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REIDEMEISTER TORSION AND LENS SURGERIES ON KNOTS IN HOMOLOGY 3-SPHERES I

TERUHISA KADOKAMI

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Abstract

Let $\Sigma(K; p/q)$ be the result of p/q -surgery along a knot K in a homology 3-sphere Σ . We investigate the Reidemeister torsion of $\Sigma(K; p/q)$. Firstly, when the Alexander polynomial of K is the same as that of a torus knot, we give a necessary and sufficient condition for the Reidemeister torsion of $\Sigma(K; p/q)$ to be that of a lens space. Secondly, when the Alexander polynomial of K is of degree 2, we show that if the Reidemeister torsion of $\Sigma(K; p/q)$ is the same as that of a lens space, then $\Delta_K(t) = t^2 - t + 1$.

1. Introduction

We investigate when the result of Dehn surgery along a knot in a homology 3-sphere is a lens space. In this paper, we call such a surgery *lens surgery* not only for hyperbolic knots but also for any knots in homology 3-spheres. Many authors have studied lens surgery by geometric method (see [1, 2, 5, 7, 9, 10, 17, 22, 23, 24, 29, 31]). We approach the problem by algebraic method (see [13, 14, 15, 18, 19]).

K. Reidemeister [20] and W. Franz [8] classified lens spaces completely by using the Reidemeister torsion. Franz provided a useful lemma called *Franz's lemma* (see Theorem 3.1) which is deduced by a result of L -function (see [30]) from Number Theory. We apply the lemma in the present paper. J. Milnor [16] pointed out that the Reidemeister torsion was closely related to the Alexander polynomial. Our method is based on the surgery formula for Reidemeister torsions due to V.G. Turaev (see [21, 25, 26, 27] and Section 2.2). Our results are mentioned in terms of the Alexander polynomials.

We define that an oriented closed 3-manifold M is of *lens type* (or of (p, q) -*lens type*) if it has the same Reidemeister torsion as the lens space (or as $L(p, q)$) (for a precise definition, see Section 2.3). By algebraic and number theoretic study, we obtain necessary conditions for the Alexander polynomial of a knot having a lens type surgery. The multiplicative group $(\mathbf{Z}/n\mathbf{Z})^\times$ plays important roles in our study because it is the Galois group of the n -th cyclotomic field, and the second term q of a lens space $L(p, q)$ is an element of $(\mathbf{Z}/p\mathbf{Z})^\times$.

We point out the following lemma which states a property of the Alexander polynomial of a knot having a lens type surgery. We prove it in Section 3.2. Our two main theorems are obtained by using this lemma. Let $\Sigma(K; p/q)$ be the result of p/q -surgery along a knot K in a homology 3-sphere Σ . If Σ is the 3-sphere S^3 , then we denote $\Sigma(K; p/q)$ by $(K; p/q)$.

Lemma 1.1. *Let K be a knot in a homology 3-sphere Σ , and $\Delta_K(t)$ the Alexander polynomial of K . Let $d (\geq 2)$ be a divisor of p , and ζ a primitive d -th root of unity. If the p/q -surgery $\Sigma(K; p/q)$ is of lens type for $p \geq 2$ and $q \neq 0$, then*

$$N(\Delta_K(\zeta)) = \begin{cases} \pm 1 & (d = 2), \\ 1 & (d \geq 3), \end{cases}$$

where $N(\alpha)$ is the norm of α in the d -th cyclotomic field $\mathbf{Q}(\zeta)$ over \mathbf{Q} .

For the *norm* in a Galois extension, see Section 3.1. In [17], L. Moser firstly showed that non-trivial knots can yield a lens space by Dehn surgery. In fact, Moser determined all rational surgery along all torus knots, and proved that every torus knot yields a lens space as below.

Theorem 1.2 (Moser [17]). *Let K be the (r, s) -torus knot in S^3 , and $M = (K; p/q)$ the result of p/q -surgery along K , where $p, r, |s| \geq 2$ and $q \neq 0$. Then there are three cases:*

- (1) *If $|p - qrs| \neq 0, 1$, then M is a Seifert fibered space with three singular fibers of multiplicities $r, |s|$ and $|p - qrs|$.*
- (2) *If $|p - qrs| = 1$, then M is the lens space $L(p, qr^2)$.*
- (3) *If $|p - qrs| = 0$ (i.e., $p/q = rs$), then M is the connected sum of two lens spaces, $L(r, s) \sharp L(s, r)$.*

Our first main theorem is an algebraic “translation” of the theorem. The following is the first main theorem of this paper.

Main Theorem 1. *Let K be a knot in a homology 3-sphere Σ whose Alexander polynomial is the same as the (r, s) -torus knot, and $M = \Sigma(K; p/q)$, where $p, r, s \geq 2$ and $q \neq 0$. Then M is of lens type if and only if the following (1) and (2) hold.*

- (1) $\gcd(p, r) = 1$ and $\gcd(p, s) = 1$,
- (2) $r \equiv \pm 1 \pmod{p}$ or $s \equiv \pm 1 \pmod{p}$ or $qrs \equiv \pm 1 \pmod{p}$.

We prove Main Theorem 1 by using Franz’s lemma (see [6, 8, 27]) in Section 3.

H. Goda and M. Teragaito [9] showed that if a genus one knot in S^3 yields a lens space, then the knot is the trefoil. Our second main theorem is included by Goda-Teragaito’s theorem in the restricted case $\Sigma = S^3$, but extends theirs to the case of

knots in any homology 3-spheres from the algebraic view point. When we say “the degree of $\Delta_K(t)$,” we take a regularization that $\Delta_K(t)$ is in $\mathbf{Z}[t]$ and $\Delta_K(0) \neq 0$. We prove Main Theorem 2 by using norm of an algebraic number (see [3, 30]) in Section 4.

Main Theorem 2. *Let K be a knot in a homology 3-sphere Σ whose Alexander polynomial $\Delta_K(t)$ is of degree 2. If a Dehn surgery $\Sigma(K; p/q)$ is of lens type for $p \geq 2$ and $q \neq 0$, then $\Delta_K(t) = t^2 - t + 1$.*

Recently, P. Ozsváth and Z. Szabó [18] obtained a necessary condition on the Alexander polynomial of a knot in S^3 which yields a lens space by using the knot Floer homology (see [13, 19], and Appendix). Note that Main Theorem 2 extends a special case ($m = 1$, $s_1 = 1$) of Ozsváth-Szabó’s result to a rational surgery along a knot in a homology 3-sphere.

In Section 2, we recall Turaev’s definition [26] of the Reidemeister torsion, prepare the surgery formula due to Turaev [25, 26] and Sakai [21], and give a precise definition of an oriented closed 3-manifold of *lens type*. In Section 3, we prove Lemma 1.1 and Main Theorem 1. In Section 4, we prove Main Theorem 2. In Appendix, we show that the Alexander polynomial of (p, q) -torus knot satisfies Ozsváth-Szabó’s condition only by deformations of the polynomials from the well-known expression.

Lemma 1.1 may have many applications. We applied it in the papers [11, 12] which are joint works with Yuichi Yamada.

2. Surgery formula of Reidemeister torsion

2.1. Torsion of chain complex. We review the *torsion* of a chain complex.

Let V be an n -dimensional vector space over a field F , and $\mathbf{b} = (b_1, \dots, b_n)$ and $\mathbf{c} = (c_1, \dots, c_n)$ two bases of V . Then b_i can be expressed by a linear combination of c_1, \dots, c_n .

$$b_i = \sum_{j=1}^n a_{ij} c_j \quad (i = 1, \dots, n)$$

The matrix $A = (a_{ij})$ is the transition matrix from \mathbf{c} to \mathbf{b} . We denote the determinant of A by

$$[\mathbf{b}/\mathbf{c}].$$

It is a non-zero element of F . If $n = 0$, then $[\emptyset/\emptyset] = 1$.

Let \mathbf{C}_* be a finitely generated free chain complex over a field F :

$$\mathbf{C}_* : 0 \rightarrow C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \rightarrow 0$$

In the case that \mathbf{C}_* is *acyclic*, we define the *torsion* of \mathbf{C}_* as follows: Let $\mathbf{c}_i = (c_i^{(1)}, \dots, c_i^{(p_i)})$ ($i = 0, \dots, m$) be a basis of C_i . We denote the kernel of ∂_{i-1} (resp. the image of ∂_i) by Z_i (resp. B_i). Then $Z_i = B_i$ and

$$C_i \cong Z_i \oplus B_{i-1} = B_i \oplus B_{i-1}.$$

We take bases of B_i as $\mathbf{b}_i = (b_i^{(1)}, \dots, b_i^{(q_i)})$. A lift of \mathbf{b}_{i-1} in C_i is denoted by $\tilde{\mathbf{b}}_{i-1}$. Then $\mathbf{b}_i \tilde{\mathbf{b}}_{i-1}$ is a basis of C_i . The *torsion* of \mathbf{C}_* with respect to $\mathbf{c} = (\mathbf{c}_0, \dots, \mathbf{c}_m)$ is defined by

$$\tau(\mathbf{C}_*; \mathbf{c}) = \prod_{i=0}^m [\mathbf{b}_i \tilde{\mathbf{b}}_{i-1} / \mathbf{c}_i]^{(-1)^{i+1}}.$$

The torsion $\tau(\mathbf{C}_*; \mathbf{c})$ is a non-zero element of F , and does not depend on the choices of \mathbf{b}_i and $\tilde{\mathbf{b}}_{i-1}$. When \mathbf{c} is clear, we denote $\tau(\mathbf{C}_*; \mathbf{c})$ by $\tau(\mathbf{C}_*)$. In the other case, i.e., if \mathbf{C}_* is not acyclic, then we define $\tau(\mathbf{C}_*) = 0$ formally. When \mathbf{C}_* is a finitely generated free chain complex over an integral domain R , we define $\tau(\mathbf{C}_*)$ by

$$\tau(\mathbf{C}_*) = \tau(\mathbf{C}_* \otimes \mathcal{Q}(R)),$$

where $\mathcal{Q}(R)$ is the quotient field of R .

2.2. Reidemeister torsion of CW-complex and surgery formula. We define the *Reidemeister torsion* of a CW-complex, and prepare a surgery formula, which is a main tool of the present paper.

Let X be a connected finite CW-complex, H the first homology group $H_1(X; \mathbf{Z})$, and $\mathbf{Z}[H]$ the group ring generated by H over \mathbf{Z} . Let $p: \hat{X} \rightarrow X$ be the maximal abelian covering whose covering transformation group is H . Then \hat{X} is also a connected CW-complex whose CW-structure is naturally induced by X . Let $\mathbf{C}_*(\hat{X})$ be the cellular chain complex of \hat{X} , where $C_i(\hat{X})$ is the set of formal linear combinations of oriented i -cells of \hat{X} with integer coefficients. Since H acts on \hat{X} as the covering transformation group, H also acts on $\mathbf{C}_*(\hat{X})$. We may regard $\mathbf{C}_*(\hat{X})$ as a module over $\mathbf{Z}[H]$. This module is free. A *fundamental family* of cells is a family of cells in \hat{X} such that over each cell of X lies exactly one cell of this family. We can observe that each ordered fundamental family of ordered cells determines basis in $\mathbf{C}_*(\hat{X})$ over $\mathbf{Z}[H]$. Let F be a field, and $\varphi: \mathbf{Z}[H] \rightarrow F$ a ring homomorphism. We define

$$\mathbf{C}_*^\varphi(X) = \mathbf{C}_*(\hat{X}) \otimes_{\mathbf{Z}[H]} F,$$

and the *Reidemeister torsion* of X , $\tau^\varphi(X)$, associated to a ring homomorphism $\varphi: \mathbf{Z}[H] \rightarrow F$.

$$\tau^\varphi(X) = \begin{cases} \tau(\mathbf{C}_*^\varphi(X)) \in F - \{0\} & \text{if } H_*(\mathbf{C}_*^\varphi(X)) = 0, \\ 0 \in F & \text{if } H_*(\mathbf{C}_*^\varphi(X)) \neq 0. \end{cases}$$

The invariant $\tau^\varphi(X)$ is a simple-homotopy invariant determined up to a multiplication of an element in $\pm\varphi(H)$. A simple-homotopy invariant is a topological invariant by A. Chapman [4]. For a finite CW-pair (X, Y) , we can also define $\tau^\varphi(X, Y)$ associated to $C_*^\varphi(X, Y) = C_*(\hat{X}, p^{-1}(Y))$.

The following theorems are fundamental to compute the Reidemeister torsion. We denote the first homology group $H_1(X; \mathbf{Z})$ of X over \mathbf{Z} by $H_1(X)$ for short.

Theorem 2.1 (Turaev [26]; the excision theorem). *Let X_1 and X_2 be sub-complexes of X whose union is X , and whose intersection is Y . Let $j: \mathbf{Z}[H_1(Y)] \rightarrow \mathbf{Z}[H_1(X)]$ and $j_i: \mathbf{Z}[H_1(X_i)] \rightarrow \mathbf{Z}[H_1(X)]$ ($i = 1, 2$) be homomorphisms induced by the natural inclusions. If $\tau^{\varphi \circ j}(Y) \neq 0$, then*

$$\tau^\varphi(X) = \tau^{\varphi \circ j_1}(X_1) \tau^{\varphi \circ j_2}(X_2) [\tau^{\varphi \circ j}(Y)]^{-1}.$$

For example, if t is a generator of $H_1(S^1) \cong \mathbf{Z}$, then $\tau(S^1) = (t - 1)^{-1}$ and $\tau(S^1 \times S^1) = 1$. The following theorem is a special case of more general result [26].

Theorem 2.2 (Milnor [16], Turaev [26]). *Let K be a knot in a homology 3-sphere Σ , t a generator of $H_1(\overline{\Sigma - N(K)}) \cong \mathbf{Z}$ where $N(K)$ is a tubular neighborhood of K , and $\Delta_K(t)$ the Alexander polynomial of K . Then*

$$\tau(\overline{\Sigma - N(K)}) = \Delta_K(t)(t - 1)^{-1}.$$

Since any homology lens space is obtained by a p/q -surgery along a knot K in a homology 3-sphere Σ , where $p \geq 2$ and $q \neq 0$, we can compute the Reidemeister torsion of it by Theorem 2.1 and Theorem 2.2. By $(\mathbf{Z}/n\mathbf{Z})^\times$ we denote the multiplicative group of invertible elements in the ring $\mathbf{Z}/n\mathbf{Z}$ with respect to the multiplicity. For an element x of $(\mathbf{Z}/n\mathbf{Z})^\times$, we denote the inverse element of x by \bar{x} .

Theorem 2.3 (Turaev [25, 26, 27]; Sakai [21]). *Let K be a knot in a homology 3-sphere Σ , $\Delta_K(t)$ the Alexander polynomial of K , and $M = \Sigma(K; p/q)$, where $p \geq 2$ and $q \neq 0$. Let d (≥ 2) be a divisor of p , $\zeta = \zeta_d$ a primitive d -th root of unity, and $\varphi_d: \mathbf{Z}[t, t^{-1}]/(t^p - 1) \rightarrow \mathbf{Q}(\zeta)$ a homomorphism such that $\varphi_d(t) = \zeta$. Then the Reidemeister torsion of M , $\tau^{\varphi_d}(M)$, associated to φ_d is*

$$\tau^{\varphi_d}(M) = \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}.$$

Theorem 2.4 (Reidemeister [20]; Franz [8]). *Let $L(p, q)$ be the lens space of type (p, q) , t a generator of the first homology group $H_1(L(p, q))$, d (≥ 2) a divisor of p , $\zeta = \zeta_d$ a primitive d -th root of unity, and $\varphi_d: \mathbf{Z}[t, t^{-1}]/(t^p - 1) \rightarrow \mathbf{Q}(\zeta)$ a homomorphism such that $\varphi_d(t) = \zeta$. Then the Reidemeister torsion of $L(p, q)$, $\tau^{\varphi_d}(L(p, q))$, associated to φ_d is*

$$\tau^{\varphi_d}(L(p, q)) = (\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}.$$

Lens spaces are completely classified by using Theorem 2.4 and Franz's lemma (see [8] and Section 3). We apply Franz's lemma to show Main Theorem 1 in Section 3.

2.3. Closed 3-manifold of lens type. Let M be an oriented closed 3-manifold whose first homology group $H_1(M)$ is a finite cyclic group of order p (i.e., M is a *homology lens space*), and t a generator of $H_1(M)$. Let d (≥ 2) be a divisor of p , ζ a primitive d -th root of unity, and $\varphi_d: \mathbf{Z}[H_1(M)] \rightarrow \mathbf{Q}(\zeta)$ a ring homomorphism such that $\varphi_d(t) = \zeta$. A homology lens space M is of *lens type* if its Reidemeister torsion $\tau^{\varphi_d}(M)$ has the form $(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1}$ for every d where i and j are coprime to p , and do not depend on d . In particular, a homology lens space M is of (p, q) -lens type if $\bar{i}j \equiv \pm q$ or $\pm \bar{q}$ (mod p).

It is clear that the lens space $L(p, q)$ is of (p, q) -lens type. If a homology lens space of (p, q) -lens type is a lens space, then it is homeomorphic to $L(p, \pm q)$ or $L(p, \pm \bar{q})$. If a Dehn surgery along a knot in a homology 3-sphere yields a 3-manifold of lens type, then we call it *lens type surgery*. It is clear that a lens surgery is a lens type surgery.

3. Proof of Main Theorem 1

In this section we show Main Theorem 1, which states a necessary and sufficient condition for the Reidemeister torsion of $\Sigma(K; p/q)$ to be of lens type in the case that the Alexander polynomial $\Delta_K(t)$ of K is equal to that of the (r, s) -torus knot.

3.1. Franz's lemma and norm of an algebraic number. We prepare Franz's lemma and some results about algebraic numbers.

Theorem 3.1 (Franz [8]). *Let ζ be a primitive n -th root of unity, and $\{a_i$ ($i \in (\mathbf{Z}/n\mathbf{Z})^\times$) $\}$ the set of integers satisfying the following conditions:*

- (1) $a_{-i} = a_i$,
- (2) $\sum_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} a_i = 0$,
- (3) $\prod_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} (\zeta^i - 1)^{a_i} = 1$.

Then $a_i = 0$ for all $i \in (\mathbf{Z}/n\mathbf{Z})^\times$.

Let F be a finite Galois extension over \mathbf{Q} , and α an element of F . We denote the norm of α over \mathbf{Q} by $N_{F/\mathbf{Q}}(\alpha)$, or simply $N(\alpha)$.

$$N_{F/\mathbf{Q}}(\alpha) = \prod_{\sigma \in \text{Gal}(F/\mathbf{Q})} \sigma(\alpha)$$

The followings are fundamental facts in Number Theory (see [3, p.89], [30]).

Proposition 3.2. *In the situation above, we have the followings.*

- (1) $N(\alpha)$ is a rational number, and $N(\alpha) = 0$ if and only if $\alpha = 0$.
- (2) If α is an algebraic integer, then $N(\alpha)$ is an integer.
- (3) An algebraic integer α is a unit in the ring of algebraic integers if and only if $N(\alpha) = \pm 1$.

3.2. Proof of Lemma 1.1. If $\Sigma(K; p/q)$ is of lens type, then there are integers i, j and m such that

$$\Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1} = \pm \zeta^m(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1},$$

where i and j are coprime to p .

By taking the norms of both sides, we have

$$N(\Delta_K(\zeta)) = N(\pm \zeta^m),$$

because

$$N(\zeta - 1) = N(\zeta^{\bar{q}} - 1) = N(\zeta^i - 1) = N(\zeta^j - 1) \neq 0.$$

Since

$$N(\pm \zeta^m) = \begin{cases} \pm 1 & (d = 2), \\ 1 & (d \geq 3), \end{cases}$$

we have the result. □

3.3. Proof of Main Theorem 1. Let $\Delta_{r,s}(t)$ be the Alexander polynomial of the (r, s) -torus knot

$$\Delta_{r,s}(t) = \frac{(t^{rs} - 1)(t - 1)}{(t^r - 1)(t^s - 1)},$$

$d (\geq 2)$ a divisor of p , ζ a primitive d -th root of unity, and $\varphi_d: \mathbf{Z}[t]/(t^p - 1) \rightarrow \mathbf{Q}(\zeta)$ a ring homomorphism such that $\varphi_d(t) = \zeta$. Since $M = \Sigma(K; p/q)$ is the p/q -surgery along a knot K whose Alexander polynomial is $\Delta_{r,s}(t)$, we have

$$\tau^{\varphi_d}(M) = \Delta_{r,s}(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}$$

by Theorem 2.3. Suppose M is of lens type, then there are integers i, j and m such that

$$\Delta_{r,s}(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1} = \pm \zeta^m(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1},$$

where i and j are coprime to p .

Suppose $\gcd(p, r) \geq 2$, we take $d = \gcd(p, r)$. Then $\gcd(d, s) = 1$ because $\gcd(r, s) = 1$. We set $p = p'd$ and $r = r'd$. Then

$$\begin{aligned}\Delta_{r,s}(t) &= \frac{(t^{rs} - 1)(t - 1)}{(t^r - 1)(t^s - 1)} = \frac{(t^{r'sd} - 1)(t - 1)}{(t^{r'd} - 1)(t^s - 1)} \\ &= (t^{(s-1)r'd} + t^{(s-2)r'd} + \cdots + t^{2r'd} + t^{r'd} + 1) \cdot \frac{t - 1}{t^s - 1},\end{aligned}$$

and therefore

$$N(\Delta_{r,s}(\zeta)) = s^{\varphi(d)}.$$

By Lemma 1.1, M is not of lens type. Thus we have the conclusion (1).

We assume $\gcd(p, r) = 1$ and $\gcd(p, s) = 1$, and take any divisor d (≥ 2) of p . Then

$$\Delta_{r,s}(\zeta) = \frac{(\zeta^{rs} - 1)(\zeta - 1)}{(\zeta^r - 1)(\zeta^s - 1)}.$$

If M is of lens type, then

$$(\zeta^{rs} - 1)(\zeta^i - 1)(\zeta^j - 1) = \pm \zeta^m (\zeta^{\bar{q}} - 1)(\zeta^r - 1)(\zeta^s - 1).$$

Multiplying the complex conjugates to both sides, we have

$$\begin{aligned}(\zeta^{rs} - 1)(\zeta^i - 1)(\zeta^j - 1)(\zeta^{-rs} - 1)(\zeta^{-i} - 1)(\zeta^{-j} - 1) \\ = (\zeta^{\bar{q}} - 1)(\zeta^r - 1)(\zeta^s - 1)(\zeta^{-\bar{q}} - 1)(\zeta^{-r} - 1)(\zeta^{-s} - 1).\end{aligned}$$

By the same argument as the proof of the classification of lens spaces (see [6, 8, 20, 27], and Theorem 3.1),

$$\{rs, i, j\} = \{\bar{q}, r, s\} \quad \text{in } (\mathbf{Z}/d\mathbf{Z})/\{\pm 1\}.$$

There are two cases.

(i) $rs \equiv \pm \bar{q} \pmod{d}$.

This is equivalent to $qrs \equiv \pm 1 \pmod{d}$.

(ii) $rs \equiv \pm s \pmod{d}$ or $rs \equiv \pm r \pmod{d}$.

This is equivalent to $r \equiv \pm 1 \pmod{d}$ or $s \equiv \pm 1 \pmod{d}$.

If $qrs \equiv \pm 1$ (resp. $r \equiv \pm 1$, $s \equiv \pm 1$) \pmod{p} holds, then $qrs \equiv \pm 1$ (resp. $r \equiv \pm 1$, $s \equiv \pm 1$) \pmod{d} holds for any d . So we state only the case of $d = p$.

The converse is obvious. This completes the proof. \square

4. Proof of Main Theorem 2

In this section we show Main Theorem 2, which states a necessary and sufficient condition for the Reidemeister torsion of $\Sigma(K; p/q)$ to be of lens type in the case that the Alexander polynomial $\Delta_K(t)$ of K is of degree 2.

We prepare some lemmas.

Let n be a positive integer, $\varphi(n)$ the Euler function, ζ a primitive n -th root of unity, and

$$\Phi_n(x) = \prod_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} (x - \zeta^i)$$

the n -th cyclotomic polynomial. Then $\Phi_n(x)$ is an irreducible monic symmetric polynomial over \mathbf{Z} with degree $\varphi(n)$, and

$$\Phi_n(1) = \begin{cases} 0 & (n = 1), \\ p & (n = p^r, p: \text{prime}), \\ 1 & (\text{otherwise}). \end{cases}$$

The Alexander polynomial of a knot in a homology sphere with degree 2 has the following form for some integer $n \neq 0$:

$$(4.1) \quad \Delta_n(t) = nt^2 - (2n - 1)t + n = t + n(t - 1)^2 \quad (n \neq 0)$$

Let ζ be a primitive p -th root of unity, and α_1 and α_2 the roots of $\Delta_n(t) = 0$. Then we have

$$(4.2) \quad N(\Delta_n(\zeta)) = \prod_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} n(\zeta^i - \alpha_1)(\zeta^i - \alpha_2) = n^{\varphi(p)} \Phi_p(\alpha_1) \Phi_p(\alpha_2)$$

We regard the right-hand side of (4.2) as a polynomial of n over \mathbf{Z} depending on p , denote it by $f_p(n)$ (i.e., $f_p(n) \in \mathbf{Z}[n]$), and call it *the p -th norm polynomial* or simply *the norm polynomial*.

Lemma 4.1. (1) If $n \leq -1$, then $f_p(n) \neq \pm 1$.

(2) If $|n| \geq 2$ and p is a prime number, then $f_p(n) \neq \pm 1$.

Main Theorem 2 is proved by Lemma 1.1 and Lemma 4.1: By the assumption that $\Sigma(K; p/q)$ is of lens type, by definition, $N(\Delta_K(\zeta)) = \pm 1$ holds not only in the p -th cyclotomic field but also in the d -th cyclotomic field for any divisor d of p . In the case that $n \geq 2$ and p is not prime, we study $f_d(n)$ for a prime divisor d of p in Lemma 4.1 (2).

In the case that $p = 2$, Lemma 4.1 holds because $f_2(n) = 4n - 1$. From now on, we assume $p \geq 3$. To show Lemma 4.1, we study properties of $f_p(n)$.

Proposition 4.2. (1) The degree of $f_p(n)$ is $\varphi(p)$.

(2) If $p \geq 3$, then there exists a polynomial of n , $g_p(n)$, over \mathbf{Z} with degree $\varphi(p)/2$ such that $f_p(n) = \{g_p(n)\}^2$.

Proof. (1) Since $\Delta_n(\zeta) = (1 - \zeta)^2 n + \zeta$, the degree of $f_p(n)$ is $\varphi(p)$.

(2) Firstly we note

$$\Delta_n(\zeta) = \zeta^2 \Delta_n(\zeta^{-1}).$$

From this equation,

$$\delta(\zeta) = \frac{\Delta_n(\zeta)}{\zeta}$$

satisfies $\delta(\zeta^{-1}) = \delta(\zeta)$, and $\delta(\zeta)$ is an element of $\mathbf{Q}(\zeta + \zeta^{-1})$. Since $\zeta \neq \zeta^{-1}$, we have $[\mathbf{Q}(\zeta) : \mathbf{Q}(\zeta + \zeta^{-1})] = 2$ and $[\mathbf{Q}(\zeta + \zeta^{-1}) : \mathbf{Q}] = \varphi(p)/2$. If we set

$$g_p(n) = N_{\mathbf{Q}(\zeta + \zeta^{-1})/\mathbf{Q}}(\delta(\zeta)),$$

then $g_p(n)$ is a polynomial of n over \mathbf{Z} with degree $\varphi(p)/2$ such that $f_p(n) = \{g_p(n)\}^2$. \square

We write down $f_p(n)$ and $g_p(n)$ in the following form:

$$f_p(n) = \sum_{i=0}^{\varphi(p)} a_i n^i, \quad g_p(n) = \sum_{j=0}^{\varphi(p)/2} b_j n^j.$$

Let $F(n) = s_0 n^m + s_1 n^{m+1} + s_2 n^{m+2} + \dots + s_d n^{m+d}$ ($s_0 \neq 0$, $s_d \neq 0$) be a polynomial of n over \mathbf{R} . If (i) $d = 0$ or (ii) $d \geq 1$ and $s_{i-1} s_i < 0$ ($i = 1, 2, \dots, d$), then we say that $F(n)$ is an *alternating polynomial*. We note that if all roots of $F(n) = 0$ are positive real numbers or 0, then $F(n)$ is an alternating polynomial.

Lemma 4.3. (1) *The polynomials $f_p(n)$ and $g_p(n)$ are alternating polynomials.*

(2) $a_{\varphi(p)} = \{\Phi_p(1)\}^2$ and $a_0 = 1$.

(3) $b_{\varphi(p)/2} = (-1)^{\varphi(p)/2} \Phi_p(1)$ and $b_0 = 1$.

(4) $a_1 = 2T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) - 2\varphi(p)$, where $T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta)$ is the trace of ζ in $\mathbf{Q}(\zeta)/\mathbf{Q}$. In particular, if p is an odd prime number, then $a_1 = -2p$ and $b_1 = -p$.

Proof. (1) Firstly we note

$$\delta(\zeta) = \frac{\Delta_n(\zeta)}{\zeta} = 1 - \{2 - (\zeta + \zeta^{-1})\}n.$$

Since $2 - (\zeta + \zeta^{-1}) > 0$, the polynomials $f_p(n)$ and $g_p(n)$ are alternating polynomials.

(2) $a_{\varphi(p)} = N_{\mathbf{Q}(\zeta)/\mathbf{Q}}((1 - \zeta)^2) = \{\Phi_p(1)\}^2$ and $a_0 = N_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) = 1$.

(3) It is clear by (1) and (2).

(4) By the definition,

$$\begin{aligned} a_1 &= \sum_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} (1 - \zeta^i)^2 \cdot \frac{N_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta)}{\zeta^i} = \sum_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} (\zeta^i + \zeta^{-i} - 2) \\ &= 2 \sum_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} \zeta^i - 2\varphi(p) = 2T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) - 2\varphi(p), \end{aligned}$$

If p is a prime number, then $T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) = -1$ and $\varphi(p) = p - 1$. Therefore $a_1 = -2p$. This completes the proof. \square

Proof of Lemma 4.1 (1). Assume $n \leq -1$. From Lemma 4.3 (1) and (2), we see

$$f_p(n) \geq \{\Phi_p(1)\}^2 + 1 \geq 2.$$

This completes the proof. \square

To prove Lemma 4.1 (2), we need the following lemma.

Lemma 4.4. (1) *If p is a prime number, then*

$$f_p(n) = n^p (\alpha_1^p - 1)(\alpha_2^p - 1),$$

where α_1 and α_2 are roots of $\Delta_n(t)$ in (4.1).

(2) *If p is an odd prime number, then $b_j \equiv 0 \pmod{p}$ for $j = 1, 2, \dots, \varphi(p)/2$.*

Proof. (1) From $\varphi(p) = p - 1$ and $t^p - 1 = (t - 1)\Phi_p(t)$ if p is a prime, we have

$$\frac{n^p(\alpha_1^p - 1)(\alpha_2^p - 1)}{n^{\varphi(p)}\Phi_p(\alpha_1)\Phi_p(\alpha_2)} = n(\alpha_1 - 1)(\alpha_2 - 1) = \Delta_n(1) = 1.$$

By (4.2), we have the equality.

(2) Let (p) be an ideal in the polynomial ring $\mathbf{Z}[n]$ generated by p , and α_1 and α_2 roots of the Alexander polynomial $\Delta_n(t)$ with degree 2 in (4.2). Since $\Delta_n(t)$ is a polynomial over \mathbf{Z} and p is an odd prime number, we have

$$n^p(\alpha_1 - 1)^p(\alpha_2 - 1)^p \equiv n^p(\alpha_1^p - 1)(\alpha_2^p - 1) \pmod{(p)}.$$

By (1),

$$f_p(n) \equiv n^p(\alpha_1 - 1)^p(\alpha_2 - 1)^p = \{n(\alpha_1 - 1)(\alpha_2 - 1)\}^p = 1 \pmod{(p)}.$$

Since $\mathbf{Z}[n]/(p) = (\mathbf{Z}/p\mathbf{Z})[n]$ is a unique factorization domain,

$$g_p(n) \equiv b_0 = 1 \pmod{(p)}.$$

This means $b_j \equiv 0 \pmod{p}$ for $j = 1, 2, \dots, \varphi(p)/2$. \square

Proof of Lemma 4.1 (2). Let p be an odd prime number, and $h_p(n)$ a polynomial of n satisfying

$$g_p(n) = pn h_p(n) + 1.$$

Then $h_p(n)$ is a polynomial over \mathbf{Z} by Lemma 4.4 (2), and $f_p(n) = 1$ if and only if $h_p(n) = 0$. We write down

$$h_p(n) = \sum_{k=0}^{\varphi(p)/2-1} c_k n^k.$$

By Lemma 4.3 (3) and (4), $c_{\varphi(p)/2-1} = \pm 1$ and $c_0 = -1$. From this, if $h_p(n) = 0$, then $n = \pm 1$. Therefore if $|n| \geq 2$, then $h_p(n) \neq 0$. This completes the proof of Lemma 4.1 (2). \square

For example, $h_3(n) = -1$, $h_5(n) = n - 1$, $h_7(n) = -(n - 1)^2$, $h_{11}(n) = -(n - 1)(n^3 - 4n^2 + 3n - 1)$.

Corollary 4.5. *Let K be a knot in a homology 3-sphere Σ , $\Delta_K(t)$ the Alexander polynomial of K , and $M = \Sigma(K; p/q)$ the p/q -surgery for $p \geq 2$ and $q \neq 0$. If $\Delta_K(t)$ is divisible by $nt^2 - (2n - 1)t + n$ ($n \in \mathbf{Z}; n \neq 0, 1$), then M is not of lens type.*

Appendix

We introduce a result in [18] due to Ozsváth and Szabó, which is a necessary condition on the Alexander polynomial of a knot in S^3 which yields a lens space. They show it by using knot Floer homology ([13, 18, 19]).

Theorem [Ozsváth-Szabó [18]]. *Let K be a knot in S^3 , and $M = (K; p)$, where p is an integer. If M is a lens space, then the Alexander polynomial of K is of the following form*

$$\Delta_K(t) = (-1)^m + \sum_{j=1}^m (-1)^{m-j} (t^{s_j} + t^{-s_j}),$$

where $0 < s_1 < s_2 < \cdots < s_m$.

By Moser's theorem (Theorem 1.2), the Alexander polynomial of a torus knot satisfies the condition above. We can check it easily as follows.

Proposition. *The Alexander polynomial of a torus knot has the form in Ozsváth-Szabó's theorem.*

Proof. Let $\Delta_{p,q}(t)$ be the Alexander polynomial of (p, q) -torus knot. We may assume $2 \leq q \leq p$. There are integers c_k and d_k ($k = 0, 1, \dots, q-1$) such that $pk = qc_k + d_k$ ($0 \leq d_k \leq q-1$). The integers d_0, \dots, d_{q-1} are mutually distinct, because p and q are coprime integers. It is clear that $c_0 = d_0 = 0$.

We list the following formulas which are proved easily.

$$(1) \quad \Delta_{p,q}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)} = \frac{(t^{p(q-1)} + t^{p(q-2)} + \dots + t^p + 1)(t - 1)}{t^q - 1}$$

$$(2) \quad t^{pk} = t^{d_k}(t^{qc_k} - 1) + t^{d_k}$$

$$(3) \quad \frac{t^{pk}(t - 1)}{t^q - 1} = t^{d_k}(t^{q(c_k-1)} + t^{q(c_k-2)} + \dots + t^q + 1)(t - 1) + \frac{t^{d_k}(t - 1)}{t^q - 1}$$

$$(4) \quad \sum_{k=0}^{q-1} \frac{t^{d_k}(t - 1)}{t^q - 1} = \frac{(t^{q-1} + t^{q-2} + \dots + t + 1)(t - 1)}{t^q - 1} = 1$$

$$(5) \quad \Delta_{p,q}(t) = 1 + \sum_{k=1}^{q-1} \sum_{l=0}^{c_k-1} (t^{ql+d_k+1} - t^{ql+d_k})$$

Equation (5) is obtained from equations (1), (2), (3) and (4). If two pairs (k, l) and (k', l') are distinct, then two numbers $ql + d_k$ and $q'l' + d_{k'}$ are distinct. Therefore $\Delta_{p,q}(t)$ has the form in Ozsváth-Szabó's theorem. \square

In [12], we characterized the Alexander polynomial of a knot in any homology 3-sphere having a lens type surgery. Infinitely many knots in S^3 having lens surgery appear as certain families or sequences (see [1]), and no counterexample is discovered. The author thinks such a deformation of the Alexander polynomial above is related to the structure of each family or sequence not only in S^3 but also in any homology 3-sphere.

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References

- [1] J. Berge: Some Knots with Surgeries Yielding Lens Spaces, Unpublished manuscript, 1990.
- [2] S. Bleiler and R. Litherland: *Lens spaces and Dehn surgery*, Proc. Amer. Math. Soc. **107** (1989), 1127–1131.
- [3] Z.I. Borevich and I.R. Shafarevich: Number Theory, Transl. by N. Greenleaf, New York, Academic Press, 1966.
- [4] T.A. Chapman: *Topological invariance of Whitehead torsion*, Amer. J. Math. **96** (1974), 488–497.
- [5] M. Culler, M. Gordon, J. Luecke and P. Shalen: *Dehn surgery on knots*, Ann. of Math. **125** (1987), 237–300.
- [6] M.M. Cohen: *A Course in Simple-Homotopy Theory*, Springer-Verlag, 1972.
- [7] R. Fintushel and R.J. Stern: *Constructing lens spaces by surgery on knots*, Math. Z. **175** (1980), 33–51.
- [8] W. Franz: *Über die Torsion einer Überdeckung*, J. Reine Angew. Math. **173** (1935), 245–254.
- [9] H. Goda and M. Teragaito: *Dehn surgeries on knots which yield lens spaces and genera of knots*, Math. Proc. Cambridge Philos. Soc. **129** (2000), 501–515.
- [10] C.McA. Gordon: *Dehn surgery and satellite knots*, Trans. Amer. Math. Soc. **275** (1983), 687–708.
- [11] T. Kadokami and Y. Yamada: *Reidemeister torsion and lens surgeries on $(-2, m, n)$ -pretzel knots*, Kobe J. of Math. **23** (2006), 65–78.
- [12] T. Kadokami and Y. Yamada: *A deformation of the Alexander polynomials of knots yielding lens spaces*, preprint, (2004).
- [13] P. Kronheimer, T. Mrowka, P. Ozsváth and Z. Szabó: *Monopoles and lens space surgeries*, (2003), 1–76, math.GT/0310164.
- [14] N. Maruyama: *On Dehn surgery along a certain family of knots*, J. of Tsuda College **19** (1987), 261–280.
- [15] T. Mattman: *Cyclic and finite surgeries on pretzel knots*, J. Knot Theory Ramifications **11** (2002), 891–902.
- [16] J.W. Milnor: *A duality theorem for Reidemeister torsion*, Ann. of Math. (2) **76** (1962), 137–147.
- [17] L. Moser: *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 737–745.
- [18] P. Ozsváth and Z. Szabó: *On knot Floer homology and lens space surgeries*, Topology **44** (2005), 1281–1300.
- [19] J. Rasmussen: *Lens space surgeries and a conjecture of Goda and Teragaito*, Geometry and Topology **8** (2004), 1013–1031.
- [20] K. Reidemeister: *Homotopieringe und Linsenräume*, Abh. Math. Sem. Univ. Hamburg **11** (1935), 102–109.
- [21] T. Sakai: *Reidemeister torsion of a homology lens space*, Kobe J. Math. **1** (1984), 47–50.
- [22] T. Saito: *Dehn surgery and $(1, 1)$ -knots in lens spaces*, preprint, (2004).
- [23] M. Shimozaawa: *Dehn surgery on torus knots*, Master Thesis, Osaka City University, (2004), (in Japanese).
- [24] W.P. Thurston: *The Geometry and Topology of Three-Manifolds*, Electronic version 1.0 (www.msri.org/gt3m/), 1997.
- [25] V.G. Turaev: *Reidemeister torsion and the Alexander polynomial*, Math. USSR-Sbornik **30** (1976), 221–237.
- [26] V.G. Turaev: *Reidemeister torsion in knot theory*, Russian Math. Surveys **41-1** (1986), 119–182.
- [27] V.G. Turaev: *Introduction to Combinatorial Torsions*, Birkhäuser Verlag, 2001.
- [28] V.G. Turaev: *The Alexander Polynomials and Torsions of 3-Manifolds*, Lecture at RIMS, 2000.
- [29] S. Wang: *Cyclic surgery on knots*, Proc. Amer. Math. Soc. **107** (1989), 1091–1094.
- [30] L.C. Washington: *Introduction to Cyclotomic Fields*, Graduate Texts in Mathematics **83**, Springer-Verlag, (1982).
- [31] Y.Q. Wu, *Cyclic surgery and satellite knots*, Topology Appl. **36** (1990), 205–208.

Department of Mathematics
Osaka City University
Sugimoto 3-3-138, Sumiyoshi-ku
Osaka, 558-8585
Japan
e-mail: kadokami@sci.osaka-cu.ac.jp