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Author(s)	Ohsawa, Takeo; Tarama, Daisuke
Citation	Osaka Journal of Mathematics. 2010, 47(3), p. 731-737
Version Type	VoR
URL	<a href="https://doi.org/10.18910/11321">https://doi.org/10.18910/11321</a>
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## A REMARK ON THE EMBEDDING THEOREM ASSOCIATED TO COMPLEX CONNECTIONS OF MIXED TYPE

TAKEO OHSAWA and DAISUKE TARAMA

(Received April 1, 2009)

### Abstract

Let  $M$  be a compact complex manifold and let  $(L, H)$  be a holomorphic Hermitian line bundle over  $M$  such that the curvature form of  $h$  is nondegenerate and splits into the difference  $\Theta_+ - \Theta_-$  of two semipositive forms  $\Theta_+$  and  $\Theta_-$  whose null spaces define mutually transverse holomorphic foliations  $\mathcal{F}_-$  and  $\mathcal{F}_+$ , respectively. Then  $L^m$  admits, for sufficiently large  $m \in \mathbb{N}$ ,  $C^\infty$  sections whose ratio embeds  $M$  into  $\mathbb{CP}^N$  holomorphically (resp. antiholomorphically) along  $\mathcal{F}_+$  (resp. along  $\mathcal{F}_-$ ).

### Introduction

In the theory of complex manifolds, geometric structures defined by the subbundles of tangent bundles are basic in analyzing submanifolds and holomorphic maps. Foliations is one of such structures.

In [6], an embedding theorem was established for those manifolds equipped with two mutually transverse holomorphic foliations. Namely, it was proved that, given a compact complex manifold  $M$  with holomorphic foliations  $\mathcal{F}_+$  and  $\mathcal{F}_-$  such that the tangent bundle of  $M$  is the direct sum of those of  $\mathcal{F}_+$  and  $\mathcal{F}_-$ ,  $M$  is embeddable into  $\mathbb{CP}^N$  by a  $C^\infty$  map in such a way that the map is holomorphic along  $\mathcal{F}_+$  and antiholomorphic along  $\mathcal{F}_-$ , if  $M$  admits a holomorphic line bundle  $L$  whose curvature form is everywhere nondegenerate and splits into the difference of two semipositive forms say  $\Theta_+$  and  $\Theta_-$ , in such a way that  $\Theta_+$  (resp.  $\Theta_-$ ) is definite along  $\mathcal{F}_+$  (resp.  $\mathcal{F}_-$ ) and zero along  $\mathcal{F}_-$  (resp.  $\mathcal{F}_+$ ).

Example of such quadruples  $(M, \mathcal{F}_+, \mathcal{F}_-, L)$  arose naturally in the study of bundle-valued  $\bar{\partial}$ -cohomology groups on complex tori (cf. [5]). For other examples, see Section 3 below.

In such a situation, it was proved that there exist  $m \in \mathbb{N}$  and  $C^\infty$  sections  $s_0, \dots, s_N$  of  $K_+ \otimes \bar{K}_- \otimes L^m$  such that the ratio  $(s_0 : \dots : s_N)$  gives such an embedding, where  $K_\pm$  denote the canonical bundles of  $\mathcal{F}_\pm$  and  $\bar{K}_-$  the complex conjugate of  $K_-$ .

In view of this result, a natural question is whether or not, under the same situation as above, one can embed  $M$  similarly by  $L^m$ . A significance of this question lies in the fact that the method of [6] consisting in applying an  $L^2$  estimate for the twisted  $\bar{\partial}$ -operator does not work any more. The purpose of the present note is to overcome

this difficulty by establishing an analogue of Serre's vanishing theorem for algebraic cohomology groups on projective algebraic varieties. For the statement of our vanishing theorem and the resulting embedding theorem, see Section 1. The proofs of these results will be given in Section 2.

## 1. Preliminaries and results

Let  $M$  be a compact complex manifold of dimension  $n$  and let  $TM$  denote the tangent bundle of  $M$ . We shall say that a foliation  $\mathcal{F}$  on  $M$  is holomorphic if the tangent bundle  $T\mathcal{F}$  of  $\mathcal{F}$  is a holomorphic subbundle of  $TM$ . We shall say that  $M$  is holomorphically woven if  $M$  is endowed with two holomorphic foliations say  $\mathcal{F}_+$  and  $\mathcal{F}_-$  whose tangent bundles  $T\mathcal{F}_\pm \subset TM$  satisfy  $T\mathcal{F}_+ + T\mathcal{F}_- = TM$  and  $T\mathcal{F}_+ \cap T\mathcal{F}_- = \{0\} \times M$ .

Typical examples of holomorphically woven manifolds are naturally induced from orthogonal decompositions of the tangent bundles of complex tori, with respect to constant nondegenerate indefinite  $(1, 1)$ -forms (cf. [5]).

Let  $L \rightarrow M$  be a holomorphic line bundle equipped with a  $C^\infty$  fiber metric  $h$ . Let  $T_+^{0,1}$  (resp.  $T_-^{1,0}$ ) denote the antiholomorphic (resp. holomorphic) tangent bundle of  $\mathcal{F}_+$  (resp.  $\mathcal{F}_-$ ), and let  $F^{p,q}(L)$  ( $= F^{p,q}(M, L)$ ) be the set of  $C^\infty$  sections over  $M$  of the bundle  $L \otimes \bigwedge^p (T_+^{0,1})^* \otimes \bigwedge^q (T_-^{1,0})^*$ , where  $(\cdot)^*$  denote the duals.

Then we define the differential operators

$$\bar{\partial}_+ : F^{p,q}(L) \rightarrow F^{p+1,q}(L)$$

and

$$\partial_{-,h} : F^{p,q}(L) \rightarrow F^{p,q+1}(L)$$

as the leafwise complex exterior differentiations for the  $L$ -valued forms, of type  $(0, 1)$  (resp.  $(1, 0)$ ) with respect to  $\mathcal{F}_+$  (resp.  $\mathcal{F}_-$  and  $h$ ).

We put  $\check{\partial} = \bar{\partial}_+ + \partial_{-,h}$ . Then  $\check{\partial}$  maps the space  $\bigoplus_{p,q} F^{p,q}(L)$  to itself. We shall say that  $(L, h)$  is  $\mathcal{F}_\pm$ -integrable if  $\check{\partial}^2 = 0$ .

Clearly  $\check{\partial}^2 = 0$  holds if and only if the curvature form of  $h$  splits into the sum of a section of  $\overline{(T_+^{0,1})^*} \otimes (T_+^{0,1})^*$  say  $\Theta_+$  and that of  $(T_-^{1,0})^* \otimes \overline{(T_-^{1,0})^*}$  say  $\Theta_-$ . Here  $\overline{(T_+^{0,1})^*} \otimes (T_+^{0,1})^*$  and  $(T_-^{1,0})^* \otimes \overline{(T_-^{1,0})^*}$  are naturally identified with subbundles of  $\bigwedge^2 (TM \otimes \mathbb{C})^*$  and  $\overline{(\cdot)}$  denote the complex conjugates.

**Theorem 1.1.** *Let  $(M, \mathcal{F}_\pm)$  be a compact holomorphically woven manifold and let  $(L, h)$  be a Hermitian holomorphic line bundle over  $M$  which is  $\mathcal{F}_\pm$ -integrable. Suppose that the curvature form  $\Theta_+ + \Theta_-$  of  $h$ , where  $\Theta_\pm$  are as above, satisfies  $\Theta_+|_{\mathcal{F}_+} > 0$  and  $\Theta_-|_{\mathcal{F}_-} < 0$ . Then there exists  $m_0 \in \mathbb{N}$  such that one can find, for any integer  $m \geq m_0$ ,  $C^\infty$  sections  $s_0, \dots, s_N$  of  $L^m$  satisfying  $\check{\partial}s_k = 0$  ( $0 \leq k \leq N$ )*

such that the ratio  $(s_0 : \cdots : s_N)$  embeds  $M$  into  $\mathbb{CP}^N$ . Here  $L^m$  is equipped with the fiber metric  $h^m$ .

The proof of this assertion is based on the following vanishing theorem with  $L^2$  norm estimates.

**Theorem 1.2.** *Let  $(M, \mathcal{F}_\pm)$  and  $(L, h)$  be as in Theorem 1.1, and let  $g$  be any  $C^\infty$  Hermitian metric on  $M$ . Then there exist  $m_1 \in \mathbb{N}$  and  $C > 0$  such that, for any integer  $m \geq m_1$  and for any  $v \in \text{Ker } \check{\partial} \cap \bigoplus_{p+q=r} F^{p,q}(L^m)$  with  $r \geq 1$ , one can find  $u \in \bigoplus_{p+q=r-1} F^{p,q}(L^m)$  satisfying  $\check{\partial}u = v$  and  $m\|u\|^2 \leq C\|v\|^2$ . Here  $\|\cdot\|$  denotes the  $L^2$  norm with respect to  $h^m$  and  $g$ .*

## 2. Solving the $\check{\partial}$ -equation

Admitting the validity of Theorem 1.2, the proof of Theorem 1.1 is carried over similarly as in the case of  $K_+ \otimes \bar{K}_- \otimes L^m$  in [6, §3. Proof of Theorem 0.1], so that we shall dispense with this part.

Proof of Theorem 1.2. Let  $v \in \bigoplus_{p+q=r} F^{p,q}(L^m) \cap \text{Ker } \check{\partial}$  ( $r \geq 1$ ), where  $m$  will be specified later.

Let  $U \subset M$  be a local coordinate neighbourhood with a holomorphic coordinate  $(z_+, z_-): U \rightarrow \mathbb{D}^n$ , where  $\mathbb{D} = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ , such that  $T_+^{0,1}|U = \text{Ker } d\bar{z}_-$  and  $T_-^{1,0}|U = \text{Ker } dz_+$  hold. We choose  $z_\pm$  in such a way that they are holomorphic on a neighbourhood of the closure of  $U$  in  $M$ . We put  $z_+ = (z_1, \dots, z_s)$ ,  $z_- = (z_{s+1}, \dots, z_n)$  and  $v|U = \sum_{I,J} v_{IJ} d\bar{z}_I \otimes dz_J$ . We note that  $|I| + |J| = r$  and the components of multi-indices  $I$  (resp.  $J$ ) are contained in  $\{1, \dots, s\}$  (resp.  $\{s+1, \dots, n\}$ ).

Since  $\check{\partial}v = 0$ , we have

$$(\bar{\partial}_+ + \partial_{-,h}) \sum_{I,J} v_{IJ} (dz_1 \wedge \cdots \wedge dz_s \wedge d\bar{z}_I) \otimes (d\bar{z}_{s+1} \wedge \cdots \wedge d\bar{z}_n \wedge dz_J) = 0.$$

Here  $\bar{\partial}_+$  and  $\partial_{-,h}$  are defined leafwise with respect to  $\mathcal{F}_\pm$  as before.

On the other hand, with respect to this extension of the mixed complex exterior derivative  $\check{\partial} = \bar{\partial}_+ + \partial_{-,h}$ , the leafwise application on Nakano's identity yields similarly as in [6, §2] an estimate

$$(2.1) \quad m\|\phi\|_*^2 \leq \|\check{\partial}\phi\|_*^2 + \|\check{\partial}^*\phi\|_*^2$$

for compactly supported  $C^\infty$   $L^m$ -valued differential forms  $\phi$  on  $U$  of the form

$$\phi = \sum_{|I|+|J|=r} \phi_{IJ} (dz_1 \wedge \cdots \wedge dz_s \otimes d\bar{z}_I) \otimes (d\bar{z}_{s+1} \wedge \cdots \wedge d\bar{z}_n \wedge dz_J) \quad (r \geq 1).$$

Here  $\check{\partial}^*$  denotes the adjoint of  $\check{\partial}$  and the  $L^2$  norms  $\|\cdot\|_*$  are defined with respect to  $h^m$  and  $\Theta_+ - \Theta_-$ . We have used the assumptions that  $\Theta_+|_{\mathcal{F}_+} > 0$  and  $\Theta_-|_{\mathcal{F}_-}$  to identify  $\Theta_+ - \Theta_-$  with a Hermitian metric on  $U$  and exploited the fact that the canonical bundles of  $\mathcal{F}_\pm$  are trivial on  $U$ . (For the computation, see (2.2)–(2.10) in [6].)

Since  $U$  is equivalent to the product of the polydisc  $\mathbb{D}^s$  in the  $z_+$ -coordinate and the polydisc  $\mathbb{D}^{n-s}$  in the  $z_-$ -coordinate, by a standard argument as in [4] and [1] we obtain from (2.1), that there exists a smooth,  $L^m$ -valued form  $u$  on  $U$  satisfying  $\check{\partial}u = v|_U$  and  $m\|u\|_*^2 \leq C_0\|v|_U\|_*^2$ , where

$$C_0 = \frac{\sup_U |dz_1 \wedge \cdots \wedge dz_s|^2 |dz_{s+1} \wedge \cdots \wedge dz_n|^2}{\inf_U |dz_1 \wedge \cdots \wedge dz_s|^2 |dz_{s+1} \wedge \cdots \wedge dz_n|^2}.$$

Here the length of the forms is measured with respect to  $\Theta_+ - \Theta_-$ .

Since  $M$  is compact, there exist finitely many such coordinate neighbourhoods, say  $U_i$  ( $i = 1, 2, \dots, k$ ) such that  $M = \bigcup_{i=1}^k U_i$ . Let  $\{\chi_i\}_{i=1}^k$  be a  $C^\infty$  partition of unity subordinate to  $\{U_i\}$ . Since the given metric  $g$  is equivalent to  $\Theta_+ - \Theta_-$ , there exists a constant  $C_1 > 0$  such that

$$C_1^{-1}\|\phi\|_*^2 \leq \|\phi\|^2 \leq C_1\|\phi\|_*^2$$

hold for all  $\phi \in F^{p,q}(L^m)$ , where the norm  $\|\cdot\|$  is with respect to  $h^m$  and  $g$ .

Hence there exists a constant  $C_2 > 0$  such that, for any  $i$  and  $m \in \mathbb{N}$ , one can find  $u_i \in \bigoplus_{p+q=r-1} F^{p,q}(U_i, L^m|_{U_i})$  satisfying

$$(2.2) \quad \check{\partial}u_i = v|_{U_i}$$

and

$$(2.3) \quad m\|u_i\|^2 \leq C_2\|v\|^2.$$

We put

$$u_1 = \sum_{i=1}^k \chi_i u_i,$$

where  $\chi_i u_i$  are defined to be zero outside  $U_i$ . Then, since  $\sum \chi_i \check{\partial}u_i = \sum \chi_i v|_{U_i} = v$ , we have

$$\check{\partial}u_1 = v + \sum_{i=1}^k \check{\partial}\chi_i \wedge u_i,$$

where the wedge product of  $\check{\partial}\chi_i$  and  $u_i$  is defined in the obvious manner.

In view of (2.3) we have

$$\begin{aligned}
 & \|v - \check{\partial}u_1\| \\
 & \leq \left\| \sum_{i=1}^k \check{\partial}\chi_i \wedge u_i \right\| \\
 (2.4) \quad & \leq k \sup_i \left( \sup_{U_i} |\check{\partial}\chi_i| \cdot \|u_i\| \right) \\
 & \leq k \sqrt{\frac{C_2}{m}} \left( \sup_i \sup_{U_i} |\check{\partial}\chi_i| \right) \|v\|.
 \end{aligned}$$

Hence one can find  $m_1 \in \mathbb{N}$  and  $C < 4m_1$  such that if  $m \geq m_1$ ,

$$(2.5) \quad 4m \|u_1\|^2 \leq C \|v\|^2$$

and

$$(2.6) \quad \|v - \check{\partial}u_1\| \leq \frac{1}{2} \|v\|$$

hold for  $v$ .

We put  $v_0 = v$ ,  $v_1 = v - \check{\partial}u_1$ , and define  $u_\mu \in \bigoplus_{p+q=r-1} F^{p,q}(L^m)$  and  $v_\mu \in \bigoplus_{p+q=r} F^{p,q}(L^m)$  inductively for a fixed  $m \geq m_1$  and for  $\mu = 2, 3, \dots$  by requiring

$$(2.7) \quad 4m \|u_\mu\|^2 \leq C \|v_{\mu-1}\|^2,$$

$$(2.8) \quad \|v_{\mu-1} - \check{\partial}u_\mu\| \leq \frac{1}{2} \|v_{\mu-1}\|$$

and

$$(2.9) \quad v_\mu = v_{\mu-1} - \check{\partial}u_\mu.$$

Then, by letting  $u_\infty = \sum_{\mu=1}^\infty u_\mu$  we have  $\check{\partial}u_\infty = v$  and  $m \|u_\infty\|^2 \leq C \|v\|^2$ . Replacing  $u_\infty$  by the solution of  $\check{\partial}u = v$  satisfying  $u \perp \text{Ker } \check{\partial}$ , we obtain a required smooth solution for  $\check{\partial}u = v$ .  $\square$

**REMARK.** Similarly as above, given any holomorphic vector bundle  $E \rightarrow M$ , one can solve the  $\check{\partial}$ -equations for the  $E \otimes L^m$ -valued forms for sufficiently large  $m$ . A more refined successive approximation has already been used in [7] to prove a similar embedding theorem for “singly” complex-foliated manifolds.

### 3. Application

Theorem 1.1 implies the existence of nonzero sections of sufficiently high tensor powers of the product of a semipositive line bundle and a seminegative line bundle

which are holomorphic (resp. antiholomorphic) along the positively curved (resp. negatively curved) directions, on certain fiber spaces. Such examples are described below.

Let  $X_j$  ( $j = 1, 2$ ) be two compact complex manifolds equipped with positive line bundles  $L_j \rightarrow X_j$ . Let  $\tilde{X}_1 \xrightarrow{\pi} X_1$  be the universal covering of  $X_1$ , let  $\rho$  be a homomorphism from the group of covering transformations of  $\pi$ , say  $\Gamma$ , to the group of biholomorphic automorphisms of  $X_2$ , and let  $M$  be the quotient space of the product  $\tilde{X}_1 \times X_2$  by the equivalence relation

$$(x, y) \sim (\gamma_x, \rho(\gamma)y) \quad (x \in \tilde{X}_1, y \in X_2 \text{ and } \gamma \in \Gamma).$$

Then  $M$  is a flat fiber bundle over  $X_1$  with fiber  $X_2$ , and carries two mutually transverse holomorphic foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  such that the leaves of  $\mathcal{F}_1$  are the images of  $\tilde{X}_1 \times \{y\}$  ( $y \in X_2$ ) in  $M$  and those of  $\mathcal{F}_2$  are the fibers of the bundle  $f: M \rightarrow X_1$ .

If  $L_2$  is invariant under the action of  $\rho(\Gamma)$ , which is always the case if  $X_2$  is simply connected, then  $L_2$  naturally induces a semipositive line bundle say  $\hat{L}_2$  over  $M$ .

It is obvious that the foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  and the line bundles  $f^*L_1 \otimes \hat{L}_2^{-1}$  and  $f^*L_1^{-1} \otimes \hat{L}_2$  both satisfy the assumptions of Theorem 1.1, up to the roles of  $\mathcal{F}_1$  and  $\mathcal{F}_2$ .

It might be an intriguing question to describe a relation between the two embeddings with respect to  $L^m$  and  $L^{-m}$  in the situation of Theorem 1.1.

REMARK. Classification results are known for holomorphic foliations on some complex manifolds. For the case of ruled surfaces, see [3]. Holomorphic foliations of codimension one on complex tori are classified by Ghys [2].

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### References

- [1] J.-P. Demailly: *Estimations  $L^2$  pour l'opérateur  $\bar{\partial}$  d'un fibré vectoriel holomorphe sémi-positif au-dessus d'une variété kählérienne complète*, Ann. Sci. École Norm. Sup. (4) **15** (1982), 457–511.
- [2] É. Ghys: *Feuilletages holomorphes de codimension un sur les espaces homogènes complexes*, Ann. Fac. Sci. Toulouse Math. (6) **5** (1996), 493–519.
- [3] X. Gómez-Mont: *Holomorphic foliations in ruled surfaces*, Trans. Amer. Math. Soc. **312** (1989), 179–201.
- [4] L. Hörmander:  *$L^2$  estimates and existence theorems for the  $\bar{\partial}$  operator*, Acta Math. **113** (1965), 89–152.
- [5] Y. Matsushima: *On the intermediate cohomology group of a holomorphic line bundle over a complex torus*, Osaka J. Math. **16** (1979), 617–631.
- [6] T. Ohsawa: *A generalization of Matsushima's embedding theorem*, J. Math. Kyoto Univ. **48** (2008), 383–388.
- [7] T. Ohsawa: *On the projectively embeddable complex-foliated structures*, preprint.

Takeo Ohsawa  
Graduate School of Mathematics  
Nagoya University  
Chikusaku Furocho 464–8603 Nagoya  
Japan  
e-mail: ohsawa@math.nagoya-u.ac.jp

Daisuke Tarama  
Department of Applied Mathematics and Physics Graduate School of Informatics  
Kyoto University  
Yoshida-honmachi Sakyo-ku Kyoto, 606–8501  
Japan  
e-mail: dsktrm@amp.i.kyoto-u.ac.jp