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## WEAKLY REGULAR MODULES

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Let  $R$  be a ring with an identity. Following Ramamurthi [2], we call  $R$  a *left weakly regular ring* if  $R$  satisfies one of the following equivalent conditions: 1)  $a \in RaRa$  for every  $a \in R$ ; 2)  $R/\alpha$  is right  $R$ -flat for any two-sided ideal  $\alpha$  of  $R$ ; 3)  $\alpha^2 = \alpha$  for any left ideal  $\alpha$  of  $R$ . In this paper, we shall introduce the notion of a weakly regular (right) module: A right  $R$ -module  $M$  is called a *weakly regular module* if  $m \in \text{Hom}_R(M, M)(m) \text{Hom}_R(M, R)(m) = \{\sum_i s_i(m) f_i(m) \mid s_i \in \text{Hom}_R(M, M), f_i \in \text{Hom}_R(M, R)\}$  for every  $m \in M$ . Needless to say,  $R$  is a left weakly regular ring if and only if  $R_R$  is weakly regular. We shall give a list of equivalent conditions for  $M_R$  to be weakly regular including the condition that  $M_R$  is locally projective and  $T\alpha = T\alpha^2$  for any left ideal  $\alpha$  of  $R$ , where  $T$  is the trace ideal of  $M_R$  (Theorem 7). We shall show also that if  $M_R$  is a finitely generated (abbr. f.g.) weakly regular module, then  $\text{Hom}_R(M, M)$  is a left weakly regular ring (Theorem 8). The author would like to express his thanks to Prof. H. Tominaga for his helpful suggestion.

### 1. Preliminaries

Throughout this paper,  $R$  will represent an associative ring with 1, and  $M$  a unitary right  $R$ -module. Every (right or left) module is unitary and unadorned  $\otimes$  means  $\otimes_R$ , unless otherwise stated. We set  $M^* = \text{Hom}_R(M, R)$  and  $S = \text{Hom}_R(M, M)$ . For any  $S$ - $R$ -submodule  $N$  of  $M$ , we set  $T_N = \sum_{f \in M^*} f(N) = \text{Hom}_R(M, R)(N)$ .  $T = T_M$  is the trace ideal of  $M_R$ . Given  ${}_R A$ ,  $U_{S(N \otimes A)}$  will denote the set of all  $S$ -submodules of  $N \otimes A$ . Further,  $U_{T_N({}_R A)}$  will denote the set of all  $R$ -submodules  $A'$  of  $A$  with  $T_N A' = A'$ . Especially,  $U_{T({}_R R)}$  is the set of all left ideals  $\alpha$  of  $R$  such that  $T\alpha = \alpha$ . Finally, let  $\Gamma_R(M, A): M \otimes A \rightarrow \text{Hom}_R({}_R M^*, {}_R A)$  be the unique map such that  $\Gamma_R(M, A) \cdot (m \otimes a)(U) = U(m)a$  for  $m \in M$ ,  $a \in A$  and  $U \in M^*$  (see [1]).

A right  $R$ -module  $M$  is called a *weakly regular module* (abbr. *w. regular module*) if  $m \in S(m)M^*(m)$  for every  $m \in M$ . A submodule  $N_R$  of  $M_R$  is said to be *ideal pure* if  $N \cap M\alpha = N\alpha$  for every left ideal  $\alpha$  of  $R$ , or equivalently,  $i \otimes 1: N \otimes R/\alpha \rightarrow M \otimes R/\alpha$  is monic for every left ideal  $\alpha$  of  $R$ , where  $i: N \rightarrow M$  is the inclusion (see [1]).

**Proposition 1.** *The following conditions are equivalent:*

- 1)  $\Gamma_R(M, A)$  is monic for every  ${}_R A$ .
- 2)  $m \in MM^*(m)$  for every  $m \in M$ .
- 3) If  $\beta: G_R \rightarrow M_R$  is a map such that  $\beta(G)$  is ideal pure in  $M$ , then for each  $x_1, x_2, \dots, x_n$  in  $G$  there exists some  $\phi: M_R \rightarrow G_R$  such that  $\beta\phi\beta(x_i) = \beta(x_i)$  for  $i=1, 2, \dots, n$ .
- 4) For each  $m_1, m_2, \dots, m_k \in M$  there exist some  $x_1, x_2, \dots, x_n \in M$  and  $f_1, f_2, \dots, f_n \in M^*$  such that  $m_i = \sum_j x_j f_j(m_i)$  for  $i=1, 2, \dots, k$ .
- 5) The lattice homomorphism  $U_{T(R)R} \rightarrow U_{S(S)M}$ ;  $\alpha \rightarrow M\alpha$ , is bijective.

Proof. See [1, Theorem 3.2] and [4, Theorems 2.1 and 3.1].

A right  $R$ -module  $M$  is said to be *locally projective* (abbr. 1. *projective*) if  $M$  satisfies any of the equivalent conditions in Proposition 1.

One may remember that every projective module is 1. projective and every 1. projective module is flat [1].

## 2. Weakly regular modules

We shall begin this section with the following.

**Proposition 2.** *If  $M_R$  is w. regular, then there hold the following:*

- (1)  $M_R$  is 1. projective.
- (2) If  $N$  is an  $S$ - $R$ -submodule of  $M$ , then  $N_R$  is w. regular.
- (3) If  $R$  is a regular ring, then  $M_R$  is regular in the sense of Zelmanowitz [3].
- (4) If  $S = S_1 \oplus S_2 \oplus \dots \oplus S_n$  with simple rings  $S_i$ , then  $M = S_1(M) \oplus S_2(M) \oplus \dots \oplus S_n(M)$  and  $S_i(M)$  is  $S$ - $R$ -simple.

Proof. (1), (2) and (3) are immediate from Proposition 1 and [4].

(4) Obviously,  $M$  is the direct sum of  $S$ - $R$ -submodules  $S_i(M)$ . Let  $m$  be an arbitrary non-zero element of  $S_i(M)$ . By the usual way,  $mM^*$  may be regarded as a subset of  $S$ . Since  $S_j S(mM^*) = S_j(mM^*) = 0$  if  $i \neq j$ ,  $S(mM^*)$  is an ideal of  $S$  included in  $S_i$ . By hypothesis,  $SmM^*(m)$  contains non-zero  $m$ . Hence the non-zero ideal  $S(mM^*)$  coincides with  $S_i$ , and  $SmR \supseteq SmM^*(m) = S_i(M)$ , proving that  $S_i(M)$  is  $S$ - $R$ -simple.

EXAMPLE 1. Let  $R$  be a left w. regular ring. Then, by Proposition 2(2), every two-sided ideal of  $R$  is w. regular as a right  $R$ -module.

**Proposition 3.** (1)  $M_R$  is w. regular if and only if for any  $S$ -submodule  ${}_S N$  of  $M$  there holds  $N = NM^*(N)$ .

(2) Let  $M_i (i \in I)$  be right  $R$ -modules. Then  $\sum_{i \in I} \oplus M_i$  is w. regular if and only if each  $M_i$  is w. regular.

Proof. (1) is evident from the definition.

(2) We assume  $M = \sum_i \oplus M_i$  is w. regular. Let  $p_i: M \rightarrow M_i$  be the projection, and take an arbitrary element  $m_i \in M_i$ . As is easily seen,  $p_i S p_i = \text{Hom}_R(M_i, M_i)$  and  $\text{Hom}_R(M, R)(m_i) = \text{Hom}_R(M_i, R)(m_i)$ . Now, recalling that  $M$  is w.regular, we obtain  $m_i = p_i m_i \in p_i S(m_i) \text{Hom}_R(M, R)(m_i) = p_i S(p_i m_i) \text{Hom}_R(M_i, R)(m_i) = \text{Hom}_R(M_i, M_i)(m_i) \text{Hom}_R(M_i, R)(m_i)$ . The converse is almost evident.

**Lemma 4.** Let  $\alpha$  be in the center of  $S$ . Then there exists an element  $\beta$  in the center of  $S$  with  $\alpha\beta\alpha = \alpha$  if and only if  $M = \alpha M \oplus \ker \alpha$ .

Proof. See [3, Lemma 3.3].

**Proposition 5.** If  $M_R$  is w.regular, then there hold the following:

- (1)  $S$  is a semiprime ring.
- (2) The center of  $S$  is a regular ring.

Proof. The proofs of (1) and (2) are similar to those of [3, (3.2)] and [3, Theorem 3.4], respectively. Here, we shall prove (2) only. Let  $\alpha$  be in the center of  $S$ . According to Lemma 4, it suffices to show that  $M = \alpha M \oplus \ker \alpha$ . For each  $m \in M$ , we have  $\alpha m = \sum_i s_i(\alpha m) f_i(\alpha m)$  with some  $s_i \in S$  and  $f_i \in M^*$ . Setting  $t = \sum_i s_i(m f_i) \in S$ , we obtain  $\alpha m = \alpha^2 t m$ , so that  $m - \alpha t m \in \ker \alpha$ . Since  $m = \alpha t m + (m - \alpha t m)$ , it follows  $M = \alpha M + \ker \alpha$ . If  $\alpha m' (m' \in M)$  is in  $\ker \alpha$  then, as we have seen above, there exists some  $t' \in S$  such that  $\alpha m' = \alpha^2 t' m' = t' \alpha^2 m' = 0$ . Hence,  $M = \alpha M \oplus \ker \alpha$ .

**Lemma 6.** If  $M_R$  is 1.projective and  $N_R$  is an ideal pure submodule of  $M$ , then for each  $n_1, \dots, n_k \in N$  there exist  $x_1, \dots, x_n \in N$  and  $f_1, \dots, f_n \in M^*$  such that  $n_i = \sum_j x_j f_j(n_i) (i=1, \dots, k)$ .

Proof. As is well known, there exists an  $R$ -homomorphism of a free  $R$ -module  $G_R$  onto  $N_R$ . By Proposition 1 (3), there exists  $\phi \in \text{Hom}_R(M, G)$  such that  $\beta\phi(n_i) = n_i (i=1, \dots, k)$ . Choose a finitely generated free direct summand  $F$  of  $G_R$  including  $\phi(n_i) (i=1, \dots, k)$ . Let  $y_1, \dots, y_n$  be a free  $R$ -basis of  $F$ , and  $y = \sum_j y_j v_j(y)$  with coordinate functions  $v_j$ . Let  $\pi: G_R \rightarrow F_R$  be the projection,  $\theta = \pi\phi$  and  $\alpha: F_R \rightarrow N_R$  the restriction of  $\beta$ . If we set  $x_j = \alpha(y_j)$  and  $f_j = v_j\theta$ , then  $\sum_j x_j f_j(n_i) = \alpha \sum_j y_j v_j\theta(n_i) = \alpha\theta(n_i) = \alpha\pi\phi(n_i) = \beta\phi(n_i) = n_i$ .

Now, we are at a position to state our first principal theorem.

**Theorem 7.** The following conditions are equivalent:

- 1)  $M_R$  is a w.regular module.
- 2)  $M_R$  is 1.projective and every  $S$ - $R$ -submodule of  $M$  is ideal pure.
- 3)  $M_R$  is 1.projective and  $SmR_R$  is ideal pure for each  $m \in M$ .
- 4) For any  $S$ - $R$ -submodule  $N$  of  $M$ ,  $N_R$  is flat and each left  $R$ -module  $A$

the lattices  $U_{T_N}({}_R A)$  and  $U_S({}_S N \otimes A)$  are isomorphic via the inverse assignments  $\psi: U_{T_N}({}_R A) \rightarrow U_S({}_S N \otimes A)$ ;  $A' \mapsto N \otimes A'$  and  $\Phi: U_S({}_S N \otimes A) \rightarrow U_{T_N}({}_R A)$ ;  ${}_S B \mapsto \{\sum_i f_i(n_i) a_i \mid f_i \in M^*, n_i \otimes a_i \in B\}$ .

5) For any  $S$ - $R$ -submodule  $N$  of  $M$ , the lattice isomorphism  $U_{T_N}({}_R R) \rightarrow U_S(N_S)$ ;  $\alpha \mapsto N\alpha$ , is surjective.

6)  $M_R$  is 1. projective and  $\mathfrak{b} = \alpha\mathfrak{b}$  for each pair  $\alpha, \mathfrak{b} \in U_T({}_R R)$  such that  $\alpha \supseteq \mathfrak{b}$  and  $\alpha$  is a two sided ideal of  $R$ .

7)  $M_R$  is 1. projective and  $T\alpha = T\alpha^2$  for each left ideal  $\alpha$  of  $R$ .

Proof. 1) $\Rightarrow$ 2).  $M_R$  is 1. projective by Proposition 2(1). Take an arbitrary  $S$ - $R$ -submodule  $N$  of  $M$ . Let  $\mathfrak{b}$  be an arbitrary left ideal, and consider the diagram

$$(7.1) \quad N \otimes R/\mathfrak{b} \xrightarrow{i \otimes 1} M \otimes R/\mathfrak{b} \xrightarrow{\Gamma_R(M, R/\mathfrak{b})} \text{Hom}_R({}_R M^*, {}_R(R/\mathfrak{b})),$$

where  $i: N \rightarrow M$  is the inclusion. If  $(i \otimes 1)(n \otimes \bar{1}) = 0$  for some  $n \otimes \bar{1} \in N \otimes R/\mathfrak{b}$ , then  $\Gamma_R(M, R/\mathfrak{b})(i \otimes 1)(n \otimes \bar{1})(M^*) = \bar{0}$ , and hence  $M^*(n) \subseteq \mathfrak{b}$ . We note that  $N \otimes R/\mathfrak{b} \cong N/N\mathfrak{b}$  and  $n \otimes \bar{1}$  corresponds to  $n + N\mathfrak{b}$  under this isomorphism. Since  $M_R$  is w. regular, there holds  $n \in SnM^*(n) = SnRM^*(n) \subseteq N\mathfrak{b}$ , which means that  $n \otimes \bar{1} = 0$ . Hence,  $i \otimes 1$  is monic, and  $N$  is ideal pure.

2) $\Rightarrow$ 3). Trivial.

3) $\Rightarrow$ 1). Let  $n$  be an arbitrary element of  $M$ , and consider the following diagram

$$(7.2) \quad \begin{array}{ccc} SnR \otimes R/M^*(n) & \xrightarrow{i \otimes 1} & M \otimes R/M^*(n) \xrightarrow{\Gamma_R(M, R/M^*(n))} \\ & & \text{Hom}_R({}_R M^*, {}_R(R/M^*(n))) \end{array}$$

Then  $\Gamma_R(M, R/M^*(n))(i \otimes 1)(n \otimes \bar{1})(M^*) = M^*(n)\bar{1} = \bar{0}$ . Since  $SnR_R$  is ideal pure and  $M_R$  is 1. projective,  $\Gamma_R(M, R/M^*(n))(i \otimes 1)$  is monic by Proposition 1 (1). Hence  $n \otimes \bar{1} = 0$ . Now, recalling that  $n \otimes \bar{1}$  corresponds to  $n + SnM^*(n)$  under the isomorphism  $SnR \otimes R/M^*(n) \cong SnR/SnM^*(n)$ , we see that  $n \in SnM^*(n)$ .

1) $\Rightarrow$ 4) (cf. [4]). Let  $N$  be an arbitrary  $S$ - $R$ -submodule of  $M$ . Then  $N_R$  is flat by Proposition 2(1), (2) and the remark at the end of § 1. Hence, for each  $A' \in U_{T_N}({}_R A)$ ,  $N \otimes A'$  is included naturally in  $N \otimes A$  as an  $S$ -module, and so  $\psi$  is well-defined. Next, we shall show that  $\Phi$  is well-defined. Since  $M^*$  is a left  $R$ -module,  $L = \{\sum_i f_i(n_i) a_i \mid f_i \in M^*, n_i \otimes a_i \in B\}$  is a left  $R$ -module. By 1), 2) and Lemma 6, for each  $\sum_i f_i(n_i) a_i \in L$ , we have  $n_i = \sum_{p=1}^i x_p g_p(n_i)$  with some  $x_p \in N$  and  $g_p \in M^*$ . Then  $\sum_i f_i(n_i) a_i = \sum_i f_i(\sum_p x_p g_p(n_i)) a_i = \sum_{i,p} f_i(x_p) g_p(n_i) a_i \in T_N L$ . Hence,  $L = T_N L$  and  $L$  is in  $U_{T_N}({}_R A)$ . We have therefore seen that  $\Phi$  is well-defined. Now, we shall show that  $\Phi\psi(A') = A'$  for each  $A' \in U_{T_N}({}_R A)$ . Obviously,  $\Phi\psi(A')$  is included in  $A'$ . On the other hand,  $A' = T_N A' \subseteq \Phi\psi(A')$ , and hence  $\Phi\psi(A') = A'$ . Finally, we shall show that  $\psi\Phi(B) = B$  for each  $S$ -

submodule  $B$  of  $N \otimes A$ . Since  $\psi\Phi(B) = N \otimes L$  with  $L = \{\sum_i f(n_i)a_i \mid f_i \in M^*, n_i \otimes a_i \in B\}$ , it suffices to prove that  $N \otimes L = B$ . Every element of  $N \otimes L$  is a finite sum of  $x \otimes (\sum_i f_i(n_i)a_i)$  with  $x \in N$ ,  $f_i \in M^*$  and  $n_i \otimes a_i \in B$ . Since  $x \otimes (\sum_i f_i(n_i)a_i) = \sum_i x f_i(n_i) \otimes a_i = \sum_i (x f_i)(n_i \otimes a_i) \in B$  by  $x f_i \in S$ , we see that  $N \otimes L \subset B$ . Conversely, let  $b = \sum_i n_i \otimes a_i$  be an arbitrary element of  $B$ . Then again by 1), 2) and Lemma 6, there exist  $x_p \in N$  and  $g_p \in M^*$  such that  $n_i = \sum_p x_p g_p(n_i)$  for all  $i$ . It is immediate that  $b = \sum_i \sum_p x_p g_p(n_i) \otimes a_i = \sum_p x_p \otimes (\sum_i g_p(n_i)a_i)$  and  $x_p \otimes \sum_i g_p(n_i)a_i = (x_p g_p)b \in B$  by  $x_p g_p \in S$ . This means that we may assume from the beginning that every  $n_i \otimes a_i$  is in  $B$ . Hence,  $b = \sum_p x_p \otimes (\sum_i g_p(n_i)a_i) \in N \otimes L$ , whence it follows  $B \subset N \otimes L$ .

4)  $\Rightarrow$  5). Trivial.

5)  $\Rightarrow$  1). Given  $m \in M$ , the map  $U_{T_{SmR}}({}_R R) \rightarrow U_S(SmR)$ ;  $\alpha \mapsto Sma$ , is surjective by assumption. There exists therefore some  $\alpha \in U_{T_{SmR}}({}_R R)$  such that  $Sm = Sma = Sm(T_{SmR}\alpha) = SmM^*(SmR)\alpha = SmM^*(Sma) = SmM^*(Sm) = SmM^*(m)$ , which shows that  $M_R$  is w.regular.

1)  $\Rightarrow$  6). By Proposition 2(1),  $M_R$  is 1.projective. Let  $\alpha, \mathfrak{b} \in U_{T({}_R R)}$  be such that  $\alpha \supseteq \mathfrak{b}$  and  $\alpha$  is a two-sided ideal of  $R$ , and let  $N$  be the  $S$ - $R$ -submodule  $M\alpha$  of  $M$ . Since  $N$  is ideal pure by 2), there holds  $M\mathfrak{b} \cap N = N\mathfrak{b} = M\alpha\mathfrak{b}$ . Combining this with  $\alpha \supseteq \mathfrak{b}$ , we obtain  $M\mathfrak{b} = M\mathfrak{b} \cap N = M\alpha\mathfrak{b}$ . Now, by Proposition 1 (5) we readily obtain  $\mathfrak{b} = \alpha\mathfrak{b}$ .

6)  $\Rightarrow$  5). If  $N$  is an  $S$ - $R$ -submodule of  $M$ , then  $N = M\alpha$  with some  $\alpha \in U_{T({}_R R)}$  by Proposition 1 (5). Since  $\alpha = T\alpha = M^*(M)\alpha = M^*(N)$  and  $N$  is a right  $R$ -module,  $\alpha$  is a two-sided ideal. It suffices therefore to show that each  $L \in U_S({}_S N)$  there exists some  $\mathfrak{b} \in U_{T_N}({}_R R)$  such that  $L = N\mathfrak{b}$ . Again by Proposition 1 (5),  $L = M\mathfrak{b}$  with some  $\mathfrak{b} \in U_{T({}_R R)}$ . Then,  $\alpha = T\alpha = M^*(N) \supseteq M^*(L) = M^*(M)\mathfrak{b} = T\mathfrak{b} = \mathfrak{b}$ . Hence,  $\mathfrak{b} = \alpha\mathfrak{b} = T_N\mathfrak{b}$  by hypothesis, and so  $L = M\mathfrak{b} = M\alpha\mathfrak{b} = N\mathfrak{b}$  with  $\mathfrak{b} \in U_{T_N}({}_R R)$ .

6)  $\Rightarrow$  7). If  $\alpha$  is a left ideal of  $R$ , then the two-sided ideal  $T\alpha R$  includes  $T\alpha$ . As is easily seen,  $T\alpha$  and  $T\alpha R$  are in  $U_{T({}_R R)}$ . Hence,  $T\alpha = (T\alpha R)T\alpha \subseteq T\alpha^2$  by assumption, proving  $T\alpha = T\alpha^2$ .

7)  $\Rightarrow$  6). Let  $\alpha, \mathfrak{b} \in U_{R({}_T R)}$  be such that  $\alpha \supseteq \mathfrak{b}$  and  $\alpha$  is a two-sided ideal of  $R$ . Then,  $\mathfrak{b} = T\mathfrak{b} = T\mathfrak{b}^2 \subseteq T\alpha\mathfrak{b} = \alpha\mathfrak{b}$ , that is,  $\mathfrak{b} = \alpha\mathfrak{b}$ .

**EXAMPLE 2.** If  $R$  is not left w.regular, then  $R_R$  is not w.regular but (locally) projective. Next, let  $R$  be the ring  $\mathbf{Z}$  of rational integers, and  $M = \mathbf{Z}/(p)$ ,  $p$  a prime. Then  $M^* = 0$ . Hence,  $M_R$  is not w.regular but every  $S$ - $R$ -submodule of  $M$  is trivially ideal pure. According to Theorem 7, above examples enable us to see that the local projectivity of  $M_R$  and the property that each  $S$ - $R$ -submodule of  $M$  is ideal pure are independent.

The next corresponds to a theorem of Ware concerning regular modules (see [3, Corollary 4.2]).

**Theorem 8.** *If  $M_R$  is f.g. w.regular, then  $S$  is a left w.regular ring.*

Proof. Let  $M = m_1R + \cdots + m_pR$ , and  $a = a_1$  an arbitrary element of  $S$ . By hypothesis,  $a_1m_1 = \sum_{i=1}^l g_i(a_1m_1)f_i(a_1m_1)$  with some  $g_i \in S$  and  $f_i \in M^*$ . Setting  $b_1 = \sum_i g_i a_1(m_1 f_i) a_1 \in Sa_1Sa_1$ , we obtain  $a_1(m_1) = b_1(m_1)$ , and so  $\ker(a_1 - b_1) \supseteq m_1R$ . Repeating the above argument for  $a_2 = a_1 - b_1$  instead of  $a_1$ , we find  $b_2 \in Sa_2Sa_2 (\subseteq Sa_1Sa_1)$  such that  $\ker(a_2 - b_2) \supseteq m_2R$ . Since  $a_3 = a_2 - b_2 \in Sa_2$ , there holds further  $\ker a_3 \supseteq m_1R + m_2R$ . Continuing the above procedure, we obtain eventually  $a_1 = a, \dots, a_p, a_{p+1} \in Sa_1$  and  $b_1, \dots, b_p \in Sa_1Sa_1$  such that  $a_{k+1} = a_k - b_k$  and  $\ker a_{k+1} \supseteq m_1R + \cdots + m_kR$  ( $k = 1, 2, \dots, p$ ). Since  $a_{p+1} = 0$  by  $\ker a_{p+1} \supseteq m_1R + \cdots + m_pR = M$ , it follows  $a = b_1 + \cdots + b_p \in SaSa$ , completing the proof.

**Corollary 9.** *Let  $N$  be an  $S$ - $R$ -submodule of  $M$ . If  $M_R$  is w.regular and  $M/N_R$  is f.g., then  $\text{Hom}_R(M/N, M/N)$  is a left w.regular ring.*

Proof. By Proposition 2 (1) and Proposition 1 (5),  $N = M\alpha$  with some  $\alpha \in U_{T(R)}$ . Since  $\alpha = T\alpha = M^*(M)\alpha = M^*(N)$  and  $N$  is a right  $R$ -module,  $\alpha$  is a two-sided ideal of  $R$ . It is easy to see that  $M/M\alpha$  is a w.regular module as an f.g. right  $R/\alpha$ -module. Then  $\text{Hom}_R(M/N, M/N) = \text{Hom}_{R/\alpha}(M/M\alpha, M/M\alpha)$  is a left w.regular ring by Theorem 8.

EXAMPLE 3. Let  $R$  be a commutative regular ring with countably infinite set of orthogonal idempotents  $e_i$ . We consider  $M = \sum_{i=1}^{\infty} \oplus R_i$ ;  $R_i = R$ . As usual,  $S$  can be regarded as the ring of column finite matrices over  $R$  with matrix units  $e_{ij}$ . If  $a = \sum_{i=1}^{\infty} e_i e_{1i}$ , then  $Sa$  consists of all elements of the form  $\sum_{j=1}^{\infty} \sum_i a_j e_i e_{ji}$ . Now, we can easily see that  $a \notin SaSa$ , which means that  $S$  is not left w.regular.

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