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ON G-h-COBORDISMS BETWEEN G-HOMOTOPY SPHERES

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1. Introduction

We work in the smooth category, and G will be a finite group in the present paper. This paper is concerned with free G-actions on homotopy spheres.

Let us recall Milnor's theorem:

Theorem 1.1 ([6; Corollary 12.13]). *Any h-cobordism W between lens spaces L and L' must be diffeomorphic to $L\times[0,1]$ if the dimension of L is greater than or equal to* 5.

Let V be a unitary G-representation space of complex dimension n. When we consider a unit sphere $S(V)$ in V, we call it *a linear G-sphere* or we say that it has *a linear G-action.* In particular, if the G-action is free, $S(V)$ is called *a free linear G-sphere.* We see that Theorem 1.1 is put in another form as follows:

Theorem 1.2. *Let $S(V)$ and $S(V')$ be free linear G-spheres of dimension $2n-1\geqq 5$. If G is cyclic and W is a G-h-cobordism between $S(V)$ and $S(V')$, then W must be G-diffeomorphic to $S(V)\times I$, where $I=[0,1]$.*

As a generalization of Theorem 1.2, we proved the following result and gave some examples in [11].

Proposition 1.3 ([11; Proposition 3.1]). *Let G be a finite group such that $SK_1(\boldsymbol{Z}[G])=0$. Then the following hold*:

(1) *If X is a free G-homotopy sphere of dimension $2n-1\geqq 5$, any G-h-cobordism W between X and itself must be G-diffeomorphic to $X\times I$.*

(2) *If $S(V)$* and *$S(V')$ are free linear G-spheres of dimension $2n-1\geqq 5$, any G-h-cobordism W' between $S(V)$ and $S(V')$ must be G-diffeomorphic to $S(V)\times I$.*

The purpose of this paper is to extend this result to a much more general case. Let $Wh(G)$ be the Whitehead group of G, $L^s_m(G)$ and $L^h_m(G)$ the Wall groups. $\boldsymbol{Z}[G]$ is the integral group ring with involution - defined by $\overline{\Sigma a_g g}=\Sigma a_g g^{-1}$ where $a_g \in \boldsymbol{Z}$ and $g \in G$. For a matrix (x_{ij}) with coefficients in $\boldsymbol{Z}[G]$, $\overline{(x_{ij})}$ is defined by $(\overline{x_{ji}})$. Then $Wh(G)$ has the induced involution also denoted by -. We define a subgroup $\tilde{A}_m(G)$ of $Wh(G)$ by

$$\tilde{A}_m(G)=\{\tau \in Wh(G) | \bar{\tau}=(-1)^m \tau\}.$$

Put

$$A_m(G)=\tilde{A}_m(G)/\{\tau+(-1)^m \bar{\tau} | \tau \in Wh(G)\}.$$

Let c: $A_{2n+1}(G) \rightarrow L^s_{2n}(G)$ be the map in the Rothenberg exact suquence

$$\cdots \rightarrow A_{2n+1}(G) \xrightarrow{c} L^s_{2n}(G) \xrightarrow{d} L^h_{2n}(G) \rightarrow \cdots,$$

and $\tilde{c}$: $\tilde{A}_{2n+1}(G) \rightarrow L^s_{2n}(G)$ the map determining c. Suppose G acts freely on an odd-dimensional homotopy sphere X. We note that the action of each element $g \in G$ preserves the orientation of X. Then we have

Theorem A. *Let G be a finite group, and X a free G-homotopy sphere of dimension $2n-1 \geqq 5$. Then the following* (1) *and* (2) *are equivalent.*

(1) *Any G-h-cobordism W between X and itself must be G-diffeomorphic to $X \times I$.*

(2) $\ker \tilde{c}$ *is trivial.*

Remark 1.4. Since G acts freely on X, we can use the s-cobordism theorem ([3]), thereby the condition (1) is equivalent to the condition that any G-h-cobordism W between X and itself must be a G-s-cobordism.

Corollary B. *Suppose* $\ker \tilde{c}=0$. *Let $S(V)$ and $S(V')$ be free linear G-spheres of dimension $2n-1 \geqq 5$. Then a G-h-cobordism W between $S(V)$ and $S(V')$ must be G-diffeomorphic to $S(V) \times I$.*

Proof. Let C be a cyclic subgroup of G. By Theorem 1.2, $\operatorname{res}_C V=\operatorname{res}_C V'$ as real C-modules. Thus $V=V'$ as real G-modules, and then $S(V')$ is G-diffeomorphic to $S(V)$. Since $\ker \tilde{c}=0$, the conclusion now follows from Theorem A. □

Let G be a finite group which has periodic cohomology. Sondow [9; Theorem 3] showed that $\tilde{A}_{2n+1}(G)$ is isomorphic to $SK_1(\boldsymbol{Z}[G])$. Since $SK_1(\boldsymbol{Z}[G])=0$ implies $\ker\tilde{c}=0$, Proposition 1.3 is a special case of Theorem A and Corollary B. Therefore, it is important to construct an example of a group G which satisfies the condition (2) of Theorem A, although $SK_1(\boldsymbol{Z}[G])\neq 0$. Let G^3 be a finite group which acts freely and linearly on 3-dimensional spheres. We note that G^3 also acts freely and linearly on $S^{4N-1}(N=2,3,\cdots)$. Recently, Kwasik and Schultz [4] showed that the forgetful map $L_1^s(G^3)\to L_1^h(G^3)$ is onto, and the involution - acts trivially on $Wh(G^3)$. Hence, we see that $\tilde{A}_{4N+1}(G^3)\cong A_{4N+1}(G^3)$, and by the Rothenberg exact sequence

$$\cdots\to L_{4N+1}^s(G^3)\xrightarrow{a} L_{4N+1}^h(G^3)\xrightarrow{b} A_{4N+1}(G^3)\xrightarrow{c} L_{4N}^s(G^3)\xrightarrow{d} L_{4N}^h(G^3)\to\cdots$$

we have $\ker\tilde{c}=0$. Then we have the following corollary:

Corollary C. *Let G^3 be a finite group which can act freely and linearly on S^3. Then any G^3-h-cobordism W between a free G^3-homotopy sphere X of dimension $4N-1\geqq 7$ and itself must be G^3-diffeomorphic to $X\times I$.*

EXAMPLE 1.5. Let p be an odd prime, q a prime such that $q\geqq 5$. Let G denote one of the groups $\boldsymbol{Q}_8\times\boldsymbol{Z}_p$, $\boldsymbol{T}^*\times\boldsymbol{Z}_q$, and $\boldsymbol{O}^*\times\boldsymbol{Z}_q$, where $\boldsymbol{Q}_8$, $\boldsymbol{T}^*$, and $\boldsymbol{O}^*$ denote the quaternionic group, the binary tetrahedral group, and the binary octahedral group respectively. By [10; Theorem] we see that $SK_1(\boldsymbol{Z}[G])\cong\boldsymbol{Z}_2$, and see that G satisfies the condition of Corollary C.

This paper is organized as follows: In Section 2 we prepare some notations and definitions which are necessary for proving our theorem. In Section 3, we prove that (1) implies (2) in Theorem A. In Section 4, we prove that (2) implies (1).

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2. Preliminaries

Let R be a ring with unit, G a finite group. Put $GL(R)=\varinjlim GL_n(R)$ and $E(R)=[GL(R),GL(R)]$ the commutator subgroup of $GL(R)$. Then $K_1(R)$ denotes the quotient group $GL(R)/E(R)$. Let $\boldsymbol{Z}$ be the ring of integers and $\boldsymbol{Q}$ the ring of rational numbers. Let $\boldsymbol{Z}[G]$ and $\boldsymbol{Q}[G]$ denote the group rings of G over $\boldsymbol{Z}$ and $\boldsymbol{Q}$ respectively. The Whitehead group of G is the quotient group

$$Wh(G)=K_1(\boldsymbol{Z}[G])/<\pm g\colon g\in G>.$$

The natural inclusion map $i\colon GL(\boldsymbol{Z}[G])\to GL(\boldsymbol{Q}[G])$ gives rise to a group homomorphism $i_*\colon K_1(\boldsymbol{Z}[G])\to K_1(\boldsymbol{Q}[G])$. Then $SK_1(\boldsymbol{Z}[G])$ is defined by setting

$$SK_1(\boldsymbol{Z}[G])=\ker[i_*\colon K_1(\boldsymbol{Z}[G])\to K_1(\boldsymbol{Q}[G])].$$

Let F be a free $\boldsymbol{Z}[G]$-module, and let $\mathscr{B}=\{b_1,\cdots,b_k\}$, $\mathscr{C}=\{c_1,\cdots,c_k\}$ be two different bases for F. Setting $c_j=\sum_{i=1}^{k} b_i a_{ij}$, we obtain a non-singular matrix (a_{ij}) with coefficients in $\boldsymbol{Z}[G]$. The corresponding element of $Wh(G)$ will be denoted by $[\mathscr{C}/\mathscr{B}]$.

Next, we recall the algebraic definitions of Wall's even-dimensional surgery obstruction groups. (These definitions and notations are based on Bak's book [1].) For fixed n, put

$$min=\{a-(-1)^n\bar{a}\mid a\in\boldsymbol{Z}[G]\},$$

which is an additive subgroup of $\boldsymbol{Z}[G]$. Let M be a right $\boldsymbol{Z}[G]$-module. *A sesquilinear form* on M is a biadditive map $B\colon M\times M\to\boldsymbol{Z}[G]$ such that $B(xa,yb)=\bar{a}B(x,y)b$ for $a,b\in\boldsymbol{Z}[G]$ and $x,y\in M$. A sesquilinear form B is called *a* $(-1)^n$*-hermitian form* if $B(x,y)=(-1)^n\overline{B(y,x)}$ for $x,y\in M$. *A min-quadratic module* means a triple $(M,<,>,q)$ *of a finitely generated projective right* $\boldsymbol{Z}[G]$-module M, a $(-1)^n$-hermitian form $<,>\colon M\times M\to\boldsymbol{Z}[G]$ and a map $q\colon M\to\boldsymbol{Z}[G]/min$ which satisfies the following conditions:

(1) $q(xa)=\bar{a}q(x)a\quad(a\in\boldsymbol{Z}[G],x\in M)$

(2) $q(x+y)-q(x)-q(y)\equiv<x,y>\quad(\mathrm{mod}\,min,\ x,y\in M)$

(3) $\tilde{q}(x)+(-1)^n\overline{\tilde{q}(x)}=<x,x>$ for any lifting $\tilde{q}(x)\in\boldsymbol{Z}[G]$ of $q(x)\in\boldsymbol{Z}[G]/min$.

The map $q\colon M\to\boldsymbol{Z}[G]/min$ above is called *a min-quadratic form*. A morphism $(M,<,>,q)\to(M',<,>',q')$ of *min*-quadratic modules is a $\boldsymbol{Z}[G]$-linear map $M\to M'$ which preserves the hermitian and quadratic forms. We say that two *min*-quadratic modules $(M,<,>,q)$ and $(M',<,>',q')$ are *isomorphic* if there exists a morphism $f\colon(M,<,>,q)\to(M',<,>',q')$ such that $f\colon M\to M'$ is bijective. We say that $(M,<,>,q)$ is *non-singular* if the map $M\to M^{\#}=\mathrm{Hom}_{\boldsymbol{Z}[G]}(M,\boldsymbol{Z}[G])$ defined by $x\mapsto<x,\ >$ is an isomorphism. Here $M^{\#}$ is regarded as a right $\boldsymbol{Z}[G]$-module by $(f\cdot a)(x)=\bar{a}(f(x))$ for $f\in M^{\#}$, $a\in \boldsymbol{Z}[G]$ and $x\in M$. Since M is projective over $\boldsymbol{Z}[G]$, so is $M^{\#}$. If P is a finitely generated projective right $\boldsymbol{Z}[G]$-module, we define *the hyperbolic module* $\mathbf{H}(P)=(P\oplus P^{\#},<,>,q)$, where $<(x,f),(y,g)>=f(y)\oplus(-1)^n\overline{g(x)}$, and $q((x,f))=[f(x)]\in \boldsymbol{Z}[G]/min$. It is a non-sigular min-quadratic module. $\mathbf{H}(\boldsymbol{Z}[G])$ is called *the hyperbolic plane*. The standard preferred basis for its underlying module $\boldsymbol{Z}[G]\oplus\boldsymbol{Z}[G]^{\#}$ is the set $\{e=(1,0),f=(0,identity)\}$. If $\mathbf{H}(\boldsymbol{Z}[G])$ has the standard preferred basis, we denote it by $\mathbf{H}(\boldsymbol{Z}[G])_{based}$ and call it *the based hyperbolic plane*. Let Y be a bar operation invariant subgroup of $K_1(\boldsymbol{Z}[G])$ including $\{\pm 1\}$. We define the category $\mathbf{Q}(\boldsymbol{Z}[G],min)_{based-Y}$ as follows. The objects are all non-singular min-quadratic modules $(M,<,>,q)$ such that M is free module with a preferred basis $\{e_1,\cdots,e_m\}$ such that the $m\times m$ matrix $(<e_i,e_j>)$ vanishes in $K_1(\boldsymbol{Z}[G])/Y$. When we emphasize the preferred basis $\mathscr{C}$ of M, we also denote it by $(M,\mathscr{C},<,>,q)$. Let $(M,\mathscr{C},<,>,q)$ and $(M',\mathscr{C}',<,>',q')$ be two objects of $\mathbf{Q}(\boldsymbol{Z}[G],min)_{based-Y}$ with $\operatorname{rank}M=\operatorname{rank}M'$. Put $\mathscr{C}=\{e_1,\cdots,e_m\}$ and $\mathscr{C}'=\{e_1',\cdots,e_m'\}$. A $\boldsymbol{Z}[G]$-isomorphism f: $M\to M'$ and the preferred bases $\mathscr{C}$ and $\mathscr{C}'$ determine a matrix A by a formula

$$(f(e_1),\cdots,f(e_m))=(e'_1,\cdots,e'_m)A.$$

Then a morphism f: $(M,\mathscr{C},<,>,q)\to(M',\mathscr{C}',<,>',q')$ is an isomorphism of min-quadratic modules such that the matrix A given above vanishes in $K_1(\boldsymbol{Z}[G])/Y$. If $(M,<,>,q)$ (resp.$(M',<,>',q')$) has a preferred basis $\{e_1,\cdots,e_m\}$ (resp.$\{e'_1,\cdots,e'_m\}$), we define the orthogonal sum $(M,<,>,q)\perp(M',<,>',q')=(M\oplus M',<,>\oplus<,>',q\oplus q')$ such that $M\oplus M'$ has the preferred basis $\{e_1,\cdots,e_m,e'_1,\cdots,e'_m\}$. It is clear that $\mathbf{H}(\boldsymbol{Z}[G])_{based}\in\mathbf{Q}(\boldsymbol{Z}[G],min)_{based-Y}$. We define the Grothendieck group under the orthogonal sum

$$KQ_0(\boldsymbol{Z}[G],min)_{based-Y}=K_0(\mathbf{Q}(\boldsymbol{Z}[G],min)_{based-Y}.$$

We also define *the Witt group*

$$WQ_0\big(\boldsymbol{Z}[G],min\big)_{based-Y}=KQ_0(\boldsymbol{Z}[G],min)_{based-Y}/[\mathbf{H}(\boldsymbol{Z}[G])_{based}].$$

The even-dimensional surgery obstruction groups are defined by

$$L^h_{2n}(G)=WQ_0(\boldsymbol{Z}[G],min)_{based-K_1(\boldsymbol{Z}[G])},$$

and

$$L^s_{2n}(G)=WQ_0(\boldsymbol{Z}[G],min)_{based-[\pm G]}.$$

Finally, we recall the definition of $\tilde{c}$: $\tilde{\mathscr{A}}_{2n+1}(G)\to L^s_{2n}(G)$. For $\tau\in\tilde{\mathscr{A}}_{2n+1}(G)$, let $A=(a_{ij})$ be a $2m\times 2m$ matrix representing τ. Put $\mathbf{H}(\boldsymbol{Z}[G]^m)_{based}=((\boldsymbol{Z}[G]^m)\oplus(\boldsymbol{Z}[G]^m)^{\#},\mathscr{B},<,>,q)$, where $\mathscr{B}=\{e_1,\cdots,e_m,f_1,\cdots,f_m\}$ is the standard preferred basis. By applying A to $\{e_1,\cdots,e_m,f_1,\cdots,f_m\}$ by $(e_1,\cdots,e_m,f_1,\cdots,f_m)A$, we get a newly based module M' with the same underlying $\boldsymbol{Z}[G]$-module as $\mathbf{H}(\boldsymbol{Z}[G]^m)_{based}$ and the basis $\mathscr{B}'$ of M' is given by

$$\mathscr{B}'=\{\sum_{i=1}^{m}e_ia_{i1}+\sum_{i=1}^{m}f_ia_{m+i\,1},\cdots,\sum_{i=1}^{m}e_ia_{i\,2m}+\sum_{i=1}^{m}f_ia_{m+i\,2m}\}.$$

Since $\tau+\bar{\tau}=0$, $(M',\mathscr{B}',<,>,q)$ is a non-singular *min*-quadratic module in $\mathbf{Q}(\boldsymbol{Z}[G],min)_{based-[\pm G]}$. We define $\tilde{c}(\tau)=[(M',\mathscr{B}',<,>,q)]$, where [ ] denotes the equivalence class of $(M',\mathscr{B}',<,>,q)$ in $L^s_{2n}(G)$. We see that this defines a homomorphism $\tilde{c}$: $\tilde{A}_{2n+1}(G)\rightarrow L^s_{2n}(G)$.

3. Proof of the part (1) implying (2).

Let τ be an element of $\tilde{A}_{2n+1}(G)$. Then there exist framed immersions $\{\eta_i$: $S^{n-1}\times I\rightarrow X^{2n-1}\times I\}$ with boundary embedded ($1\leqq i\leqq r$ for some large r), such that the resulting G-normal map from the *G-h-cobordism W*

$$(f;b):(W,\partial W;T(W))\rightarrow(X\times I,X\times\{0,1\};\ f^*T(X\times I))$$

has the G-surgery obstruction $\tilde{c}(\tau)$. As Wall discussed in [12; Proof of Theorem 5.8], $(f;b)$ is obtained as follows. Suppose a $2r\times 2r$ matrix A represents τ, for sufficiently large r. Let $\mathbf{H}(\boldsymbol{Z}[G]^r)_{based}$ be the hyperbolic module whose underlying module is $(\boldsymbol{Z}[G]^r)\oplus(\boldsymbol{Z}[G]^r)^{\#}$ with the standard preferred basis

$$\mathscr{S}=\{e_1,\cdots,e_r,f_1,\cdots,f_r\}.$$

Let Q be the quadratic module $\mathbf{H}(\boldsymbol{Z}[G]^r)$ with the basis $\mathscr{B}=A(\mathscr{S})$, where $A(\mathscr{S})$ means the basis obtained by applying A to $\mathscr{S}$ as in the definition of the map c. Put $\mathscr{B}=\{b_1,\cdots,b_{2r}\}$. Then the values $<b_i,b_j>$ and $q(b_i)$ are determined. We can choose framed immersions η_i: $S^{n-1}\times I\rightarrow X^{2n-1}\times I, i=1,\cdots,r$, such that the diagram

$$\begin{tikzcd}[column sep=small] S^{n-1}\times I \arrow[rr, "\eta_i"] \arrow[dr] & & X^{2n-1}\times I \arrow[dl] \\ & I & \end{tikzcd}$$

commutes, the restrictions η_i: $S^{n-1}\times\{0,1\}\to X^{2n-1}\times\{0,1\}$ are embeddings, and the equivariant intersection numbers and equivariant self-intersection numbers among η_i's are equal to $<b_i,b_j>$ and $q(b_i)$. Furthermore, we may regard that η_i's are regular homotopies form the trivial embeddings h_i^0: $S^{n-1}\times D^n\to X$ to embeddings h_i^1: $S^{n-1}\times D^n\to X$. Now we attach n-handles to $X\times I$ with attaching maps $h_i^1\times\{1\}$. Let W be the resulting manifold. Then, W is the trace of G-surgery of id: $X\to X$ along h_i^1: $S^{n-1}\times D^n\to X$. (f: $W\to X\times I$ and b: $T(W)\to f^*T(X\times I)$ are simultaneously obtained.) Put $K_q(W)=\ker[f_*$: $H_q(W)\to H_q(X\times I)]$ and $K_q(W,\partial W)=\ker[f_*$: $H_q(W,\partial W)\to H_q(X\times I,X\times\{0,1\})]$, then the module $K_n(W)$ is isomorphic to $K_n(W,\partial W)$ and $K_n(W,\partial W)$ has the class of the cores of the attached handles as the preferred basis. We complete these to n-dimensional spheres S_i by adjoining the images in $X\times I$ of the η_i, and the disks in the D_i^{2n-1} spanning the images of the h_i^0, and rounding the resulting corners. Denote by $\mathscr{C}$ the basis $\{S_1,\cdots,S_{2r}\}$. Then the quadratic module $(K_n(W),\mathscr{C},<,>,q)$ is isomorphic to the Q given above. We say that $\mathscr{C}$ is *a basis of $K_n(W)$ which is given by the definition of the G-surgery obstruction $\sigma(f;b)$*. Thus, the resulting G-normal map

$$(f;b)\colon (W,\partial W;T(W))\to(X\times I,X\times\{0,1\};\ f^*T(X\times I))$$

has the G-surgery obstruction $\tilde{c}(\tau)$. Here $\partial W=X\amalg X'$, f is a degree one G-map satisfying $f(X)\subset X\times\{0\}$ and $f(X')\subset X\times\{1\}$, and b is a G-vector bunble isomorphism. Moreover it follows from the construction that $f|_X$: $X\to X\times\{0\}=X$ coincides with the identity map on X, and $f|_{X'}$: $X'\to X\times\{1\}$ is a G-simple homotopy equivalence

Suppose that τ lies in the kernel of $\tilde{c}$. We denote the isomorphism above from Q to $(K_n(W),\mathscr{C},<,>,q)$ by ψ. Now we consider a set $\{\psi(e_1),\cdots,\psi(e_r),\psi(f_1),\cdots,\psi(f_r)\}$. This gives another basis of $K_n(W)$, and satisfies $<\psi(e_i),\psi(e_j)>=0$ and $q(\psi(e_i))=0$ for $1\le i\le r$. Then the set $\{\psi(e_i)\}$ determines a subkernel in the category $\mathbf{Q}(\boldsymbol{Z}[G],min)_{based-K_1(\boldsymbol{Z}[G])}$. Hence we can perform G-surgery of $(f;b)$ along $\psi(e_i)$'s and obtain a G-homotopy equivalence

$$(f';b')\colon (W',\partial W';T(W'))\to(X\times I,X\times\{0,1\};\ f'^*T(X\times I)).$$

Then the Whitehead torsion $\tau(f')$ of f': $W'\to X\times I$ is computed as follows. At first, we note that W is obtained from G-surgery along the embeddings α_i: $S^{n-1}\times D^{n+1}\to W'$ dual to the $\psi(e_i)$. The cores of α_i's are the boundaries of the disks which are obtained by removing the open disks around the embedded spheres (corresponding to) $\psi(e_i)$ from the embedded spheres(corresponding to) $\psi(f_i)$. Thus α_i's are trivial embeddings and

$$\mathscr{C}_0=\{\psi(f_1),\cdots,\psi(f_r),(-1)^n\psi(e_1),\cdots,(-1)^n\psi(e_r)\}$$

is the standard basis of $K_n(W)$ when we construct W by taking the equivariant connected sum of W' with r-copies of $G\times S^n\times S^n$: that is, if $G\times S_i^n\times S_i^n$ $(1\leqq i\leqq r)$ denote the r-copies of $G\times S^n\times S^n$, $\mathscr{C}_0$ is represented by

$$\{\{1\}\times\{*_1\}\times S_1^n,\cdots,\{1\}\times\{*_r\}\times S_r^n,\{1\}\times S_1^n\times\{*_1\},\cdots,\{1\}\times S_r^n\times\{*_r\}\}$$

where $\{1\}\times\{*_i\}\times S_i^n$ and $\{1\}\times S_i^n\times\{*_i\}$ intersect at the point $(1,*_i,*_i)$, and they generate the homology groups $H_n(S_i^n\times S_i^n)$ $(1\leqq i\leqq r)$. We note that $\mathscr{C}_0$ is an s-basis of $K_n(W')$. For these two bases $\mathscr{C}$ and $\mathscr{C}_0$ of $K_n(W)$, the following lemma holds.

Lemma 3.1. *Let X be a homotopy sphere, and f': $W'\to X\times I$ a G-homotopy equivalence. We suppose that we can obtain a G-normal map*

$$(f;b)\colon (W,X\amalg X';T(W))\to(X\times I,X\times\{0,1\};\ f^*T(X\times I))$$

with G-surgery kernel $K_n(W)\cong \boldsymbol{Z}[G]^{2r}$ as a free $\boldsymbol{Z}[G]$-module when we construct W by taking the equivariant connected sum of W' with r copies of $G\times S^n\times S^n$. Let $\mathscr{C}_0$ be a basis of $K_n(W)$ which is given by

$$\mathscr{C}_0=\{\{1\}\times\{*_1\}\times S_1^n,\cdots,\{1\}\times\{*_r\}\times S_r^n,\{1\}\times S_1^n\times\{*_1\},\cdots,\{1\}\times S_r^n\times\{*_r\}\}$$

where $\{1\}\times\{*_i\}\times S_i^n$ and $\{1\}\times S_i^n\times\{*_i\}$ intersect at the point $(1,*_i,*_i)$, and they generate the homology groups $H_n(S_i^n\times S_i^n)$ $(1\leqq i\leqq r)$. Let $\mathscr{C}$ be the basis of $K_n(W)$ which is given by the definition of the G-surgery obstruction $\sigma(f;b)$. Then we have

$$[\mathscr{C}/\mathscr{C}_0]=(-1)^n\tau(f'),$$

where $[\mathscr{C}/\mathscr{C}_0]$ is the element of $Wh(G)$ defined in the previous section.

Proof. Let f'_*: $C_*(W')\to C_*(X\times I)$ be the chain map induced from the G-homotopy equivalence f': $W'\to X\times I$. $C_*(f')$ denotes the mapping cone of f'_*. Let F_* be the chain complex such that $F_n=\boldsymbol{Z}[G]^{2r}$ with the standard basis as the preferred basis and $F_k=0$ for $k\neq n$. We put

$$g_*=f'_*\oplus 0_*\colon\ C_*(W')\oplus F_*\to C_*(X\times I)$$

and denote by $C_*(g)$ the mapping cone of g_*. We claim that $C_*(g)=C_*(f')\oplus F_{*-1}$ as stably-based chain complexes. In fact, $C_k(g)=C_{k-1}(W')\oplus F_{k-1}\oplus C_k(X\times I)=C_k(f')\oplus F_{k-1}$ and if $\partial_*,\partial'_*,d_*$, and d'_* denote the boundary operators of $C_*(g),C_*(f'),C_*(W')\oplus F_*$ and $C_*(X\times I)$ respectively,

$$\begin{aligned}\partial_k(a+b+c) &= -d_{k-1}(a+b)+g_*(a+b)+d'_k(c)\\ &= -d_{k-1}(a)+f'_*(a)+d'_k(c)\\ &= \partial'_k(a+c)\end{aligned}$$

for $a \in C_{k-1}(W')$, $b \in F_{k-1}$ and $c \in C_k(X \times I)$. Since the boundary operator of F_* is the 0-map, the claim is established.

Let f_*: $C_*(W) \to C_*(X \times I)$ be the chain map induced from the G-normal map f: $W \to X \times I$. By considering the way to construct W from W', g_* can be regarded as f_*. The module F_n represents the n-cycles of the chain complex of r-copies of $G \times S^n \times S^n$ which were attached to W' in the procedure of obtaining W from W'. We can check that f: $W \to X \times I$ satisfies $H_i(C_*(f))=0$ $(i \neq n+1)$, and $H^{n+2}(C_*(f), L)=0$ for any $\boldsymbol{Z}[G]$-module L. Then as in the proof of [5; Theorem 4], we can choose a stable basis for $H_{n+1}(C_*(f))$ so that $\tau(C_*(f))=0$. This defines an equivalence class of preferred bases for $H_{n+1}(C_*(f))$. By [5; P. 128] $H_{n+1}(C_*(f))=$ $K_n(W)$ as finitely generated stably free $\boldsymbol{Z}[G]$-modules with a preferred equivalence class of basis. Hence this base is $\mathscr{C}$.

For calculating $\tau(C_*(f))$, we consider the following short exact sequence in the category of chain complexes and chain maps;

$$0 \to C_*(f') \to C_*(g) \to F_{*-1} \to 0.$$

Now we see that $H_q(C_*(f))=0$ for all q and the homology groups of $C_*(g)$ and F_* have preferred bases. Since we can regard $C_*(g)$ as $C_*(f)$, we have $H_q(C_*(g))=0$ if $q \neq n+1$, and the preferred basis of $H_{n+1}(C_*(g)) = H_{n+1}(C_*(f))$ is $\mathscr{C}$. On the other hand, it holds that $H_q(F_*)=0$ if $q \neq n$ and the preferred basis of $H_n(F_*)$ is $\mathscr{C}_0$ as mentioned above. Then the exact homology sequence

$$\cdots \to H_r(C_*(f')) \to H_r(C_*(g)) \to H_{r-1}(F_*) \to \cdots$$

can be thought of as a free acyclic chain complex $\mathscr{H}$ of dimension $6n+2$. Hence the torsion $\tau(\mathscr{H})$ is defined. The calculation of $\tau(\mathscr{H})$ is reduced to the calculation of the torsion of

$$0 \to H_{n+1}(C_*(f)) \to H_n(F_*) \to 0.$$

Since $\mathscr{H}_{3n+3}=H_n(F_*)$ and $\mathscr{H}$ is acyclic, we have

$$\tau(\mathscr{H})=(-1)^{3n+3}[\mathscr{C}/\mathscr{C}_0]=(-1)^{n+1}[\mathscr{C}/\mathscr{C}_0].$$

Hence by [6; Theorem 3.2]

$$\tau(C_*(f))=\tau(C_*(g))$$

$$=\tau(C_*(f'))+\tau(F_*)+\tau(\mathscr{H}).$$

Since $F_k=0$ for $k\neq n$, we have $\tau(F_*)=0$. Since $\mathscr{C}$ is taken to satisfy $\tau(C_*(f))=0$, we obtain $[\mathscr{C}/\mathscr{C}_0]=(-1)^n\tau(f')$. □

By the definition of $\tilde{c}$, we have $\tau=[\mathscr{C}/\psi(\mathscr{S})]$. Since $[\mathscr{C}_0/\psi(\mathscr{S})]=0$, we get $\tau=(-1)^n\tau(f')$.

Lemma 3.2. *If f': $W'\to X\times I$ is a G-homotopy equivalence, then*

$$\tau(f')=-f'_*\tau(W',X).$$

Proof. Since W' is G-homotopic to $X\times I$, W' is a G-h-cobordism between X and X'. Let r: $W'\to X$ be a strong G-deformation retract. Then we have a G-homotopy commutative diagram of G-homotopy equivalences

$$\begin{tikzcd} W' \arrow[r, "f'"] \arrow[d, "r"] & X\times I \arrow[d, "\rho"] \\ X \arrow[r, "id"] & X, \end{tikzcd}$$

where ρ: $X\times I\to X$ is the canonical strong G-deformation retract. Therefore,

$$\begin{aligned} \tau(\rho\circ f')&=\tau(id\circ r)\\ \tau(\rho)+\rho_*\tau(f')&=\tau(id)+id_*\tau(r)\\ \rho_*\tau(f')&=\tau(r). \end{aligned}$$

Let i: $X\to W'$ be the canonical inclusion. Clearly i is a G-homotopy inverse of r. By [6; Lemma 7.6], we have $\tau(i)=\tau(W',X)$ and by [2; (22.5)], we have $\tau(r)=-r_*\tau(i)$. Then we have

$$\begin{aligned} \tau(f')&=\rho_*^{-1}\tau(r)\\ &=-\rho_*^{-1}\circ r_*\tau(i)\\ &=-f'_*\tau(W',X). \end{aligned}$$ □

By this lemma, we have $\tau(W',X)=(-1)^{n+1}f'^{-1}_*(\tau)$. Now we claim that there exists a G-s-cobordism W'' such that W' is G-cobordant to W'' relative to boundary. In fact, since $\tilde{c}(\tau)=0$, we can do surgery on

(f',b') leaving the boundary fixed, thereby we obtain a G-normal map (f'',b''), where $f''\colon (W'', \partial W'') \to (X\times I, X\times\{0,1\})$ is a G-simple homotopy equivalence and $b''\colon T(W'') \to f''^{*}T(X\times I)$ is a G-vector bundle isomorphism. Let $i_0\colon X \to W''$ and $i_1\colon X' \to W''$ be the inclusion maps. It is sufficient to show that $\tau(i_0)=0$. Let $\iota\colon X \to X\times I$ be the inclusion map into the 0-level. Then we have a G-homotopy commutative diagram of G-homotopy equivalences

$$\begin{array}{ccc} W'' & \xrightarrow{\ f''\ } & X\times I \\ \uparrow{\scriptstyle i_0} & & \uparrow{\scriptstyle \iota} \\ X & \xrightarrow{\ id\ } & X. \end{array}$$

Since $f''\colon W'' \to X\times I$ is a G-simple homotopy equivalence,

$$\begin{aligned} \tau(\iota\circ id) &= \tau(f''\circ i_0) \\ \tau(\iota)+\iota_*\tau(id) &= \tau(f'')+f''_*\tau(i_0) \\ 0 &= f''_*\tau(i_0). \end{aligned}$$

Since f''_* is a group isomorphism, we have $\tau(i_0)=0$. Similarly, it holds that $\tau(i_1)=0$, thereby W'' is a G-s-cobordism between X and X'. Thus our claim is established. Since G acts freely on X, by the s-cobordism theorem, W'' is G-diffeomorphic to $X\times I$, that is, X' is G-diffeomorphic to X. Hence W' is a G-h-cobordism between X and itself with the Whitehead torsion $\tau(W',X)=(-1)^{n+1}f'^{-1}_*(\tau)$.

The assumption, which says that any G-h-cobordism must be a G-s-cobordism, implies $\tau(W',X)=0$. Now we get $\tau=0$, and have completed the proof of the part: (1) $\Rightarrow$ (2).

4. Proof of the part (2) implying (1).

In the case $|G|\leqq 2$, since it holds that $Wh(G)=0$, the conclusion follows from the h-cobordism theorem. Our proof will be given in the case $|G|\geqq 3$. Let W be a G-h-cobordism between X and itself, with $\dim W=2n\geqq 6$. To distinguish the inclusions of X to W, we put $\partial W = X\amalg X'$, where X' is a copy of X.

At first, we show that $\tau=\tau(W,X)$ lies in $\tilde{A}_{2n+1}(G)$. We prepare the following lemma.

Lemma 4.1. *Let G be a finite group of order $|G| \geqq 3$. If G acts freely on a homotopy sphere X with* $\dim X \geqq 5$, *any G-self-homotopy equivalence of X is G-homotopic to the identity map.*

Proof. Let φ be a G-self-homotopy equivalence between X and itself, then $\deg\varphi = \pm 1$. Since G acts freely on X, we have $\deg\varphi \equiv 1 \bmod |G|$. Since $|G| \geqq 3$, we have $\deg\varphi = 1$, thereby φ is homotopic to the identity map. Now, let $[X,X]_G$ denote the set of all G-homotopy classes of G-maps from X to itself. We see that

$$[X,X]_G \cong H^n(X/G; \underline{\pi_n(X)}) \cong H^n(X/G; \underline{\boldsymbol{Z}}),$$

where $\underline{\pi_n(X)}$ is the coefficient bundle with fiber $\pi_n(X)$ derived from the bundle $X \underset{G}{\times} X$ over X/G. Since the G-action preserves the orientation of X, $\underline{\boldsymbol{Z}}$ is $\boldsymbol{Z}$ and $H^n(X/G; \boldsymbol{Z})$ is isomorphic to $\boldsymbol{Z}$. We consider $[X,X]$, the set of all homotopy classes of maps from X to itself. Then,

$$[X,X] \cong H^n(X; \pi_n(X)) \cong H^n(X; \boldsymbol{Z}) \cong \boldsymbol{Z}.$$

Let tr: $H^n(X; \boldsymbol{Z}) \to H^n(X/G; \boldsymbol{Z})$ denote the transfer map, and p^*: $H^n(X/G; \boldsymbol{Z}) \to H^n(X; \boldsymbol{Z})$ the homomorphism induced by the canonical projection. Since it holds that $p^* \circ tr(x) = |G| \cdot x$ for $x \in H^n(X; \boldsymbol{Z})$ and both tr and p^* are homomorphisms from $\boldsymbol{Z}$ to $\boldsymbol{Z}$, we see that p^* is injective. Thus, φ is G-homotopic to the identity map. $\square$

Lemma 4.2. *For the G-h-cobordism* (W, X, X'), it holds that

$$\tau(W, X) = \tau(W, X').$$

Proof. Let r be a G-homotopy inverse of i. Let i: $X \to W$ and i': $X' \to W$ be the inclusion maps. By Lemma 4.1, we have

$$\tau(r \circ i') = \tau(id) = 0.$$

On the other hand,

$$\begin{aligned} \tau(r \circ i') &= \tau(r) + r_* \tau(i') \\ &= -r_* \tau(i) + r_* \tau(i') \\ &= r_*(\tau(i') - \tau(i)). \end{aligned}$$

Thus we have $\tau(i') = \tau(i)$, which proves the lemma. $\square$

By the duality theorem ([6; p.394]), we also get

$$\tau(W,X')=-\overline{\tau(W,X)}.$$

Hence by these formulae, we see that $\tau=-\bar{\tau}$, that is, τ is an element of $\tilde{A}_{2n+1}(G)$.

Lemma 4.3. *There exists a G-homotopy equivalence* $f\colon W\to X\times I$ *such that* $\tau(f)=-f_*\tau$, $f(X)\subset X\times\{0\}$, $f(X')\subset X\times\{1\}$, $f|_X$ *and* $f|_{X'}$ *are the identity maps on* X.

Proof. By using a strong G-deformation retract $r\colon W\to X$, we can construct a G-homotopy equivalence $f_0\colon W\to X\times I$ satisfying that $f_0(X)\subset X\times\{0\}$, $f_0(X')\subset X\times\{1\}$, $f_0|_X$ and $f_0|_{X'}$ are G-homotopy equivalences. By Lemma 4.1, $f_0|_X$ and $f_0|_{X'}$ are G-homotopic to the identity map. Hence there exists a G-homotopy equivalence $f\colon W\to X\times I$ such that $f(X)\subset X\times\{0\}$, $f(X')\subset X\times\{1\}$, $f|_X$ and $f|_{X'}$ are the identity maps. Now it follows from Lemma 3.2 that $\tau(f)=-f_*\tau$. □

Let g be a G-homotopy inverse of f relative to the boundary such that $\partial g\colon \partial(X\times I)\to\partial W$ is the identity map. Then the G-vector bundle $g^*T(W)$ over $\partial(X\times I)$ can be identified with $T(X\times I)|_{\partial(X\times I)}$, *and* $f^*g^*T(W)$ is isomorphic to $T(W)$. Thus we can get a G-normal map $(f;b)$, where $b\colon T(W)\to f^*g^*T(W)$ is a G-vector bundle isomorphism such that its restriction to the boundary is the identity map.

Lemma 4.4. *The G-surgery obstruction* $\sigma(f;b)\in L^s_{2n}(G)$ *is* $(-1)^{n+1}\tilde{c}(\tau)$.

Proof. Let F be the free $\boldsymbol{Z}[G]$-module $\boldsymbol{Z}[G]^{2r}$ of rank $2r$, for sufficiently large r. Taking equivariant connected sum of W with r copies of $G\times S^n\times S^n$, we obtain a G-normal map

$$(f'';b'')\colon (W'',X\amalg X';T(W''))\to(X\times I,X\times\{0,1\};\ f''^*g^*T(W))$$

with G-surgery kernel $K_n(W'')\cong F$ as a $\boldsymbol{Z}[G]$-module. We can consider two bases of the G-surgery kernel $K_n(W'')$. One is given by

$$\mathscr{B}_0=\{\{1\}\times\{*_1\}\times S^n_1,\cdots,\{1\}\times\{*_r\}\times S^n_r,\{1\}\times S^n_1\times\{*_1\},\cdots,\{1\}\times S^n_r\times\{*_r\}\},$$

where $\{1\}\times\{*_i\}\times S^n_i$ and $\{1\}\times S^n_i\times\{*_i\}$ intersect at the point $(1,*_i,*_i)$, and they generate the homology groups $H_n(S^n_i\times S^n_i)$ $(1\leqq i\leqq r)$. The other is $\mathscr{B}$ which is given by the definition of the G-surgery obstruction $\sigma(f'';b'')$ as we mentioned in the previous section. We note that $\mathscr{B}$ is taken to satisfy $\tau(C_*(f''))=0$. By Lemma 3.1, we have $[\mathscr{B}/\mathscr{B}_0]=(-1)^n\tau(f)$. Since f satisfies $\tau(f)=-f_*(\tau)$ by Lemma 4.3, we have

$$[\mathscr{B}/\mathscr{B}_0]=(-1)^{n+1}f_*(\tau).$$

Then, we obtain that

$$(-1)^{n+1}\tilde{c}(\tau)=\tilde{c}([\mathscr{B}/\mathscr{B}_0])=\sigma(f'';b'')=\sigma(f;b). \qquad \square$$

Lemma 4.5. *The G-surgery obstruction $\sigma(f;b)$ is obtained as a G-surgery obstruction of a closed G-manifold with free action.*

Proof. At first, we take a point $x_0 \in X$, and consider $G\times\{x_0\}\times I$ in $X\times I$. Since $f|_{\partial W}:\partial W\to X\amalg X$ is the identity map, we have

$$f^{-1}|_X(x_0)=f^{-1}|_{X'}(x_0)=x_0.$$

Then, $f^{-1}(G\times\{x_0\}\times I)$ is G-diffeomorphic to $G\times\{x_0\}\times I\bigcup \overset{l}{\underset{i=1}{\cup}} A_i$, where $A_i\subset W$ is a 1-dimensional submanifold of W. By [7; Proposition 1.3], there exisits a G-map f': $W\to X\times I$ such that f' is G-homotopic to f, f' is transverse regular to $G\times\{x_0\}\times\ I$, $f'^{-1}(G\times\{x_0\}\times I)=G\times\{x_0\}\times I$, and $f'|$: $\partial W\to X\times\{0,1\}$ is the identity map. By the transverse regularity of f' to $G\times\{x_0\}\times I$, if we make a G-tubular neighbourhood $G\times D_{x_0}\times I$ of $G\times\{x_0\}\times I$ small enough, then we may assume that $f'|$: $f'^{-1}(G\times D_{x_0}\times I)\to G\times D_{x_0}\times I$ is a linear map on each fiber. Since $f'|$: $f'^{-1}(G\times\{x_0\}\times I)\to G\times\{x_0\}\times I$ is G-homotopic to a G-diffeomorphism relative to $G\times\{x_0\}\times\{0,1\}$, by using the equivariant homotopy covering property, we may regard that $f'|_{f'^{-1}(G\times D_{x_0}\times I)}$ is a G-diffeomorphism. We note that $f'|$: $\partial W\to X\times\{0,1\}$ is the identity map. Put $W_0=\text{Closure}(W-f'^{-1}(G\times D_{x_0}\times I))$ and $(X\times I)_0=\text{Closure}(X\times I-(G\times D_{x_0}\times I))$. Furthermore, there exists a G-homotopy inverse g': $X\times I\to W$ of f' such that $g'|_{\partial(X\times I)\cup(G\times D_{x_0}\times I)}$: $\partial(X\times I)\cup(G\times D_{x_0}\times I)\to(X\cup X')\cup f'^{-1}(G\times D_{x_0}\times I)$ is strictly the inverse of $f'|_{(X\cup X')\cup f'^{-1}(G\times D_{x_0}\times I)}$ and $g'((X\times I)_0)\subset W_0$. Thus we get a G-normal map

$$(f';b')\colon (W,X\amalg X';T(W))\to(X\times I,X\times\{0,1\};f'^*g'^*T(W))$$

such that the restriction to $(W_0;T(W_0))$

$$(f'|_{W_0};b'|_{T(W_0)})\colon (W_0,\partial W_0;T(W_0))\to((X\times I)_0,\partial(X\times I)_0;f'^*|_{W_0}g'^*|_{(X\times I)_0}T(W_0))$$

is also a G-normal map. Let $V=W_0\bigcup_{f'|_{\partial W_0}}(X\times I)_0$ be the identification space $[W_0\amalg(X\times I)_0/x=f'(x)$ if $x\in\partial W_0]$. Since $f'|_{\partial W_0}$ is a diffeomorphism, V is a smooth G-manifold. Similarly we make $V'=(X\times I)_0\bigcup_{id|_{\partial(X\times I)_0}}$

$(X\times I)_0$ and define a degree 1 G-map F: $V\to V'$ by $F(x)=f'(x)$ if $x\in W_0$ and $F(x)=id(x)$ if $x\in(X\times I)_0$. Now we construct a G-normal map

$$(F;B)\colon (V;T(V))\to(V';F^*T(V')),$$

where B is a G-vector bundle isomorphism defined by $B|_{T(W_0)}=b'|_{T(W_0)}$ and $B|_{T((X\times I)_0)}=id$. Then for $\sigma(F;B)$ the G-surgery obstruction of $(F;B)$, it is easy to see that

$$\sigma(F;B)=\sigma(f'|_{W_0};b'|_{T(W_0)})=\sigma(f';b')=(f;b).$$

This completes the proof. □

Let P be the 2-Sylow subgroup of G. Since G has periodic cohomology, the 2-Sylow subgroups of G are cyclic or quaternonic. Hence by [8; p. 14, Example 2] it holds that $SK_1(\boldsymbol{Z}[P])=0$. Thus $\mathrm{res}_P f$ is a P-simple homotopy equivalence because $\mathrm{res}_P\tau\in SK_1(\boldsymbol{Z}[P])=0$. This yields $\mathrm{res}_P\sigma(f;b)=0$. Since by Lemma 4.5 we can use Wall's transfer theorem ([13; Theorem 12]), we have $\sigma(f;b)=0$. By Lemma 4.4, we have $\tilde{c}(f_*(\tau))=0$, that is, $\tau=0$. We have completed the proof of the part: $(2)\Rightarrow(1)$.

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