leq \chi(L')$. This shows that $R$ is a minimal genus Seifert surface for $L$ in $M$, a contradiction. This completes the proof of Theorem 1.

4. Fiber surfaces and pre-fiber surfaces

In this section, we give the definition of pre-fiber surfaces, and show that if there is a fiber surface $F$ whose monodromy has a certain property, then we can get a pre-fiber surface by removing a band from $F$ (Proposition 4.5). And, by using the result, we prove Theorem 2 (1).

Let $S$ be a connected surface in a 3-manifold, and $(N^c, \delta^c)$ the complementary sutured manifold for $S$. $S$ is a pre-fiber surface, if there are pairwise disjoint compressing disks $D^+$, $D^-$ for $R_+(\delta^c)$, $R_-(\delta^c)$ respectively in $N^c$ such that $(\tilde{N}, \delta^c)$ is homeomorphic to the product sutured manifold, where $\tilde{N}$ is obtained from $N^c$ by doing a surgery along $D^+ \cup D^-$. Then $S$ has two compressing disks $\tilde{D}^+$, $\tilde{D}^-$ such that $\text{Int } \tilde{D}^+ \cap \text{Int } \tilde{D}^- = \phi$, $\tilde{D}^+ \cap N^c = D^+$, $\tilde{D}^- \cap N^c = D^-$. We say that $\tilde{D}^+$, $\tilde{D}^-$ is a pair of canonical compressing disks for a pre-fiber surface $S$.

Remark. We note that $N(\partial \tilde{D}^+; \tilde{D}^+)$ lies in the — side of $S$.

We say that a pre-fiber surface $S$ is of type 1 (type 2 resp.) if $\partial D^+$ is non-separating (separating resp.) in $R_+ (\delta^c)$. It is easy to see that if $S$ is of type 1, then $(N^c, \delta^c)$ is homeomorphic to $(D^c \times S^1 \cup_{d_+(S'(\times I))} D^c \times S^1, \partial S'(\times I))$, where $S'$ is a connected surface, $\cup$ denotes a boundary connected sum, and $d_+$ ($d_-$ resp.) denotes a disk in $S'(\times \{1\})$ ($S'(\times \{0\})$ resp.).

Example 4.1. Let $T$ be a genus 1 Heegaard surface for a lens space [6], and $D^2$ a disk in $T$. Let $S=T-\text{Int } D^2$. Then $S$ is a pre-fiber surface of type 1. In fact, the complementary sutured manifold for $S$ is homeomorphic to $(D^c \times S^1 \cup (D^c \times I)) \cup D^c \times S^1, \partial D^c \times I)$.

Let $A$ be an unknotted, untwisted annulus in $S^3$. Then $A$ is a pre-fiber
surface of type 2. In fact, the complementary sutured manifold for $A$ is homeomorphic to $(D^2 \times S^1, \gamma)$, where $s(\gamma)$ consists of two essential loops in $\partial(D^2 \times S^1)$ which are contractible in $D^2 \times S^1$.

The next proposition shows that pairs of canonical compressing disks for a pre-fiber surface are unique.

**Proposition 4.2.** Let $S$ be a pre-fiber surface, and $D^+, D^-, \bar{D}^+, \bar{D}^-$ as above. Let $\bar{D}^+, \bar{D}^-$ be a pair of canonical compressing disks for $S$ such that $N(\partial \bar{D}^+; \partial \bar{D}^-)$ (or $N(\partial \bar{D}^+; \partial \bar{D}^-)$ resp.) lies in the $-$ side ($+$-side resp.) of $S$. Then $\bar{D}^+(\bar{D}^-$ resp.) is isotopic to $\bar{D}^+$ ($\bar{D}^-$ resp.) by an ambient isotopy of the 3-manifold respecting $S$.

For the proof of Proposition 4.2, we prepare two lemmas. Let $(N, \delta)$ be a connected sutured manifold such that $N$ is obtained from a (possibly disconnected) product sutured manifold $(N', \delta')$ with $N'$ irreducible by attaching a 1-handle along disks in $R_+(\delta')$, and $\delta$ is the image of $\delta'$. Let $D$ be the dual core of the 1-handle. Then;

**Lemma 4.3.** Suppose that $N'$ is disconnected. Let $D_1$ be a compressing disk for $R_+(\delta_1)$. Then $D_1$ is isotopic to $D$ by an ambient isotopy of $N$ respecting $\delta$.

Proof. Since $N'$ is irreducible, $N$ is irreducible. Hence, by using standard innermost disk arguments, we may suppose that no component of $D \cap D_i$ is a simple loop. Suppose that $D \cap D_1 = \phi$. Then $\partial D_1$ bounds a disk $D'$ in $R_+(\delta')$. Since $D_1$ is a compressing disk, we see that $D'$ contains a component of $N' \cap (1\text{-handle})$, so that $D_1$ is parallel to $D$. Suppose that $D \cap D_1 = \phi$. Let $\Delta(\subset D_1)$ be an outermost disk, i.e. $\Delta \cap D = \partial \Delta \cap D = \alpha$ an arc, and $\Delta \cap \partial D_1 = \beta$ an arc such that $\alpha \cup \beta = \partial \Delta$. Let $\Delta'$ be the image of $\Delta$ in $N'$. Then $\partial \Delta' \subset R_+(\delta')$, and $\partial \Delta'$ bounds a disk $D'$ in $R_+(\delta')$ such that $\Delta'$ is parallel to $D'$. Hence we can remove $\alpha$ by moving $D_1$ by an ambient isotopy of $N$ respecting $\delta$. Then by the induction on $\#(D \cap D_i)$, we have the conclusion.

**Lemma 4.4.** Let $(N, \delta)$, $(N', \delta')$ be as above. Suppose that $N'$ is connected. Let $D_1$ be a compressing disk for $R_+(\delta_1)$ such that $\partial D_1$ is non separating in $R_+(\delta')$. Then $D_1$ is isotopic to $D$ by an ambient isotopy of $N$ respecting $\delta$.

Proof. Let $D_1, D^2$ be the disks in $R_+(\delta')$ along which the 1-handle is attached. We may suppose that no component of $D \cap D_1$ is a simple loop (see the proof of Lemma 4.3). We see that if $D \cap D_1 = \phi$, then we have the conclusion (see the proof of Lemma 4.3). Suppose that $D \cap D_1 = \phi$. Let $\Delta(\subset D_1)$ be an outermost disk, and $\alpha = \Delta \cap D, \beta = \Delta \cap \partial D_1$. Let $\Delta'$ be the image of $\Delta$ in $N'$. Without loss of generality, we may suppose that $\partial \Delta' \cap D^2 = \phi$, and $\partial \Delta' \cap D^1$ consists of an arc $\alpha'$ parallel to $\alpha$ in $D_1$. Let $\beta'$ be the image of $\beta$ in $N'$. Then $\partial \Delta' = \alpha' \cup \beta'$, and $\partial \Delta'$ bounds a disk $D'$ in $R_+(\delta')$ such that $\Delta'$ is parallel
If $D'$ does not contain $D^2$ then we can move $D_1$ by an isotopy to reduce $\#(D \cap D_1)$. Suppose that $D'$ contains $D^2$. Then trace the arc $\bar{\alpha} = \partial D_1 - \beta$ from one endpoint to the other. It is easy to see that there is a subarc $\alpha^*$ of $\bar{\alpha}$ such that $\alpha^* \cap D = \partial \alpha^*$, the image of $\alpha^*$ in $N'$ is an arc contained in $D'$, and the endpoints of the image of $\alpha^*$ is contained in $\partial D^2$ (Figure 4.1). This shows that, by moving $D_1$ by an isotopy, we can remove $\alpha^*$. Hence, by the induction on $\#(D_1 \cap D)$, we have the conclusion.

![Diagram](Fig. 4.1)

**Proof of Proposition 4.2.** We prove Proposition 4.2 for $\bar{D}^+$ and $\bar{D}^{+\prime}$. The other case is essentially the same. Let $(N^e, \delta^e), (\bar{N}, \delta')$ be as above. Then we may suppose that $D^{+\prime} = \bar{D}^{+\prime} \cap N^e$ is a disk. Let $S_{1/2}$ be the surface in $N^e$ corresponding to $S \times \{1/2\} \subset (\bar{N}, \delta') \cong (S \times I, \partial S \times I)$. Then by using standard innermost disk arguments, we may suppose that $D^{+\prime} \cap S_{1/2} = \phi$. Then, by Lemma 4.3 or Lemma 4.4, we see that $D^{+\prime}$ is ambient isotopic to $D^+$ in $N^e$. This shows that $\bar{D}^{+\prime}$ is isotopic to $\bar{D}^+$ by an ambient isotopy respecting $S$.

This completes the proof of Proposition 4.2.

Let $F$ be a fiber surface in a 3-manifold $M$, and $\varphi: F \to F$ a monodromy map. Suppose that there is an arc $a(\subset S)$ such that;

1. $a \cap \varphi(a) = \partial a = \partial \varphi(a)$, and
2. the components of $N(\partial \varphi(a); \varphi(a))$ lie in one side of $a$ (Figure 4.2).

The purpose of this section is to prove;

**Proposition 4.5.** Let $F, \varphi, a$ be as above. If $M$ is a rational homology
3-sphere, and a does not separate \( F \), then the surface obtained from \( F \) by cutting along \( a \) is a pre-fiber surface.

In case when \( a \) separates \( F \), we have:

**Proposition 4.6.** Let \( F, \varphi, a \) be as above. If \( a \) separates \( F \), then there is a separating 2-sphere \( S^2 \) in \( M \) such that \( S^2 \cap F = a \), i.e. \( F \) is a connected sum of two fiber surfaces.

Proof of Proposition 4.6. Suppose that \( a \) separates \( F \) into \( F_1 \) and \( F_2 \). Since \( \varphi|_{S^2} = \text{id}_{S^2} \) and \( \varphi \) is a homeomorphism, we see that \( \varphi(F_i) \) is rel \( \partial \) isotopic to \( F_i \). Hence, we may suppose that \( \varphi(a) = a \). Take a pp disk \( \Box \) such that \( \partial_+ \Box = \partial_+ \Box = a \). Then \( \Box \) is topologically an annulus. Then, by adding two meridian disks to \( \Box \), we get a 2-sphere \( S^2 \) in \( M \), which intersects \( F \) in \( a \).

Assume that \( S^2 \) does not separate \( M \). Let \( M' \) be the 3-manifold obtained from \( M \) by cutting along \( S^2 \), and then capping off the boundary by two 3-cells. We note that the complementary sutured manifold \(( \Lambda \Pi, \delta') \) for the disconnected surface \( F_1 \cup F_2 \) in \( M' \) is homeomorphic to the sutured manifold obtained from the complementary sutured manifold \(( N, \delta) \) of \( F \) by decomposing along the product disk \( \Box \cap N \). Hence \( F_1 \cup F_2 \) is a fiber surface in a connected 3-manifold \( M' \), contradicting Lemma 2.1.

Proof of Proposition 4.5. Let \( a_1 \) and \( a_2 \) be the components of \( \text{Fr}_F N(a; F) \). We may suppose that \( a_1 \cap \varphi(a) \) consists of two points, and \( a_2 \cap \varphi(a) = \varphi \). See Figure 4.2. Let \( \alpha \) be the subarc of \( a_1 \) such that \( \partial \alpha = a_1 \cap \varphi(a) \), and \( \lambda = (\varphi(a) - N(a; F)) \cup \alpha \). Then \( \lambda \) is a simple loop on \( F \).

**Claim 4.1.** There exists a disk \( D \) in \( M \) such that \( \partial D = \lambda \), and \( (\text{Int} D) \cap F = a \).

Proof. Let \( \Box \) be a pp disk for \( F \) such that \( \partial_+ \Box = a, \partial_+ \Box = \varphi(a) \). We note that \( \Box \cap \partial E(L) \) consists of two meridian loops. Let \( D_1, D_2 \) be meridian disks for \( L \) such that \( \partial D_1 \cup \partial D_2 = \Box \cap \partial E(L) \), and \( \Box = \Box \cup D_1 \cup D_2 \). Then we identify \( F \cap E(L) \) to \( F \). Let \( B \) be the rectangle in \( F \) such that one edge is \( a \), two edges are the components of \( \varphi(a) \cap N(a; F) \), and the last edge is \( \alpha \). Then \( \tilde{D} = \Box \cup B \) is topologically a disk such that \( \partial \tilde{D} = \lambda \), and \( \tilde{D} \cap F = B \cup \lambda \). Then, by deforming \( \tilde{D} \) by pushing \( B - (\alpha \cup a) \) slightly to the \( - \)-side of \( F \), we get a disk \( D \) satisfying the conclusion.

Let \( S_i \) be the surface obtained from \( F \) by cutting along \( a \) and \( D \) as in Claim 4.1. Then \( D \cap S_i = \partial D = \lambda \), and we have;

**Claim 4.2.** No component of the surface obtained from \( S_i \) by doing a surgery along \( D \), is closed.

Proof. If \( \lambda \) is non-separating in \( S_i \), then Claim 4.2 is clear. Hence assume that \( \lambda \) separates \( S_i \) into \( S' \) and \( S'' \) such that \( S' \cup D \) is a closed surface. Since
$a$ is non-separating in $F$, there is a simple loop $m$ on $F$ such that $m \cap l = \phi$, and $m$ intersects $a$ in one point. Then $m$ intersects the closed surface $S' \cup D$ in one point, contradicting the fact that $M$ is a rational homology 3-sphere.

Let $a'$ be the component of $\text{Fr}_F N(\varphi(a); F)$ such that $a' \cap l = \phi$. Then we have;

**Claim 4.3.** There is a properly embedded arc $a'' \subset (F)$ such that $a'' \cap (a \cup a') = \phi$, $a'' \cap l = \phi$, and $a \cup a' \cup a''$ cuts off an annulus $A$ from $F$ such that $l$ is a core of $A$.

**Proof.** Let $F'$ be the component of the surface obtained from $F$ by cutting along $a \cup a'$ such that $l \subset F'$. Then $l$ is parallel to the component of $\partial F'$ which meets $a \cup a'$. By Claim 4.2, there is a component $l'$ of $\partial F$ such that $l' \subset F'$. Let $\beta$ be an arc in $F'$ such that $\beta \cap l' = \partial \beta \cap l'$ consists of one point, the other endpoint of $\beta$ is contained in $l$, and $\text{Int} \beta \cap l = \phi$. Then $\text{Fr}_{F'} N(\beta \cup l; F')$ consists of two components such that one is a simple loop parallel to $l$, and the other is an arc $a''$ properly embedded in $F'$. It is easy to see that $a''$ satisfies the conclusion.

**Claim 4.4.** Let $a', a'', A$ be as in Claim 4.3. Then there is a 3-ball $B^3$ in $M$ such that $B^3 \cap F = A$, and $A$ looks as in Figure 4.3 in $B^3$.

![Fig. 4.3](image)

**Proof.** Let $\overline{D}, B$ be as in the proof of Claim 4.1. Then $N(A \cup \overline{D}; M)$ is a 3-ball, and $A, \overline{D}$ looks as in Figure 4.4 in the 3-ball. Since $D$ is obtained from

![Fig. 4.4](image)
fibred links and unknotting operations

Let $D$, $a'$, $a''$, $B^3$ be as in Claim 4.4. By Figure 4.3, we see that the complementary sutured manifold $(N_F', \delta_f')$ for $F$ looks as in Figure 4.5 (i) in $B^3$. Let $\Box$ be a pp disk for $F$ such that $\partial \Box = a$. Then we may suppose that $\Box \subset B^3$, and $\Delta = \Box \cap N_F' \subset \partial (N_F', \delta_f')$ (Figure 4.5 (i)). Let $(\bar{N}_1, \delta_1)$ be the product sutured manifold obtained from $(N_F', \delta_f')$ by a product decomposition along $\Delta$, $\bar{D}^-$, $\bar{D}^+$ the disks properly embedded in $\text{cl}(E(L) - \bar{N}_1)$ as in Figure 4.5 (ii). Let $S_2$ be the surface obtained from $S_1$ by doing surgery along $D$. See Figure 4.6. Finally, let $(\bar{N}_1, \delta_1)$ ($(N_1, \delta_1)$ resp.) be the sutured manifold obtained from $S_1$ (the complementary sutured manifold for $S_1$ resp.).

Since $S_i$ is obtained from $F$ by cutting along $a_i$ and $(N_F', \delta_f')$ is properly isotopic in $E(\partial F)$ to the sutured manifold obtained from $F$ (note that $F$ is a fiber surface), we see that $(\bar{N}_1, \delta_1)$ is ambient isotopic to $(\bar{N}_1, \delta_1)$ in $M$. Hence, hereafter, we identify $(\bar{N}_1, \delta_1)$ to $(\bar{N}_1, \delta_1)$, and we identify $S_i$ to $S_i \times \{1/2\}$ ($\subset S_i \times I = \bar{N}_1$). Then $\bar{D}^+$, $\bar{D}^-$ are compressing disks for $R_+(\delta_1)$, $R_-(\delta_1)$ in $N_i(=\text{cl}(E(\partial S_1) - \bar{N}_1))$ respectively. Let $\bar{N}^*$ be the manifold obtained from $\bar{N}_1$ by doing surgery along $\bar{D}^+ \cup \bar{D}^-$. Then $(\bar{N}^*, \delta_1)$ is ambient isotopic to the sutured manifold obtained from $S_2$ (see Figure 4.6). This shows that $S_1$ is a
pre-fiber surface, and this completes the proof of Proposition 4.5.

As a consequence of Proposition 4.5, we have;

Proof of Theorem 2(1). Let \( D \) be the crossing disk for \( L \). Then, by Theorem 1, we see that \( S \) looks as in Figure 1.1. Then \( S_0 \) looks as in Figure 4.7 (i). Let \( S^* \) be the surface obtained from \( S_0 \) by adding a band \( b \) as in Figure 4.7 (ii). We note that \( S_0 \) is a plumbing of \( F \) and a fiber surface \( A^* \) in \( S^3 \) (Example 2.4). Hence \( S^* \) is a fiber surface. Moreover, by Figure 4.7 (ii), it is directly observed that the arc \( \alpha \) in Figure 4.7 (ii) satisfies the assumptions of Proposition 4.5 (cf. Figure 4.3). Hence, by Proposition 4.5, we see that \( S_0 \) is a pre-fiber surface.

\[ \text{Fig. 4.7} \]

Let \( S_0 \) be as in Theorem 2, \( a_0 \) as in Figure 1.2, and \( D^+, D^- \) a pair of canonical compressing disks for the pre-fiber surface \( S_0 \). Then the next lemma will be used in section 6 to prove Proposition 6.1.

**Lemma 4.7.** Let \( S_0, a_0, D^+, \text{and } D^- \) be as above. Then \( \partial D^+, \text{and } \partial D^- \) are ambient isotopic in \( S_0 \) to a loop intersecting \( a_0 \) in one point.

Proof. Without loss of generality we may suppose that the Hopf band \( A \) is attached to the + side of \( F \) (Figure 1.1). Then there is a compressing disk \( \tilde{D}^- \) for \( S_0 \) such that \( \partial \tilde{D}^- \) corresponds to the core curve of \( A \), and \( N(\partial \tilde{D}^-; \tilde{D}^-) \) lies in the + side of \( S_0 \). Then by the proof of Theorem 2 (1) (Figure 4.7), and the proof of Proposition 4.5 (Figures 4.5, 4.6), we see that \( \tilde{D}^- \) is a component of a pair of canonical comporesing disks for \( S_0 \). Hence, by Proposition 4.2, we see that \( \partial D^- \) is ambient isotopic to a loop intersecting \( a_0 \) in one point. Let \( a(\subset S) \) be the arc corresponding to \( a_0 \) (Figure 4.8). Then it is directly observed from Figure 4.8 that there is a pp disk \( \square \) such that \( \partial \square = a \), \( \partial_+ \square \cap \partial_- \square = \partial a \), and the components of \( N(\partial a; \partial_- \square) \) lie in pairwise different sides of \( a \). Hence there is a monodromy map \( \psi: S \to S \) such that \( \psi^{-1}(a) \cap a = \partial a \), and the components of \( N(\partial \psi^{-1}(a); \psi^{-1}(a)) \) lie in pairwise different sides of \( a \).
Let $\square'$ be a disk such that $\partial_- \square' = a$, $\partial_+ \square' = \psi(a)$. Roughly speaking, $\square' = \psi(\square)$. Then $\square'$ looks as in Figure 4.9 in the 3-ball $B = N(a; M)$.

Let $b_0$ be an unknotted band, and $\Delta_0$ a disk in a 3-ball $B_0$ as in Figure 4.10. Let $h: \partial B \to \partial B_0$ be a homeomorphism such that $h(S \cap \partial B) = h(b_0 \cap \partial B_0)$, and $h(\square' \cap \partial B) = h(\Delta_0 \cap \partial B_0)$. Then $(M - \text{Int } B) \cup_k B_0 = M$, and it is easy to see that $(S - \text{Int } B) \cup b_0 = S_0$ and $\bar{D}^+ = (\square' - \text{Int } B) \cup \Delta_0$ is a compressing disk for $S_0$ such that $N(\partial \bar{D}^+; \bar{D}^+)$ lies in the side of $S_0$.

By definition, it is easy to see that $\partial \bar{D}^+$ is ambient isotopic to a loop corresponding to $\psi(\text{the core curve of } A)$. Hence $\bar{D}^+$ is a component of a pair of canonical compressing disks for $S_0$. Hence, by Proposition 4.2, $\partial D^+$ is ambient isotopic to a loop intersecting $a_0$ in one point.
5. Propositions

In this section, we prove some technical propositions. For the statement of the results, we give some definitions.

Let $M$ be a compact 3-manifold, $\mu$ a subsurface of $\partial M$. For a connected surface $S$ properly embedded in $(M, \mu)$, let
\[ \chi(S) = \max\{0, -\chi(S)\}. \]
When $S$ is a union of connected surfaces $S_1, \ldots, S_n$, let
\[ \chi(S) = \sum_{i=1}^{n} \chi(S_i). \]
Then, we define the function $x: H_2(M, \mu) \to \mathbb{Z}$ by
\[ x(a) = \min \{\chi_-(S) | S \text{ is an embedded surface representing } a\}. \]

We say that $S$ is norm minimizing if $\chi_-(S) = x([S])$, where $[S]$ denotes the homology class in $H_2(M, \mu)$ represented by $S$.

Let $S'$ be a compact, connected surface with $\partial S' \neq \emptyset$, $\bar{l}_0$, $\bar{l}_1$ non separating simple loops in $S'$. Let $N = S' \times I$, $\delta = \partial S' \times I$, and $L_0 = \bar{l}_0 \times \{0\}$, $L_1 = \bar{l}_1 \times \{1\}$ ($\subset \partial N$). Let $\bar{N}_0$ be the manifold obtained from $N$ by attaching a 2-handle $\partial_0$ along $L_0$, $\bar{N}$ the manifold obtained from $N$ by attaching two 2-handles along $L_0 \cup L_1$. We may regard that $\bar{N}$ is obtained from $\bar{N}_0$ by attaching a 2-handle $\partial_1$ along $L_1$. $\bar{\xi}_0$, $\bar{\xi}_1$ denote the images of $\delta$ in $\bar{N}_0$, $\bar{N}$ respectively. Then $(N, \delta)$, $(\bar{N}_0, \bar{\xi}_0)$, $(\bar{N}, \bar{\xi}_1)$, have mutually coherent sutured manifold structures. The purpose of this section is to prove Propositions 5.1 and 5.2 below.

**Proposition 5.1.** Suppose that $R_{\pm}(\delta)$ are not norm minimizing in $H_2(\bar{N}, \bar{\xi})$. Then $\bar{l}_0$ is ambient isotopic to a loop disjoint from $\bar{l}_1$.

**Remark.** It is easily observed that if $\bar{l}_0$ and $\bar{l}_1$ are disjoint, and not parallel then $R_{\pm}(\delta)$ is not norm minimizing in $H_2(\bar{N}, \bar{\xi})$.

**Proposition 5.2.** Suppose that $(\bar{N}, \bar{\xi})$ is a product sutured manifold. Then $\bar{l}_0$ is ambient isotopic to a loop intersecting $\bar{l}_1$ in one point.

As a consequence of Proposition 5.1, we have;

**Corollary 5.4.** Let $S_0$ be a pre-fiber surface of type 1 in a rational homology 3-sphere $M$, $D^+$, $D^-$ a pair of canonical compressing disks for $S_0$, and $S_i$ the surface obtained from $S_0$ by doing a surgery along $D^+$. Suppose that $\chi(L) > \chi(S_0) + 2$, where $L = \partial S_0$. Then $\partial D^+$ is ambient isotopic in $S_0$ to a loop disjoint from $\partial D^-$, and $S_i$ is a pre-fiber surface, where $D^-$ is a component of a pair of canonical com-
pressing disks for $S$.

The proof of Proposition 5.1 is done by using the outermost fork argument of M. Scharlemann [11]. And the proof of Proposition 5.2 is done by using the Haken type argument of Casson-Gordon [1].

For the proof of the propositions, we prepare one lemma. Let $(E, \xi)$ be a connected sutured manifold. Suppose that there is a non separating compressing disk $C$ for $R_+(\xi)$ such that $(\tilde{E}, \tilde{\xi})$ is a product sutured manifold, where $\tilde{E}$ is obtained from $E$ by cutting along $C$, and $\tilde{\xi}$ the image of $\xi$ in $E$. Let $A$ be an incompressible product annulus in $(\tilde{E}, \tilde{\xi})$. Then;

Lemma 5.3. $A$ is ambient isotopic to an annulus disjoint from $D$ by an ambient isotopy of $E$ respecting $\xi$.

The proof of Lemma 5.3 is done by using the same arguments as that of Lemma 4.4. Hence we omit it.

Proof of Proposition 5.1. Let $F$ be a norm minimizing surface in $(N, \partial)$ such that $[F] = [R_+(\xi)] \in H_2(N, \partial)$. Since $[F] = [R_+(\xi)]$, by piping the boundary components of $F$, if necessary, we may suppose that $\partial F = s(\xi)$ (Figure 5.1).

![Fig. 5.1](image)

The next claim will be used in the proof of Corollary 5.4.

Claim 5.0. $\tilde{N}$ is irreducible.

Proof. Assume not. Let $F$ be a surface in $\tilde{N}$ corresponding to $S' \times \{1/2\}$, and $V_1, V_2$ the closure of the components of $\tilde{N} - F$. Then $(V_1, V_2)$ is a Heegaard splitting of $\tilde{N}$ in terms of $[1]$. Hence, by [1, Lemma 1.1], we see that there is an essential 2-sphere $S_i$ in $\tilde{N}$ such that $V_i \cap S_i$ consists of a disk. Then it is easy to see that $\tilde{N}$ is a connected sum of a lens space and a product sutured manifold. But this contradicts the fact that $R_+(\xi)$ are not norm minimizing.

Claim 5.1. $F \cap D_1 = \emptyset$.

Proof. Assume that $F \cap D_1 = \emptyset$. Then we can regard that $F \subset \tilde{N}_0$. Let
Let $D$ be the disk properly embedded in $\tilde{N}_0$ such that $D = (I \times I) \cup \text{(the core of $D_0$)}$. Then the manifold $N_0$ obtained by cutting $\tilde{N}_0$ along $D$ is homeomorphic to $R_-(\delta_0) \times I$, where $R_-(\delta_0) \times \{0\}$ corresponds to $R_-(\delta_0)$. Since $\tilde{N}_0$ is irreducible, by using standard innermost disk arguments, we may suppose that $F \cap D = \emptyset$. Hence we may regard that $F \subset N_0$. Then, by [15, Corollary 3.2], we see that $F$ is a parallel to $R_-(\delta_0)$. Hence $\chi(F) = \chi(S') + 2 = -\chi(R_-(\delta))$, a contradiction.

We may suppose that $F$ intersects $D$ in horizontal disks $E_1, \cdots, E_n$ in this order. Let $F_0 = \text{cl}(F - (E_1 \cup \cdots \cup E_n))$ and $A_i (i = 1, \cdots, n-1)$ the annulus in $\partial N_0$ bounded by $\partial E_i \cup \partial E_{i+1}$. Let $D$ be as in the proof of Claim 5.1. We suppose that $\#(\partial F_0 \cap \partial D)$ is minimal among all disks ambient isotopic to $D$ in $N_0$. Let $\alpha$ be the dual core of the 2-handle $D$. Then $\alpha$ is an arc in $\tilde{N}$ such that $\alpha \cap \partial \tilde{N} = \alpha \cap R_+(\delta_0) = \partial \alpha$. Since $F$ is norm minimizing, by [12, 3.5 Lemma b)], we may suppose that $F$ separates $\tilde{N}$ into two components $M_0, M_1$ such that $M_0 \supseteq R_-(\delta), M_1 \supseteq R_+(\delta)$. This shows that $\alpha$ intersects $F$ an even number of times and the signs of the intersections are alternately different on $\alpha$. Hence we have;

**Claim 5.2.** $n$ is an even number, and the orientations on $\partial E_1, \cdots, \partial E_n$ induced from $F_0$ are alternately different in $\partial \tilde{N}_0$.

**Claim 5.3.** If $n=2$, then $I_0$ is ambient isotopic to a loop disjoint from $I_1$.

Proof. Let $F_1 = (F - (E_1 \cup E_2)) \cup A_1$. Then $\chi(F_1) = \chi(F) - 2$. By the argument of the proof of Claim 5.1, we see that $F_1$ is parallel to $R_-(\delta_0)$. Hence, there is a product annulus $A$ in $\tilde{N}_0$ such that $A \cap R_+(\delta_0) = I_0$. Let $D \subset \tilde{N}_0$ be as in the proof of Claim 5.1. Then $D$ cuts $(\tilde{N}_0, \delta_0)$ into a product sutured manifold. Hence, by Lemmas 5.3, we may suppose that $D$ and $A$ are disjoint. We note that $A_0 = D \cap N$ is the product annulus $I_0 \times I$ in $(N, \delta)$. Hence $I_0 \times \{1\}$ and $I_1$ are disjoint, and we have the conclusion.

By Claim 5.3, hereafter, we suppose that $n \geq 4$. Let $D$ be as above. Then, by using standard cut and paste arguments, we may suppose that $D \cap F_0$ consists of arcs. We suppose that $\#(\partial D' \cap I_1)$ is minimal among all disks ambient isotopic to $D$ in $\tilde{N}_0$. Then;

**Claim 5.4.** No component of $D \cap F_0$ is an inessential arc in $F_0$.

Proof. Assume that a component $\beta$ of $D \cap F_0$ is an inessential arc in $F_0$. Then there is a disk $\Delta_0$ in $F_0$ such that $\text{Fr}_F \Delta_0 = \beta$. By doing $\partial$-compression on $D$ along $\Delta_0$ in $\tilde{N}_0$, we get two disks $D', D''$ whose boundaries lie in $R_+(\delta_0)$. Since $\partial D$ is non separating in $R_+(\delta_0)$, at least one of the disks, say $D'$, is non separating in $\tilde{N}_0$. By Lemma 4.4, we see that $D'$ is ambient isotopic to $D$. On
the other hand, by moving \( D' \) by an ambient isotopy, we have \( \#(\partial D' \cap l_i) < \#(\partial D \cap l_i) \), a contradiction.

We get a planar tree \( T \) by corresponding each component of \( D - F_0 \) to a vertex, and each component of \( D \cap F_0 \) to an edge. We regard that \( T \) is embedded in \( D \) and each edge of \( T \) intersects \( D \cap F_0 \) in one point which is contained in the corresponding component of \( D \cap F_0 \). See Figure 5.2. Let \( \gamma \) be a component of \( D \cap F_0 \), and \( e_\gamma \) the edge of \( T \) corresponding to \( \gamma \). Then \( \gamma \cap e_\gamma \) consists of a point, which separates \( \gamma \) into two arcs \( \gamma_1 \) and \( \gamma_2 \). One endpoint of \( \gamma_1 \) lies in \( \bigcup_{j \neq 1} \partial E_j \). Labell the corresponding side of \( e_\gamma \) by \( k \) if the endpoint lies in \( \partial E_k \). Then we can label the each side of the edges of \( T \) by \( \{1, \ldots, n\} \).

In general, for a tree \( \mathcal{L} \), an outermost vertex is a vertex with valency 1. An edge adjacent to an outermost vertex is called an outermost edge. A fork is a vertex with valency \( \geq 3 \). Let \( \mathcal{F} \) be the collection of the forks of \( \mathcal{L} \). Let \( \mathcal{L}' \) be the tree obtained by removing all components of \( \mathcal{L} - \mathcal{F} \) which contains an outermost vertex. An outermost vertex for \( \mathcal{L}' \) is an outermost fork of \( \mathcal{L} \). If \( \mathcal{F} = \emptyset \), then \( \mathcal{L} \) does not contain an outermost fork. If \( v \) is an outermost fork, then the components of \( \mathcal{L} - v \) which contain no forks are called outermost lines of \( v \). If \( v_0 \) (\( e_0 \) resp.) is a vertex (an edge resp.) which is contained in an outermost line of \( v \), then we say that \( v_0 \) (\( e_0 \) resp.) is dominated by \( v \). Then we have;

**Claim 5.5.** *If there is an outermost edge of \( T \) which is labelled by \( i \) and \( i+1 \) for some \( i \in \{1, \ldots, n-1\} \), then there is a norm minimizing surface \( F' \) in \( (\tilde{N}, \tilde{S}) \) such that \( [F'] = [F] \) and, \( \#(F' \cap \partial_1) = \#(F \cap \partial_1) - 2 \).*

**Proof.** Let \( \Delta \) be the closure of the component of \( D - F_0 \) corresponding to the outermost vertex adjacent to the outermost edge. Let \( F_i := (F - (E_i \cup E_{i+1})) \cup A_i \). By Claim 5.2, we see that \( F_i \) is orientable. Then \( [F_i] = [F] \in H_2(\tilde{N}, \tilde{S}) \), and \( \chi(F_i) = \chi(F) - 2 \). Since the core arc of \( A_i \) intersects \( \partial \Delta \) in one point, \( \partial \Delta \) is an essential loop in \( F_i \). Hence \( \Delta \) is a compressing disk. Let \( F' \) be the surface obtained from \( F_i \) by doing a surgery along \( \Delta \). By moving \( F' \) by a tiny
isotopy, we see that $F'$ satisfies the conclusion.

**Claim 5.6.** Suppose that there is a vertex $v$ of $T$ such that $v$ is not an outermost vertex, and the adjacent edges of $v$ are labelled alternately by $i$ and $i+1$ (Figure 5.3). Then there is a norm minimizing surface $F'$ in $(\bar{N}, \partial)$ such that $[F'] = [F]$, and $\#(F' \cap D) = \#(F \cap D) - 2$.

![Fig. 5.3](image-url)

**Proof.** Let $\Delta$ be the closure of the component of $D - F_0$ corresponding to $v$, and $F_1 = (F - (E_1 \cup E_{i+1})) \cup A_i$. Then $F_1$ is orientable (see the proof of Claim 5.5), $[F_1] = [F]$, and $\chi(F_1) = \chi(F) - 2$. $\Delta \cap F_1 = \partial \Delta$, and the absolute value of the algebraic intersection number of $\partial \Delta$ with the core of $A_i$ is the number of the edges adjacent to $v$. Hence $\Delta$ is a compressing disk for $F_1$. Let $F'$ be the surface obtained from $F_1$ by doing surgery along $\Delta$. By moving $F'$ by a tiny isotopy, we see that $F'$ satisfies the conclusion.

**Claim 5.7.** If there is an outermost line with the pattern as in Figure 5.4, then there is a norm minimizing surface $F'$ in $(\bar{N}, \partial)$ such that $[F'] = [F]$, and $\#(F' \cap D) = \#(F \cap D) - 2$.

![Fig. 5.4](image-url)

**Proof.** Suppose that there is a pattern of Figure 5.4 (i). The other case is essentially the same. Let $\Delta$ be the closure of the component of $D - F_0$ corresponding to $v$ (Figure 5.4), and $F_1 = (F - (E_1 \cup E_2)) \cup A_i$. Then $\Delta \cap F_1 = \partial \Delta$. Hence if $\partial \Delta$ is not contractible in $F_1$, then, by compressing $F_1$ along $\Delta$, we have a surface $F'$ satisfying the conclusions. Hence, in the rest of the proof, we suppose that $\partial \Delta$ is contractible in $F_1$. Then $\Delta \cap cl(F - (E_1 \cup E_2))$ consists of two
inessential arcs $\beta_1, \beta_2$ in $\partial(F-(E_1 \cup E_2))$ such that $\partial \beta_i \subset \partial E_i$ ($i=1, 2$). Hence there are two planar surfaces $P_1, P_2$ in $F_0$ such that $\text{Fr}_{P_0}P_1=\beta_i$ (Figure 5.5). By Claim 5.4, we see that $P_i$ is not a disk.

**Subclaim 1.** $T$ contains a fork.

Proof. Assume that $T$ does not contain a fork. Then, by tracing the edges of $T$ from $v_1$ (Figure 5.4), we see that there are $n$ components $\beta_1, \beta_2, \beta_3, \ldots, \beta_n$ of $D \cap F_0$ such that $\partial \beta_i \subset \partial E_i$ ($i=1, \ldots, n$), where $\beta_1, \beta_2$ are as above. Then it is easy to see that some $\beta_j$ contained in $P_1$ is an inessential arc in $F_0$, contradicting Claim 5.4.

Let $v_0$ be the outermost fork which dominates $v_1$, $v_2$ an outermost vertex dominated by $v_0$, and located next to $v_1$. By using the argument of the proof of Subclaim 1, we have;

**Subclaim 2.** The outermost line between $v_0$ and $v_1$ contains at most $n-1$ edges.

**Subclaim 3.** Either the conclusions of Claim 5.7 holds or the outermost edge adjacent to $v_2$ is labelled by $1$ and $n$.

Proof. Suppose that the outermost edge is not labelled by $1$ and $n$. Then, by Claim 5.5, we see that either the conclusions of Claim 5.7 hold or the edge is labelled by two $1$'s or two $n$'s. Suppose that the second case occurs. If the outermost line between $v_0$ and $v_2$ contains more than $n-1$ edges, then we have a contradiction as in the proof of Subclaim 1. Hence the outermost line contains at most $n-1$ edges, and this fact together with Subclaim 2 show that there are exactly $n$ edges between $v_1$ and $v_2$ in $T$, and the outermost edge adjacent to $v_2$ is labelled by two $n$'s (Figure 5.6). Then, by tracing the edges in $T$ from $v_1$ to $v_2$, we again have a contradiction as in the proof of Subclaim 1.

Suppose that the second conclusion of Subclaim 3 holds. If the outermost line between $v_0$ and $v_2$ contains more than $n/2$ edges, then we have a pattern of Figure 5.3 in the outermost line, so that we have the conclusion of Claim 5.7.
by Claim 5.6. Assume that the outermost line contains \(j(\leq n/2)\) edges. By Subclaim 2, we see that there are exactly \(n\) edges between \(v_1\) and \(v_2\) in \(T\) (Figure 5.7).

Let \(\beta_1, \beta_2, \beta_3, \ldots, \beta_n\) be the components of \(D\cap F_0\) corresponding to the edges between \(v_1\) and \(v_2\) in \(T\). Then, for \(i \leq n-j\), \(\partial \beta_i \subset \partial E_i\). Then fix some \(\beta_k(k \leq n-j)\) such that \(\beta_k d P\) and \(\beta_k\) is innermost, i.e. \(\beta_k\) cuts off a planar surface \(P_k\) from \(F_0\) such that no component of \(\partial E_1 \cup \partial E_2 \cup \cdots \cup \partial E_{n-j}\) is contained in \(P_k\) (Figure 5.8).

By Claim 5.4, we see that some \(\partial E_m(m \geq n-j+1)\) is contained in \(\partial P_k\). Since \(j \leq n/2\) and \(\beta_k\) is innermost, we see that \(\beta_m\) joins \(\partial E_k\) and \(\partial E_m\). This shows that \(m = n+1-k\), so that \(P_k\) is an annulus. Then, by Claim 5.4, we see that every
component of $D \cap F_0$ which meets $\partial E_m$ joins $\partial E_m$ and $\partial E_k$. But this contradicts the fact that $\#(\partial D \cap \partial E_m) = \#(\partial D \cap \partial E_k)$, and this completes the proof of Claim 5.7.

Completion of the proof of Proposition 5.1. We suppose that $\#(F \cap D_i)$ is minimal among all norm minimizing surfaces representing $[R_+ (\delta_i)]$. If $\#(F \cap D_i) = 2$, then, by Claim 5.3, we have the conclusion. Assume that $n > 2$. By Claim 5.5, we see that each outermost edge is labelled by either two $1$'s, two $n$'s or $1$ and $n$.

Suppose that $T$ does not have a fork. If an outermost edge is labelled by two $1$'s or two $n$'s, then we have a contradiction by Claim 5.7. If an outermost edge is labelled by $1$ and $n$, then we have a pattern of Figure 5.3 in $T$, so that we have a contradiction by Claim 5.6. Hence $T$ has a fork.

Let $v$ be an outermost fork for $T$. If all the outermost edges dominated by $v$ are labelled by $1$ and $n$, then by Claim 5.6, we see that each outermost line contains at most $n/2$ edges. Hence the adjacent edges of $v$ are labelled alternately by $n/2$ and $n/2 + 1$, contradicting Claim 5.6. Hence we may suppose that some outermost edge dominated by $v$ is labelled by two $1$'s. Then, by Claim 5.7, we see that $v$ is adjacent to the edge. Let $v_1$ be an outermost vertex which is dominated by $v$ and next to the outermost edge. By Claim 5.5, we see that there are at least $n - 1$ edges in the outermost line between $v$ and $v_1$. Then, by Claims 5.5 and 5.7, we see that the edge adjacent to $v_1$ is labelled by $1$ and $n$. Hence we have a pattern of Figure 5.3 in the outermost line, contradicting Claim 5.6, and this completes the proof.

Proof of Corollary 5.4. Let $(N_i, \delta_i)((N_0, \delta_0))$ resp. be the sutured manifold obtained from $S_i$ (the complementary sutured manifold for $S_i$ resp.) $i=0, 1$. Then we may suppose that $D_i^+ = D_i^+ \cap N_i^*$ are disks properly embedded in $N_i^*$, and $(cl(N_i^* - N(D_i^+ \cup D_i^-); N_i^*, \delta_i^*))$ is properly isotopic to $(N_i, \delta_i)$ in $E(L)$. Hence, hereafter, we identify $(N_i, \delta_i)$ to $(cl(N_i^* - N(D_i^+ \cup D_i^-); N_i^*, \delta_i^*))$. Then $(N_i, \delta_i)$ is obtained from $(N_0, \delta_0)$ by attaching two $2$-handles $N(D_i^+; N_0^*), N(D_i^-; N_0^*)$ along the simple loops $\partial D^+ \times \{1\}, \partial D^- \times \{0\}$ in $(N_i, \delta_i)$ ($\cong (S_0 \times I, \partial S_0 \times I)$).

Case 1. $\chi_-(S_i) > 0$.

In this case, $S_i$ is not norm minimizing. Hence, by Claim 5.0, and [5, Lemma 0.4] or [12, section 3], we see that $R_+(\delta_i)$ is not norm minimizing in $H_0(N_i^*, \delta_i^*)$. Then, by Proposition 5.1, we may suppose that $\partial D^+$ and $\partial D^-$ are disjoint. Moreover since $M$ is a rational homology $3$-sphere, they are not parallel. Let $D_i^* = D_i^+ \cup A_i^-$, where $A_i^+$, $A_i^-$ are the product annuli $\partial D^+ \times I$, $\partial D^- \times I$ in $(N_i, \delta_i)$. Then $D_i^+$, $D_i^-$ are mutually disjoint disks properly embedded in $N_i^*$ such that $D_i^+ \cup D_i^-$ cuts $(N_i^*, \delta_i^*)$ into a product sutured manifold. Hence $S_i$ is a pre-fiber surface and clearly $D^-$ corresponds to $D_i^-$. 

Case 2. $\chi_-(S_i)=0$.

Since $\chi_-(S_i)=0$, $\chi(S_0)$ is either 0, 1 or 2. Since $S_0$ is a pre-fiber surface of type 1, $S_0$ contains a non-separating loop. Hence it is easy to see that $S_0$ is either a torus with one hole, or a torus with two holes. If $S_0$ is a torus with one hole, then $S_1$ is a disk so that $\chi(L)=1=\chi(S_0)+2$, a contradiction. Suppose that $S_0$ is a torus with two holes, so that $S_1$ is an annulus. Then

Claim. There are mutually disjoint disks $E_1, E_2$ in $M$ such that $(E_1 \cup E_2) \cap S_1=\partial(E_1 \cup E_2)=L$.

Proof. Since $\chi(L) > \chi(S_0) + 2$, we see that there is a Seifert surface $\varepsilon$ for $L$ such that $\chi(\varepsilon)=2$, so that $\varepsilon$ is a union of two disks. Then, by using standard innermost disk, outermost arc arguments, we see that either $\varepsilon$ satisfies the conclusion of Claim, or $\varepsilon$ intersects $S_1$ in essential loops in $S_1$, so that $S_1$ is compressible. Suppose that the second conclusion holds. Then by doing a surgery along a compressing disk for $S_1$, and moving the resulting surface by a tiny isotopy, we get a pair of disks satisfying the conclusion.

By the above claim, we see that $E_1, E_2$ are embedded in $(N_1, \delta_1)$, so that, by regarding $E_1 \cup E_2$ as $F$, the proof of Proposition 5.1 shows that $\partial D^+$ is ambient isotopic in $S_0$ to a loop disjoint from $\partial D^-$. Hence, by the argument of Case 1, we see that the conclusion holds.

Proof of Proposition 5.2. Let $\{D_1, \ldots, D_n\}$ be a system of mutually disjoint product disks in $(\tilde{N}, \delta)$ such that $\cup D_i$ decomposes $(\tilde{N}, \delta)$ to the product sutured manifold $(D^2 \times I, \partial D^2 \times I)$. Let $\tilde{S}$ be the surface corresponding to $S' \times \{1/2\}$ in $\tilde{N}$. $\tilde{S}$ is a Heegaard surface of $(\tilde{N}, \delta)$ [1]. Then, by the arguments of the proof of [1, Lemma 1.1], and the distinguished circle argument of Ochiai [8, Lemma], we may suppose that each $D_i$ intersects $\tilde{S}$ in an arc. We note that the arguments in [1, Lemma 1.1] and [8] work for product disks. Hence the image of $\tilde{S}$ in $D^2 \times I$ is a torus with one hole $T$ with $\partial T=\partial D^2 \times \{1/2\}$. Moreover, by using the core disks of the 2-handles, we see that $T$ has two compress-

Fig. 5.9
6. Monodromy maps

Let $L, L', S, F, A, S_0, S_1,$ and $M$ be as in Theorem 2, and $b$ as in Figure 1.1. Let $\varphi : F \rightarrow F$ be a monodromy map, and $a (\subset F)$ a component of $\operatorname{Fr}_b N(b; F)$ (Figure 1.1). The purpose of this section is to prove the following proposition.

**Proposition 6.1.** If $\chi(L') > \chi(L) + 2$, then, by deforming $\varphi$ by a rel $\partial$ ambient isotopy, if necessary, we may suppose that $a_0 = a_0 \varphi$ and the components of $N(a_0 \varphi)$ lie in one side of $a$ (Figure 4.2).

**Remark.** Proposition 6.1 together with Proposition 4.6 shows that if $\chi(L') > \chi(L) + 2$, then $a$ is non separating in $F$.

Then we give a proof of Theorem 2 (2). As a consequence of Proposition 6.1, we have;

**Corollary 6.2.** Let $S$ be as in Theorem 2 (2), and $\varphi : S \rightarrow S$ a monodromy map of $S$. Then there is a non separating simple loop $l$ in $S$ such that $\varphi(l)$ is ambient isotopic in $S$ to a loop disjoint from $l$.

Proof of Proposition 6.1. Let $(N, \delta), (N_0, \delta_0), (N_1, \delta_1)$ be the sutured manifolds obtained from $S, S_0, S_1$ respectively, and $(N_0^c, \delta_0^c), (N_1^c, \delta_1^c)$ the complementary sutured manifolds for $S, S_0, S_1$ respectively. By Theorem 2 (1) (section 4), $S_0$ is a pre-fiber surface. Let $D_0^0, D_0^1$ be a pair of canonical compressing disks for $S_0$. Then we may suppose that $D_0^0$ looks as in Figure 6.1.

![Fig. 6.1](image)

By Lemma 4.7, we may suppose that $\partial D_0^0$ intersects $a_0$ of Figure 1.1 in one point. By Corollary 5.4, we may suppose that $\partial D_0^0$ and $\partial D_0^1$ are pairwise disjoint. Hence $\partial D_0^0$ looks as in Figure 6.2.

**Claim 6.1.** There is a disk $D$ in $M$ such that $D \cap S = D \cap \operatorname{Int} S_1 = \partial D$, and $D$ intersects the band $b$ in an essential arc $a_1$. 


Proof. We identify $S_1$ to the surface obtained from $S_0$ by doing a surgery along $D_5$. Let $D=D_5$. By Figure 6.3, it is directly observed that $D$ satisfies the conclusions.

Let $\square$ be a pp disk for $F$ such that $\partial_- \square=a$, $\partial_+ \square=\varphi(a)$. Suppose that $\varphi(a)$ does not run through $b$. Then it is easy to see that we have the conclusion of Proposition 6.1. Hence suppose that $\varphi(a)$ runs through $b$. Then, by deforming $\square$ by an isotopy as a pp disk, we may suppose.

(6.1) $\partial_+ \square \cap b$ consists of arcs joining the components of $\text{Fr}_F b$, and $\# \{(\partial_+ \square \cap b) \cap a_i\}$ is minimal among the rel $\partial$ isotopy class in $b$, and

(6.2) If $\alpha$ is a component of $\partial_+ \square \cap (F-b)$ such that $\partial \alpha \subset \text{Fr}_F b$, then $\alpha$ is not rel $\partial$ isotopic in $\text{cl}(F-b)$ to a subarc of $\text{Fr}_F b$.

Since $\partial_- \square \cap D=a \cap D=\phi$, we see that each component of $\square \cap D$ is either an arc whose endpoints lie in $\partial_+ \square$, or a simple loop. Then;

Claim 6.2. If necessary, by applying cut and paste on $D$, we may suppose that $\square \cap D$ consists of arcs.

Proof. Let $(N^+_F, \delta^+_F)$ be the complementary sutured manifold for $F$. Then we may suppose that $\square \cap N^+_F$ is a product disk. Suppose that a component $l$ of $\square \cap D$ is a simple loop. We may suppose that $l \subset (\square \cap N^+_F)$. Then $l$ bounds a disk in $\square$. Hence, we can apply a cut and paste on $D$, by using the disk, to remove $l$. Do the same until all the simple loops are removed.

Let $p: F \times I \rightarrow E(\partial F)$ be a natural map (section 2), and $\square$ the product disk
in \((F \times I, \partial F \times I)\) such that \(p(\mathcal{D}) = \emptyset\). Then, by Claim 6.2, we see that \(p^{-1}(D)\) consists of arcs whose endpoints lie in \(\mathcal{D} \cap (F \times \{1\})\). Then let \(\Delta\) be the closure of an outermost component of \(\mathcal{D} - p^{-1}(D)\) which does not intersect \(\mathcal{D} \cap (F \times \{0\})\) (Figure 6.4). Then \(\beta = p(\Delta) \cap D(= p(\partial \mathcal{D}))\) is an arc with \(\beta \cap a_1 = \partial \beta\). Let \(\alpha\) be the subarc of \(a_1\) such that \(\partial \alpha = \partial \beta\). Then \(\alpha \cup \beta\) bounds a disk \(D^*\) in \(D\). If \(D^*\) does not contain \(\partial a_1\) (Figure 6.5 (i)), then, by (6.2), \(p(\Delta) \cup D^*\) is a compressing disk for \(F\), a contradiction. Hence \(\partial a_1 \subset D^*\) (Figure 6.5 (ii)). Then \(D = D^* \cup p(\Delta)\) is a pp disk for \(F\) such that \(\partial_D D^* = a_1\). Since \(\partial_D D^* = (a_1 - \alpha) \cup (p(\Delta) \cap F)\), by moving \(D^*\) by a tiny isotopy as a pp disk, we get a pp disk \(D^*\) such that \(\partial_D D^*\) is properly isotopic to \(a\) in \(F\) (in fact, it moves through \(b\)), and \(\partial_D D^*\) does not go through \(b\). Since \(\partial_D D^*\) is ambient isotopic to \(a_1\), we have the conclusion of Proposition 6.1.

![Fig. 6.4](image)

![Fig. 6.5](image)

**Proof of Theorem 2 (2).** By the remark of Proposition 6.1, we see that \(S_0\) is a type 1 pre-fiber surface. Hence, by Corollary 5.4, we see that \(S_1\) is a pre-fiber surface.

**Proof of Corollary 6.2.** Let \(l\) be a non separating simple loop in \(S\) corresponding to \(\partial D^*\) of Figure 6.1. By [3], we see that \(\psi = \psi_2 \circ \psi_1\), where \(\psi_1: S \to S\) is an orientation preserving homeomorphism such that \(\psi_1|_A\) is a Dehn twist along \(l\), \(\psi_1|_{d(s-A)} = \text{id}\), \(\psi_2|_F = \varphi\), and \(\psi_2|_{d(s-F)} = \text{id}\). Then, by Proposition 6.1, it is easy to see that \(\psi(l)\) is ambient isotopic to a loop disjoint from \(l\).
7. Proof of Theorem 3

In this section, we prove Theorem 3 stated in section 1.

Firstly, we prepare some notations. Let \( S \) be a surface in a 3-manifold \( M \), and \( a \) (\( \subset M \)) an arc such that \( a \cap S = \partial a \) (\( \subset \text{Int} \ S \)), and the components of \( N(a; \partial a) \) lie in one side of \( S \). Let \( A \) be the component of \( \partial N(a; M) - S \) which is an open annulus. Then \( S_a = (S - \text{Int} N(a; M)) \cup A \) is a surface, and has the orientation coherent to \( S \). See Figure 7.1. We say that \( S_a \) is obtained from \( S \) by adding a pipe along \( a \).

\[ \text{Fig. 7.1} \]

Let \( S, a, S_a \) be as above, and \( (N^r, \delta^r) \) the complementary sutured manifold for \( S \). Then we may suppose that \( a' = a \cap N^r \) is an arc such that \( \partial a' \subset R_+(\delta^r) \) or \( \partial a' \subset R_-(\delta^r) \). We suppose that \( \partial a' \subset R_-(\delta^r) \). The other case is essentially the same. Let \( (N^r, \delta^r) \) be the complementary sutured manifold for \( S_a \). Then, by Figure 7.2, we immediately have;

**Lemma 7.1.** \( (N^r, \delta^r) \) is homeomorphic to \( (N', \delta') \), where \( N' \) is obtained from \( cl(N^r - N(a'; N^r)) \) by adding a 1-handle along disks in \( R_+(\delta) \), and \( \delta' \) is the image of \( \delta^r \) in \( N' \).

\[ \text{Fig. 7.2} \]

Then we give the definition of the surface \( \Sigma_a \) in \( S^3 \) (see section 1). Let \( D \) be a disk in \( S^3 \). Fix a \( D^2 \)-bundle structure with \( D \) a fiber on \( E(\partial D) \). Then we define a sequence of arcs \( a_1, a_2, \ldots \) as follows.

Let \( a_1 \) be an arc in \( S^3 \) such that \( N(\partial a_1; a_1) \) lies in the - side of \( D \), \( a_1 \cap D = \)}
\[ \partial a_i (\subset \text{Int } D) \], and there is a disk \( \Delta \) such that \( a_i \subset \partial \Delta, \Delta \cap \text{Int } D = \partial \Delta - \text{Int } a_i = \beta \)

an arc in \( D \). Clearly \( a_i \) is unique up to ambient isotopy of \( S^3 \) respecting \( D \).

\[ \begin{align*}
\text{Fig. 7.3}
\end{align*} \]

Suppose that \( a_k \) has defined. Then let \( a_{k+1} \) be an arc such that \( N(\partial a_{k+1}; a_{k+1}) \) lies in the \(-\) side of \( D \), \( a_{k+1} \cap \text{Int } \Delta = \phi \), \( a_k \subset \text{Int } a_{k+1} \) (so that \( cl(a_{k+1}-a_k) \) consists of two arcs), \( cl(a_{k+1}-a_k) \cap D = \partial(a_{k+1}-a_k) \), and each component of \( a_{k+1}-a_k \) is transverse to the fibration on \( E(\partial D) \). By the induction on \( i \), it is not hard to see that \( a_i \) is unique up to the ambient isotopy of \( S^3 \) respecting \( D \).

Let \( \Sigma_i \) be the surface obtained from \( D \) by adding a pipe along \( a_i \). Then \( a_2 \cap \Sigma_i = \partial a_2 \) and we let \( \Sigma_2 \) be the surface obtained from \( \Sigma_i \) by adding a pipe along \( a_2 \), and so on. We note that each \( \Sigma_n \) has two compressing disks \( D^n, D^- \) corresponding to a meridian of \( a_n \), and \( \Delta \) respectively. Then \( \partial D^n, D^- \) are \( I^\pm \) of Figure 1.3. Then we have;

**Proposition 7.2.** \( \Sigma_n \) is a pre-fiber surface of type 1, and \( D^n, D^- \) is a pair of canonical compressing disks for \( \Sigma_n \).

Proof. The proof is done by the induction on \( n \). By the observation in Example 4.1, we see that \( \Sigma_i \) is a pre-fiber surface of type 1, and \( D^i, D^- \) is a pair of canonical compressing disks for \( \Sigma_i \).

Suppose, by induction, that \( \Sigma_n \) satisfies the conclusion of Proposition 7.2. Let \( (N_n, \delta_2)(N_n', \delta_2') \) be the sutured manifold obtained from \( \Sigma_n \) (the complementary sutured manifold for \( \Sigma_n \) resp.) Let \( \tilde{D}^\pm = D^\pm \cap N_n, \tilde{D}^\pm = N(\tilde{D}^\pm ; N_n') \), and \( N_{n-1} = cl(N_n' - (\partial N_n' \cup D^\pm)) \). Then \( (N_{n-1}, \delta_2') \) is ambient isotopic to the product sutured manifold obtained from \( \Sigma_{n-1} \). Hence \( N_{n-1} \) has a \( \Sigma_{n-1} \)-bundle
structure such that each fiber corresponds to $\Sigma_{n-1} \times \{x\} (x \in I)$. We regard $\mathcal{D}^\pm_n$ are 1-handles attached to $N_n$. By definition we may suppose that $\alpha = a_{n+1} \cap N_n^+$ is an arc such that $\alpha \cap \mathcal{D}^+_n = \phi$, and $\alpha \cap \mathcal{D}^-_n$ is a vertical arc in $\mathcal{D}_n^\pm (\cong D^2 \times I)$. Hence $\alpha \cap N_{n-1}$ consists of two arcs $\alpha_1, \alpha_2$.

Claim. By moving $a_{n+1}$ by an ambient isotopy of $S^3$ respecting $\Sigma_n$, if necessary, we may suppose that $\alpha_1, \alpha_2$ are transverse to the fibration on $N_{n-1}$ (Figure 7.5).

Proof. By Figure 7.6, we may suppose that each component of $a_{n+1} - a_n$ is close to a meridian loop in $\partial E(\partial \Sigma_n)$. Since the fibration on $\partial E(\partial \Sigma_n)$ induced from the fibration on $N_{n-1}$ is a fibration by longitudes, we see that the components of $a_{n+1} - a_n$ are transverse to the fibration. Hence $\alpha_1, \alpha_2$ are transverse to the fibration on $N_{n-1}$.

The complementray sutured manifold $(N^+_n, \delta^+_n)$ for $\Sigma_{n+1}$ is obtained from $(N^+_n, \delta^+_n)$, and $\alpha$ as in Lemma 7.1. Hence, by Figure 7.5, we easily see that $\Sigma_{n+1}$ is a pre-fiber surface, and $D^+_{n+1}, D^-_{n+1}$ is a pair of canonical compressing disks.

This completes the proof of Proposition 7.2.

Proof of Theorem 3. The proof is done by the induction on $n = (\chi(L) - \chi(S_i))/2$. Let $D^+, D^-$ be a pair of canonical compressing disks for $S_1$ and $S_2$ the surface obtained from $S_i$ by doing a surgery along $D^+, (N_i, \delta_i)((N_i', \delta_i)$ resp.) the sutured manifold obtained from $S_i$ (the complementary sutured manifold for $S_i$, resp.) $(i=1, 2)$.

Claim 7.1. If $\chi(S_i) = \chi(L) - 2$, then $S_1$ is a connected sum of $S_2$ and $\Sigma_i$. 
Proof. By Proposition 5.2, we may suppose that \( \partial D^+ \) intersects \( \partial D^- \) in one point. Let \( \alpha \) be an arc in \( S_1 \) such that one endpoint of \( \alpha \) lies in \( \partial S_1 \), the other endpoint is \( \partial D^+ \cap \partial D^- \), and \( \text{Int} \alpha \cap (\partial D^+ \cup \partial D^-) = \emptyset \). Then the regular neighborhood \( B \) of \( \alpha \cup D^+ \cup D^- \) in \( M \) is a 3-ball such that \( B \cap S_1 \) is a regular neighborhood of \( \alpha \cup \partial D^+ \cup \partial D^- \) in \( S_1 \). \( \partial B \) desums \( S_1 \) into \( S_2 \) and \( \Sigma_1 \).

Claim 7.2. If \( \chi(S_1) < \chi(L) - 2 \), then \( S_2 \) is a pre-fiber surface of type 1.

Proof. By Corollary 5.4, we see that \( S_2 \) is a pre-fiber surface. Assume that \( S_2 \) is of type 2. Then, by Corollary 5.4, we may suppose that \( D^+ \cap D^- = \emptyset \), and \( \partial D^- \) is a separating loop in \( S_2 \), i.e. \( \partial D^+ \cup \partial D^- \) separates \( S_1 \). Let \( S_3 \) be the surface obtained from \( S_1 \) by doing surgery along \( D^+ \cup D^- \).

Subclaim. No component of \( S_3 \) is closed.

Proof. Assume that a component \( \bar{S} \) of \( S_3 \) is closed. Let \( l \subset S_1 \) be a simple loop which intersects \( \partial D^+ \) in one point. Then, by pushing \( l \) to the \(-\) side of \( S_1 \), we get a simple loop intersecting \( \bar{S} \) in one point, contradicting the fact that \( M \) is a rational homology 3-sphere.

By Subclaim, we see that \( S_3 \) is a disconnected Seifert surface for \( L \). Then, by doing compressions on \( S_3 \) as much as possible, we get a disconnected, incompressible Seifert surface \( S^* \) for \( L \). By Lemma 2.2, we see that \( S^* \) is a fiber surface, contradicting Lemma 2.1.

Completion of the proof. Claim 7.1 shows that if \( n = 1 \), then the conclusion holds. Suppose that \( n > 1 \). By Claim 7.2 and the induction, we see that \( S_2 \) is a connected sum of a fiber surface and \( \Sigma_{n-1} \) (Figure 7.7). Let \( S_3 \) be as in the proof of Subclaim.

\[
(N_1, \delta_1) \text{ is homeomorphic to } (D^2 \times S^1 \sqcup (S_2 \times I) \sqcup D^2 \times S^1, \partial S_2 \times I). \] Let \( D^+_1 = D^2 \cap N_1 \), and \( D^-_1 = N(D^+_1 ; N_1) \). Then, we may identify \( (\text{cl}(N_1 - (D^+_1 \cup \partial D_1)), \delta_1) \) to \( (N_2, \delta_2) \), where \( S_2 \times \{1/2\} \) corresponds to \( S_2 \). We regard \( D^+_1, D^-_1 \).
are 2-handles attached to \((N_1, \delta_1)\). Then \((N_1 \cup \Delta^+_1 \cup \Delta^+_7, \delta_1)\) is properly isotopic to \((N_2, \delta_2^+; N_2^1)\). Hence we identify \((N_2^1, \delta_2^+)\) to \((N_1^1, \Delta^+_1, \Delta^+_7, \delta_1)\). Let \(A^+, A^-\) be pairwise disjoint product annuli in \((N_1^1, \delta_1)\), such that \(A^+ \cap R^-_1(\delta_1) = \partial D^+_1\), \(A^- \cap R^+_1(\delta_1) = \partial D^+_1\). Let \(D^+_2 = A^+ \cup D^+_1\), \(D^+_3 = A^- \cup D^+_1\), and \(D^+_2 = N(D^+_1; \delta_1)\) (Figure 7.8). Then \(D^+_2, D^+_3\) represents a pair of canonical compressing disks for \(S_2\), and \((d(N_2^1 = (D^+_2 \cup D^+_3)), \delta_1)\) is ambient isotopic the sutured manifold \((\Delta^+_3, \delta_3)\) obtained from \(\Delta^+_3\). Hence we may regard that \(N_2^1\) is obtained from \(N_1^1\) by attaching two 1-handles \(\Delta^+_2, \Delta^+_3\). Then fix a 2-bundle structure on \(D^+_2 \times [0, 1]\), and \(\delta^+_3\)-bundle structure on \(N_3^1 = S^3 \times [0, 1]\). Let \(\alpha\) be an arc in \(N_3^1\) such that \(\alpha \cap \partial N_3^1 = \alpha \cap R^+_3(\delta_3) = \partial \alpha\), \(\alpha \cap D^+_2 = \phi\), \(\alpha \cap D^+_3\) is an arc transverse to the fibers, and \(\alpha \cap N_3^1\) consists of two arcs transverse to the fibers. It is easy to see that the arcs with the above properties are unique up to the ambient isotopies of \(N_3^1\) respecting the fibers. Let \(\alpha_1\) be an arc as in Figure 7.7. Then, by the arguments of the proof of Proposition 7.2 (see Figure 7.6), we see that the arc \(\alpha_1 \cap N_3^1\) has the above properties. Moreover, by Figure 7.8, we see that \(S_3\) is obtained from \(S_3\) by adding a pipe along \(\alpha_1\). This shows that \(S_3\) is a connected sum of a fiber surface and \(\Sigma_n\), and it is easy to see that a pair of canonical compressing disks for \(S_3\) corresponds to that of \(\Sigma_n\).

This completes the proof of Theorem 3.

**Fig. 7.8**

8. **Arcs and bands for pre-fiber surfaces**

In this section, we study the converse to Theorem 2. For the statement of the result, we prepare some notations. Let \(S\) be a surface in a 3-manifold such that \(\partial S \neq \phi\). Let \(\alpha\) be an arc properly embedded in \(S\), \(D\) a disk such that \(D \cap S = \alpha\), and \(S'\) the image of \(S\) after \(\pm 1\) surgery along \(\partial D\). We say that \(S'\) is obtained from \(S\) by adding a twist along \(\alpha\). Let \(\beta : I \times I \to N\) be an embedding such that \(\beta^{-1}(S) = \beta^{-1}(\partial S) = (\{0\} \times I) \cup (\{1\} \times I)\), and the orientation on \(I \times \{0, 1\}\) is coherent with that of \(\partial S\). Then we say that the surface \(S \cup \beta(I \times I)\) is obtained from \(S\) by adding a band \(b = \beta(I \times 1)\). The arc \(\beta(I \times \{1/2\})\) is called the core arc of the band \(b\).

Let \(T\) be a pre-fiber surface in a closed 3-manifold \(M\), possibly \(\dim H_1(M; Q) > 0\), and \(D^+, D^-\) a pair of canonical compressing disks for \(T\).
Then we have the following two propositions.

**Proposition 8.1.** Suppose that a properly embedded arc \( a \) \((\subset T)\) intersects \( \partial D^+, \partial D^- \) in one points. Then the surface \( T' \) obtained from \( T \) by adding a twist along \( a \) is a fiber surface.

**Remark.** Let \( S \) be a fiber surface in a rational homology 3-sphere. Lemma 4.7 shows that if we get a pre-fiber surface \( S' \) from \( S \) by adding a twist along an arc \( a \), then the arc on \( S' \) corresponding to \( a \) satisfies the assumptions of Proposition 8.1.

Let \( (N'^c, \delta') \) be the complementary sutured manifold for \( T \). Then we may suppose that \( \alpha \cap N'^c \) is an arc \( \alpha' \) such that \( \partial \alpha' \subset \text{Int} \delta' \) for a core arc \( \alpha \).

**Proposition 8.2.** Let \( b \) be a band attached to \( T \) with the following properties.

1. The core arc \( \alpha \) of \( b \) intersects \( \partial D^+, \partial D^- \) in one points.
2. There is a disk \( \Delta \) in \( N'^c \) such that \( \alpha' \subset \partial \Delta, \Delta \cap \partial N'^c = \partial \Delta \cap \partial N'^c = cl(\partial \Delta - \alpha') \), and \( \partial \Delta \cap R_+((\delta'))(\partial \Delta \cap R_-((\delta')) \text{ resp.}) \) consists of an arc.

Then the surface \( T' \) obtained from \( T \) by adding the band \( b \) is a fiber surface.

**Remark.** Let \( S_i \) be a pre-fiber surface in a rational homology 3-sphere as in Theorem 2 (2), and \( b \) a band for \( S_i \) as in Figure 1.2. Proposition 6.1, Figures 4.5, and 8.1 shows that the core arc of \( b \) has the properties (1), (2) of Proposition 8.2.

**Remark.** We note that if \( F \) is fibered and the band \( b \) satisfies the above conditions (1), (2), then the twists on the band is not essential. In fact, by doing Stallings twists \([13]\) along \( \partial D^+ \), we see that the bands obtained from \( b \) by adding twists also produce fiber surfaces.

![Fig. 8.1](image)

Let \( D_i^+=D^+ \cap N'^c, D_i^-=D^- \cap N'^c \).

Proof of Proposition 8.1. Let \( D \) be a disk in \( M \) such that \( D \cap T = a \). Then the image of \( D \) in \( N'^c \) is an annulus \( A \) such that one boundary component \( l \) of \( A \)
is contained in $\text{Int } N^e$ and the other is a simple loop in $\partial N^e$ intersecting $s(\delta')$ in two points (Figure 8.2). Then, by the assumption, we may suppose that $l$ intersects $D_\Gamma$, $D_\Gamma^-$ in one points. Moreover, by taking sufficiently small $D$, if necessary, we may suppose that $(D_\Gamma^+ \cup D_\Gamma^-) \cap A$ consists of two essential arcs in $A$.

Fig. 8.2

Let $\bar{N} = \text{cl}(N^e - N(D_\Gamma^+ \cup D_\Gamma^-; N^e))$, and $\delta$ the image of $\delta'$ in $\partial \bar{N}$. Then $(\bar{N}, \delta)$ is a product sutured manifold. Let $\mathcal{D}^{++}, \mathcal{D}^{+-}$ be the disks in $R_+(\delta)$ corresponding to $\text{Fr}_{N^e} N(D_\Gamma^+; N^e)$, $\mathcal{D}^{+-}$, $\mathcal{D}^{--}$ the disks in $R_-(\delta)$ corresponding to $\text{Fr}_{N^e} N(D_\Gamma^-; N^e)$. Then, by the above, we may suppose that $A \cap \bar{N}$ consists of two disks $\Delta_1, \Delta_2$ (Figure 8.2) such that $\Delta_1 \cap (\mathcal{D}^{--} \cup \mathcal{D}^{--}) = \phi$, $\Delta_1 \cap (\mathcal{D}^{++} \cup \mathcal{D}^{++}) = \phi$. The we may suppose that $\Delta_1, \Delta_2$ have the following properties with respect to the $I$-bundle structure on $(\bar{N}, \delta)$.

(8.1) $\Delta_i$ is a union of fibers.

(8.2) $N(\Delta_1; \bar{N}) \supset (\mathcal{D}^{++} \cup \mathcal{D}^{+-}), N(\Delta_2; \bar{N}) \supset (\mathcal{D}^{+-} \cup \mathcal{D}^{--})$.

Let $P_1 = \text{Fr}_{\bar{N}} N(\Delta_1; \bar{N}), P_2 = \text{Fr}_{\bar{N}} N(\Delta_2; \bar{N})$. Then, by (8.1), and (8.2), $P_1, P_2$ are regarded as product disks in in $(N^e, \delta')$, and $P_1 \cup P_2$ decomposes $(N^e, \delta')$ into the union of a product sutured manifold $(\bar{N}', \delta')$ homeomorphic to $(\bar{N}, \delta)$ and $(D^2 \times S^1, \gamma)$, where $s(\gamma)$ consists of two essential loops in $\partial(D^2 \times S^1)$ which are contractible in $D^2 \times S^1$. We note that $l$ is a core curve of $D^2 \times S^1$ and if we do $\pm 1$ surgery on $(D^2 \times S^1, \gamma)$ along $l$ then we get product sutured manifold

Fig. 8.3
(\(D^2 \times S^1, \gamma'\)) \(\text{(Figure 8.3)}\). Since the complementary sutured manifold for \(T'\) is obtained from \((\bar{N}, \delta')\) and \((D^2 \times S^1, \gamma')\) by summing them along product disks corresponding to \(P_1, P_2\), it is a product sutured manifold. Hence \(T'\) is a fiber surface.

**Proof of Proposition 8.2.** We may suppose that \(\Delta \cap D_i^+ (\Delta \cap D_i^- \text{ resp.})\) consists of an arc with one endpoint lies in \(\partial D_i^+ (\partial D_i^- \text{ resp.})\). Let \(\bar{N} = cl(N - (N(D_1^+ \cup D_1^-); N'))\), and \(\delta\) the image of \(\delta^c\) in \(\delta \bar{N}\). \((\bar{N}, \delta)\) is a product sutured manifold. Then, by the above, \(\Delta \cap \bar{N}\) consists of three disks \(\Delta_1, \Delta_2, \Delta_3\) such that \(\Delta_1 \cap R_-(\delta) = \phi, \Delta_3 \cap R_+ (\delta) = \phi \) \(\text{(Figure 8.4)}\). Let \(\mathcal{D}^{++}, \mathcal{D}^{+-}\) be the disks in \(R_+(\delta)\) corresponding to \(Fr_{\bar{N}^*} N(D_i^+; N^*), \mathcal{D}^{+-}, \mathcal{D}^{--}\) the disks in \(R_-(\delta)\) corresponding to \(Fr_{\bar{N}^*} N(D_i^-; N^*)\) such that \(\mathcal{D}^{++} \cap \Delta_1 = \phi, \mathcal{D}^{+-} \cap \Delta_3 = \phi, \mathcal{D}^{--} \cap \Delta_2 = \phi\). Then we may suppose that \(\Delta_1, \Delta_2, \Delta_3\) have the following properties with respect to the product structures on \((\bar{N}, \delta)\).

\[\text{(8.3) There are mutually disjoint disks } D_i, D_2, D_3 \text{ in } \bar{N} \text{ such that } D_i \text{ is a union of fibers } (i = 1, 2, 3), D_j \supset \Delta_j \text{ } (j = 1, 3), D_2 = \Delta_2, D_1 \cap R_+(\delta) = \Delta_1 \cap R_+(\delta), D_3 \cap R_-(\delta) = \Delta_3 \cap R_-(\delta).\]
(8.4) \( N(D_i; \overline{N}) \supseteq \mathcal{D}^+, \quad N(D_j; \overline{N}) \supseteq (\mathcal{D}^+ \cup \mathcal{D}^-), \quad N(D_k; \overline{N}) \supseteq \mathcal{D}^-. \)

Let \( P_i = \text{Fr}_{\overline{N}} N(D_i; \overline{N}) \) \((i = 1, 2, 3)\). Then, by (8.4), \( P_1, P_2, P_3 \) are regarded as product disks in \( (N^e, \delta^e) \), and \( P_1 \cup P_2 \cup P_3 \) decomposes \( (N^e, \delta^e) \) into a union of a sutured manifold \( (\overline{N}', \delta') \) homeomorphic to \( (\overline{N}, \delta) \) and a sutured manifold \( (B, \gamma_1 \cup \gamma_2 \cup \gamma_3) \), where \( B \) is a 3-ball, and \( s(\gamma_1), s(\gamma_2), s(\gamma_3) \) are sutures as in Figure 8.5.

Let \( (N'^e, \delta'^e) \) be the complementary sutured manifold for \( T' \). Then \( N'^e \) is obtained from \( N^e \) by removing \( \text{Int} N(b; N^e) \), and \( s(\delta'^e) \) is obtained from \( s(\delta^e) - N(b; N^e) \) by adding two arcs in \( \text{Fr}_{N^e} N(b; N^e) \) corresponding to \( \partial b \cap \partial T' \). See Figure 8.6. Hence \( P_1, P_2, P_3 \) are regarded as product disks for \( (N'^e, \delta'^e) \), and \( P_1 \cup P_2 \cup P_3 \) decomposes \( (N'^e, \delta'^e) \) into a union of a product sutured manifold homeomorphic to \( (\overline{N}, \delta) \) and \( (D^2 \times S^1, \gamma) \), where \( (D^2 \times S^1, \gamma) \) is obtained from \( (B, \gamma_1 \cup \gamma_2 \cup \gamma_3) \) by using \( b \). Then, by Figures 8.5 and 8.6, it is directly observed that \( (D^2 \times S^1, \gamma) \) is a product sutured manifold. Hence \( (N'^e, \delta'^e) \) is a product sutured manifold, so that \( T' \) is a fiber surface.

9. Unknotting number 1 fibered knots

In this section, we study unknotting number 1 fibered knots in rational homology 3-spheres. Firstly, we prove Theorem 4 stated in section 1. Then we show that, for each \( g > 1 \), every lens space contains an unknotting number 1 fibered knot of genus \( g \) (Proposition 9.2). In Proposition 9.1 we show that a rational homology 3-sphere \( M \) contains an unknotting number 1 fibered knot of genus 1 if and only if \( M \) is a lens space of type \( L_{m,1} \).
Proof of Theorem 4. Suppose that $M$ contains an unknotting number 1 fibered knot of genus $g$. Then, by Theorem 2(1), we see that $M$ contains a pre-fiber surface $S_0$ of genus $g$ such that $\partial S_0$ is a trivial knot. Let $D^+, D^-$ be a pair of canonical compressing disks for $S_0$.

Claim 9.1. $S_0$ is a type 1 pre-fibered surface.

Proof. By Figure 6.1, we see that there is a properly embedded arc in $S_0$ which intersects $\partial D^+$ in one point. Since $S_0$ has one boundary component, this shows that $\partial D^+$ is non separating in $S_0$. Hence $S_0$ is of type 1.

Claim 9.2. If $M$ contains a type 1 pre-fiber surface $S_\ast$ of genus 1, then $M$ is a lens space.

Proof. The complementary sutured manifold $(N^\ast, \delta^\ast_\ast)$ for $S_\ast$ is homeomorphic to $(D^2 \times S^1 \sqcup (D^2 \times I) \sqcup D^2 \times S^1, \partial D^2 \times I)$ (cf. Example 4.1). Since $(N^\ast, \delta^\ast_\ast)$ is the complementary sutured manifold, there is a homeomorphism $f: R_+(\delta^\ast_\ast) \to R_-(\delta^\ast_\ast)$ such that the manifold obtained from $N^\ast$ by identifying the points in $R(\delta^\ast_\ast)$ by $f$ is homeomorphic to $E(\partial S_\ast)$. Let $D$ be a disk in $N^\ast_\ast$ corresponding to $D^2 \times \{1/2\}$. Then $D$ cuts $N^\ast_\ast$ into two components $N^+, N^-$ such that $N^+, N^-$ are solid tori, and $R_+(\delta^\ast_\ast) \subset \partial N^+, R_-(\delta^\ast_\ast) \subset \partial N^-$. There is a homeomorphism $h: \partial N^+ \to \partial N^-$ such that $h$ is an extension of $f$ and $N^+ \cup_k N^-$ is homeomorphic to $M$. Hence $M$ admits a Heegaard splitting of genus 1.

By Claims 9.1, and 9.2, we see that if $g=1$, then $M$ is a lens space. Hereafter we suppose that $g>1$. Then, by Claim 9.1 and Corollary 5.4, we may suppose that $\partial D^+$ and $\partial D^-$ are disjoint.

Claim 9.3. $\partial D^+ \cup \partial D^-$ does not separate $S_0$.

Proof. Assume that $\partial D^+ \cup \partial D^-$ separates $S_0$. Let $S_\ast$ be the component of $S_0-(\partial D^+ \cup \partial D^-)$ which does not contain $\partial S_0$. Then $\tilde{S} = S_\ast \cup D^+ \cup D^-$ is a closed surface in $M$. By Claim 9.1, there is a simple loop $l$ in $S_0$ which intersects $\partial D^+$ in one point. Then, by pushing $l$ slightly to the $-$side, we see that there is a simple loop in $M$ which intersects $\tilde{S}$ in one point, contradicting the fact that $M$ is a rational homology 3-sphere.

Let $S_1$ be the surface obtained from $S_0$ by doing surgery along $D^+$. By Corollary 5.4, we see that $S_1$ is a pre-fiber surface. Then;

Claim 9.4. $S_1$ is a type 1 pre-fiber surface.

Proof. By Claim 9.3, we see that $\partial D^-$ is non separating in $S_1$. Hence, by Corollary 5.4, we see that $S_1$ is of type 1.

By Claim 9.4, and the induction on $g$, we see that $M$ contains a pre-fiber
surface of type 1 and of genus 1. Then, by Claim 9.2, we see that $M$ admits a Heegaard splitting of genus 1.

**Proposition 9.1.** A rational homology 3-sphere $M$ contains an unknotting number 1, genus 1 fibered knot if and only if $M$ is a lens space of type $L_{m,1}$ for some $m \in \mathbb{Z} - \{0\}$.

For the notation of the lens spaces, see [6].

**Proof.** Suppose that $M$ contains an unknotting number 1, genus 1 fibered knot $K$. Then, by Theorem 1, we see that there is a minimal genus Seifert surface $S$ for $K$ such that $S$ is a plumbing of a surface $F$ in $M$ and a Hopf band. Since genus $(S)=1$, $F$ is an annulus, so that $E(\partial F)$ is homeomorphic to $T^2 \times I$, where $T^2$ is a 2-dimensional torus. Hence $M$ is obtained from $T^2 \times I$ and two solid tori $T_1, T_2$ by identifying their boundaries. Let $A$ be the annulus in $E(\partial F)$ corresponding to the fiber $F$, and $l_0=A \cap (T^2 \times \{0\})$, $l_1=A \cap (T^2 \times \{1\})$. Then meridian loop of $T_i$ intersects $l_i$ in one point ($i=1, 2$). Hence it is easy to see that $M$ is a lens space of type $L_{m,1}$.

Suppose that $M$ is a lens space of type $L_{m,1}$. Then it is observed in [7] that the knots $K_1, K_2$ of Figure 9.1 are fibered. It is easy to see that both $K_1$ and $K_2$ have unknotting number 1.

This completes the proof of Proposition 9.1.

![Fig. 9.1](image)

**Proposition 9.2.** If $M$ is a lens space, possibly $\dim H_1(M; \mathbb{Q}) > 0$, then, for each $g>1$, there is an unknotting number 1 fibered knot of genus $g$ in $M$.

**Remark.** If $M$ is a lens space with $\dim H_1(M; \mathbb{Q}) > 0$, then $M$ is homeomorphic to $S^2 \times S^1$.

**Proof.** By Example 4.1, there is a genus 1 pre-fiber surface $T$ in $M$ such that $\partial T$ is a trivial knot. Let $D^+, D^-$ be a pair of canonical compressing disks for $T$, $T^+, T^-$ a pair of properly embedded arcs in $T$ such that $T^+ \cap \partial D^+$ consists of one point, $T^- \cap \partial D^-$ consists of one point, and $\partial T^+ \cap \partial T^-$ consists of one
point \( p \). Let \( l^+(l^- \text{ resp.}) \) be the arc obtained from \( I^+ (I^- \text{ resp.}) \) by pushing \( \text{Int } I^+ (\text{Int } I^- \text{ resp.}) \) slightly to the—side (+side resp.) of \( T \). \( l = l^+ \cup l^- \) is an embedded arc in \( M \) such that \( l \cap T = \partial l \cup p \). Then deform \( l \) by an ambient isotopy in a small neighborhood of \( p \) so that \( l \cap T = \partial l \). Clearly \( l \) satisfies the conditions (1), (2) of Proposition 8.2. Hence there is a band \( b \) for \( T \) such that the surface \( F \) obtained from \( T \) by attaching \( b \) is a fiber surface. Then, by a plumbing of \( F \) and a Hopf band along \( b \), we have a genus 2, fiber surface which bounds an unknotting number 1 fibered knot (Figure 9.2).

\[
\text{trivial knot}
\]

\[
\text{trivial knot}
\]

Suppose that \( g > 2 \). Let \( F_n \) be the surface in \( S^3 \) as in Figure 9.3. It is observed in [9] that \( F_n \) is a fiber surface. In fact, \( F_n \) is obtained from one Hopf band and \( n \) copies of the fiber surface of Figure 9.4. Then, by a plumbing of the above \( F \) and \( F_{g-2} \) along \( b \) and \( E \) of Figure 9.3, we get a genus \( g \) fiber surface \( S_g \) [4]. It is directly observed from Figure 9.3 that if we apply a crossing change on \( \partial S_g \) along the crossing disk \( D \) of Figure 9.3, then we get a trivial knot. Hence \( u(\partial S_g) = 1 \).
This completes the proof of Proposition 9.2.

References


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