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1. Introduction

Let $FG$ be the group algebra of a finite group $G$ over an algebraically closed field $F$ of characteristic $p > 0$. We call an $FG$-module $V$ monomial if $V$ is induced from some 1-dimensional $FH$-module for some subgroup $H$ of $G$. An ordinary character $\chi$ of $G$ is called monomial if $\chi$ is induced from some linear character of some subgroup of $G$. We call $G$ an $M_p$-group if every irreducible $FG$-module is monomial. We call $G$ an $M$-group if every irreducible ordinary character of $G$ is monomial. For details, see a paper of Bessenrodt [1] and a book of Isaacs [4]. It is well known that $M$-groups are solvable (15.7 in [2]). $M_p$-groups are also solvable (3.8 in [6]). By Fong-Swan’s theorem, $M$-groups are $M_p$-groups for any prime $p$. But $M_p$-groups need not be $M$-groups. For example, $SL(2, 3)$ is an $M_p$-group but not an $M$-group. So we investigate conditions for $M_p$-groups to be $M$-groups. Namely,

**Theorem 3.** Let $G$ be a $p$-nilpotent group. Then $G$ is an $M$-group if and only if $G$ is an $M_p$-group.

Throughout this paper, groups are finite groups, $F$ is an algebraically closed field of characteristic $p > 0$, $FG$-modules are finitely generated right $FG$-modules, and characters are ordinary characters. Let $\chi$ be a character of a group $G$. We write $\chi^*$ for the Brauer character corresponding to $\chi$. Let $H$ be a subgroup of $G$ and $\varphi$ be a character of $H$. We write $\chi_H^*$ for the restriction of $\chi$ to $H$ and $\varphi^G$ for the induced character from $\varphi$. We use the same notation for modules. When $M$ and $N$ are $FG$-modules, we write $N \mid M$ if $N$ is a direct summand of $M$. We write $\text{Irr}(G)$ for the set of all irreducible characters of $G$. For the other notation and terminology we shall refer to books of Dornhoff [2] and Nagao and Tsushima [5].

We wish to thank S. Koshitani for many helpful conversations during the course of this work.
2. Consequences

Let \( H \) be a normal subgroup of \( G \) and \( \varphi \) be an irreducible character of \( H \). We denote the inertia group of \( \varphi \) in \( G \) by \( I_G(\varphi) \). When \( \varphi \) is irreducible, we put

\[
\text{Irr}(G \mid \varphi) = \{ \chi \in \text{Irr}(G) \mid (\chi_H, \varphi) \neq 0 \}.
\]

Next theorem will be a powerful tool if we consider conditions for \( M_p \)-groups to be \( M \)-groups.

**Theorem 1.** Let \( G \) be a finite group. Assume that \( G \) has a normal \( p' \)-subgroup \( N \) such that \( G \) and \( N \) satisfy the followings.

(a) \( G \) is an \( M_p \)-group.

(b) \( G/N \) is an \( M \)-group.

(c) Every proper subgroup of \( G \) containing \( N \) is an \( M \)-group.

(d) Every \( G \)-invariant irreducible character of \( N \) is extendible to \( G \).

Then \( G \) is an \( M \)-group.

Proof. Let \( \chi \in \text{Irr}(G \mid \varphi) \) where \( \varphi \in \text{Irr}(N) \). If \( I_G(\varphi) \) is a proper subgroup of \( G \) then there exists \( \xi^G \in \text{Irr}(I_G(\varphi) \mid \varphi) \) such that \( \xi^G = \chi \). From (c), \( \xi \) is monomial so is \( \chi \).

Assume \( I_G(\varphi) = G \). From (d), \( \varphi \) is extendible to \( G \). Let \( \chi_0 \) be an extension of \( \varphi \). Because \( (\chi_0^G)^N = \varphi^\ast \) and \( N \) is a \( p' \)-group, \( \chi_0^G \) is an irreducible Brauer character of \( G \). Since \( G \) is an \( M_p \)-group, there exist a subgroup \( H \) of \( G \) and a linear character \( \lambda \) of \( H \) such that \( (\lambda^G)^N = \chi_0^G \). Since \( (\lambda^G)^N = \chi_0^G \) is irreducible, \( \lambda^G \) is irreducible and an extension of \( \varphi \). By 3.5.12 in [5],

\[
\text{Irr}(G \mid \varphi) = \{ \lambda^G \eta \mid \eta \in \text{Irr}(G/N) \}.
\]

Now every \( \eta \) is monomial, so is \( \lambda^G \eta \). So \( \chi \) is monomial. The proof is completed.

Generally, normal subgroups of \( M_p \)-groups need not be \( M_p \)-groups. But next theorem holds.

**Theorem 2.** Let \( G \) be an \( M_p \)-group and \( N \) be a normal subgroup of \( G \) such that \( |G : N| = p \). Then \( N \) is an \( M_p \)-group.

Proof. Let \( U \) be an irreducible \( FN \)-module. Since \( N \) is normal in \( G \), there exists an irreducible \( FG \)-module \( V \) such that \( U \mid V_N \). Since \( G \) is an \( M_p \)-group, there exist a subgroup \( H \) and a 1-dimensional \( FH \)-module \( W \) such that \( V = W^G \). If the inertia group of \( U \) in \( G \) is \( G \) then \( U \) is extendible to \( G \). Thus we may assume \( U = V_N \). By Mackey's decomposition,

\[
U = V_N = (W^G)_N = \bigoplus_{\lambda \in \text{Irr}(G/N)} (W_{H^\lambda \cap N})^N.
\]
But $U$ is irreducible. So $G=HN$ and $U$ is monomial. We may assume that the inertia group of $U$ in $G$ is $N$. Then by Clifford’s theorem, $V_N = \oplus_{U \in G/N} U^t$. If $H$ is contained in $N$ then by Mackey’s decomposition,

$$V_N = (W^G)_N = \oplus_{t \in H \cap N} (W^G_{H \cap N})^N = \oplus_{t \in G/N} (W^G_t)^N.$$ 

Since $U$ is irreducible, $U \cong (W^G_t)^N$ for some $t \in G/N$. So $U$ is monomial. We may assume that $H$ is not contained in $N$. So $G=HN$. Let $Q$ be a vertex of $W$. Since $\dim_p W = 1$, $Q$ is a Sylow $p$-subgroup of $H$. Since $V=W^G$ and $V=U^G$, $Q$ is in $H \cap N$. Now

$$p = |G:N| = |HN:N| = \left| H:H \cap N \right| |H:Q|.$$ 

But $Q$ is a Sylow $p$-subgroup of $H$, a contradiction. Hence $U$ is monomial. So $N$ is an $M_p$-group.

Next theorem is our main result.

**Theorem 3.** Let $G$ be a $p$-nilpotent group. Then $G$ is an $M$-group if and only if $G$ is an $M_p$-group.

Proof. We know that $M$-groups are $M_p$-groups. So we shall show that $G$ is an $M$-group if $G$ is an $M_p$-group by induction on $|G|$. Since $G$ is $p$-nilpotent $G$ has a normal $p$-complement $N$. We show that $G$ and $N$ satisfy the conditions in Theorem 1. Now (a) and (b) are satisfied. By 3.5.11 in [5], (d) is satisfied. Let $H$ be a proper subgroup of $G$ containing $N$. Since $G/N$ is a $p$-group, $H$ is an $M$-group by Theorem 2. Then $H$ is an $M$-group by inductive hypothesis. So (c) is satisfied. Then $G$ is an $M$-group.

**Corollary 4.** Let $G$ be an $M$-group and $p$-nilpotent. Then a subgroup $H$ of $G$ such that $|G:H|$ is $p$-power is an $M$-group.

Proof. This is immediate from Theorem 2 and Theorem 3.

References


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