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WEAKLY REGULAR MODULES

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Let R be a ring with an identity. Following Ramamurthi [2], we call R a *left weakly regular ring* if R satisfies one of the following equivalent conditions: 1) $a \in RaRa$ for every $a \in R$; 2) $R/\mathfrak{a}$ is right R-flat for any two-sided ideal $\mathfrak{a}$ of R; 3) $\mathfrak{a}^2 = \mathfrak{a}$ for any left ideal $\mathfrak{a}$ of R. In this paper, we shall introduce the notion of a weakly regular (right) module: A right R-module M is called a *weakly regular module* if $m \in \mathrm{Hom}_R(M, M)(m)\,\mathrm{Hom}_R(M, R)(m) = \{\Sigma_i s_i(m) f_i(m) \mid s_i \in \mathrm{Hom}_R(M, M), f_i \in \mathrm{Hom}_R(M, R)\}$ for every $m \in M$. Needless to say, R is a left weakly regular ring if and only if R_R is weakly regular. We shall give a list of equivalent conditions for M_R to be weakly regular including the condition that M_R is locally projective and $T\mathfrak{a} = T\mathfrak{a}^2$ for any left ideal $\mathfrak{a}$ of R, where T is the trace ideal of M_R (Theorem 7). We shall show also that if M_R is a finitely generated (abbr. f.g.) weakly regular module, then $\mathrm{Hom}_R(M, M)$ is a left weakly regular ring (Theorem 8). The author would like to express his thanks to Prof. H. Tominaga for his helpful suggestion.

1. Preliminaries

Throughout this paper, R will represent an associative ring with 1, and M a unitary right R-module. Every (right or left) module is unitary and unadorned $\otimes$ means $\otimes_R$, unless otherwise stated. We set $M^* = \mathrm{Hom}_R(M, R)$ and $S = \mathrm{Hom}_R(M, M)$. For any S-R-submodule N of M, we set $T_N = \Sigma_{f \in M^*} f(N) = \mathrm{Hom}_R(M, R)(N)$. $T = T_M$ is the trace ideal of M_R. Given ${}_RA$, $U_S({}_SN \otimes A)$ will denote the set of all S-submodules of $N \otimes A$. Further, $U_{T_N}({}_RA)$ will denote the set of all R-submodules A' of A with $T_N A' = A'$. Especially, $U_T({}_RR)$ is the set of all left ideals $\mathfrak{a}$ of R such that $T\mathfrak{a} = \mathfrak{a}$. Finally, let $\Gamma_R(M, A)$: $M \otimes A \to \mathrm{Hom}_R({}_RM^*, {}_RA)$ be the unique map such that $\Gamma_R(M, A) \cdot (m \otimes a)(U) = U(m)a$ for $m \in M$, $a \in A$ and $U \in M^*$ (see [1]).

A right R-module M is called a *weakly regular module* (abbr. *w. regular module*) if $m \in S(m)M^*(m)$ for every $m \in M$. A submodule N_R of M_R is said to be *ideal pure* if $N \cap M\mathfrak{a} = N\mathfrak{a}$ for every left ideal $\mathfrak{a}$ of R, or equivalently, $i \otimes 1$: $N \otimes R/\mathfrak{a} \to M \otimes R/\mathfrak{a}$ is monic for every left ideal $\mathfrak{a}$ of R, where i: $N \to M$ is the inclusion (see [1]).

Proposition 1. *The following conditions are equivalent:*

1) $\Gamma_R(M, A)$ *is monic for every* ${}_RA$.

2) $m \in MM^*(m)$ *for every* $m \in M$.

3) *If* $\beta\colon G_R \to M_R$ *is a map such that* $\beta(G)$ *is ideal pure in* M, *then for each* $x_1, x_2, \cdots, x_n$ *in* G *there exists some* $\phi\colon M_R \to G_R$ *such that* $\beta\phi\beta(x_i) = \beta(x_i)$ *for* $i = 1, 2, \cdots, n$.

4) *For each* $m_1, m_2, \cdots, m_k \in M$ *there exist some* $x_1, x_2, \cdots, x_n \in M$ *and* $f_1, f_2, \cdots, f_n \in M^*$ *such that* $m_i = \Sigma_j x_j f_j(m_i)$ *for* $i = 1, 2, \cdots, k$.

5) *The lattice homomorphism* $U_T({}_RR) \to U_S({}_SM)$; $\mathfrak{a} \to M\mathfrak{a}$, *is bijective.*

Proof. See [1, Theorem 3.2] and [4, Theorems 2.1 and 3.1].

A right R-module M is said to be *locally projective* (abbr. *l. projective*) if M satisfies any of the equivalent conditions in Proposition 1.

One may remember that every projective module is l. projective and every l. projective module is flat [1].

2. Weakly regular modules

We shall begin this section with the following.

Proposition 2. *If* M_R *is w. regular, then there hold the following:*

(1) M_R *is l. projective.*

(2) *If* N *is an* S-R-*submodule of* M, *then* N_R *is w.regular.*

(3) *If* R *is a regular ring, then* M_R *is regular in the sense of Zelmanowitz* [3].

(4) *If* $S = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ *with simple rings* S_i, *then* $M = S_1(M) \oplus S_2(M) \oplus \cdots \oplus S_n(M)$ *and* $S_i(M)$ *is* S-R-*simple.*

Proof. (1), (2) and (3) are immediate from Proposition 1 and [4].

(4) Obviously, M is the direct sum of S-R-submodules $S_i(M)$. Let m be an arbitrary non-zero element of $S_i(M)$. By the usual way, mM^* may be regarded as a subset of S. Since $S_jS(mM^*) = S_j(mM^*) = 0$ if $i \neq j$, $S(mM^*)$ is an ideal of S included in S_i. By hypothesis, $SmM^*(m)$ contains non-zero m. Hence the non-zero ideal $S(mM^*)$ coincides with S_i, and $SmR \supseteq SmM^*(m) = S_i(M)$, proving that $S_i(M)$ is S-R-simple.

EXAMPLE 1. Let R be a left w. regular ring. Then, by Proposition 2(2), every two-sided ideal of R is w. regular as a right R-module.

Proposition 3. (1) M_R *is w. regular if and only if for any* S-*submodule* ${}_SN$ *of* M *there holds* $N = NM^*(N)$.

(2) *Let* $M_i (i \in I)$ *be right* R-*modules. Then* $\Sigma_{i \in I} \oplus M_i$ *is w. regular if and only if each* M_i *is w. regular.*

Proof. (1) is evident from the definition.

(2) We assume $M=\Sigma_i \oplus M_i$ is w. regular. Let $p_i\colon M \to M_i$ be the projection, and take an arbitrary element $m_i \in M_i$. As is easily seen, $p_i S p_i = \mathrm{Hom}_R(M_i, M_i)$ and $\mathrm{Hom}_R(M, R)(m_i) = \mathrm{Hom}_R(M_i, R)(m_i)$. Now, recalling that M is w.regular, we obtain $m_i = p_i m_i \in p_i S(m_i)\mathrm{Hom}_R(M, R)(m_i) = p_i S(p_i m_i)\mathrm{Hom}_R(M_i, R)(m_i) = \mathrm{Hom}_R(M_i, M_i)(m_i)\mathrm{Hom}_R(M_i, R)(m_i)$. The converse is almost evident.

Lemma 4. Let α be in the center of S. Then there exists an element β in the center of S with $\alpha\beta\alpha=\alpha$ if and only if $M=\alpha M \oplus \ker \alpha$.

Proof. See [3, Lemma 3.3].

Proposition 5. *If M_R is w.regular, then there hold the following*:
(1) *S is a semiprime ring.*
(2) *The center of S is a regular ring.*

Proof. The proofs of (1) and (2) are similar to those of [3, (3.2)] and [3, Theorem 3.4], respectively. Here, we shall prove (2) only. Let α be in the center of S. According to Lemma 4, it suffices to show that $M=\alpha M \oplus \ker \alpha$. For each $m \in M$, we have $\alpha m = \Sigma_i s_i(\alpha m) f_i(\alpha m)$ with some $s_i \in S$ and $f_i \in M^*$. Setting $t=\Sigma_i s_i(m f_i) \in S$, we obtain $\alpha m = \alpha^2 t m$, so that $m - \alpha t m \in \ker \alpha$. Since $m = \alpha t m + (m - \alpha t m)$, it follows $M = \alpha M + \ker \alpha$. If $\alpha m'$ ($m' \in M$) is in $\ker \alpha$ then, as we have seen above, there exists some $t' \in S$ such that $\alpha m' = \alpha^2 t' m' = t' \alpha^2 m' = 0$. Hence, $M = \alpha M \oplus \ker \alpha$.

Lemma 6. *If M_R is 1.projective and N_R is an ideal pure submodule of M, then for each $n_1, \cdots, n_k \in N$ there exist $x_1, \cdots, x_n \in N$ and $f_1, \cdots, f_n \in M^*$ such that $n_i = \Sigma_j x_j f_j(n_i)$ $(i=1, \cdots, k)$.*

Proof. As is well known, there exists an R-homomorphism of a free R-module G_R onto N_R. By Proposition 1 (3), there exists $\phi \in \mathrm{Hom}_R(M, G)$ such that $\beta\phi(n_i) = n_i$ $(i=1, \cdots, k)$. Choose a finitely generated free direct summand F of G_R including $\phi(n_i)$ $(i=1, \cdots, k)$. Let $y_1, \cdots, y_n$ be a free R-basis of F, and $y = \Sigma_j y_j v_j(y)$ with coordinate functions v_j. Let $\pi\colon G_R \to F_R$ be the projection, $\theta = \pi\phi$ and $\alpha\colon F_R \to N_R$ the restriction of β. If we set $x_j = \alpha(y_j)$ and $f_j = v_j\theta$, then $\Sigma_j x_j f_j(n_i) = \alpha \Sigma_j y_j v_j \theta(n_i) = \alpha\theta(n_i) = \alpha\pi\phi(n_i) = \beta\phi(n_i) = n_i$.

Now, we are at a position to state our first principal theorem.

Theorem 7. *The following conditions are equivalent*:
1) *M_R is a w.regular module.*
2) *M_R is 1.projective and every S-R-submodule of M is ideal pure.*
3) *M_R is 1.projective and SmR_R is ideal pure for each $m \in M$.*
4) *For any S-R-submodule N of M, N_R is flat and each left R-module A*

the lattices $U_{T_N}({}_RA)$ *and* $U_S({}_SN\otimes A)$ *are isomorphic via the inverse assignments* ψ: $U_{T_N}({}_RA)\to U_S({}_SN\otimes A)$; $A'\mapsto N\otimes A'$ *and* Φ: $U_S({}_SN\otimes A)\to U_{T_N}(A)$; ${}_SB\mapsto \{\Sigma_i f_i(n_i)a_i \mid f_i\in M^*,\ n_i\otimes a_i\in B\}$.

5) *For any S-R-submodule N of M, the lattice isomorphism* $U_{T_N}({}_RR)\to U_S(N_S)$; $\mathfrak{a}\mapsto N\mathfrak{a}$, *is surjective.*

6) M_R *is* 1. *projective and* $\mathfrak{b}=\mathfrak{a}\mathfrak{b}$ *for each pair* $\mathfrak{a}$, $\mathfrak{b}\in U_T({}_RR)$ *such that* $\mathfrak{a}\supseteq\mathfrak{b}$ *and* $\mathfrak{a}$ *is a two sided ideal of R.*

7) M_R *is* 1. *projective and* $T\mathfrak{a}=T\mathfrak{a}^2$ *for each left ideal* $\mathfrak{a}$ *of R.*

Proof. 1)⇒2). M_R is 1.projective by Proposition 2(1). Take an arbitrary S-R-submodule N of M. Let $\mathfrak{b}$ be an arbitrary left ideal, and consider the diagram

$$(7.1)\qquad N\otimes R/\mathfrak{b}\xrightarrow{i\otimes 1} M\otimes R/\mathfrak{b}\xrightarrow{\Gamma_R(M,\ R/\mathfrak{b})} \mathrm{Hom}_R({}_RM^*,\ {}_R(R/\mathfrak{b}))\,,$$

where i: $N\to M$ is the inclusion. If $(i\otimes 1)(n\otimes\bar{1})=0$ for some $n\otimes\bar{1}\in N\otimes R/\mathfrak{b}$, then $\Gamma_R(M,R/\mathfrak{b})\,(i\otimes 1)(n\otimes\bar{1})(M^*)=\bar{0}$, and hence $M^*(n)\subseteq\mathfrak{b}$. We note that $N\otimes R/\mathfrak{b}\cong N/N\mathfrak{b}$ and $n\otimes\bar{1}$ corresponds to $n+N\mathfrak{b}$ under this isomorphism. Since M_R is w. regular, there holds $n\in SnM^*(n)=SnRM^*(n)\subseteq N\mathfrak{b}$, which means that $n\otimes\bar{1}=0$. Hence, $i\otimes 1$ is monic, and N is ideal pure.

2)⇒3). Trivial.

3)⇒1). Let n be an arbitrary element of M, and consider the following diagram

$$(7.2)\qquad SnR\otimes R/M^*(n)\xrightarrow{i\otimes 1} M\otimes R/M^*(n)\xrightarrow{\Gamma_R(M,\ R/M^*(n))} \mathrm{Hom}_R({}_RM^*,\ {}_R(R/M^*(n)))\,.$$

Then $\Gamma_R(M,R/M^*(n))(i\otimes 1)(n\otimes\bar{1})(M^*)=M^*(n)\bar{1}=\bar{0}$. Since SnR_R is ideal pure and M_R is 1. projective, $\Gamma_R(M,R/M^*(n))\,(i\otimes 1)$ is monic by Proposition 1 (1). Hence $n\otimes\bar{1}=0$. Now, recalling that $n\otimes\bar{1}$ corresponds to $n+SnM^*(n)$ under the isomorphism $SnR\otimes R/M^*(n)\cong SnR/SnM^*(n)$, we see that $n\in SnM^*(n)$.

1)⇒4) (cf. [4]). Let N be an arbitrary S-R-submodule of M. Then N_R is flat by Proposition 2(1), (2) and the remark at the end of § 1. Hence, for each $A'\in U_{T_N}(A)$, $N\otimes A'$ is included naturally in $N\otimes A$ as an S-module, and so ψ is well-defined. Next, we shall show that Φ is well-defined. Since M^* is a left R-module, $L=\{\Sigma_i f_i(n_i)a_i \mid f_i\in M^*,\ n_i\otimes a_i\in B\}$ is a left R-module. By 1), 2) and Lemma 6, for each $\Sigma_i f_i(n_i)a_i\in L$, we have $n_i=\Sigma_{p=1}^{t} x_p g_p(n_i)$ with some $x_p\in N$ and $g_p\in M^*$. Then $\Sigma_i f_i(n_i)a_i=\Sigma_i f_i(\Sigma_p x_p g_p(n_i))a_i=\Sigma_{i,p} f_i(x_p)g_p(n_i)a_i\in T_NL$. Hence, $L=T_NL$ and L is in $U_{T_N}(A)$. We have therefore seen that Φ is well-defined. Now, we shall show that $\Phi\psi(A')=A'$ for each $A'\in U_{T_N}(A)$. Obviously, $\Phi\psi(A')$ is included in A'. On the other hand, $A'=T_NA'\subseteq\Phi\psi(A')$, and hence $\Phi\psi(A')=A'$. Finally, we shall show that $\psi\Phi(B)=B$ for each S-

submodule B of $N\otimes A$. Since $\psi\Phi(B)=N\otimes L$ with $L=\{\Sigma_i f(n_i)a_i \mid f_i\in M^*, n_i\otimes a_i\in B\}$, it suffices to prove that $N\otimes L=B$. Every element of $N\otimes L$ is a finite sum of $x\otimes(\Sigma_i f_i(n_i)a_i)$ with $x\in N, f_i\in M^*$ and $n_i\otimes a_i\in B$. Since $x\otimes(\Sigma_i f_i(n_i)_i a)=\Sigma_i xf_i(n_i)\otimes a_i=\Sigma_i (xf_i)(n_i\otimes a_i)\in B$ by $xf_i\in S$, we see that $N\otimes L\subset B$. Conversely, let $b=\Sigma_i n_i\otimes a_i$ be an arbitrary element of B. Then again by 1), 2) and Lemma 6, there exist $x_p\in N$ and $g_p\in M^*$ such that $n_i=\Sigma_p x_p g_p(n_i)$ for all i. It is immediate that $b=\Sigma_i \Sigma_p x_p g_p(n_i)\otimes a_i=\Sigma_p x_p\otimes(\Sigma_i g_p(n_i)a_i)$ and $x_p\otimes\Sigma_i g_p(n_i)a_i =(x_p g_p)b\in B$ by $x_p g_p\in S$. This means that we may assume from the beginning that every $n_i\otimes a_i$ is in B. Hence, $b=\Sigma_p x_p\otimes(\Sigma_i g_p(n_i)a_i)\in N\otimes L$, whence it follows $B\subseteq N\otimes L$.

4)$\Rightarrow$5). Trivial.

5)$\Rightarrow$1). Given $m\in M$, the map $U_{T_{SmR}}({}_RR)\to U_S(SmR)$; $\mathfrak{a}\mapsto Sm\mathfrak{a}$, is surjective by assumption. There exists therefore some $\mathfrak{a}\in U_{T_{SmR}}({}_RR)$ such that $Sm=Sm\mathfrak{a}=Sm(T_{SmR}\mathfrak{a})=SmM^*(SmR)\mathfrak{a}=SmM^*(Sm\mathfrak{a})=SmM^*(Sm)=SmM^*(m)$, which shows that M_R is w.regular.

1)$\Rightarrow$6). By Proposition 2(1), M_R is 1.projective. Let $\mathfrak{a}$, $\mathfrak{b}\in U_T({}_RR)$ be such that $\mathfrak{a}\supseteq\mathfrak{b}$ and $\mathfrak{a}$ is a two-sided ideal of R, and let N be the S-R-submodule $M\mathfrak{a}$ of M. Since N is ideal pure by 2), there holds $M\mathfrak{b}\cap N=N\mathfrak{b}=M\mathfrak{a}\mathfrak{b}$. Combining this with $\mathfrak{a}\supseteq\mathfrak{b}$, we obtain $M\mathfrak{b}=M\mathfrak{b}\cap N=M\mathfrak{a}\mathfrak{b}$. Now, by Proposition 1 (5) we readily obtain $\mathfrak{b}=\mathfrak{a}\mathfrak{b}$.

6)$\Rightarrow$5). If N is an S-R-submodule of M, then $N=M\mathfrak{a}$ with some $\mathfrak{a}\in U_T({}_RR)$ by Proposition 1 (5). Since $\mathfrak{a}=T\mathfrak{a}=M^*(M)\mathfrak{a}=M^*(N)$ and N is a right R-module, $\mathfrak{a}$ is a two-sided ideal. It suffices therefore to show that each $L\in U_S({}_SN)$ there exists some $\mathfrak{b}\in U_{T_N}({}_RR)$ such that $L=N\mathfrak{b}$. Again by Proposition 1 (5), $L=M\mathfrak{b}$ with some $\mathfrak{b}\in U_T({}_RR)$. Then, $\mathfrak{a}=T\mathfrak{a}=M^*(N)\supseteq M^*(L) =M^*(M)\mathfrak{b}=T\mathfrak{b}=\mathfrak{b}$. Hence, $\mathfrak{b}=\mathfrak{a}\mathfrak{b}=T_N\mathfrak{b}$ by hypothesis, and so $L=M\mathfrak{b}= M\mathfrak{a}\mathfrak{b}=N\mathfrak{b}$ with $\mathfrak{b}\in U_{T_N}({}_RR)$.

6)$\Rightarrow$7). If $\mathfrak{a}$ is a left ideal of R, then the two-sided ideal $T\mathfrak{a}R$ includes $T\mathfrak{a}$. As is easily seen, $T\mathfrak{a}$ and $T\mathfrak{a}R$ are in $U_T({}_RR)$. Hence, $T\mathfrak{a}=(T\mathfrak{a}R)T\mathfrak{a} \subseteq T\mathfrak{a}^2$ by assumption, proving $T\mathfrak{a}=T\mathfrak{a}^2$.

7)$\Rightarrow$6). Let $\mathfrak{a}$, $\mathfrak{b}\in U_R({}_TR)$ be such that $\mathfrak{a}\supseteq\mathfrak{b}$ and $\mathfrak{a}$ is a two-sided ideal of R. Then, $\mathfrak{b}=T\mathfrak{b}=T\mathfrak{b}^2\subseteq T\mathfrak{a}\mathfrak{b}=\mathfrak{a}\mathfrak{b}$, that is, $\mathfrak{b}=\mathfrak{a}\mathfrak{b}$.

EXAMPLE 2. If R is not left w.regular, then R_R is not w.regular but (locally) projective. Next, let R be the ring $\boldsymbol{Z}$ of rational integers, and $M=\boldsymbol{Z}/(p)$, p a prime. Then $M^*=0$. Hence, M_R is not w.regular but every S-R-submodule of M is trivially ideal pure. According to Theorem 7, above examples enable us to see that the local projectivity of M_R and the property that each S-R-submodule of M is ideal pure are independent.

The next corresponds to a theorem of Ware concerning regular modules (see [3, Corollary 4.2]).

Theorem 8. *If M_R is f.g. w.regular, then S is a left w.regular ring.*

Proof. Let $M=m_1R+\cdots+m_pR$, and $a=a_1$ an arbitrary element of S. By hypothesis, $a_1m_1=\Sigma_{i=1}^{l}g_i(a_1m_1)f_i(a_1m_1)$ with some $g_i\in S$ and $f_i\in M^*$. Setting $b_1=\Sigma_i g_ia_1(m_1f_i)a_1\in Sa_1Sa_1$, we obtain $a_1(m_1)=b_1(m_1)$, and so $\ker(a_1-b_1)\supseteq m_1R$. Repeating the above argument for $a_2=a_1-b_1$ instead of a_1, we find $b_2\in Sa_2Sa_2$ ($\subseteq Sa_1Sa_1$) such that $\ker(a_2-b_2)\supseteq m_2R$. Since $a_3=a_2-b_2\in Sa_2$, there holds further $\ker a_3\supseteq m_1R+m_2R$. Continuing the above procedure, we obtain eventually $a_1=a, \cdots, a_p, a_{p+1}\in Sa_1$ and $b_1, \cdots, b_p\in Sa_1Sa_1$ such that $a_{k+1}=a_k-b_k$ and $\ker a_{k+1}\supseteq m_1R+\cdots+m_kR$ $(k=1, 2, \cdots, p)$. Since $a_{p+1}=0$ by $\ker a_{p+1}\supseteq m_1R+\cdots+m_pR=M$, it follows $a=b_1+\cdots+b_p\in SaSa$, completing the proof.

Corollary 9. *Let N be an S-R-submodule of M. If M_R is w.regular and M/N_R is f.g., then $Hom_R(M/N, M/N)$ is a left w.regular ring.*

Proof. By Proposition 2 (1) and Proposition 1 (5), $N=M\mathfrak{a}$ with some $\mathfrak{a}\in U_T({}_RR)$. Since $\mathfrak{a}=T\mathfrak{a}=M^*(M)\mathfrak{a}=M^*(N)$ and N is a right R-module, $\mathfrak{a}$ is a two-sided ideal of R. It is easy to see that $M/M\mathfrak{a}$ is a w.regular module as an f.g. right $R/\mathfrak{a}$-module. Then $\mathrm{Hom}_R(M/N, M/N)=\mathrm{Hom}_{R/\mathfrak{a}}(M/M\mathfrak{a}, M/M\mathfrak{a})$ is a left w.regular ring by Theorem 8.

EXAMPLE 3. Let R be a commutative regular ring with countably infinite set of orthogonal idempotents e_i. We consider $M=\Sigma_{i=1}^{\infty}\oplus R_i$; $R_i=R$. As usual, S can be regarded as the ring of column finite matrices over R with matrix units e_{ij}. If $a=\Sigma_{i=1}^{\infty} e_ie_{1i}$, then Sa consists of all elements of the form $\Sigma_{j=1}^{n}\Sigma_i a_je_ie_{ji}$. Now, we can easily see that $a\notin SaSa$, which means that S is not left w.regular.

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