Kobayashi, T.
Osaka J. Math.
26 (1989), 699–742

FIBERED LINKS AND UNKNOTTING OPERATIONS

Dedicated to Professor Kunio Murasugi on his sixtieth birthday

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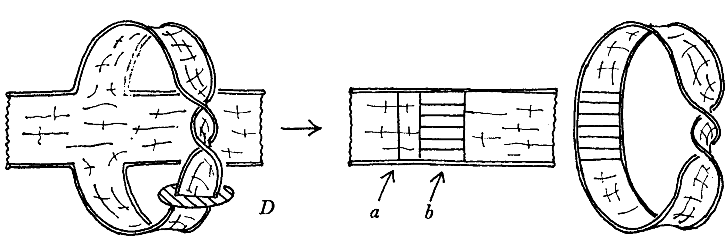
(Received December 1, 1988)

1. Introduction

Let L be an oriented link in a 3-manifold M. A *Seifert surface* S for L is a compact oriented surface, without closed components, such that $\partial S=L$. $\chi(L)$ denotes the maximal Euler characteristic of all Seifert surfaces for L. L is a *fibered link* if the exterior $E(L)$ of L is a surface bundle over S^1 such that a Seifert surface represents a fiber. An oriented surface F in M is a *fiber surface* if ∂F is a fibered link, and $F\cap E(\partial F)$ is a fiber. Let D be a disk in M, which intersects L in two points of opposite orientations, L' the image of L after ± 1 surgery along ∂D. We say that L' is obtained from L by a *crossing change,* and D (∂D resp.) is called the *crossing disk* (*crossing link* resp.). For the links in the 3-sphere S^3, Scharlemann-Thompson [**14**] proved that if L' is obtained from L by a single crossing change along a crossing disk D, and $\chi(L')>\chi(L)$, then there is a minimal genus Seifert surface S for L such that S is a plumbing of a surface F and a Hopf band A with $F\cap D=\phi$, and $A\cap D$ an essential arc in A. See Figure 1.1.



S F A

(i) (ii)

Fig. 1.1

This work was supported by Grant-in Aid for Scientific Research, The Ministry of Education, Science and Culture.

In this paper, firstly, we show that a similar result holds for links in rational homology 3-spheres if L is a fibered link.

Theorem 1. *Let L be a fibered link in a rational homology 3-sphere M. Suppose that L' is obtained from L by a single crossing change along a crossing disk D, and that $\chi(L')>\chi(L)$. Then there is a minimal genus Seifert surface S for L such that S is a plumbing of a surface F in M and a Hopf band A with $F\cap D=\phi$, and $A\cap D$ an essential arc in A.*

REMARK. We note that S and F are fiber surfaces (Lemma 2.2, [6, Theorem 7.4]).

Let S_0 be the image of S in Theorem 1 after the ± 1 surgery along ∂D, and $S_1=cl(S-A)$. Then S_0, S_1 are Seifert surfaces for L' (Figure 1.2). In section 4, we study the surfaces S_0, S_1.

Theorem 2. *Let S_0, S_1 be as above. Then*

(1) *S_0 is a pre-fiber surface,*

(2) *if $\chi(L')>\chi(L)+2$ (i.e. S_1 is not a minimal genus Seifert surface), then S_1 is also a pre-fiber surface.*

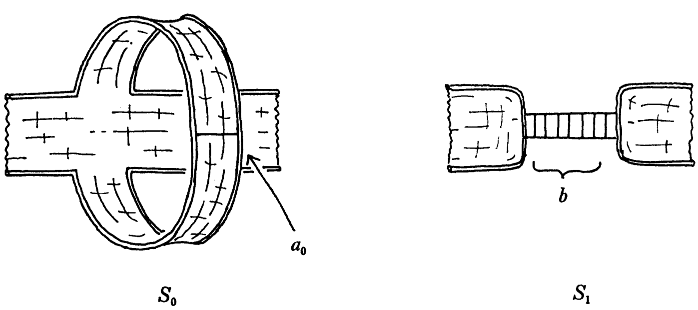


Fig. 1.2

For the definition of pre-fiber surface, see section 4. We prove Theorem 2 in sections 4,5, and 6. In section 7, we give a characterization of a class of pre-fiber surfaces in case when they bound fibered links. For the statement of the result, we prepare some notations. Let Σ_n be the genus $n(\geq 1)$ Seifert surface for a trivial knot in S^3 as in Figure 1.3. For the precise definition of Σ_n, see section 7. Then we have;

Theorem 3. *Suppose that a surface S_1 in a rational homology 3-sphere M is a pre-fiber surface of type 1 with $L=\partial S_1$ a fibered link. Then S_1 is a connected sum of a fiber surface for L and Σ_n, where $n=(\chi(L)-\chi(S_1))/2$. Moreover a pair of canonical compressing disks for S_1 corresponds to that of Σ_n.*

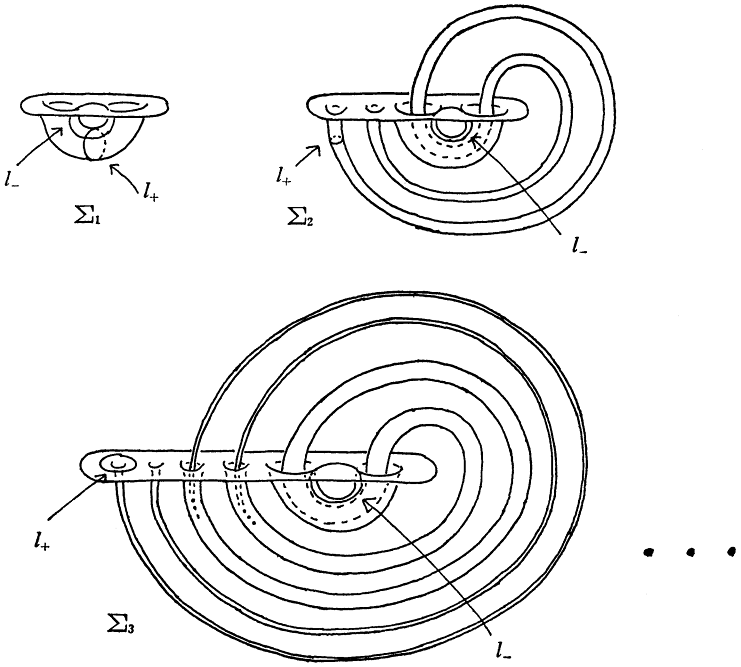


Fig. 1.3

Theorem 2 shows that we can get a pre-fiber surface from a fiber surface S by adding a twist along a properly embedded arc in S, or by removing a band from S (Figure 1.2). In section 8, we study the converse to this. Namely, we give a characterization of the arcs in a pre-fiber surface S_* the twists along which produce fiber surfaces, and a characterization of the bands for S_* to produce fiber surfaces in case when the ambient manifold is a rational homology 3-sphere. See the remarks of section 8.

We say that a knot in a 3-manifold M is *trivial* if it bounds a non-singular disk in M. Suppose that a knot K is contractible in M. Then it is easy to see that K is tranformed into a trivial knot by a finite number of crossing changes. The *unknotting number* $u(K)$ is the minimal number of crossing changes that are necessary to transform K into a trivial knot. Let Σ_n, l_+, l_- $(\subset\Sigma_n)$ be as in Figure 1.3. Then, as consequences of the above results, we have;

Corollary 1. *A genus g (≥ 1) surface S in S^3 is a fiber surface with ∂S an unknotting number* 1 *knot if and only if S is obtained from Σ_g by adding a twist along an arc a $(\subset\Sigma_g)$ such that a intersects l_+ and l_- transversely in one points.*

Corollary 2. *A genus g (>1) surface S in S^3 is a fiber surface with ∂S an unknotting number* 1 *knot if and only if S is obtained from Σ_{g-1} by adding a band satisfying the properties* (1), (2) *of Proposition* 8.2, *and then plumbing a Hopf band along b.*

REMARK. Quach [9] proved that if $A(t)(\neq 1)$ is an Alexander polynomial with leading coefficient ± 1, then there exists an unknotting number 1, fibered knot K in S^3 with $\Delta_K(t)=A(t)$, where $\Delta_K(t)$ denotes the Alexander polynomial of K. The result implies that, for each $g(>1)$, there are infinitely many unknotting number 1, fibered knots of genus g.

In section 9, by using Theorem 2, we study the rational homology 3-spheres containing unknotting number 1 fibered knots. We say that a 3-manifold is a *lens space* if it admits a Heegaard splitting of genus 1 [6]. Then we have;

Theorem 4. *If a rational homology 3-sphere M contains an unknotting number 1 fibered knot, then M is a lens space.*

REMARK. Moreover we will show that, for each $g(>1)$, every lens space contains an unknotting unmber 1 fibered knot of genus g, and we will give the list of lens spaces containing genus 1, unknotting number 1, fibered knots. We note that there exist lens spaces which do not contain genus 1 fibered knots [7].

As an immediate consequence of Theorem 4, we have;

Corollary 3. *If an integral homology 3-sphere Σ^3 contains an unknotting number 1 fibered knot, then Σ^3 is homeomorphic to S^3.*

2. Preliminaries

Throughout this paper, we work in the piecewise linear category, all manifolds, including knots, links, and Seifert surfaces are oriented, and all submanifolds are in general position unless otherwise specified. For the definitions of standard terms of 3-dimensional topology, knot and link theory, see [6], and [10]. For a topological space B, $\# B$ denotes the number of the components of B. Let H be a subcomplex of a complex K. Then $N(H; K)$ denotes a regular neighborhood of H in K. Let N be a manifold embedded in a manifold M with $\dim N=\dim M$. Then $\mathrm{Fr}_M N$ denotes the frontier of N in M. An arc a properly embedded in a surface S is *inessential* if it is rel ∂ isotopic to an arc in ∂S. If a is not inessential, then it is *essential.*

Let S be a surface properly embedded in a 3-manifold M. A disk D in M is a *compressing disk* for S if $D\cap S=\partial D$, and ∂D is not contractible in S. If there does not exist a compressing disk for S, then S is *incompressible.*

Let S_i be a surface with boundary in a 3-manifold M_i (i=1, 2). Let B_i be a 3-ball in M_i such that $B_i\cap \partial S_i$ is an arc, and $B_i\cap S_i$ is a disk (Figure 2.1). Let $h: \partial B_1\to\partial B_2$ be an orientation reversing homeomorphism such that $h(\partial B_1\cap S_1)=h(\partial B_2\cap S_2)$. Then $(M_1-\mathrm{Int}\,B_1)\cup_h(M_2-\mathrm{Int}\,B_2)$ is a *connected sum* of M_1 and M_2, and is denoted by $M_1\# M_2$. The image of $S_1\cup S_2$ in $M_1\# M_2$ is called a *connected sum* of S_1 and S_2.

A *sutured manifold* (M, γ) is a compact 3-manifold M together with a set

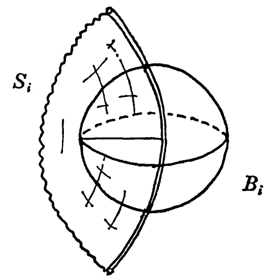


Fig. 2.1

$\gamma(\subset\partial M)$ of mutually disjoint annuli $A(\gamma)$ and tori $T(\gamma)$ [2]. In this paper, we mainly treat the case of $T(\gamma)=\phi$. The core curves of $A(\gamma)$, $s(\gamma)$, are the *sutures*. Every component of $R(\gamma)=\partial M-\operatorname{Int}\gamma$ is oriented, and $R_+(\gamma)(R_-(\gamma)$ resp.) denotes the union of the components whose normal vector point of (into resp.) M. Moreover the orientation of $R(\gamma)$ must be coherent with respect to $s(\gamma)$. We say that a sutured manifold (M, γ) is a *product sutured manifold* if (M, γ) is homeomorphic to $(F\times I, \partial F\times I)$ with $R_+(\gamma)=F\times\{1\}$, where F is a surface, and I is the unit interval [0, 1].

Let (M, γ) be a sutrued manifold. A properly embedded annulus A in M is a *product annulus* if one boundary component of A is contained in $R_+(\gamma)$, and the other is contained in $R_-(\gamma)$. A properly embedded disk D in M is a *product disk* if $\partial D\cap\gamma$ consists of two essential arcs in $A(\gamma)$. A *product decomposition* $(M, \gamma)\to(M', \gamma')$ is a sutured manifold decomposition [2] along a product disk. See Figure 2.2.

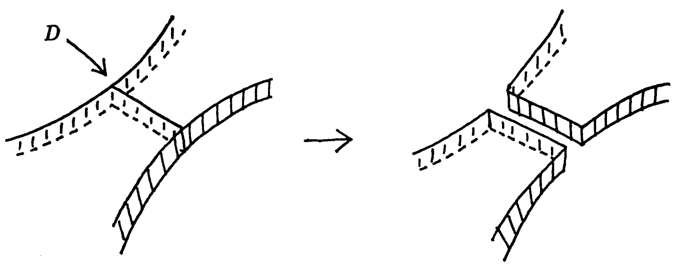


Fig. 2.2

Let L be a link in a 3-manifold M. The *exterior* $E(L)$ of L is the closure of the complement of $N(L; M)$. A *meridian loop* for L is a non-contractible simple loop in $\partial E(L)$, which bounds a disk in $N(L; M)$. Let S be a Seifert surface for L. Then we often abbreviate $S\cap E(L)$ to S. S is a *minimal genus Seifert surface* if $\chi(S)=\chi(L)$.

Let S be a Seifert surface for L. Then $(N, \delta)=(N(S; E(L)), N(\partial S; \partial E(L)))$ has a product sutured manifold structure $(S\times I, \partial S\times I)$. (N, δ) is called the *sutured manifold obtained from* S. Then the sutured manifold $(N^c, \delta^c)=$

$(cl(E(L)-N), cl(\partial E(L)-\delta))$ with $R_+(\delta^c)=R_-(\delta)$ is the *complementary sutured manifold* for S. We say that a surface S in a 3-manifold is a *fiber surface,* if ∂S is a fibered link with S a fiber. It is easy to see that S is a fiber surface if and only if the complementary sutured manifold for S is a product sutured manifold.

Then we easily see;

Lemma 2.1. *Every fiber surface in a connected* 3*-manifold is connected.*

Let L be a link with a Seifert surface in a rational homology 3-sphere. It is easy to see that Seifert surfaces for L determine a unique non trivial element of $H_2(E(L), \partial E(L))$, so that the cyclic covering space for L is well defined. Then the next lemma follows from the fact that the infinite cyclic covering space of a fibered link is homeomorphic to $(surface)\times R$, and details of the proof are left to the reader.

Lemma 2.2. *For a surface S in a rational homology* 3*-sphere, with $L=\partial S$ a fibered link, the following three conditions are equivalent.*

(1) *S is a fiber surface.*

(2) *S is a minimal genus Seifert surface for L.*

(3) *S is incompressible.*

Let S be a fiber surface. Then there is an orientation preserving homeomorphism φ of S such that $\varphi|_{\partial S}=\mathrm{id}_{\partial S}$, and $E(L)$ is homeomorphic to $S\times I/\sim$, where $(x, 1)\sim(\varphi(x), 0)$ $(x\in S)$. φ is called a *monodromy map.* $\partial S\times I$ has an I-bundle structure such that each fiber projects to a meridian loop of $\partial E(L)$. Let p: $S\times I\to E(L)$ be a natural map, $D(\subset S\times I)$ a product disk for the product sutured manifold $(S\times I, \partial S\times I)$ such that each component of $\partial D\cap(\partial S\times I)$ is a fiber. Then the 2-complex $\square=p(D)$ is called a *projected product disk* (or *pp disk* for short). For the pp disk $\square$, $\partial_-\square$, $\partial_+\square$ denotes $p(D\cap(S\times\{0\}))$, $p(D\cap(S\times\{1\}))$ respectively. Suppose that there is an ambient

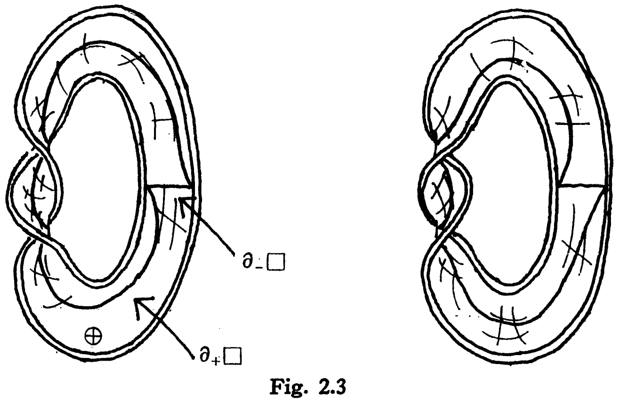


Fig. 2.3

isotopy f_t for $S\times I$ such that $f_0=\mathrm{id}$, $f_t(D)$ is a product disk such that $\partial f_t(D)\cap(\partial S\times I)$ consists of fibers of $\partial S\times I$. Then we say that the pp disk $\square'=p(f_1(D))$ is isotopic to $\square$ by an isotopy as a pp disk.

EXAMPLE 2.3. A *Hopf band* A is a ± 1 twisted unknotted annulus in S^3 (Figure 2.3). A is a fiber surface, and a monodromy map for A is a right or left hand Dehn twist along the core curve of A.

EXAMPLE 2.4. The genus 0 surface A^* of Figure 2.4 is a connected sum of two Hopf bands, and hence, by [3] or [13], is a fiber surface.

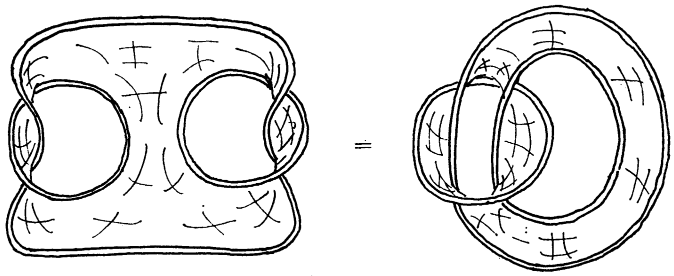


Fig. 2.4

3. Theorem 1

In this section, we prove Theorem 1 stated in section 1. We assume that the reader is familiar with [5], and [14].

Let L, L', and D be as in Theorem 1. Let S be a minimal genus Seifert surface for L in M. Let L_1 be the link obtained from L by splitting it as in Figure 3.1, D_1 the disk as in Figure 3.1, and R_1 a minimal genus Seifert surface for L_1 *in* $E(\partial D_1)$. By the arguments of the proof of [14, 1.4 Theorem], we may suppose that R_1 intersects D_1 in an arc a_1 (Figure 3.2 (i)). Let R be the Seifert surface for L obtained from R_1 by plumbing a Hopf band as in Figure 3.2 (ii).

Claim 3.0. *If $E(\partial D_1\cup L_1)$ is not irreducible, then the conclusion of Theorem* 1 *holds.*

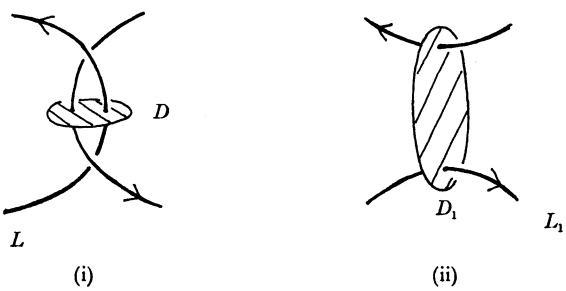


Fig. 3.1

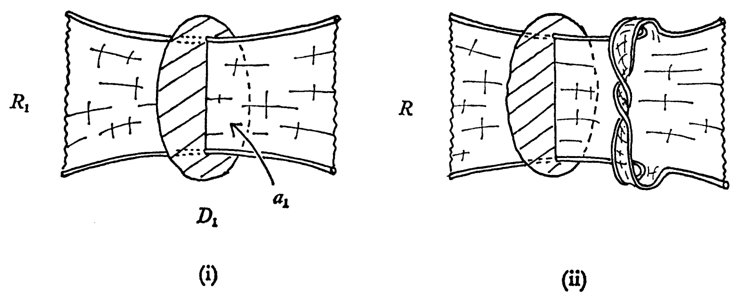


Fig. 3.2

Proof. Let $P=D_1\cap E(\partial D_1\cup L_1)$. Then P is a disk with two holes, with two boundary components l_1, l_2 are meridian loops of L_1, and the rest boundary component l_3 is parallel to ∂D_1 in D_1. Let S_1 be an essential 2-sphere in $E(L_1\cup\partial D_1)$.

Subclaim 1. $S_1\cap P\neq\phi$.

Proof. Assume that $S_1\cap P=\phi$. Then, by Figure 3.2, we may suppose that S_1 is embedded in $E(\partial D_1\cup L)$, and $\partial D_1\cup L$ is contained in a component of $M-S_1$. Since $E(L)$ is irreducible, S_1 bounds a 3-ball in $E(\partial D_1\cup L)$, so that S_1 bounds a 3-ball in $E(\partial D_1\cup L_1)$, a contradiction.

Then we suppose that $\#(S_1\cap P)$ is minimal among all essential 2-spheres in $E(\partial D_1\cup L_1)$. Let $V(\subset S_1)$ be an innermost disk, i.e. $V\cap P=\partial V$. By the minimality of $\#(S_1\cap P)$, we see that ∂V is not contractible in P.

Subclaim 2. *∂V is parallel to l_3 in P.*

Proof. Assume not. Then ∂V is parallel to l_1 or l_2. Let D^* be the disk in D_1 such that $\partial D^*=\partial V$, and $S_2=V\cup D^*$. S_2 is a 2-sphere, and intersects L_1 in one point. Then, by plumbing a Hopf band to R_1 in the right or left side of D_1 in Figure 3.2, we may suppose that $S_2\cap L$ consists of one point. This shows that a meridian loop for L is contractible in $E(L)$, contradicting the fact that L is a fibered link.

Subclaim 3. *R_1 is of minimal genus in M.*

Proof. Let D^* be the disk in D_1 such that $\partial D^*=\partial V$, and $S_2=D^*\cup V$. By Subclaim 2, S_2 is a 2-sphere in M which intersects in L_1 in two points. Let R_1^* be a minimal genus Seifert surface for L_1 in M. Since $S_2\cap L_1$ consists of two points, by applying cut and paste arguments on S_2, we may suppose that $S_2\cap R_1^*=D_1\cap R_1^*$ consists of an arc whose endpoints are $S_2\cap L_1$. This shows that $\chi(R_1)\geq\chi(R_1^*)$. Clearly $\chi(R_1^*)\geq\chi(R_1)$. Hence $\chi(R_1)=(R_1^*)$, so that R_1 is of minimal genus in M.

Subclaim 4. *$E(L_1)$ is irreducible.*

Proof. Assume not. Let S_3 be an essential 2-sphere in $E(L_1)$. Since R_1 is incompressible (Subclaim 3), by using standard innermost disk arguments, we may suppose that $S_3 \cap R_1 = \phi$. Hence we may suppose that $S_3 \cap L = \phi$. It is easy to see that S_3 is an essential 2-sphere in $E(L)$, contradicting the irreducibility of $E(L)$.

By Subclaims 3 and 4, we see that R_1 is taut in terms of [**2**]. Hence, by [**2**, Theorem 5.5] and the argument of the proof of [**3**, Theorem 1.1], we see that $E(L)$ posseses a taut foliation such that R is a leaf of the foliation. Hence R is a minimal genus Seifert surface for L in M, and this completes the proof of Claim 3.0.

By Claim 3.0, hereafter, we suppose that $E(\partial D_1 \cup L_1)$ is irreducible. Then, by the argument in the last paragraph of the proof of Claim 3.0, we see that $E(\partial D_1 \cup L)$ posseses a taut foliation such that R is a leaf of the foliation, so that $E(\partial D_1 \cup L)$ is irreducible, and R is a minimal genus Seifert sufrace for L in $E(\partial D_1)$. Then we have the following two cases.

Case 1. *$E(L)$ is $R_{\partial D_1}$-atoroidal.*

If R is a minimal genus Seifert surface for L in M, then we have the conclusion of Theorem 1. Suppose that R is not of minimal genus in M. Then by [**5**, Theorem 1.8] or [**12**, 5.1 Theorem], and by the arguments of the proof of [**14**, 1.14 Theorem], we see that the surface R^* obtained from R by cutting along a_1 is of minimal genus in M (Figure 3.3 (i)). Hence we see that the Seifert surface S' for L' obtained from R^* by removing the Hopf band is of minimal genus in M (Figure 3.3 (ii)). We note that $\chi(S')$ $(=\chi(L'))=\chi(R)+2$. Since $\chi(L')>\chi(L)$ (i.e. $\chi(L')\geq\chi(L)+2$), this shows that R is a minimal genus Seifert surface for L in M, a contradiction.

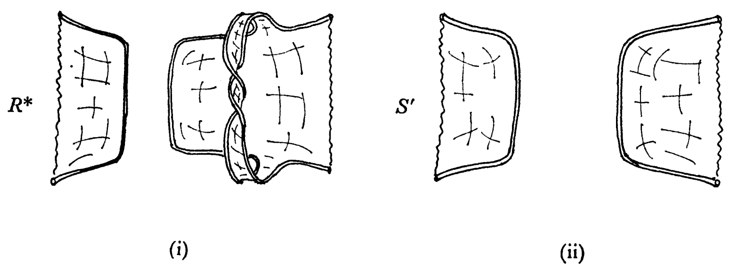


Fig. 3.3

Case 2. *$E(L)$ is not $R_{\partial D_1}$-atoroidal.*

Since $E(L)$ is not $R_{\partial D_1}$-atoroidal, there is an incompressible, non-boundary parallel torus T in $E(\partial D_1 \cup L)$ with the following properties.

(3.1) T separates $E(\partial D_1)$ into V_1 and V_2 with $\partial E(\partial D_1)\subset V_1$, and $R\subset V_2$, and
(3.2) $i_*: H_1(T)\to H_1(V_1)$ is injective.

Let T_1, T_2 be incompressible, non-boundary parallel tori satisfying the above conditions (3.1), (3.2). We say that $T_1\leq T_2$ if T_1 is isotopic to T_1' such that $T_1'\cap T_2=\phi$, and $V^1\subset V^2$, where V^1 (V^2 resp.) denotes the closure of the component of $E(L)-T_1'$ ($E(L)-T_2$ resp.) which contains ∂D_1. Clearly $\leq$ is an order on the tori with the above properties (3.1), (3.2). Then we suppose that T is maximal with respect to the order.

Claim 3.1. *If T is incompressible in $E(L)$, then R is a minimal genus Seifert surface for L in M.*

Proof. Since $E(L)$ is irreducible, and S is incompressible, by using standard innermost disk arguments, we may suppose that T intersects S in essential loops, so that each component of $T\cap N^c$ is an annulus, where (N^c, γ^c) is the complementary sutured manifold for S in M. Since (N^c, γ^c) is a product sutured manifold, by [**15**, Corollary 3.2], we may suppose, by moving T by an ambient isotopy, that each component of $T\cap N^c$ is a product annulus.

Since T is incompressible, and $T\cap R=\phi$, we may suppose that T intersects D_1 in essential loops in the annulus $cl(D_1-N(a_1; D_1))$. Suppose that some component of $T\cap D_1$ is contractible in T. Then, by using cut and paste arguments, we see that ∂D_1 bounds a disk in $E(L)$, contradicting the fact that $E(\partial D_1\cup L)$ is irreducible. Hence we see that ∂D_1 is ambient isotopic to an essential loop l on T. Then, by the above, we may suppose that either l is ambient isotopic to a component of $T\cap S$ or each component of $l\cap N^c$ runs from $R_-(\gamma^c)$ to $R_+(\gamma^c)$. Then since $lk(l, L)=lk(\partial D_1, L)=0$, we see that l is ambient isotopic to a component of $T\cap S$. Hence we may suppose that $\partial D_1\cap S=\phi$. This shows that $\chi(S)\leq\chi(R)$. Clearly $\chi(S)\geq\chi(R)$. Hence $\chi(S)=\chi(R)$, and R is a minimal genus Seifert surface for L in M.

Claim 3.2. *If T is compressible in $E(L)$, then T bounds a solid torus in $E(L)$.*

Proof. Since $E(L)$ is irreducible and T separates $E(L)$, we see that T bounds either a solid torus or a 3-manifold homeomorphic to the exterior of a non-trivial knot in S^3 such that the boundary of the compressing disk is a meridian loop. Assume that T bounds the exterior E of a non-trivial knot with a compressing disk C for T such that ∂C is a meridian loop for E. Then $\partial D_1\subset E$. Then $B=E\cup N(C; E(L))$ is a 3-ball such that $\partial D_1\subset B$, contradicting the irreducibility of $E(\partial D_1\cup L)$.

Claim 3.3. *If T is compressible in $E(L)$, then R is a minimal genus Seifert surface for L in M.*

Proof. Assume that R is not a minimal genus Seifert surface for L in M.

By Claim 3.2, T bonuds a solid torus τ such that $\partial D_1 \subset \tau$. Since $E(\partial D_1 \cup L)$ is irreducible, and T is incompressible in $E(\partial D_1 \cup L)$, we may suppose that T intersects D_1 in essential loops in the annulus $D_1 - N(a_1; D_1)$. By the argument of the second paragraph of the proof of Claim 3.1, we see that every component of $T \cap D_1$ is an essential loop in T. Then ∂D_1 is ambient isotopic to an essential loop l on T.

Let m be an essential simple loop on T. Then $M(m)$ denotes the manifold obtained from $D^2 \times S^1$ and $M - \mathrm{Int}\, \tau$ by identifying their boundaries by a homeomorphism which takes $\partial(D^2 \times pt.)$ to m. Clearly $M(m)$ is obtained from N by doing a Dehn surgery along the core curve c of τ. Then $R(m)$ denotes the image of R in $M(m)$. Let m_0 be a simple loop on T such that $M(m_0) = M$, and $R(m_0) = R$.

Subclaim 1. *The absolute value of the intersection number of m_0 and l in T is greater than one.*

Proof. Assume that m_0 does not intersect l, i.e. m_0 and l are parallel. Then l bounds a disk in τ, contradicting the fact that $E(\partial D_1 \cup L)$ is irreducible. Assume that m_0 intersects l in one point. Then l is isotopic to c in τ, contradicting the fact that T is not boundary parallel in $E(\partial D_1 \cup L)$.

Let l^* be a simple loop in T intersecting l in one point. By Subclaim 1, we see that M is homeomorphic to the connected sum of $M(l^*)$ and a non-trivial lens space L_n (Figure 3.4).

Since T is incompressible, and $E(\partial D_1 \cup L)$ is irreducible, $E(c \cup L)$ ($\cong E(L) - \mathrm{Int}\, \tau$) is irreducible. By the maximality of T, it is easy to see that $E(L)$ is R_c-actoroidal. By Subcalim 1, l is not ambient isotopic to m_0. Since $R(m_0)$ is not of minimal genus, by [**5**, Theorem 1.8] or [**12**, 5.1 Theorem], we see that $R(l)$ is taut, so that of minimal genus.

Let $\bar{R}^*$ be the image of R^* (Figure 3.3 (i)) in $M(l^*)$. Then;

Subclaim 2. *$\bar{R}^*$ is a minimal genus Seifert surface in $M(l^*)$.*

Proof. The idea of the following proof can be found in [**14**]. Let (N^0, δ^0), (N^l, δ^l), (N^*, δ^*) be the complementary sutured manifolds for $R(=R(m_0))$, $R(l)$, $R(l^*)$ respectively. Let S^2 be a 2-sphere in $M(l)$ such that $S^2 \cap (M - \mathrm{Int}\, \tau)$ is a disk whose boundary is l, and intersecting $R(l)$ in an essential arc (Figure 3.4 (i)). Then the image of S^2 in N^l is a product disk $\mathcal{D}$ in (N^l, δ^l), and, by doing the product decomposition along $\mathcal{D}$, we get a sutured manifold $(\bar{N}, \bar{\delta})$, which is homeomorphic to the complementary sutured manifold for $\bar{R}^*$. Since $R(l)$ is taut, (N^l, δ^l) is taut. Hence, by [**2**, Lemma 3.12] or [**12**, 4.2 Lemma], $(\bar{N}, \bar{\delta})$ is taut, so that $\bar{R}^*$ is of minimal genus.

Since $M = M(l^*) \# L_n$(Figure 3.4 (ii)), Subclaim 2 shows that R^* of Figure

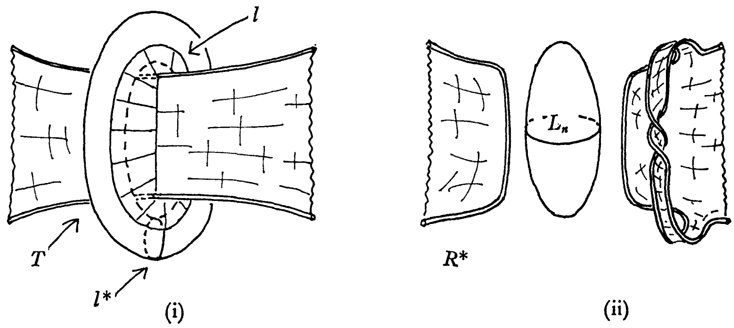


Fig. 3.4

3.3 (i) is of minimal genus. Hence S' of Figure 3.3 (ii) is of minimal genus. We note that $\chi(L')(=\chi(S'))=\chi(R)+2$, and $\chi(L)<\chi(L')$, i.e. $\chi(L)+2\leq\chi(L')$. This shows that R is a minimal genus Seifert surface for L in M, a contradiction.

This completes the proof of Theorem 1.

4. Fiber surfaces and pre-fiber surfaces

In this seation, we give the definition of pre-fiber surfaces, and show that if there is a fiber surface F whose monodromy has a certain property, then we can get a pre-fiber surface by removing a band from F (Proposition 4.5). And, by using the result, we prove Theorem 2 (1).

Let S be a *connected* surface in a 3-manifold, and (N^c, δ^c) the complementary sutured manifold for S. S is a *pre-fiber surface,* if there are pairwise disjoint compressing disks D^+, D^- for $R_+(\delta^c)$, $R_-(\delta^c)$ respectively in N^c such that $(\bar{N}, \delta^c)$ is homeomorphic to the product sutured manifold, where $\bar{N}$ is obtained from N^c by doing a surgery along $D^+\cup D^-$. Then S has two compressing disks $\bar{D}^+$, $\bar{D}^-$ such that $\operatorname{Int}\bar{D}^+\cap\operatorname{Int}\bar{D}^-=\phi$, $\bar{D}^+\cap N^c=D^+$, $\bar{D}^-\cap N^c=D^-$. We say that $\bar{D}^+$, $\bar{D}^-$ is *a pair of canonical compressing disks* for a pre-fiber surface S.

REMARK. We note that $N(\partial\bar{D}^+; \bar{D}^+)$ lies in the — side of S.

We say that a pre-fiber surface S is of *type* 1 (*type* 2 resp.) if ∂D^+ is non-separating (separating resp.) in $R_+(\delta^c)$. It is easy to see that if S is of type 1, then (N^c, δ^c) is homeomorphic to $(D^2\times S^1\natural_{d_+}(S'\times I)\natural_{d_-}D^2\times S^1, \partial S'\times I)$, where S' is a connected surface, $\natural$ denotes a boundary connected sum, and d_+ (d_- resp.) denotes a disk in $S'\times\{1\}$ ($S'\times\{0\}$ resp.).

EXAMPLE 4.1. Let T be a genus 1 Heegaard surface for a lens space [6], and D^2 a disk in T. Let $S=T-\operatorname{Int}D^2$. Then S is a pre-fiber surface of type 1. In fact, the complementary sutured manifold for S is homeomorphic to $(D^2\times S^1\natural(D^2\times I)\natural D^2\times S^1, \partial D^2\times I)$.

Let A be an unknotted, untwisted annulus in S^3. Then A is a pre-fiber surface of type 2. In fact, the complementary sutured manifold for A is homeomorphic to $(D^2\times S^1, \gamma)$, where $s(\gamma)$ consists of two essential loops in $\partial(D^2\times S^1)$ which are contractible in $D^2\times S^1$.

The next proposition shows that pairs of canonical compressing disks for a pre-fiber surface are unique.

Proposition 4.2. *Let S be a pre-fiber surface, and D^+, D^-, $\bar{D}^+$, $\bar{D}^-$ as above. Let $\bar{D}^{+\prime}$, $\bar{D}^{-\prime}$ be a pair of canonical compressing disks for S such that $N(\partial\bar{D}^{+\prime}; \bar{D}^{+\prime})$ ($N(\partial\bar{D}^{-\prime}; \bar{D}^{-\prime})$ resp.) lies in the $-$ side ($+$side resp.) of S. Then $\bar{D}^{+\prime}$($\bar{D}^{-\prime}$ resp.) is isotopic to $\bar{D}^+$ ($\bar{D}^-$ resp.) by an ambient isotopy of the 3-manifold respecting S.*

For the proof of Proposition 4.2, we prepare two lemmas. Let (N, δ) be a connected sutured manifold such that N is obtained from a (possibly disconnected) product sutured manifold (N', δ') with N' irreducible by attaching a 1-handle along disks in $R_+(\delta')$, and δ is the image of δ'. Let D be the dual core of the 1-handle. Then;

Lemma 4.3. *Suppose that N' is disconnected. Let D_1 be a compressing disk for $R_+(\delta_1)$. Then D_1 is isotopic to D by an ambient isotopy of N respecting δ.*

Proof. Since N' is irreducible, N is irreducible. Hence, by using standard innermost disk arguments, we may suppose that no component of $D\cap D_1$ is a simple loop. Suppose that $D\cap D_1=\phi$. Then ∂D_1 bounds a disk D' in $R_+(\delta')$. Since D_1 is a compressing disk, we see that D' contains a component of $N'\cap$(1-*handle*), so that D_1 is parallel to D. Suppose that $D\cap D_1\neq\phi$. Let $\Delta(\subset D_1)$ be an outermost disk, i.e. $\Delta\cap D=\partial\Delta\cap D=\alpha$ an arc, and $\Delta\cap\partial D_1=\beta$ an arc such that $\alpha\cup\beta=\partial\Delta$. Let Δ' be the image of Δ in N'. Then $\partial\Delta'\subset R_+(\delta')$, and $\partial\Delta'$ bounds a disk D' in $R_+(\delta')$ such that Δ' is parallel to D'. Hence we can remove α by moving D_1 by an ambient isotopy of N respecting δ. Then by the induction on $\#(D\cap D_1)$, we have the conclusion.

Lemma 4.4. *Let (N, δ), (N', δ') be as above. Suppose that N' is connected. Let D_1 be a compressing disk for $R_+(\delta)$ such that ∂D_1 is non separating in $R_+(\delta)$. Then D_1 is isotopic to D by an ambient isotopy of N respecting δ.*

Proof. Let D^1, D^2 be the disks in $R_+(\delta')$ along which the 1-handle is attached. We may suppose that no component of $D\cap D_1$ is a simple loop (see the proof of Lemma 4.3). We see that if $D\cap D_1=\phi$, then we have the conclusion (see the proof of Lemma 4.3). Suppose that $D\cap D_1\neq\phi$. Let $\Delta(\subset D_1)$ be an outermost disk, and $\alpha=\Delta\cap D$, $\beta=\Delta\cap\partial D_1$. Let Δ' be the image of Δ in N'. Without loss of generality, we may suppose that $\partial\Delta'\cap D^2=\phi$, and $\partial\Delta'\cap D^1$ consists of an arc α' parallel to α in D_1. Let β' be the image of β in N'. Then $\partial\Delta'=\alpha'\cup\beta'$, and $\partial\Delta'$ bounds a disk D' in $R_+(\delta')$ such that Δ' is parallel to D'. If D' does not contain D^2 then we can move D_1 by an isotopy to reduce $\#(D\cap D_1)$. Suppose that D' contains D^2. Then trace the arc $\tilde{\alpha}=\partial D_1-\beta$ from one endpoint to the other. It is easy to see that there is a subarc α^* of $\tilde{\alpha}$ such that $\alpha^*\cap D=\partial\alpha^*$, the image of α^* in N' is an arc contained in D', and the endpoints of the image of α^* is contained in ∂D^2 (Figure 4.1). This shows that, by moving D_1 by an isotopy, we can remove α^*. Hence, by the induction on $\#(D_1\cap D)$, we have the conclusion.

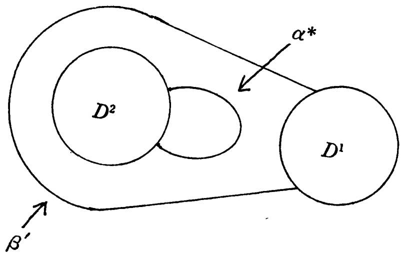


Fig. 4.1

Proof of Proposition 4.2. We prove Proposition 4.2 for $\bar{D}^+$ and $\bar{D}^{+\prime}$. The other case is essentially the same. Let (N^c, δ^c), $(\bar{N}, \delta^c)$ be as above. Then we may suppoe that $D^{+\prime}=\bar{D}^{+\prime}\cap N^c$ is a disk. Let $S_{1/2}$ be the surface in N^c corresponding to $S\times\{1/2\}$ $(\subset(\bar{N}, \delta')\cong(S\times I, \partial S\times I))$. Then by using standard innermost disk arguments, we may suppose that $D^{+\prime}\cap S_{1/2}=\phi$. Then, by Lemma 4.3 or Lemma 4.4, we see that $D^{+\prime}$ is ambient isotopic to D^+ in N^c. This shows that $\bar{D}^{+\prime}$ is isotopic to $\bar{D}^+$ by an ambient isotopy respecting S.

This completes the proof of Proposition 4.2.

Let F be a fiber surface in a 3-manifold M, and φ: $F\to F$ a monodromy map. Suppose that there is an arc $a(\subset S)$ such that;

(4.1) $a\cap\varphi(a)=\partial a=\partial\varphi(a)$, and

(4.2) the components of $N(\partial\varphi(a); \varphi(a))$ lie in one side of a (Figure 4.2).

The purpose of this section is to prove;

Proposition 4.5. *Let F, φ, a be as above. If M is a rational homology*

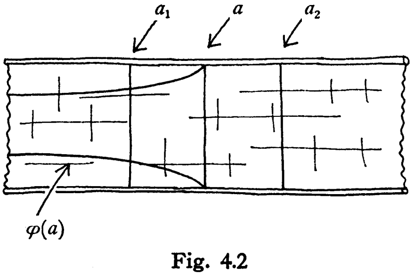


Fig. 4.2

3-sphere, and a does not separate F, then the surface obtained from F by cutting along a is a pre-fiber surface.

In case when a separates F, we have;

Proposition 4.6. *Let F, φ, a be as above. If a separates F, then there is a separating 2-sphere S^2 in M such that $S^2 \cap F = a$, i.e. F is a connected sum of two fiber surfaces.*

Proof of Proposition 4.6. Suppose that a separates F into F_1 and F_2. Since $\varphi|_{\partial F} = \mathrm{id}_{\partial F}$, and φ is a homeomorphism, we see that $\varphi(F_i)$ is rel ∂ isotopic to F_i. Hence, we may suppose that $\varphi(a) = a$. Take a pp disk $\square$ such that $\partial_-\square = \partial_+\square = a$. Then $\square$ is topologically an annulus. Then, by adding two meridian disks to $\square$, we get a 2-sphere S^2 in M, which intersects F in a.

Assume that S^2 does not separate M. Let M' be the 3-manifold obtained from M by cutting along S^2, and then capping off the boundary by two 3-cells. We note that the complementary sutured manifold (N', δ') for the disconnected surface $F_1 \cup F_2$ in M' is homeomorphic to the sutured manifold obtained from the complementary sutured manifold (N, δ) of F by decomposing along the product disk $\square \cap N$. Hence $F_1 \cup F_2$ is a fiber surface in a connected 3-manifold M', contradicting Lemma 2.1.

Proof of Proposition 4.5. Let a_1 and a_2 be the components of $\mathrm{Fr}_F N(a; F)$. We may suppose that $a_1 \cap \varphi(a)$ consists of two points, and $a_2 \cap \varphi(a) = \phi$. See Figure 4.2. Let α be the subarc of a_1 such that $\partial\alpha = a_1 \cap \varphi(a)$, and $l = (\varphi(a) - N(a; F)) \cup \alpha$. Then l is a simple loop on F.

Claim 4.1. *There exists a disk D in M such that $\partial D = l$, and* $(\mathrm{Int}\, D) \cap F = a$.

Proof. Let $\square$ be a pp disk for F such that $\partial_-\square = a$, $\partial_+\square = \varphi(a)$. We note that $\square \cap \partial E(L)$ consists of two meridian loops. Let D_1, D_2 be meridian disks for L such that $\partial D_1 \cup \partial D_2 = \square \cap \partial E(L)$, and $\bar{\square} = \square \cup D_1 \cup D_2$. Then we identify $F \cap E(L)$ to F. Let B be the rectangle in F such that one edge is a, two edges are the components of $\varphi(a) \cap N(a; F)$, and the last edge is α. Then $\bar{D} = \bar{\square} \cup B$ is topologically a disk such that $\partial\bar{D} = l$, and $\bar{D} \cap F = B \cup l$. Then, by deforming $\bar{D}$ by pushing $B - (\alpha \cup a)$ slightly to the $-$side of F, we get a disk D satisfying the conclusion.

Let S_1 be the surface obatined from F by cutting along a, and D as in Claim 4.1. Then $D \cap S_1 = \partial D = l$, and we have;

Claim 4.2. *No component of the surface obtained from S_1 by doing a surgery along D, is closed.*

Proof. If l is non-separating in S_1, then Claim 4.2 is clear. Hence assume that l separates S_1 into S' and S'' such that $S' \cup D$ is a closed surface. Since

a is non-separating in F, there is a simple loop m on F such that $m \cap l = \phi$, and m intersects a in one point. Then m intersects the closed surface $S' \cup D$ in one point, contradicting the fact that M is a rational homology 3-sphere.

Let a' be the component of $\mathrm{Fr}_F N(\varphi(a); F)$ such that $a' \cap l = \phi$. Then we have;

Claim 4.3. *There is a properly embedded arc a'' ($\subset F$) such that $a'' \cap (a \cup a') = \phi$, $a'' \cap l = \phi$, and $a \cup a' \cup a''$ cuts off an annulus $\mathcal{A}$ from F such that l is a core of $\mathcal{A}$.*

Proof. Let F' be the component of the surface obtained from F by cutting along $a \cup a'$ such that $l \subset F'$. Then l is parallel to the component of $\partial F'$ which meets $a \cup a'$. By Claim 4.2, there is a component l' of ∂F such that $l' \subset F'$. Let β be an arc in F' such that $\beta \cap l' = \partial\beta \cap l'$ consists of one point, the other endpoint of β is contained in l, and, $\mathrm{Int}\,\beta \cap l = \phi$. Then $\mathrm{Fr}_{F'} N(\beta \cup l; F')$ consists of two components such that one is a simple loop parallel to l, and the other is an arc a'' properly embedded in F'. It is easy to see that a'' satisfies the conclusion.

Claim 4.4. *Let a', a'', $\mathcal{A}$ be as in Claim 4.3. Then there is a 3-ball B^3 in M such that $B^3 \cap F = \mathcal{A}$, and $\mathcal{A}$ looks as in Figure 4.3 in B^3.*

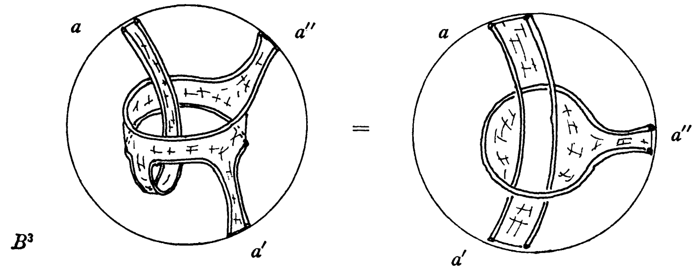


Fig. 4.3

Proof. Let $\bar{D}$, B be as in the proof of Claim 4.1. Then $N(\mathcal{A} \cup \bar{D}; M)$ is a 3-ball, and $\mathcal{A}$, $\bar{D}$ looks as in Figure 4.4 in the 3-ball. Since D is obtained from

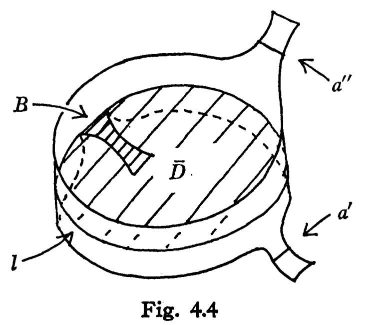
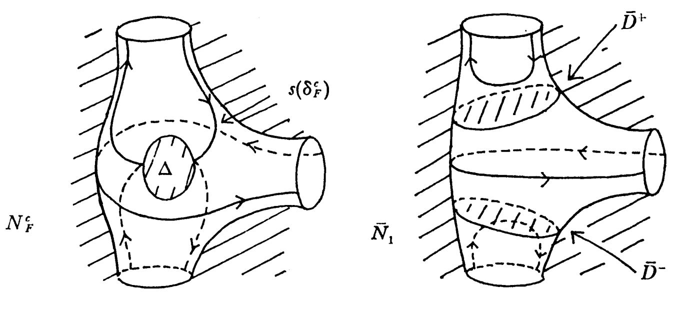


Fig. 4.4

$\bar{D}$ by pushing $B-(\alpha\cup a)$ to the $-$ side of F, it is easy to see that the conclusion holds.

Let D, a', a'', B^3 be as in Claim 4.4. By Figure 4.3, we see that the complementary sutured manifold (N_F^c, δ_F^c) for F looks as in Figure 4.5 (i) in B^3. Let $\square$ be a pp disk for F such that $\partial_-\square=a$. Then we may suppose that $\square\subset B^3$, and $\Delta=\square\cap N_F^c$ is a product disk for (N_F^c, δ_F^c) (Figure 4.5 (i)). Let $(\bar{N}_1, \bar{\delta}_1)$ be the product sutured manifold obtained from (N_F^c, δ_F^c) by a product decomposition along Δ, $\bar{D}^-$, $\bar{D}^+$ the disks properly embedded in $cl(E(L)-\bar{N}_1)$ as in Figure 4.5 (ii). Let S_2 be the surface obtained from S_1 by doing surgery along D. See Figure 4.6. Finally, let (N_1, δ_1) ((N_1^c, δ_1^c) resp.) be the sutured manifold obtained from S_1 (the complementary sutured manifold for S_1 resp.).



(i) (ii)

Fig. 4.5

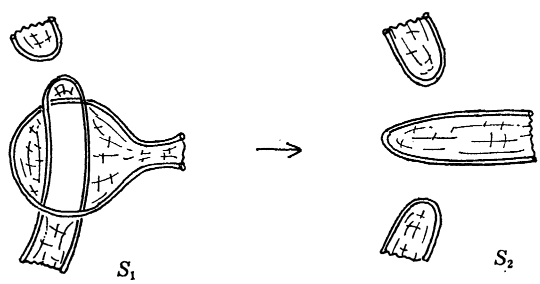
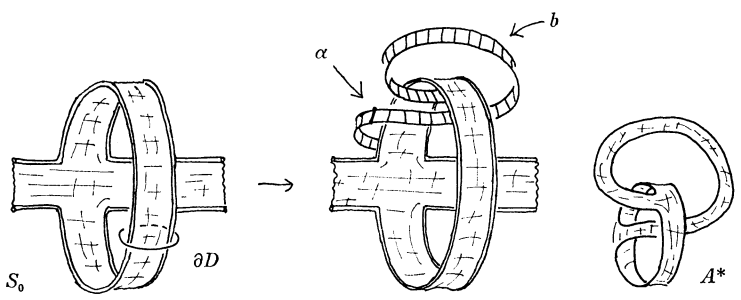


Fig. 4.6

Since S_1 is obtained from F by cutting along a, and (N_F^c, δ_F^c) is properly isotopic in $E(\partial F)$ to the sutured manifold obtained from F (note that F is a fiber surface), we see that $(\bar{N}_1, \bar{\delta}_1)$ is ambient isotopic to (N_1, δ_1) in M. Hence, hereafter, we identify (N_1, δ_1) to $(\bar{N}_1, \bar{\delta}_1)$, and we identify S_1 to $S_1\times\{1/2\}$ $(\subset S_1\times I=\bar{N}_1)$. Then $\bar{D}^+$, $\bar{D}^-$ are compressing disks for $R_+(\delta_1^c)$, $R_-(\delta_1^c)$ in $N_1^c(=cl(E(\partial S_1)-\bar{N}_1))$ respectively. Let N^* be the manifold obtained from N_1^c by doing surgery along $\bar{D}^+\cup\bar{D}^-$. Then (N^*, δ_1^c) is ambient isotopie to the sutured manifold obtained from S_2 (see Figure 4.6). This shows that S_1 is a pre-fiber surface, and this completes the proof of Proposition 4.5.

As a consequence of Proposition 4.5, we have;

Proof of Theorem 2(1). Let D be the crossing disk for L. Then, by Theorem 1, we see that S looks as in Figure 1.1. Then S_0 looks as in Figure 4.7 (i). Let S^* be the surface obtained from S_0 by adding a band b as in Figure 4.7 (ii). We note that S_0 is a plumbing of F and a fiber surface A^* in S^3 (Example 2.4). Hence S^* is a fiber surface. Moreover, by Figure 4.7 (ii), it is directly observed that the arc α in Figure 4.7 (ii) satisfies the assumptions of Proposition 4.5 (cf. Figure 4.3). Hence, by Proposition 4.5, we see that S_0 is a pre-fiber surface.



(i) (ii)

Fig. 4.7

Let S_0 be as in Theorem 2, a_0 as in Figure 1.2, and D^+, D^- a pair of canonical compressing disks for the pre-fiber surface S_0. Then the next lemma will be used in section 6 to prove Proposition 6.1.

Lemma 4.7. *Let S_0, a_0, D^+, and D^- be as above. Then ∂D^+, and ∂D^- are ambient isotopic in S_0 to a loop intersecting a_0 in one point.*

Proof. Without loss of generality we may suppose that the Hopf band A is attached to the $+$ side of F (Figure 1.1). Then there is a compressing disk $\bar{D}^-$ for S_0 such that $\partial\bar{D}^-$ corresponds to the core curve of A, and $N(\partial\bar{D}^-; \bar{D}^-)$ lies in the $+$ side of S_0. Then by the proof of Theorem 2 (1) (Figure 4.7), and the proof of Proposition 4.5 (Figures 4.5, 4.6), we see that $\bar{D}^-$ is a component of a pair of canonical comporesing disks for S_0. Hence, by Proposition 4.2, we see that ∂D^- is ambient isotopic to a loop intersecting a_0 in one point. Let $a(\subset S)$ be the arc correspondsing to a_0 (Figure 4.8). Then it is directly observed from Figure 4.8 that there is a pp disk $\square$ such that $\partial_+\square=a$, $\partial_+\square\cap\partial_-\square=\partial a$, and the components of $N(\partial a; \partial_-\square)$ lie in pairwise different sides of a. Hence there is a monodromy map ψ: $S\to S$ such that $\psi^{-1}(a)\cap a=\partial a$, and the components of $N(\partial\psi^{-1}(a); \psi^{-1}(a))$ lie in pairwise different sides of a.

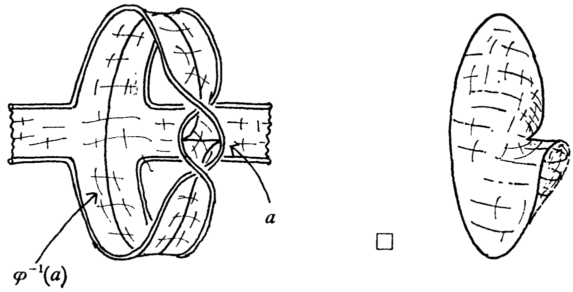


Fig. 4.8

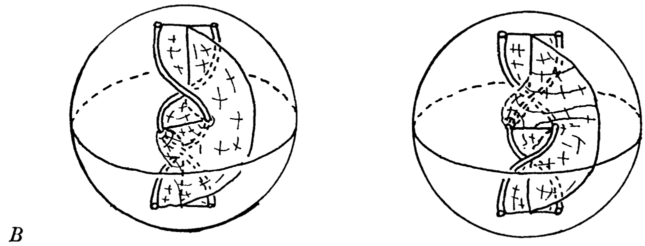


Fig. 4.9

Let $\square'$ be pp a disk such that $\partial_{-}\square'=a$, $\partial_{+}\square'=\psi(a)$. Roughly speaking, $\square'=\psi(\square)$. Then $\square'$ looks as in Figure 4.9 in the 3-ball $B=N(a;M)$.

Let b_0 be an unknotted band, and Δ_0 a disk in a 3-ball B_0 as in Figure 4.10. Let $h\colon \partial B\to\partial B_0$ be a homeomorphism such that $h(S\cap\partial B)=h(b_0\cap\partial B_0)$, and $h(\square'\cap\partial B)=h(\Delta_0\cap\partial B_0)$. Then $(M-\operatorname{Int}B)\cup_h B_0=M$, and it is easy to see that $(S-\operatorname{Int}B)\cup b_0=S_0$ and $\bar{D}^{+}=(\square'-\operatorname{Int}B)\cup\Delta_0$ is a compressing disk for S_0 such that $N(\partial\bar{D}^{+};\bar{D}^{+})$ lies in the$-$side of S_0.

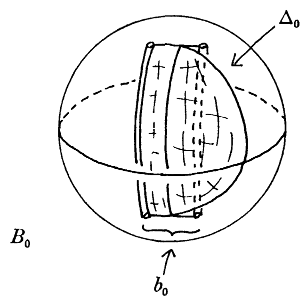


Fig. 4.10

By definition, it is easy to see that $\partial\bar{D}^{+}$ is ambient isotopic to a loop corresponding to ψ(*the core curve of* A). Hence $\bar{D}^{+}$ is a component of a pair of canonical compressing disks for S_0. Hence, by Proposition 4.2, ∂D^{+} is ambient isotopic to a loop intersecting a_0 in one point.

5. Propositions

In this section, we prove some technical propositions. For the statement of the results, we give some definitions.

Let M be a compact 3-manifold, μ a subsurface of ∂M. For a connected surface S properly embedded in (M, μ), let

$$\chi(S) = \max\{0, -\chi(S)\}.$$

When S is a union of connected surfaces $S_1, \cdots, S_n$, let

$$\chi(S) = \sum_{i=1}^{n} \chi_-(S_i).$$

Then, we define the function

$$x\colon H_2(M, \mu) \to Z$$

by

$$x(a) = \min\{\chi_-(S) \mid S \text{ is an embedded surface representing } a\}.$$

We say that S is *norm minimizing* if $\chi_-(S)=x([S])$, where $[S]$ denotes the homology class in $H_2(M, \mu)$ represented by S.

Let S' be a compact, connected surface with $\partial S' \neq \phi$, $\tilde{l}_0$, $\tilde{l}_1$ non separating simple loops in S'. Let $N=S'\times I$, $\delta=\partial S'\times I$, and $l_0=\tilde{l}_0\times\{0\}$, $l_1=\tilde{l}_1\times\{1\}$ $(\subset \partial N)$. Let $\bar{N}_0$ be the manifold obtained from N by attaching a 2-handle $\mathcal{D}_0$ along l_0, $\bar{N}$ the manifold obtained from N by attaching two 2-handles along $l_0 \cup l_1$. We may regard that $\bar{N}$ is obtained from $\bar{N}_0$ by attaching a 2-handle $\mathcal{D}_1$ along l_1. $\bar{\delta}_0$, $\bar{\delta}$ denote the images of δ in $\bar{N}_0$, $\bar{N}$ respectively. Then (N, δ), $(\bar{N}_0, \bar{\delta}_0)$, $(\bar{N}, \bar{\delta})$, have mutually coherent sutured manifold structures. The purpose of this section is to prove Propositions 5.1 and 5.2 below.

Proposition 5.1. *Suppose that $R_\pm(\bar{\delta})$ are not norm minimizing in $H_2(\bar{N}, \bar{\delta})$. Then $\tilde{l}_0$ is ambient isotopic to a loop disjoint from $\tilde{l}_1$.*

Remark. It is easily observed that if $\tilde{l}_0$ and $\tilde{l}_1$ are disjoint, and not parallel then $R_\pm(\bar{\delta})$ is not norm minimizing in $H_2(\bar{N}, \bar{\delta})$

Proposition 5.2. *Suppose that $(\bar{N}, \bar{\delta})$ is a product sutured manifold. Then $\tilde{l}_0$ is ambient isotopic to a loop intersecting $\tilde{l}_1$ in one point.*

As a consequence of Proposition 5.1, we have;

Corollary 5.4. *Let S_0 be a pre-fiber surface of type* 1 *in a rational homology* 3*-sphere M, D^+, D^- a pair of canonical compressing disks for S_0, and S_1 the surface obtained from S_0 by doing a surgery along D^+. Suppose that $\chi(L)>\chi(S_0)+2$, where $L=\partial S_0$. Then ∂D^+ is ambient isotopic in S_0 to a loop disjoint from ∂D^-, and S_1 is a pre-fiber surface, where D^- is a component of a pair of canonical com-*

pressing disks for S_1.

The proof of Proposition 5.1 is done by using the outermost fork argument of M. Scharlemann [11]. And the proof of Proposition 5.2 is done by using the Haken type argument of Casson-Gordon [1].

For the proof of the propositions, we prepare one lemma. Let $(E, \mathcal{E})$ be a connected sutured manifold. Suppose that there is a non separating compressing disk C for $R_+(\mathcal{E})$ such that $(\bar{E}, \bar{\mathcal{E}})$ is a product sutured manifold, where $\bar{E}$ is obtained from E by cutting along C, and $\bar{\mathcal{E}}$ the image of $\mathcal{E}$ in $\bar{E}$. Let A be an incompressible product annulus in $(E, \mathcal{E})$. Then;

Lemma 5.3. *A is ambient isotopic to an annulus disjoint from D by an ambient isotopy of E respecting $\mathcal{E}$.*

The proof of Lemma 5.3 is done by using the same arguments as that of Lemma 4.4. Hence we omit it.

Proof of Proposition 5.1. Let F be a norm minimizing surface in $(\bar{N}, \bar{\delta})$ such that $[F]=[R_+(\bar{\delta})]\in H_2(\bar{N}, \bar{\delta})$. Since $[F]=[R_+(\bar{\delta})]$, by piping the boundary components of F, if necessary, we may suppose that $\partial F=s(\bar{\delta})$ (Figure 5.1).

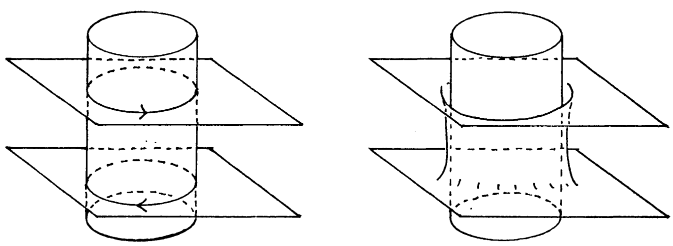


Fig. 5.1

The next claim will be used in the proof of Corollary 5.4.

Claim 5.0. *$\bar{N}$ is irreducible.*

Proof. Assume not. Let F be a surface in $\bar{N}$ corresponding to $S'\times\{1/2\}$, and V_1, V_2 the closure of the components of $\bar{N}-F$. Then (V_1, V_2) is a Heegaard splitting of $\bar{N}$ in terms of [1]. Henee, by [1, Lemma 1.1], we see that there is an essential 2-sphere S_1 in $\bar{N}$ such that $V_i\cap S_1$ consists of a disk. Then it is easy to see that $\bar{N}$ is a connected sum of a lens space and a product sutured manifold. But this contradicts the fact that $R_\pm(\bar{\delta})$ are not norm minimizing.

Claim 5.1. $F\cap\mathcal{D}_1\neq\phi$.

Proof. Assume that $F\cap\mathcal{D}_1=\phi$. Then we can regard that $F\subset\bar{N}_0$. Let

D be the disk properly embedded in $\bar{N}_0$ such that $D=(\tilde{l}_0\times I)\cup$ (*the core of* $\mathcal{D}_0$). Then the manifold N_0 obtained by cutting $\bar{N}_0$ along D is homeomorphic to $R_-(\delta_0)\times I$, where $R_-(\delta_0)\times\{0\}$ corresponds to $R_-(\delta_0)$. Since $\bar{N}_0$ is irreducible, by using standard innermost disk arguments, we may suppose that $F\cap D=\phi$. Hence we may regard that $F\subset N_0$. Then, by [**15**, Corollary 3.2], we see that F is a parallel to $R_-(\delta_0)$. Hence $\chi(F)=\chi(S')+2(=\chi(R_-(\delta)))$, a contradiction.

We may suppose that F intersects $\mathcal{D}_1$ in horizontal disks $E_1, \cdots, E_n$ in this order. Let $F_0=cl(F-\cup_{i=1}^{n} E_i)$, and A_i $(i=1, \cdots, n-1)$ the annulus in $\partial\bar{N}_0$ bounded by $\partial E_i\cup\partial E_{i+1}$. Let D be as in the proof of Claim 5.1. We suppose that $\#(\partial F_0\cap\partial D)$ is minimal among all disks ambient isotopic to D in $\bar{N}_0$. Let α be the dual core of the 2-handle $\mathcal{D}_1$. Then α is an arc in $\bar{N}$ such that $\alpha\cap\partial\bar{N}=\alpha\cap R_+(\delta)=\partial\alpha$. Since F is norm minimizing, by [**12**, 3.5 Lemma b)], we may suppose that F separates $\bar{N}$ into two components M_0, M_1 such that $M_0\supset R_-(\delta)$, $M_1\supset R_+(\delta)$. This shows that α intersects F an even number of times and the signs of the intersections are alternately different on α. Hence we have;

Claim 5.2. *n is an even number, and the orientations on $\partial E_1, \cdots, \partial E_n$ induced from F_0 are alternately different in $\partial\bar{N}_0$.*

Claim 5.3. *If $n=2$, then $\tilde{l}_0$ is ambient isotopic to a loop disjoint from $\tilde{l}_1$.*

Proof. Let $F_1=(F-(E_1\cup E_2))\cup A_1$. Then $\chi(F_1)=\chi(F)-2$. By the argument of the proof of Claim 5.1, we see that F_1 is parallel to $R_-(\delta_0)$. Hence, there is a product annulus A in $\bar{N}_0$ such that $A\cap R_+(\delta_0)=l_1$. Let $D(\subset\bar{N}_0)$ be as in the proof of Claim 5.1. Then D cuts $(\bar{N}_0, \delta_0)$ into a product sutured manifold. Hence, by Lemmas 5.3, we may suppose that D and A are disjoint. We note that $A_0=D\cap N$ is the product annulus $\tilde{l}_0\times I$ in (N, δ). Hence $\tilde{l}_0\times\{1\}$ and l_1 are disjoint, and we have the conclusion.

By Claim 5.3, hereafter, we suppose that $n\geq 4$. Let D be as above. Then, by using standard cut and paste arguments, we may suppose that $D\cap F_0$ consists of arcs. We suppose that $\#(\partial D\cap l_1)$ is minimal among all disks ambient isotopic to D in $\bar{N}_0$. Then;

Claim 5.4. *No component of $D\cap F_0$ is an inessential arc in F_0.*

Proof. Assume that a component β of $D\cap F_0$ is an inessential arc in F_0. Then there is a disk Δ_0 in F_0 such that $\mathrm{Fr}_{F_0}\Delta_0=\beta$. By doing ∂-compression on D along Δ_0 in $\bar{N}_0$, we get two disks D', D'' whose boundaries lie in $R_+(\delta_0)$. Since ∂D is non separating in $R_+(\delta_0)$, at least one of the disks, say D', is non separating in $\bar{N}_0$. By Lemma 4.4, we see that D' is ambient isotopic to D. On the other hand, by moving D' by an ambient isotopy, we have $\#(\partial D' \cap l_1) < \#(\partial D \cap l_1)$, a contradiction.

We get a planar tree T by corresponding each component of $D - F_0$ to a vertex, and each component of $D \cap F_0$ to an edge. We regard that T is embedded in D and each edge of T intersects $D \cap F_0$ in one point which is contained in the corresponding component of $D \cap F_0$. See Figure 5.2. Let γ be a component of $D \cap F_0$, and e_γ the edge of T corresponding to γ. Then $\gamma \cap e_\gamma$ consists of a point, which separates γ into two arcs γ_1 and γ_2. One endpoint of γ_i lies in $\cup_{j=1}^{n} \partial E_j$. Labell the corresponding side of e_γ by k if the endpoint lies in ∂E_k. Then we can labell the each side of the edges of T by $\{1, \cdots, n\}$.

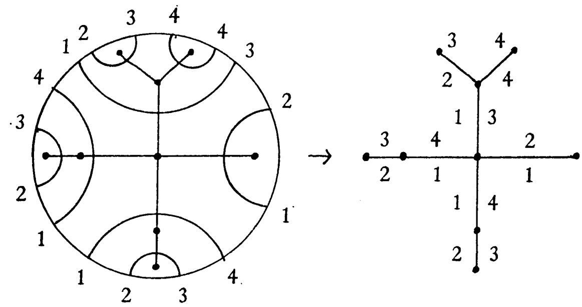


Fig. 5.2

In general, for a tree $\mathcal{T}$, an *outermost vertex* is a vertex with valency 1. An edge adjacent to an outermost vertex is called an *outermost edge*. A *fork* is a vertex with valency ≥ 3. Let $\mathcal{F}$ be the collection of the forks of $\mathcal{T}$. Let $\mathcal{T}'$ be the tree obtained by removing all components of $\mathcal{T} - \mathcal{F}$ which contains an outermost vertex. An outermost vertex for $\mathcal{T}'$ is an *outermost fork* of $\mathcal{T}$. If $\mathcal{F} = \phi$, then $\mathcal{T}$ does not contain an outermost fork. If v is an outermost fork, then the components of $\mathcal{T} - v$ which contain no forks are called *outermost lines* of v. If v_0 (e_0 resp.) is a vertex (an edge resp.) which is contained in an outermost line of v, then we say that v_0 (e_0 resp.) is *dominated* by v. Then we have;

Claim 5.5. *If there is an outermost edge of T which is labelled by i and $i+1$ for some $i \in \{1, \cdots, n-1\}$, then there is a norm minimizing surface F' in $(\bar{N}, \delta)$ such that $[F'] = [F]$ and, $\#(F' \cap \mathcal{D}_1) = \#(F \cap \mathcal{D}_1) - 2$.*

Proof. Let Δ be the closure of the component of $D - F_0$ corresponding to the outermost vertex adjacent to the outermost edge. Let $F_1 = (F - (E_i \cup E_{i+1})) \cup A_i$. By Claim 5.2, we see that F_1 is orientable. Then $[F_1] = [F] \in H_2(\bar{N}, \delta)$, and $\chi(F_1) = \chi(F) - 2$. Since the core arc of A_i intersects $\partial\Delta$ in one point, $\partial\Delta$ is an essential loop in F_1. Hence Δ is a compressing disk. Let F' be the surface obtained from F_1 by doing a surgery along Δ. By moving F' by a tiny isotopy, we see that F' satisfies the conclusion.

Claim 5.6. *Suppose that there is a vertex v of T such that v is not an outermost vertex, and the adjacent edges of v are labelled alternately by i and $i+1$ (Figure 5.3). Then there is a norm minimizing surface F' in $(\bar{N}, \bar{\delta})$ such that $[F']=[F']$, and $\#(F'\cap \mathcal{D}_1)=\#(F\cap \mathcal{D}_1)-2$.*

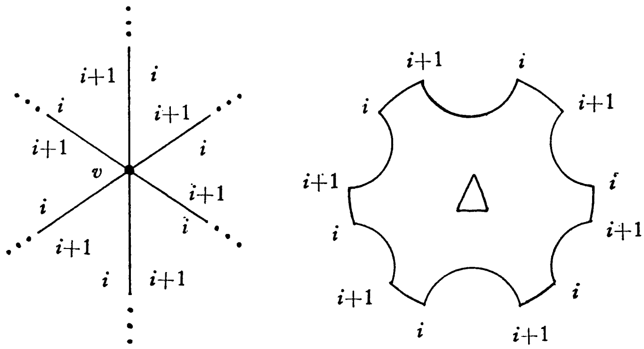
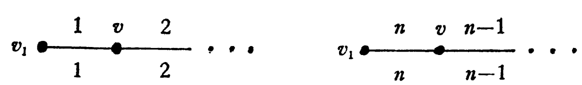


Fig. 5.3

Proof. Let Δ be the closure of the component of $D-F_0$ corresponding to v, and $F_1=(F-(E_i\cup E_{i+1}))\cup A_i$. Then F_1 is orientable (see the proof of Claim 5.5), $[F_1]=[F]$, and $\chi(F_1)=\chi(F)-2$. $\Delta\cap F_1=\partial\Delta$, and the absolute value of the algebraic intersection number of $\partial\Delta$ with the core of A_i is the number of the edges adjacent to v. Hence Δ is a compressing disk for F_1. Let F' be the surface obtained from F_1 by doing surgery along Δ. By moving F' by a tiny isotopy, we see that F' satisfies the conclusion.

Claim 5.7. *If there is an outermost line with the pattern as in Figure 5.4, then there is a norm minimizing surface F' in $(\bar{N}, \bar{\delta})$ such that $[F']=[F]$, and $\#(F'\cap \mathcal{D}_1)=\#(F\cap \mathcal{D}_1)-2$.*



(i) (ii)

Fig. 5.4

Proof. Suppose that there is a pattern of Figure 5.4 (i). The other case is essentially the same. Let Δ be the closure of the component of $D-F_0$ corresponding to v (Figure 5.4), and $F_1=(F-(E_1\cup E_2))\cup A_1$. Then $\Delta\cap F_1=\partial\Delta$. Hence if $\partial\Delta$ is not contractible in F_1, then, by compressing F_1 along Δ, we have a surface F' satisfying the conclusions. Hence, in the rest of the proof, we suppose that $\partial\Delta$ is contractible in F_1. Then $\Delta\cap cl(F-(E_1\cup E_2))$ consists of two

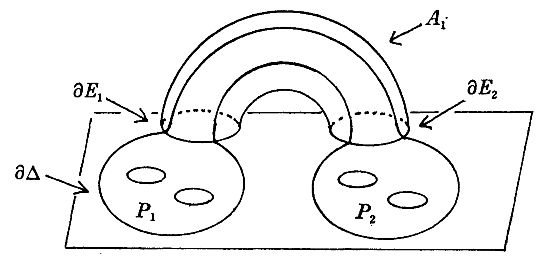


Fig. 5.5

inessential arcs β_1, β_2 in $cl(F-(E_1 \cup E_2))$ such that $\partial\beta_i \subset \partial E_i$ $(i=1, 2)$. Hence there are two planar surfaces P_1, P_2 in F_0 such that $\mathrm{Fr}_{F_0} P_i = \beta_i$ (Figure 5.5). By Claim 5.4, we see that P_i is not a disk.

Sublcaim 1. *T contains a fork.*

Proof. Assume that T does not contain a fork. Then, by tracing the edges of T from v_1 (Figure 5.4), we see that there are n components $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$ of $D \cap F_0$ such that $\partial\beta_i \subset \partial E_i$ $(i=1, \cdots, n)$, where β_1, β_2 are as above. Then it is easy to see that some β_j contained in P_1 is an inessential arc in F_0, contradicting Claim 5.4.

Let v_0 be the outermost fork which dominates v_1, v_2 an outermost vertex dominated by v_0, and located next to v_1. By using the argument of the proof of Subclaim 1, we have;

Subclaim 2. *The outermost line between v_0 and v_1 contains at most $n-1$ edges.*

Subclaim 3. *Either the conclusions of Claim* 5.7 *holds or the outermost edge adjacent to v_2 is labelled by* 1 *and n.*

Proof. Suppose that the outermost edge is not labelled by 1 and n. Then, by Claim 5.5, we see that either the conclusions of Claim 5.7 hold or the edge is labelled by two 1's or two n's. Suppose that the second case occurs. If the outermost line between v_0 and v_2 contains more than $n-1$ edges, then we have a contradiction as in the proof of Subclaim 1. Hence the outermost line contains at most $n-1$ edges, and this fact together with Subclaim 2 show that there are exactly n edges between v_1 and v_2 in T, and the outermost edge adjacent to v_2 is labelled by two n's (Figure 5.6). Then, by tracing the edges in T from v_1 to v_2, we again have a contradiction as in the proof of Subclaim 1.

Suppose that the second conclusion of Subclaim 3 holds. If the outermost line between v_0 and v_2 contains more than $n/2$ edges, then we have a pattern of Figure 5.3 in the outermost line, so that we have the conclusion of Claim 5.7

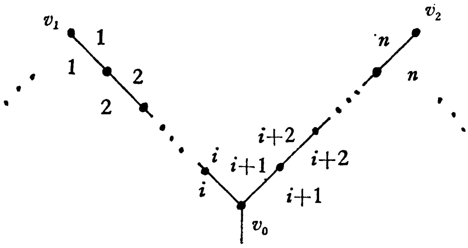


Fig. 5.6

by Claim 5.6. Assume that the outermost line contains $j(\leq n/2)$ edges. By Subclaim 2, we see that there are exactly n edges between v_1 and v_2 in T (Figure 5.7).

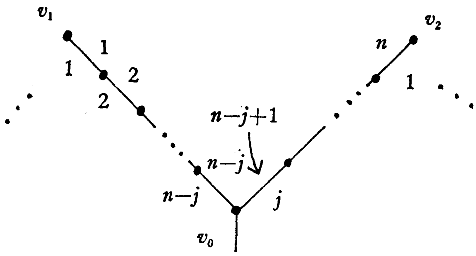


Fig. 5.7

Let $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$ be the components of $D \cap F_0$ corresponding to the edges between v_1 and v_2 in T. Then, for $i \leq n-j$, $\partial\beta_i \subset \partial E_i$. Then fix some $\beta_k(k \leq n-j)$ suc such that $\beta_k \subset P_1$, and β_k is innermost, i.e. β_k cuts off a planar surface P_k from F_0 such that no component of $\partial E_1 \cup \partial E_2 \cup \cdots \cup \partial E_{n-j}$ is contained in P_k (Figure 5.8).

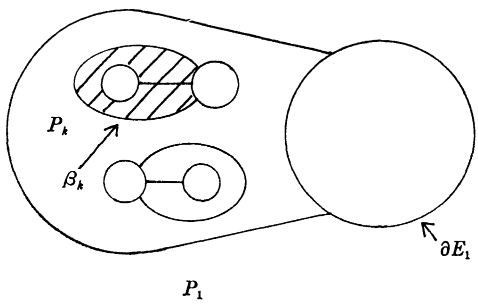


Fig. 5.8

By Claim 5.4, we see that some $\partial E_m(m \geq n-j+1)$ is contained in ∂P_k. Since $j \leq n/2$ and β_k is innermost, we see that β_m joins ∂E_k and ∂E_m. This shows that $m=n+1-k$, so that P_k is an annulus. Then, by Claim 5.4, we see that every component of $D \cap F_0$ which meets ∂E_m joins ∂E_m and ∂E_k. But this contradicts the fact that $\#(\partial D \cap \partial E_m) = \#(\partial D \cap \partial E_k)$, and this completes the proof of Claim 5.7.

Completion of the proof of Proposition 5.1. We suppose that $\#(F \cap \mathcal{D}_1)$ is minimal among all norm minimizing surfaces representing $[R_+(\delta)]$. If $\#(F \cap \mathcal{D}_1)=2$, then, by Claim 5.3, we have the conclusion. Assume that $n>2$. By Claim 5.5, we see that each outermost edge is labelled by either two 1's, two n's or 1 and n.

Suppose that T does not have a fork. If an outermost edge is labelled by two 1's or two n's, then we have a contradiction by Claim 5.7. If an outermost edge is labelled by 1 and n, then we have a pattern of Figure 5.3 in T, so that we have a contradiction by Claim 5.6. Hence T has a fork.

Let v be an outermost fork for T. If all the outermost edges dominated by v are labelled by 1 and n, then by Claim 5.6, we see that each outermost line contains at most $n/2$ edges. Hence the adjacent edges of v are labelled alternately by $n/2$ and $n/2+1$, contradicting Claim 5.6. Hence we may suppose that some outermost edge dominated by v is labelled by two 1's. Then, by Claim 5.7, we see that v is adjacent to the edge. Let v_1 be an outermost vertex which is dominated by v and next to the outermost edge. By Claim 5.5, we see that there are at least $n-1$ edges in the outermost line between v and v_1. Then, by Claims 5.5 and 5.7, we see that the edge adjacent to v_1 is labelled by 1 and n. Hence we have a pattern of Figure 5.3 in the outermost line, contradicting Claim 5.6, and this completes the proof.

Proof of Corollary 5.4. Let $(N_i, \delta_i)((N_i^c, \delta_i^c)$ resp.) be the sutured manifold obtained from S_i (the complementary sutured manifold for S_i resp.) ($i=0, 1$). Then we may suppose that $D_0^\pm = D^\pm \cap N_0^c$ are disks properly embedded in N_0^c, and $(cl(N_0^c - N(D_0^+ \cup D_0^-; N_0^c), \delta_0^c)$ is properly isotopic to (N_1, δ_1) in $E(L)$. Hence, hereafter, we identify (N_1, δ_1) to $(cl(N_0^c - N(D_0^+ \cup D_0^-; N_0^c), \delta_0^c)$. Then (N_1^c, δ_1^c) is obtained from (N_0, δ_0) by attaching two 2-handles $N(D_0^+; N_0^c)$, $N(D_0^-; N_0^c)$ along the simple loops $\partial D^+ \times \{1\}$, $\partial D^- \times \{0\}$ in (N_0, δ_0) $(\cong (S_0 \times I, \partial S_0 \times I))$.

Case 1. $\chi_-(S_1) > 0$.

In this case, S_1 is not norm minimizing. Hence, by Claim 5.0, and [**5**, Lemma 0.4] or [**12**, section 3], we see that $R_+(\delta_1^c)$ is not norm minimizing in $H_2(N_1^c, \delta_1^c)$. Then, by Proposition 5.1, we may suppose that ∂D^+ and ∂D^- are disjoint. Moreover since M is a rational homology 3-sphere, they are not parallel. Let $D_1^\pm = D_0^\pm \cup \mathcal{A}^\pm$, where $\mathcal{A}^+$, $\mathcal{A}^-$ are the product annuli $\partial D^+ \times I$, $\partial D^- \times I$ in (N_0, δ_0). Then D_1^+, D_1^- are mutually disjoint disks properly embedded in N_1^c such that $D_1^+ \cup D_1^-$ cuts (N_1^c, δ_1^c) into a product sutured manifold. Hence S_1 is a pre-fiber surface and clearly D^- corresponds to D_1^-.

Case 2. $\chi_-(S_1)=0$.

Since $\chi_-(S_1)=0$, $\chi_-(S_0)$ is either 0, 1 or 2. Since S_0 is a pre-fiber surface of type 1, S_0 contains a non separating loop. Hence it is easy to see that S_0 is either a torus with one hole, or a torus with two holes. If S_0 is a torus with one hole, then S_1 is a disk so that $\chi(L)=1=\chi(S_0)+2$, a contradiction. Suppose that S_0 is a torus with two holes, so that S_1 is an annulus. Then

Claim. *There are mutually disjoint disks E_1, E_2 in M such that $(E_1\cup E_2)\cap S_1=\partial(E_1\cup E_2)=L$.*

Proof. Since $\chi(L)>\chi(S_0)+2$, we see that there is a Seifert surface $\mathcal{E}$ for L such that $\chi(\mathcal{E})=2$, so that $\mathcal{E}$ is a union of two disks. Then, by using standard innermost disk, outermost arc arguments, we see that either $\mathcal{E}$ satisfies the conclusion of Claim, or $\mathcal{E}$ intersects S_1 in essential loops in S_1, so that S_1 is compressible. Suppose that the second conclusion holds. Then by doing a surgery along a compressing disk for S_1, and moving the resulting surface by a tiny isotopy, we get a pair of disks satisfying the conclusion.

By the above claim, we see that E_1, E_2 are embedded in (N_1^c, δ_1^c), so that, by regarding $E_1\cup E_2$ as F, the proof of Proposition 5.1 shows that ∂D^+ is ambient isotopic in S_0 to a loop disjoint from ∂D^-. Hence, by the argument of Case 1, we see that the conclusion holds.

Proof of Proposition 5.2. Let $\{D_1, \cdots, D_n\}$ be a system of mutually disjoint product disks in $(\bar{N}, \bar{\delta})$ such that $\cup D_i$ decomposes $(\bar{N}, \bar{\delta})$ to the product sutured manifold $(D^2\times I, \partial D^2\times I)$. Let $\bar{S}$ be the surface corresponding to $S'\times\{1/2\}$ in $\bar{N}$. $\bar{S}$ is a Heegaard surface of $(\bar{N}, \bar{\delta})$ [**1**]. Then, by the arguments of the proof of [**1**, Lemma 1.1], and the distinguished circle argument of Ochiai [**8**, Lemma], we may suppose that each D_i intersects $\bar{S}$ in an arc. We note that the arguments in [**1**, Lemma 1.1] and [**8**] work for product disks. Hence the image of $\bar{S}$ in $D^2\times I$ is a torus with one hole T with $\partial T=\partial D^2\times\{1/2\}$. Moreover, by using the core disks of the 2-handles, we see that T has two compress-

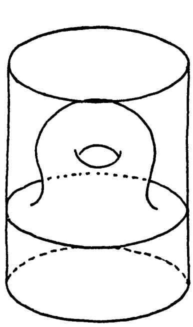


Fig. 5.9

ing disks D_0, D_1 such that ∂D_i corresponds to $\tilde{l}_i$ in S', $N(\partial D_0; D_0)$ lies in the $+$ side of T, and $N(\partial D_1; D_1)$ lies in the $-$ side of T. This fact together with Lemma 4.4 shows that $\tilde{l}_0$ is isotopic to a loop intersecting $\tilde{l}_1$ in one point. See Figure 5.9.

6. Monodromy maps

Let L, L', S, F, A, S_0, S_1, and M be as in Theorem 2, and b as in Figure 1.1. Let φ: $F \rightarrow F$ be a monodromy map, and a ($\subset F$) a component of $\mathrm{Fr}_F N(b; F)$ (Figure 1.1). The purpose of this section is to prove the following proposition.

Proposition 6.1. *If $\chi(L') > \chi(L)+2$, then, by deforming φ by a rel ∂ ambient isotopy, if necessary, we may suppose that $a \cap \varphi(a) = \partial a = \partial\varphi(a)$, and the components of $N(\partial\varphi(a); \varphi(a))$ lie in one side of a (Figure 4.2).*

REMARK. Proposition 6.1 together with Proposition 4.6 shows that if $\chi(L') > \chi(L)+2$, then a is non separating in F.

Then we give a proof of Theorem 2 (2). As a consequence of Proposition 6.1, we have;

Corollary 6.2. *Let S be as in Theorem 2 (2), and ψ: $S \rightarrow S$ a monodromy map of S. Then there is a non separating simple loop l in S such that $\psi(l)$ is ambient isotopic in S to a loop disjoint from l.*

Proof of Proposition 6.1. Let (N, δ), (N_0, δ_0), (N_1, δ_1) be the sutured manifolds obtained from S, S_0, S_1 respectively, and (N^c, δ^c), (N_0^c, δ_0^c), (N_1^c, δ_1^c) the complementary sutured manifolds for S, S_0, S_1 respectively. By Theorem 2 (1) (section 4), S_0 is a pre-fiber surface. Let D_0^+, D_0^- be a pair of canonical compressing disks for S_0. Then we may suppose that D_0^- looks as in Figure 6.1.

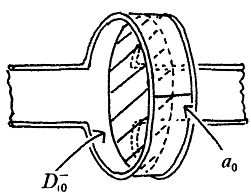


Fig. 6.1

By Lemma 4.7, we may suppose that ∂D_0^+ intersects a_0 of Figure 1.1 in one point. By Corollary 5.4, we may suppose that ∂D_0^+ and ∂D_0^- are pairwise disjoint. Hence ∂D_0^+ looks as in Figure 6.2.

Claim 6.1. *There is a disk D in M such that $D \cap S_1 = D \cap \mathrm{Int}\, S_1 = \partial D$, and D intersects the band b in an essential arc a_1.*

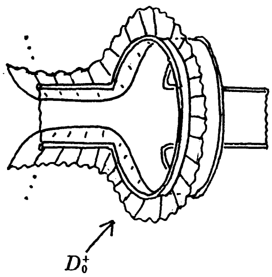


Fig. 6.2

Proof. We identify S_1 to the surface obtained from S_0 by doing a surgery along D_0^-. Let $D=D_0^+$. By Figure 6.3, it is directly observed that D satisfies the conclusions.

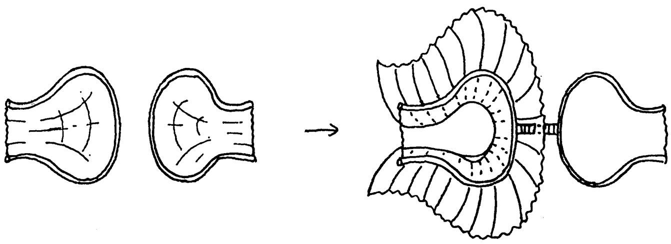


Fig. 6.3

Let $\square$ be a pp disk for F such that $\partial_-\square=a$, $\partial_+\square=\varphi(a)$. Suppose that $\varphi(a)$ does not run through b. Then it is easy to see that we have the conclusion of Proposition 6.1. Hence suppose that $\varphi(a)$ runs through b. Then, by deforming $\square$ by an isotopy as a pp disk, we may suppose;

(6.1) $\partial_+\square\cap b$ consists of arcs joining the components of $\mathrm{Fr}_F\, b$, and $\#\{(\partial_+\square\cap b)\cap a_1\}$ is minimal among the rel ∂ isotopy class in b, and

(6.2) If α is a component of $\partial_+\square\cap(F-b)$ such that $\partial\alpha\subset\mathrm{Fr}_F\, b$, then α is not rel ∂ isotopic in $cl(F-b)$ to a subarc of $\mathrm{Fr}_F\, b$.

Since $\partial_-\square\cap D=a\cap D=\phi$, we see that each component of $\square\cap D$ is either an arc whose endpoints lie in $\partial_+\square$, or a simple loop. Then;

Claim 6.2. *If necessary, by applying cut and paste on D, we may suppose that $\square\cap D$ consists of arcs.*

Proof. Let (N_F^c, δ_F^c) be the complementary sutured manifold for F. Then we may suppose that $\square\cap N_F^c$ is a product disk. Suppose that a component l of $\square\cap D$ is a simple loop. We may suppose that $l\subset(\square\cap N_F^c)$. Then l bounds a disk in $\square$. Hence, we can apply a cut and paste on D, by using the disk, to remove l. Do the same untill all the simple loops are removed.

Let p: $F\times I\to E(\partial F)$ be a natural map (section 2), and $\mathscr{D}$ the product disk in $(F\times I, \partial F\times I)$ such that $p(\mathscr{D})=\square$. Then, by Claim 6.2, we see that $p^{-1}(D)$ consists of arcs whose endpoints lie in $\mathscr{D}\cap(F\times\{1\})$. Then let Δ be the closure of an outermost component of $\mathscr{D}-p^{-1}(D)$ which does not intersect $\mathscr{D}\cap(F\times\{0\})$ (Figure 6.4). Then $\beta=p(\Delta)\cap D(=p(\mathrm{Fr}_{\mathscr{D}}\Delta))$ is an arc with $\beta\cap a_1=\partial\beta$. Let α be the subarc of a_1 such that $\partial\alpha=\partial\beta$. Then $\alpha\cup\beta$ bounds a disk D^* in D. If D^* does not contain ∂a_1 (Figure 6.5 (i)), then, by (6.2), $p(\Delta)\cup D^*$ is a compressing disk for F, a contradiction. Hence $\partial a_1\subset D^*$ (Figure 6.5 (ii)). Then $\square^*=D^*\cup p(\Delta)$ is a pp disk for F such that $\partial_-\square^*=a_1$. Since $\partial_+\square^*=(a_1-\alpha)\cup(p(\Delta)\cap F)$, by moving $\square^*$ by a tiny isotopy as a pp disk, we get a pp disk $\square^{**}$ such that $\partial_-\square^{**}$ is properly isotopic to a in F (in fact, it moves through b), and $\partial_+\square^{**}$ does not go through b. Since $\partial_-\square^{**}$ is ambient isotopic to a_1, we have the conclusion of Proposition 6.1.

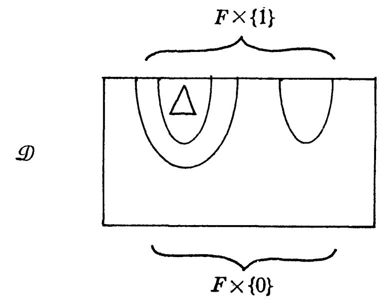


Fig. 6.4

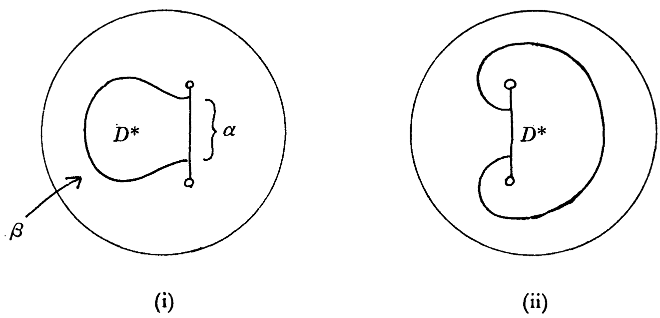


Fig. 6.5

Proof of Theorem 2 (2). By the remark of Proposition 6.1, we see that S_0 is a type 1 pre-fiber surface. Hence, by Corollary 5.4, we see that S_1 is a pre-fiber surface.

Proof of Corollary 6.2. Let l be a non separating simple loop in S corresponding to ∂D_0^- of Figure 6.1. By [3], we see that $\psi=\psi_2\circ\psi_1$, where $\psi_1\colon S\to S$ is an orientation preserving homeomorphism such that $\psi_1|_A$ is a Dehn twist along l, $\psi_1|_{cl(S-A)}=\mathrm{id.}$, $\psi_2|_F=\varphi$, and $\psi_2|_{cl(S-F)}=\mathrm{id}$. Then, by Proposition 6.1, it is easy to see that $\psi(l)$ is ambient isotopic to a loop disjoint from l.

7. Proof of Theorem 3

In this section, we prove Theorem 3 stated in section 1.

Firstly, we prepare some notations. Let S be a surface in a 3-manifold M, and a ($\subset M$) an arc such that $a \cap S = \partial a$ ($\subset \text{Int}\, S$), and the components of $N(\partial a; a)$ lie in one side of S. Let A be the component of $\partial N(a; M) - S$ which is an open annulus. Then $S_a = (S - \text{Int}\, N(a; M)) \cup A$ is a surface, and has the orientation coherent to S. See Figure 7.1. We say that S_a is obtained from S by adding a *pipe* along a.

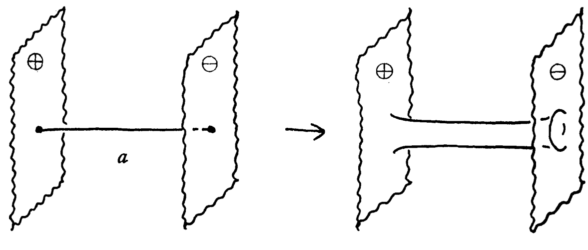


Fig. 7.1

Let S, a, S_a be as above, and (N^c, δ^c) the complementary sutured manifold for S. Then we may suppose that $a' = a \cap N^c$ is an arc such that $\partial a' \subset R_+(\delta^c)$ or $\partial a' \subset R_-(\delta^c)$. We suppose that $\partial a' \subset R_-(\delta^c)$. The other case is essentially the same. Let (N_a^c, δ_a^c) be the complementary sutured manifold for S_a. Then, by Figure 7.2, we immediately have;

Lemma 7.1. *(N_a^c, δ_a^c) is homeomorphic to (N', δ'), where N' is obtained from $cl(N_a^c - N(a'; N_a^c))$ by adding a 1-handle along disks in $R_+(\delta)$, and δ' is the image of δ^c in N'.*

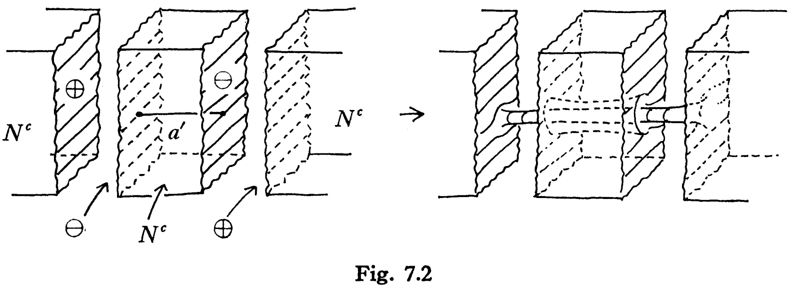


Fig. 7.2

Then we give the definition of the surface Σ_n in S^3 (see section 1). Let D be a disk in S^3. Fix a D^2-boundle structure with D a fiber on $E(\partial D)$. Then we define a sequence of arcs $a_1, a_2, \cdots$ as follows.

Let a_1 be an arc in S^3 such that $N(\partial a_1; a_1)$ lies in the $-$ side of D, $a_1 \cap D =$

$\partial a_1(\subset \mathrm{Int}\, D)$, and there is a disk Δ such that $a_1 \subset \partial\Delta$, $\Delta \cap \mathrm{Int}\, D = \partial\Delta - \mathrm{Int}\, a_1 = \beta$ an arc in D. Clearly a_1 is unique up to ambient isotopy of S^3 respecting D.

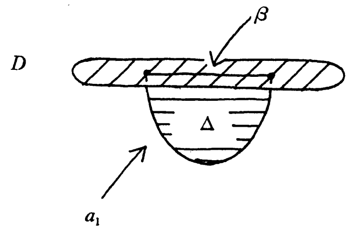


Fig. 7.3

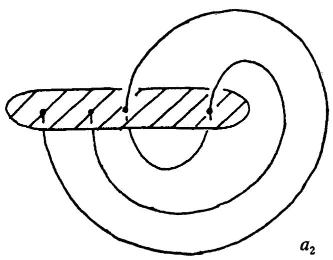


Fig. 7.4

Suppose that a_k has defined. Then let a_{k+1} be an arc such that $N(\partial a_{k+1}; a_{k+1})$ lies in the $-$ side of D, $a_{k+1} \cap \mathrm{Int}\, \Delta = \phi$, $a_k \subset \mathrm{Int}\, a_{k+1}$ (so that $cl(a_{k+1} - a_k)$ consists of two arcs), $cl(a_{k+1} - a_k) \cap D = \partial(a_{k+1} - a_k)$, and each component of $a_{k+1} - a_k$ is transverse to the fibration on $E(\partial D)$. By the induction on i, it is not hard to see that a_i is unique up to the ambient isotopy of S^3 respecting D.

Let Σ_1 be the surface obtained from D by adding a pipe along a_1. Then $a_2 \cap \Sigma_1 = \partial a_2$ and we let Σ_2 be the surface obtained from Σ_1 by adding a pipe along a_2, and so on. We note that each Σ_n has two compressing disks D_n^-, D_n^+ corresponding to a meridian of a_n, and Δ respectively. Then $\partial D_n^{\mp}$ are $l^{\pm}$ of Figure 1.3. Then we have;

Proposition 7.2. *Σ_n is a pre-fiber surface of type 1, and D_n^+, D_n^- is a pair of canonical compressing disks for Σ_n.*

Proof. The proof is done by the induction on n. By the observation in Example 4.1, we see that Σ_1 is a pre-fiber surface of type 1, and D_1^+, D_1^- is a pair of canonical compressing disks for Σ_1.

Suppose, by induction, that Σ_n satisfies the conclusion of Proposition 7.2. Let $(N_n, \delta_n)((N_n^c, \delta_n^c)$ resp.) be the sutured manifold obtained from Σ_n (the complementary sutured manifold for Σ_n resp.) Let $\bar{D}_n^{\pm} = D_n^{\pm} \cap N_n^c$, $\mathcal{D}_n^{\pm} = N(\bar{D}_n^{\pm}; N_n^c)$, and $N_{n-1} = cl(N_n^c - (\mathcal{D}_n^+ \cup \mathcal{D}_n^-))$. Then (N_{n-1}, δ_n^c) is ambient siotopic to the product surtured manifold obtained from Σ_{n-1}. Hence N_{n-1} has a Σ_{n-1}-bundle structure such that each fiber corresponds to $\Sigma_{n-1}\times\{x\}(x\in I)$. We regard $\mathcal{D}_n^{\pm}$ are 1-handles attached to N_{n-1}. By defintion we may suppose that $\alpha=a_{n+1}\cap N_n^c$ is an arc such that $\alpha\cap\mathcal{D}_n^+=\phi$, and $\alpha\cap\mathcal{D}_n^-$ is a vertical arc in $\mathcal{D}_n^-(\cong D^2\times I)$. Hence $\alpha\cap N_{n-1}$ consists of two arcs α_1, α_2.

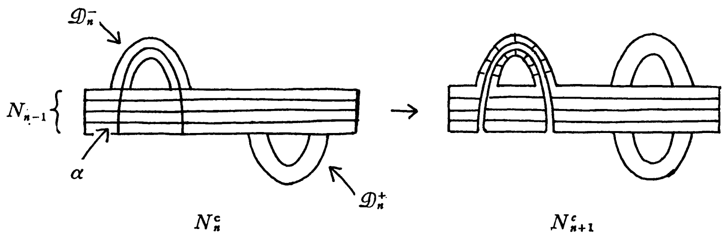


Fig. 7.5

Claim. *By moving a_{n+1} by an ambient isotopy of S^3 respecting Σ_n, if necessary, we may suppose that α_1, α_2 are transverse to the fibration on N_{n-1} (Figure 7.5).*

Proof. By Figure 7.6, we may suppose that each component of $a_{n+1}-a_n$ is close to a meridian loop in $\partial E(\partial\Sigma_n)$. Since the fibration on $\partial E(\partial\Sigma_n)$ induced from the fibration on N_{n-1} is a fibration by longitudes, we see that the components of $a_{n+1}-a_n$ are transverse to the fibration. Hence α_1, α_2 are transverse to the fibration on N_{n-1}.

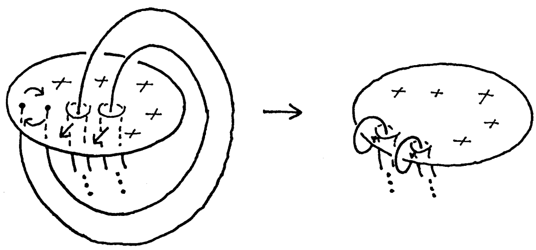


Fig. 7.6

The complementray sutured manifold $(N_{n+1}^c, \delta_{n+1}^c)$ for Σ_{n+1} is obtained from (N_n^c, δ_n^c), and α as in Lemma 7.1. Hence, by Figure 7.5, we easily see that Σ_{n+1} is a pre-fiber surface, and D_{n+1}^+, D_{n+1}^- is a pair of canonical compressing disks.

This completes the proof of Proposition 7.2.

Proof of Theorem 3. The proof is done by the induction on $n=(\chi(L)-\chi(S_1))/2$. Let D^+, D^- be a pair of canonical compressing disks for S_1 and S_2 the surface obtained from S_1 by doing a surgery along D^+, $(N_i, \delta_i)((N_i^c, \delta_i^c)$ resp.) the sutured manifold obtained from S_i (the complementary sutured manifold for S_i resp.) $(i=1, 2)$.

Claim 7.1. *If $\chi(S_1)=\chi(L)-2$, then S_1 is a connected sum of S_2 and Σ_1.*

Proof. By Proposition 5.2, we may suppose that ∂D^+ intersects ∂D^- in one point. Let α be an arc in S_1 such that one endpoint of α lies in ∂S_1, the other endpoint is $\partial D^+ \cap \partial D^-$, and Int $\alpha \cap (\partial D^+ \cup \partial D^-) = \phi$. Then the regular neighborhood B of $\alpha \cup D^+ \cup D^-$ in M is a 3-ball such that $B \cap S_1$ is a regular neighborhood of $\alpha \cup \partial D^+ \cup \partial D^-$ in S_1. ∂B desums S_1 into S_2 and Σ_1.

Claim 7.2. *If $\chi(S_1) < \chi(L) - 2$, then S_2 is a pre-fiber surface of type* 1.

Proof. By Corollary 5.4, we see that S_2 is a pre-fiber surface. Assume that S_2 is of type 2. Then, by Corollary 5.4, we may suppose that $D^+ \cap D^- = \phi$, and ∂D^- is a separating loop in S_2, i.e. $\partial D^+ \cup \partial D^-$ separates S_1. Let S_3 be the surface obtained from S_1 by doing surgery along $D^+ \cup D^-$.

Subclaim. *No component of S_3 is closed.*

Proof. Assume that a component $\bar{S}$ of S_3 is closed. Let $l (\subset S_1)$ be a simple loop which interects ∂D^+ in one point. Then, by pushing l to the $-$ side of S_1, we get a simple loop intersecting $\bar{S}$ in one point, contradicting the fact thet M is a rational homology 3-shpere.

By Subcalim, we see that S_3 is a disconnected Seifert surface for L. Then, by doing compressions on S_3 as much as possible, we get a disconnected, incompressible Seifert surface S^* for L. By Lemma 2.2, we see that S^* is a fiber surface, contradicting Lemma 2.1.

Completion of the proof. Claim 7.1 shows that if $n=1$, then the conclusion holds. Suppose that $n>1$. By Claim 7.2 and the induction, we see that S_2 is a connected sum of a fiber surface and Σ_{n-1} (Figure 7.7). Let S_3 be as in the proof of Subclaim.

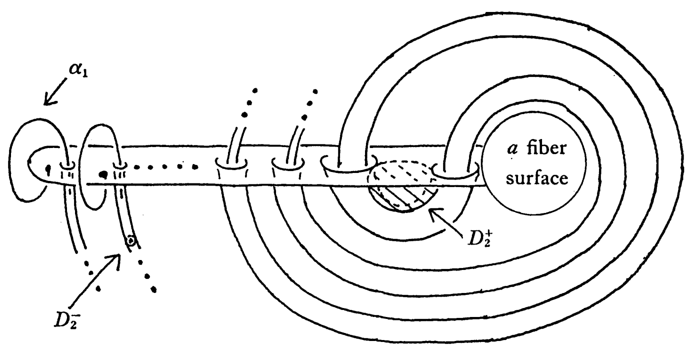


Fig. 7.7

(N_1^c, δ_1^c) is homeomorphic to $(D^2 \times S^1 \natural (S_2 \times I) \natural D^2 \times S^1, \ \partial S_2 \times I)$. Let $D_1^\pm = D^\pm \cap N_1^c$, and $\mathcal{D}_1^\pm = N(D_1^\pm; N_1^c)$. Then, we may identify $(cl(N_1^c - (\mathcal{D}_1^+ \cup \mathcal{D}_1^-)), \delta_1^c)$ to (N_2, δ_2), where $S_2 \times \{1/2\}$ corresponds to S_2. We regard $\mathcal{D}_1^+$, $\mathcal{D}_1^-$

are 2-handles attached to (N_1, δ_1). Then $(N_1 \cup \mathcal{D}_1^+ \cup \mathcal{D}_1^-, \delta_1)$ is properly isotopic to (N_2^c, δ_2^c) in $E(L)$. Hence we identify (N_2^c, δ_2^c) to $(N_1 \cup \mathcal{D}_1^+ \cup \mathcal{D}_1^-, \delta_1)$. Let A^+, A^- be pairwise disjoint product annuli in (N_1, δ_1) $(\subset (N_2^c, \delta_2^c))$, such that $A^+ \cap R_-(\delta_1) = \partial D_1^+$, $A^- \cap R_+(\delta_1) = \partial D_1^-$. Let $D_2^+ = A^+ \cup D_1^+$, $D_2^- = A^- \cup D_1^-$, and $\mathcal{D}_2^\pm = N(D_2^\pm; N_2^c)$ (Figure 7.8). Then D_2^+, D_2^- represents a pair of canonical compressing disks for S_2, and $(cl(N_2^c - (\mathcal{D}_2^+ \cup \mathcal{D}_2^-)), \delta_2^c)$ is ambient isotopic the sutured manifold (N_3, δ_3) obtained from S_3. Hence we may regard that N_2^c is obtained from N_3 by a attaching two 1-handles $\mathcal{D}_2^+$, $\mathcal{D}_2^-$. Then fix a D^2-bundle structure on $\mathcal{D}_2^- \cong D^2 \times I$, and S_3-bundle structure on $N_3 = S_3 \times I$. Let α be an arc in N_2^c such that $\alpha \cap \partial N_2^c = \alpha \cap R_+(\delta_2^c) = \partial\alpha$, $\alpha \cap \mathcal{D}_2^+ = \phi$, $\alpha \cap \mathcal{D}_2^-$ is an arc transverse to the fibers, and $\alpha \cap N_3$ consists of two arcs transverse to the fibers. It is easy to see that the arcs with the above properties are unique up to the ambient isotopies of N_2^c respecting the fibers. Let α_1 be an arc as in Figure 7.7. Then, by the arguments of the proof of Proposition 7.2 (see Figure 7.6), we see that the arc $\alpha_1 \cap N_2^c$ has the above properties. Moreover, by Figure 7.8, we see that S_1 is obtained from S_2 by adding a pipe along α_1. This shows that S_1 is a connected sum of a fiber surface and Σ_n, and it is easy to see that a pair of canonical compressing disks for S_1 corresponds to that of Σ_n.

This completes the proof of Theorem 3.

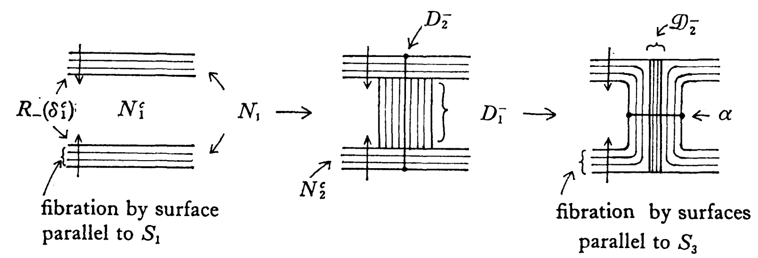


Fig. 7.8

8. Arcs and bands for pre-fiber surfaces

In this section, we study the converse to Theorem 2. For the statement of the result, we prepare some notations. Let $\mathcal{S}$ be a surface in a 3-manifold such that $\partial\mathcal{S} \neq \phi$. Let α be an arc properly embedded in $\mathcal{S}$, D a disk such that $D \cap \mathcal{S} = \alpha$, and $\mathcal{S}'$ the image of $\mathcal{S}$ after ± 1 surgery along ∂D. We say that $\mathcal{S}'$ is obtained from $\mathcal{S}$ by adding a twist along α. Let $\beta\colon I \times I \to N$ be an embedding such that $\beta^{-1}(S) = \beta^{-1}(\partial S) = (\{0\} \times I) \cup (\{1\} \times I)$, and the orientation on $I \times \{0, 1\}$ is coherent with that of ∂S. Then we say that the surface $\mathcal{S} \cup \beta(I \times I)$ is obtained from $\mathcal{S}$ by adding a band $b = \beta(I \times I)$. The arc $\beta(I \times \{1/2\})$ is called the *core arc* of the band b.

Let T be a pre-fiber surface in a closed 3-manifold M, possibly $\dim H_1(M; Q) > 0$, and D^+, D^- a pair of canonical compressing disks for T.

Then we have the following two propositions.

Proposition 8.1. *Suppose that a properly embedded arc a ($\subset T$) intersects ∂D^+, ∂D^- in one points. Then the surface T' obtained from T by adding a twist along a is a fiber surface.*

REMARK. Let S be a fiber surface in a rational homology 3-sphere. Lemma 4.7 shows that if we get a pre-fiber surface S' from S by adding a twist along an arc a, then the arc on S' corresponding to a satisfies the assumptions of Proposition 8.1.

Let (N^c, δ^c) be the complementary sutured manifold for T. Then we may suppose that $\alpha \cap N^c$ is an arc α' such that $\partial\alpha' \subset \text{Int}\, \delta^c$ for a core arc α.

Proposition 8.2. *Let b be a band attached to T with the following properties.*

(1) *The core arc α of b intersects D^+, D^- in one points.*

(2) *There is a disk Δ in N^c such that $\alpha' \subset \partial\Delta$, $\Delta \cap \partial N^c = \partial\Delta \cap \partial N^c = cl(\partial\Delta - \alpha')$, and $\partial\Delta \cap R_+(\delta^c)(\partial\Delta \cap R_-(\delta^c)$ resp.) consists of an arc.*

Then the surface T' obtained from T by adding the band b is a fiber surface.

REMARK. Let S_1 be a pre-fiber surface in a rational homology 3-sphere as in Theorem 2 (2), and b a band for S_1 as in Figure 1.2. Proposition 6.1, Figures 4.5, and 8.1 shows that the core arc of b has the properties (1), (2) of Propostion 8.2.

REMARK. We note that if F is fibered and the band b satisfies the above conditions (1), (2), then the twists on the band is not essential. In fact, by doing Stallings twists [13] along ∂D^+, we see that the bands obtained from b by adding twists also produce fiber surfaces.

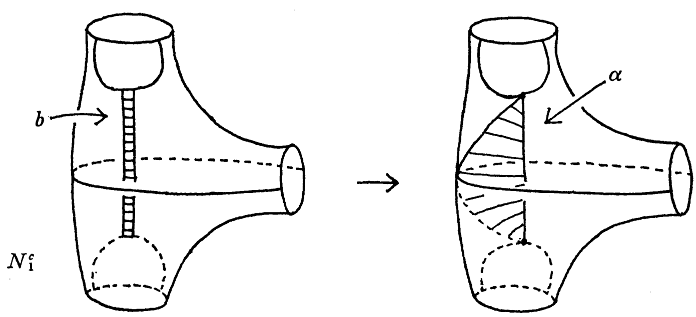


Fig. 8.1

Let $D_c^+ = D^+ \cap N^c$, $D_c^- = D^- \cap N^c$.

Proof of Proposition 8.1. Let D be a disk in M such that $D \cap T = a$. Then the image of D in N^c is an annulus A such that one boundary component l of A

is contained in Int N^c and the other is a simple loop in ∂N^c intersecting $s(\delta^c)$ in two points (Figure 8.2). Then, by the assumption, we may suppose that l intersects D_c^+, D_c^- in one points. Moreover, by taking sufficiently small D, if necessary, we may suppose that $(D_c^+ \cup D_c^-) \cap A$ consists of two essential arcs in A.

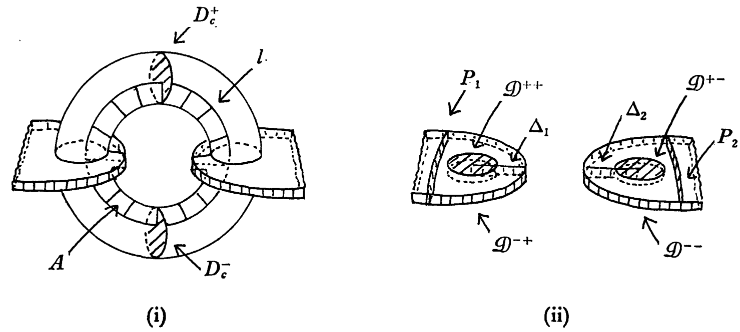


Fig. 8.2

Let $\bar{N}=cl(N^c-N(D_c^+ \cup D_c^-; N^c))$, and $\bar{\delta}$ the image of δ^c in $\partial\bar{N}$. Then $(\bar{N}, \bar{\delta})$ is a product sutured manifold. Let $\mathscr{D}^{++}$, $\mathscr{D}^{+-}$ be the disks in $R_+(\bar{\delta})$ corresponding to $\mathrm{Fr}_{N^c}N(D_c^+; N^c)$, $\mathscr{D}^{-+}$, $\mathscr{D}^{--}$ the disks in $R_-(\bar{\delta})$ corresponding to $\mathrm{Fr}_{N^c}N(D_c^-; N^c)$. Then, by the above, we may suppose that $A \cap \bar{N}$ consists of two disks Δ_1, Δ_2 (Figure 8.2) such that $\Delta_1 \cap (\mathscr{D}^{+-} \cup \mathscr{D}^{--})=\phi$, $\Delta_2 \cap (\mathscr{D}^{++} \cup \mathscr{D}^{-+})=\phi$. The we may suppose that Δ_1, Δ_2 have the following properties with respect to the I-bundle structure on $(\bar{N}, \bar{\delta})$.

(8.1) Δ_i is a union of fibers.

(8.2) $N(\Delta_1; \bar{N}) \supset (\mathscr{D}^{++} \cup \mathscr{D}^{-+})$, $N(\Delta_2; \bar{N}) \supset (\mathscr{D}^{+-} \cup \mathscr{D}^{--})$.

Let $P_1=\mathrm{Fr}_{\bar{N}}\, N(\Delta_1; \bar{N})$, $P_2=\mathrm{Fr}_{\bar{N}}\, N(\Delta_2; \bar{N})$. Then, by (8.1), and (8.2), P_1, P_2 are regarded as product disks in in (N^c, δ^c), and $P_1 \cup P_2$ decomposes (N^c, δ^c) into the union of a product sutured manifold $(\bar{N}', \bar{\delta}')$ homeomorphic to $(\bar{N}, \bar{\delta})$ and $(D^2 \times S^1, \gamma)$, where $s(\gamma)$ consists of two essential loops in $\partial(D^2 \times S^1)$ which are contractible in $D^2 \times S^1$. We note that l is a core curve of $D^2 \times S^1$ and if we do ± 1 surgery on $(D^2 \times S^1, \gamma)$ along l then we get product sutured manifold

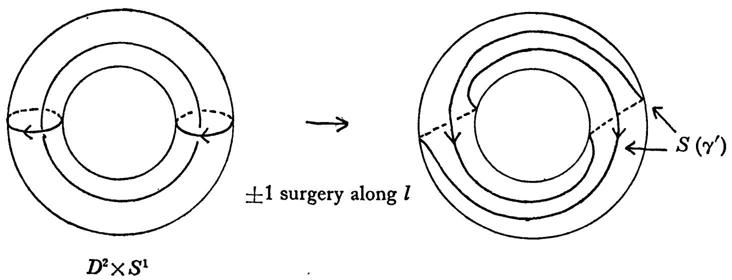
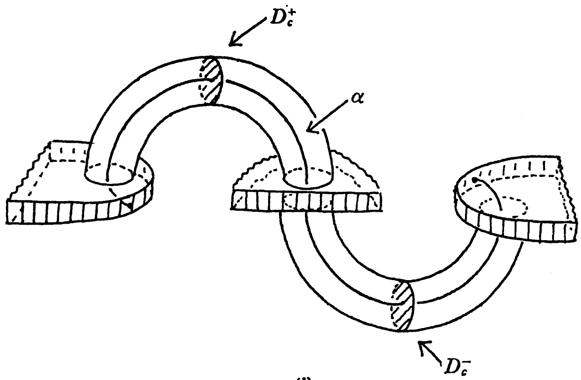


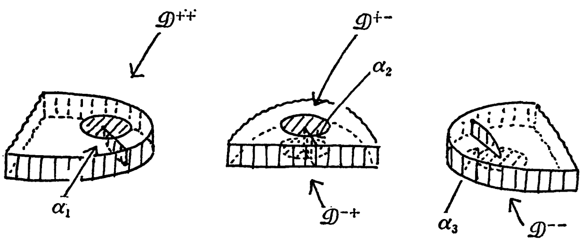
Fig. 8.3

$(D^2 \times S^1, \gamma')$ (Figure 8.3). Since the complementary sutured manifold for T' is obtained from $(\bar{N}^c, \delta^c)$ and $(D^2 \times S^1, \gamma')$ by summing them along product disks corresponding to P_1, P_2, it is a product sutured manifold. Hence T' is a fiber surface.

Proof of Proposition 8.2. We may suppose that $\Delta \cap D_c^+$ ($\Delta \cap D_c^-$ resp.) consists of an arc with one endpoint lies in ∂D_c^+ (∂D_c^- resp.). Let $\bar{N} = cl(N^c - N(D_c^+ \cup D_c^-; N^c))$, and $\bar{\delta}$ the image of δ^c in $\partial \bar{N}$. $(\bar{N}, \bar{\delta})$ is a product sutured manifold. Then, by the above, $\Delta \cap \bar{N}$ consists of three disks Δ_1, Δ_2, Δ_3 such that $\Delta_1 \cap R_-(\bar{\delta}) = \phi$, $\Delta_3 \cap R_+(\bar{\delta}) = \phi$ (Figure 8.4). Let $\mathcal{D}^{++}$, $\mathcal{D}^{+-}$ be the disks in $R_+(\bar{\delta})$ corresponding to $\mathrm{Fr}_{N^c} N(D_c^+; N^c)$, $\mathcal{D}^{-+}$, $\mathcal{D}^{--}$ the disks in $R_-(\bar{\delta})$ corresponding to $\mathrm{Fr}_{N^c} N(D_c^-; N^c)$ such that $\mathcal{D}^{++} \cap \Delta_1 \neq \phi$, $\mathcal{D}^{+-} \cap \Delta_2 \neq \phi$, $\mathcal{D}^{-+} \cap \Delta_2 \neq \phi$, $\mathcal{D}^{--} \cap \Delta_3 \neq \phi$. Then we may suppose that Δ_1, Δ_2, Δ_3 have the following properties with respect to the product structures on $(\bar{N}, \bar{\delta})$.



(i)



(ii)

Fig. 8.4

(8.3) There are mutually disjoint disks D_1, D_2, D_3 in $\bar{N}$ such that D_i is a union of fibers ($i=1, 2, 3$), $D_j \supset \Delta_j$ ($j=1, 3$), $D_2 = \Delta_2$, $D_1 \cap R_+(\bar{\delta}) = \Delta_1 \cap R_+(\bar{\delta})$, $D_3 \cap R_-(\bar{\delta}) = \Delta_3 \cap R_-(\bar{\delta})$.

(8.4) $N(D_1; \bar{N}) \supset \mathcal{D}^{++}$, $N(D_2; \bar{N}) \supset (\mathcal{D}^{+-} \cup \mathcal{D}^{-+})$, $N(D_3; \bar{N}) \supset \mathcal{D}^{--}$.

Let $P_i = \mathrm{Fr}_{\bar{N}^c} N(D_i; \bar{N})$ $(i=1, 2, 3)$. Then, by (8.4), P_1, P_2, P_3 are regarded as product disks in (N^c, δ^c), and $P_1 \cup P_2 \cup P_3$ decomposes (N^c, δ^c) into a union of a sutured manifold $(\bar{N}', \bar{\delta}')$ homeomorphic to $(\bar{N}, \bar{\delta})$ and a sutured manifold $(B, \gamma_1 \cup \gamma_2 \cup \gamma_3)$, where B is a 3-ball, and $s(\gamma_1)$, $s(\gamma_2)$, $s(\gamma_3)$ are sutures as in Figure 8.5.

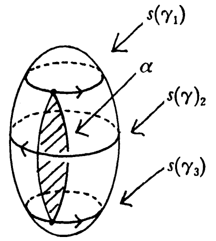


Fig. 8.5

Let $(N^{c\prime}, \delta^{c\prime})$ be the complementary sutured manifold for T'. Then $N^{c\prime}$ is obtained from N^c by removing $\mathrm{Int}\, N(b; N^c)$, and $s(\delta^{c\prime})$ is obtained from $s(\delta^c) - N(b; N^c)$ by adding two arcs in $\mathrm{Fr}_{N^c} N(b; N^c)$ corresponding to $\partial b \cap \partial T'$. See Figure 8.6. Hence P_1, P_2, P_3 are regarded as product disks for $(N^{c\prime}, \delta^{c\prime})$, and $P_1 \cup P_2 \cup P_3$ decomposes $(N^{c\prime}, \delta^{c\prime})$ into a union of a product sutured manifold homeomorphic to $(\bar{N}, \bar{\delta})$ and $(D^2 \times S^1, \gamma)$, where $(D^2 \times S^1, \gamma)$ is obtained from $(B, \gamma_1 \cup \gamma_2 \cup \gamma_3)$ by using b. Then, by Figures 8.5 and 8.6, it is directly observed that $(D^2 \times S^1, \gamma)$ is a product sutured manifold. Hence $(N^{c\prime}, \delta^{c\prime})$ is a product sutured manifold, so that T' is a fiber surface.

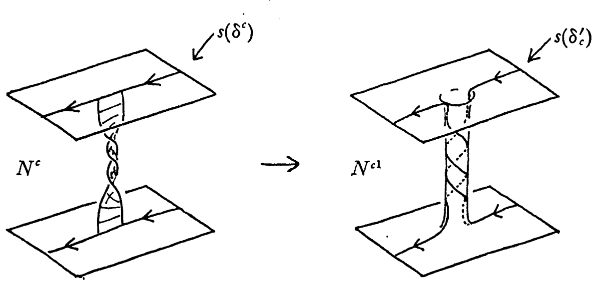


Fig. 8.6

9. Unknotting number 1 fibered knots

In this section, we study unknotting number 1 fibered knots in rational homology 3-spheres. Firstly, we prove Theorem 4 stated in section 1. Then we show that, for each $g > 1$, every lens space contains an unknotting number 1 fibered knot of genus g (Proposition 9.2). In Proposition 9.1 we show that a rational homology 3-sphere M contains an unknotting number 1 fibered knot of genus 1 if and only if M is a lens space of type $L_{m,1}$.

Proof of Theorem 4. Suppose that M contains an unknotting number 1 fibered knot of genus g. Then, by Theorem 2(1), we see that M contains a pre-fiber surface S_0 of genus g such that ∂S_0 is a trivial knot. Let D^+, D^- be a pair of canonical compressing disks for S_0.

Claim 9.1. *S_0 is a type 1 pre-fibere surface.*

Proof. By Figure 6.1, we see that there is a properly embedded arc in S_0 which intersects ∂D^+ in one point. Since S_0 has one boundary component, this shows that ∂D^+ is non separating in S_0. Hence S_0 is of type 1.

Claim 9.2. *If M contains a type 1 pre-fiber surface S_* of genus 1, then M is a lens space.*

Proof. The complementary sutured manifold (N_*^c, δ_*^c) for S_* is homeomorphic to $(D^2\times S^1 \natural (D^2\times I) \natural D^2\times S^1, \partial D^2\times I)$ (cf. Example 4.1). Since (N_*^c, δ_*^c) is the complementary sutured manifold, there is a homeomorphism $f\colon R_+(\delta_*^c)\to R_-(\delta_*^c)$ such that the manifold obtained from N_*^c by identifying the points in $R(\delta_*^c)$ by f is homeomorphic to $E(\partial S_*)$. Let D be a disk in N_*^c corresponding to $D^2\times\{1/2\}$. Then D cuts N_*^c into two components N^+, N^- such that N^+, N^- are solid tori, and $R_+(\delta_*^c)\subset\partial N^+$, $R_-(\delta_*^c)\subset\partial N^-$. There is a homeomorphism $h\colon \partial N^+\to\partial N^-$ such that h is an extension of f and $N^+\cup_h N^-$ is homeomorphic to M. Hence M admits a Heegaard splitting of genus 1.

By Claims 9.1, and 9.2, we see that if $g=1$, then M is a lens space. Hereafter we suppose that $g>1$. Then, by Claim 9.1 and Corollary 5.4, we may suppose that ∂D^+ and ∂D^- are disjoint.

Claim 9.3. *$\partial D^+\cup\partial D^-$ does not separate S_0.*

Proof. Assume that $\partial D^+\cup\partial D^-$ separates S_0. Let S_* be the component of $S_0-(\partial D^+\cup\partial D^-)$ which does not contain ∂S_0. Then $\bar{S}=S_*\cup D^+\cup D^-$ is a closed surface in M. By Claim 9.1, there is a simple loop l in S_0 which intersects ∂D^+ in one point. Then, by pushing l slightly to the $-$side, we see that there is a simple loop in M which intersects $\bar{S}$ in one point, contradicting the fact that M is a rational homology 3-sphere.

Let S_1 be the surface obtained from S_0 by doing surgery along D^+. By Corollary 5.4, we see that S_1 is a pre-fiber surface. Then;

Claim 9.4. *S_1 is a type 1 pre-fiber surface.*

Proof. By Claim 9.3, we see that ∂D^- is non separating in S_1. Hence, by Corollary 5.4, we see that S_1 is of type 1.

By Claim 9.4, and the induction on g, we see that M contains a pre-fiber surface of type 1 and of genus 1. Then, by Claim 9.2, we see that M admits a Heegaard splitting of genus 1.

Proposition 9.1. *A rational homology* 3*-sphere M contains an unknotting number* 1*, genus* 1 *fibered knot if and only if M is a lens space of type $L_{m,1}$ for some* $m \in \boldsymbol{Z}-\{0\}$.

For the notation of the lens spaces, see [6].

Proof. Suppose that M contains an unknotting number 1, genus 1 fibered knot K. Then, by Theorem 1, we see that there is a minimal genus Seifert surface S for K such that S is a plumbing of a surface F in M and a Hopf band. Since *genus* $(S)=1$, F is an annulus, so that $E(\partial F)$ is homeomorphic to $T^2 \times I$, where T^2 is a 2-dimensional torus. Hence M is obtained from $T^2 \times I$ and two solid tori T_1, T_2 by identifying their boundaries. Let A be the annulus in $E(\partial F)$ corresponding to the fiber F, and $l_0=A \cap (T^2 \times \{0\})$, $l_1=A \cap (T^2 \times \{1\})$. Then meridian loop of T_i intersects l_i in one point ($i=1, 2$). Hence it is easy to see that M is a lens space of type $L_{m,1}$.

Suppose that M is a lens space of type $L_{m,1}$. Then it is observed in [7] that the knots K_1, K_2 of Figure 9.1 are fibered. It is easy to see that both K_1 and K_2 have unknotting number 1.

This completes the proof of Proposition 9.1.

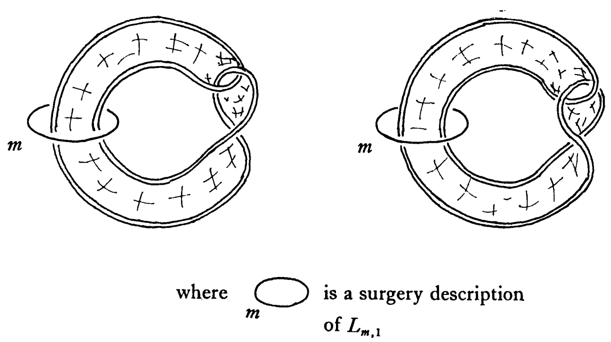


Fig. 9.1

Proposition 9.2. *If M is a lens space, possibly* $\dim H_1(M; Q)>0$, *then, for each $g>1$, there is an unknotting number* 1 *fibered knot of genus g in M.*

REMARK. If M is a lens space with $\dim H_1(M; Q)>0$, then M is homeomorphic to $S^2 \times S^1$.

Proof. By Example 4.1, there is a genus 1 pre-fiber surface T in M such that ∂T is a trivial knot. Let D^+, D^- be a pair of canonical compressing disks for T, $\tilde{l}^+$, $\tilde{l}^-$ a pair of properly embedded arcs in T such that $\tilde{l}^+ \cap \partial D^+$ consists of one point, $\tilde{l}^- \cap \partial D^-$ consists of one point, and $\partial \tilde{l}^+ \cap \partial \tilde{l}^-$ consists of one point p. Let l^+(l^- resp.) be the arc obtained from $\tilde{l}^+$ ($\tilde{l}^-$ resp.) by pushing Int $\tilde{l}^+$ (Int $\tilde{l}^-$ resp.) slightly to the $-$side ($+$side resp.) of T. $l=l^+\cup l^-$ is an embedded arc in M such that $l\cap T=\partial l\cup p$. Then deform l by an ambient isotopy in a small neighborhood of p so that $l\cap T=\partial l$. Clearly l satisfies the conditions (1), (2) of Proposition 8.2. Hence there is a band b for T such that the surface F obtained from T by attaching b is a fiber surface. Then, by a plumbing of F and a Hopf band along b, we have a genus 2, fiber surface which bounds an unknotting number 1 fibered knot (Figure 9.2).

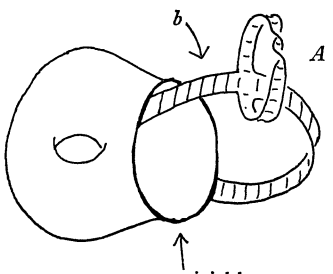


Fig. 9.2

Suppose that $g>2$. Let F_n be the surface in S^3 as in Figure 9.3. It is observed in [9] that F_n is a fiber surface. In fact, F_n is obtained from one Hopf band and n copies of the fiber surface of Figure 9.4. Then, by a plumbing of the above F and F_{g-2} along b and E of Figure 9.3, we get a genus g fiber surface $\mathcal{S}_g$ [4]. It is directly observed from Figure 9.3 that if we apply a crossing change on $\partial\mathcal{S}_g$ along the crossing disk D of Figure 9.3, then we get a trivial knot. Hence $u(\partial\mathcal{S}_g)=1$.

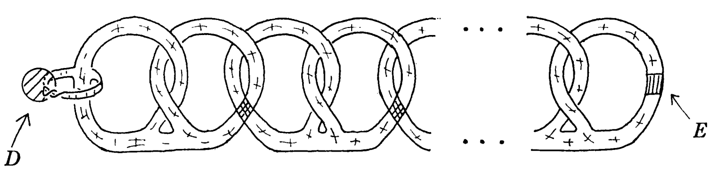


Fig. 9.3

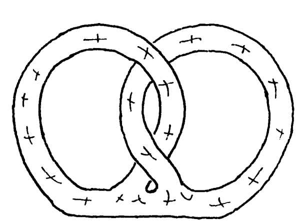


Fig. 9.4

This completes the proof of Proposition 9.2.

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