A REMARK ON $M_p$-GROUPS

Dedicated to Professor Kazuhiko Hirata on his 60th birthday

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1. Introduction

Let $FG$ be the group algebra of a finite group $G$ over an algebraically closed field $F$ of characteristic $p > 0$. We call an $FG$-module $V$ monomial if $V$ is induced from some 1-dimensional $FH$-module for some subgroup $H$ of $G$. An ordinary character $\chi$ of $G$ is called monomial if $\chi$ is induced from some linear character of some subgroup of $G$. We call $G$ an $M_p$-group if every irreducible $FG$-module is monomial. We call $G$ an $M$-group if every irreducible ordinary character of $G$ is monomial. For details, see a paper of Bessenrodt [1] and a book of Isaacs [4]. It is well known that $M$-groups are solvable (15.7 in [2]). $M_p$-groups are also solvable (3.8 in [6]). By Fong-Swan's theorem, $M$-groups are $M_p$-groups for any prime $p$. But $M_p$-groups need not be $M$-groups. For example, $SL(2, 3)$ is an $M_2$-group but not an $M$-group. So we investigate conditions for $M_p$-groups to be $M$-groups. Namely,

Theorem 3. Let $G$ be a $p$-nilpotent group. Then $G$ is an $M$-group if and only if $G$ is an $M_p$-group.

Throughout this paper, groups are finite groups, $F$ is an algebraically closed field of characteristic $p > 0$, $FG$-modules are finitely generated right $FG$-modules, and characters are ordinary characters. Let $\chi$ be a character of a group $G$. We write $\chi^*$ for the Brauer character corresponding to $\chi$. Let $H$ be a subgroup of $G$ and $\varphi$ be a character of $H$. We write $\chi_H$ for the restriction of $\chi$ to $H$ and $\varphi^G$ for the induced character from $\varphi$. We use the same notation for modules. When $M$ and $N$ are $FG$-modules, we write $N \mid M$ if $N$ is a direct summand of $M$. We write $\text{Irr}(G)$ for the set of all irreducible characters of $G$. For the other notation and terminology we shall refer to books of Dornhoff [2] and Nagao and Tsushima [5].

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2. Consequences

Let $H$ be a normal subgroup of $G$ and $\varphi$ be an irreducible character of $H$. We denote the inertia group of $\varphi$ in $G$ by $I_G(\varphi)$. When $\varphi$ is irreducible, we put

$$\text{Irr}(G|\varphi) = \{\chi \in \text{Irr}(G)| (\chi_H, \varphi) \neq 0\}.$$ 

Next theorem will be a powerful tool if we consider conditions for $M_p$-groups to be $M$-groups.

**Theorem 1.** Let $G$ be a finite group. Assume that $G$ has a normal $p'$-subgroup $N$ such that $G$ and $N$ satisfy the followings.

(a) $G$ is an $M_p$-group.
(b) $G/N$ is an $M$-group.
(c) Every proper subgroup of $G$ containing $N$ is an $M$-group.
(d) Every $G$-invariant irreducible character of $N$ is extendible to $G$.

Then $G$ is an $M$-group.

**Proof.** Let $\chi \in \text{Irr}(G|\varphi)$ where $\varphi \in \text{Irr}(N)$. If $I_G(\varphi)$ is a proper subgroup of $G$ then there exists $\xi \in \text{Irr}(I_G(\varphi)|\varphi)$ such that $\xi^G = \chi$. From (c), $\xi$ is monomial so is $\chi$.

Assume $I_G(\varphi) = G$. From (d), $\varphi$ is extendible to $G$. Let $\chi_0$ be an extension of $\varphi$. Because $(\chi_0^*)|_N = \varphi^*$ and $N$ is a $p'$-group, $\chi_0^*$ is an irreducible Brauer character of $G$. Since $G$ is an $M_p$-group, there exist a subgroup $H$ of $G$ and a linear character $\lambda$ of $H$ such that $\lambda^G = \chi_0^*$. Since $(\lambda^G)^* = \chi_0^*|_G$ is irreducible, $\lambda^G$ is irreducible and an extension of $\varphi$. By 3.5.12 in [5],

$$\text{Irr}(G|\varphi) = \{\lambda^G|\eta \in \text{Irr}(G/N)\}.$$ 

Now every $\eta$ is monomial, so is $\lambda^G\eta$. So $\chi$ is monomial. The proof is completed.

Generally, normal subgroups of $M_p$-groups need not be $M_p$-groups. But next theorem holds.

**Theorem 2.** Let $G$ be an $M_p$-group and $N$ be a normal subgroup of $G$ such that $|G:N| = p$. Then $N$ is an $M_p$-group.

**Proof.** Let $U$ be an irreducible $FN$-module. Since $N$ is normal in $G$, there exists an irreducible $FG$-module $V$ such that $U|V_N$. Since $G$ is an $M_p$-group, there exist a subgroup $H$ and a 1-dimensional $FH$-module $W$ such that $V = W^\circ$. If the inertia group of $U$ in $G$ is $G$ then $U$ is extendible to $G$. Thus we may assume $U = V_N$. By Mackey's decomposition,

$$U = V_N = (W^\circ)_N = \bigoplus_{\tau \in H\backslash G/N}(W'|_{H \cap N})^\tau_N.$$
But $U$ is irreducible. So $G=HN$ and $U$ is monomial. We may assume that the inertia group of $U$ in $G$ is $N$. Then by Clifford's theorem, $V_N = \bigoplus_{t \in G/N} U^t$. If $H$ is contained in $N$ then by Mackey's decomposition,

$$V_N = (W^G)_N = \bigoplus_{t \in H \cap N} (W^{G/H}_t N)^N = \bigoplus_{t \in G/N} (W^{G/H}_t N)^N.$$ 

Since $U$ is irreducible, $U \cong (W^{G/H}_t)^N$ for some $t \in G/N$. So $U$ is monomial. We may assume that $H$ is not contained in $N$. So $G=HN$. Let $Q$ be a vertex of $W$. Since $\dim_F W = 1$, $Q$ is a Sylow $p$-subgroup of $H$. Since $V=W^G$ and $V=U^G$, $Q$ is in $H \cap N$. Now


But $Q$ is a Sylow $p$-subgroup of $H$, a contradiction. Hence $U$ is monomial. So $N$ is an $M_p$-group.

Next theorem is our main result.

**Theorem 3.** Let $G$ be a $p$-nilpotent group. Then $G$ is an $M$-group if and only if $G$ is an $M_p$-group.

Proof. We know that $M$-groups are $M_p$-groups. So we shall show that $G$ is an $M$-group if $G$ is an $M_p$-group by induction on $|G|$. Since $G$ is $p$-nilpotent $G$ has a normal $p$-complement $N$. We show that $G$ and $N$ satisfy the conditions in Theorem 1. Now (a) and (b) are satisfied. By 3.5.11 in [5], (d) is satisfied. Let $H$ be a proper subgroup of $G$ containing $N$. Since $G/N$ is a $p$-group, $H$ is an $M_p$-group by Theorem 2. Then $H$ is an $M$-group by inductive hypothesis. So (c) is satisfied. Then $G$ is an $M$-group.

**Corollary 4.** Let $G$ be an $M$-group and $p$-nilpotent. Then a subgroup $H$ of $G$ such that $|G:H| p$-power is an $M$-group.

Proof. This is immediate from Theorem 2 and Theorem 3.

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**References**


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