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REIDEMEISTER TORSION AND LENS SURGERIES ON KNOTS IN HOMOLOGY 3-SPHERES I

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Abstract

Let $\Sigma(K; p/q)$ be the result of p/q-surgery along a knot K in a homology 3-sphere Σ. We investigate the Reidemeister torsion of $\Sigma(K; p/q)$. Firstly, when the Alexander polynomial of K is the same as that of a torus knot, we give a necessary and sufficient condition for the Reidemeister torsion of $\Sigma(K; p/q)$ to be that of a lens space. Secondly, when the Alexander polynomial of K is of degree 2, we show that if the Reidemeister torsion of $\Sigma(K; p/q)$ is the same as that of a lens space, then $\Delta_K(t) = t^2 - t + 1$.

1. Introduction

We investigate when the result of Dehn surgery along a knot in a homology 3-sphere is a lens space. In this paper, we call such a surgery *lens surgery* not only for hyperbolic knots but also for any knots in homology 3-spheres. Many authors have studied lens surgery by geometric method (see [1, 2, 5, 7, 9, 10, 17, 22, 23, 24, 29, 31]). We approach the problem by algebraic method (see [13, 14, 15, 18, 19]).

K. Reidemeister [20] and W. Franz [8] classified lens spaces completely by using the Reidemeister torsion. Franz provided a useful lemma called *Franz's lemma* (see Theorem 3.1) which is deduced by a result of L-function (see [30]) from Number Theory. We apply the lemma in the present paper. J. Milnor [16] pointed out that the Reidemeister torsion was closely related to the Alexander polynomial. Our method is based on the surgery formula for Reidemeister torsions due to V.G. Turaev (see [21, 25, 26, 27] and Section 2.2). Our results are mentioned in terms of the Alexander polynomials.

We define that an oriented closed 3-manifold M is *of lens type* (or *of* (p, q)-*lens type*) if it has the same Reidemeister torsion as the lens space (or as $L(p, q)$) (for a precise definition, see Section 2.3). By algebraic and number theoretic study, we obtain necessary conditions for the Alexander polynomial of a knot having a lens type surgery. The multiplicative group $(\mathbf{Z}/n\mathbf{Z})^{\times}$ plays important roles in our study because it is the Galois group of the n-th cyclotomic field, and the second term q of a lens space $L(p, q)$ is an element of $(\mathbf{Z}/p\mathbf{Z})^{\times}$.

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We point out the following lemma which states a property of the Alexander polynomial of a knot having a lens type surgery. We prove it in Section 3.2. Our two main theorems are obtained by using this lemma. Let $\Sigma(K; p/q)$ be the result of p/q-surgery along a knot K in a homology 3-sphere Σ. If Σ is the 3-sphere S^3, then we denote $\Sigma(K; p/q)$ by $(K; p/q)$.

Lemma 1.1. *Let K be a knot in a homology 3-sphere Σ, and $\Delta_K(t)$ the Alexander polynomial of K. Let d (≥ 2) be a divisor of p, and ζ a primitive d-th root of unity. If the p/q-surgery $\Sigma(K; p/q)$ is of lens type for $p \geq 2$ and $q \neq 0$, then*

$$N(\Delta_K(\zeta)) = \begin{cases} \pm 1 & (d = 2), \\ 1 & (d \geq 3), \end{cases}$$

where $N(\alpha)$ is the norm of α in the d-th cyclotomic field $\mathbf{Q}(\zeta)$ over $\mathbf{Q}$.

For the *norm* in a Galois extension, see Section 3.1. In [17], L. Moser firstly showed that non-trivial knots can yield a lens space by Dehn surgery. In fact, Moser determined all rational surgery along all torus knots, and proved that every torus knot yields a lens space as below.

Theorem 1.2 (Moser [17]). *Let K be the (r, s)-torus knot in S^3, and $M = (K; p/q)$ the result of p/q-surgery along K, where $p, r, |s| \geq 2$ and $q \neq 0$. Then there are three cases*:
(1) *If $|p - qrs| \neq 0, 1$, then M is a Seifert fibered space with three singular fibers of multiplicities r, $|s|$ and $|p - qrs|$.*
(2) *If $|p - qrs| = 1$, then M is the lens space $L(p, qr^2)$.*
(3) *If $|p - qrs| = 0$ (i.e., $p/q = rs$), then M is the connected sum of two lens spaces, $L(r, s) \sharp L(s, r)$.*

Our first main theorem is an algebraic "translation" of the theorem. The following is the first main theorem of this paper.

Main Theorem 1. *Let K be a knot in a homology 3-sphere Σ whose Alexander polynomial is the same as the (r, s)-torus knot, and $M = \Sigma(K; p/q)$, where $p, r, s \geq 2$ and $q \neq 0$. Then M is of lens type if and only if the following (1) and (2) hold.*
(1) $\gcd(p, r) = 1$ *and* $\gcd(p, s) = 1$,
(2) $r \equiv \pm 1 \pmod p$ *or* $s \equiv \pm 1 \pmod p$ *or* $qrs \equiv \pm 1 \pmod p$.

We prove Main Theorem 1 by using Franz's lemma (see [6, 8, 27]) in Section 3.

H. Goda and M. Teragaito [9] showed that if a genus one knot in S^3 yields a lens space, then the knot is the trefoil. Our second main theorem is included by Goda-Teragaito's theorem in the restricted case $\Sigma = S^3$, but extends theirs to the case of knots in any homology 3-spheres from the algebraic view point. When we say "the degree of $\Delta_K(t)$," we take a regularization that $\Delta_K(t)$ is in $\mathbf{Z}[t]$ and $\Delta_K(0) \neq 0$. We prove Main Theorem 2 by using norm of an algebraic number (see [3, 30]) in Section 4.

Main Theorem 2. *Let K be a knot in a homology 3-sphere Σ whose Alexander polynomial $\Delta_K(t)$ is of degree 2. If a Dehn surgery $\Sigma(K; p/q)$ is of lens type for $p \geq 2$ and $q \neq 0$, then $\Delta_K(t) = t^2 - t + 1$.*

Recently, P. Ozsváth and Z. Szabó [18] obtained a necessary condition on the Alexander polynomial of a knot in S^3 which yields a lens space by using the knot Floer homology (see [13, 19], and Appendix). Note that Main Theorem 2 extends a special case ($m = 1$, $s_1 = 1$) of Ozsváth-Szabó's result to a rational surgery along a knot in a homology 3-sphere.

In Section 2, we recall Turaev's definition [26] of the Reidemeister torsion, prepare the surgery formula due to Turaev [25, 26] and Sakai [21], and give a precise definition of an oriented closed 3-manifold *of lens type*. In Section 3, we prove Lemma 1.1 and Main Theorem 1. In Section 4, we prove Main Theorem 2. In Appendix, we show that the Alexander polynomial of (p, q)-torus knot satisfies Ozsváth-Szabó's condition only by deformations of the polynomials from the well-known expression.

Lemma 1.1 may have many applications. We applied it in the papers [11, 12] which are joint works with Yuichi Yamada.

2. Surgery formula of Reidemeister torsion

2.1. Torsion of chain complex. We review the *torsion* of a chain complex.

Let V be an n-dimensional vector space over a field F, and $\mathbf{b} = (b_1, \ldots, b_n)$ and $\mathbf{c} = (c_1, \ldots, c_n)$ two bases of V. Then b_i can be expressed by a linear combination of $c_1, \ldots, c_n$.

$$b_i = \sum_{j=1}^{n} a_{ij} c_j \quad (i = 1, \ldots, n)$$

The matrix $A = (a_{ij})$ is the transition matrix from $\mathbf{c}$ to $\mathbf{b}$. We denote the determinant of A by

$$[\mathbf{b}/\mathbf{c}].$$

It is a non-zero element of F. If $n = 0$, then $[\emptyset/\emptyset] = 1$.

Let $\mathbf{C}_*$ be a finitely generated free chain complex over a field F:

$$\mathbf{C}_* : 0 \to C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \to 0$$

In the case that $\mathbf{C}_*$ is *acyclic*, we define the *torsion* of $\mathbf{C}_*$ as follows: Let $\mathbf{c}_i = \left(c_i^{(1)}, \dots, c_i^{(p_i)}\right)$ $(i = 0, \dots, m)$ be a basis of C_i. We denote the kernel of ∂_{i-1} (resp. the image of ∂_i) by Z_i (resp. B_i). Then $Z_i = B_i$ and

$$C_i \cong Z_i \oplus B_{i-1} = B_i \oplus B_{i-1}.$$

We take bases of B_i as $\mathbf{b}_i = \left(b_i^{(1)}, \dots, b_i^{(q_i)}\right)$. A lift of $\mathbf{b}_{i-1}$ in C_i is denoted by $\tilde{\mathbf{b}}_{i-1}$. Then $\mathbf{b}_i\tilde{\mathbf{b}}_{i-1}$ is a basis of C_i. The *torsion* of $\mathbf{C}_*$ with respect to $\mathbf{c} = (\mathbf{c}_0, \dots, \mathbf{c}_m)$ is defined by

$$\tau(\mathbf{C}_*; \mathbf{c}) = \prod_{i=0}^{m} \left[\mathbf{b}_i\tilde{\mathbf{b}}_{i-1}/\mathbf{c}_i\right]^{(-1)^{i+1}}.$$

The torsion $\tau(\mathbf{C}_*; \mathbf{c})$ is a non-zero element of F, and does not depend on the choices of $\mathbf{b}_i$ and $\tilde{\mathbf{b}}_{i-1}$. When $\mathbf{c}$ is clear, we denote $\tau(\mathbf{C}_*; \mathbf{c})$ by $\tau(\mathbf{C}_*)$. In the other case, i.e., if $\mathbf{C}_*$ is not acyclic, then we define $\tau(\mathbf{C}_*) = 0$ formally. When $\mathbf{C}_*$ is a finitely generated free chain complex over an integral domain R, we define $\tau(\mathbf{C}_*)$ by

$$\tau(\mathbf{C}_*) = \tau(\mathbf{C}_* \otimes Q(R)),$$

where $Q(R)$ is the quotient field of R.

2.2. Reidemeister torsion of CW-complex and surgery formula. We define the *Reidemeister torsion* of a CW-complex, and prepare a surgery formula, which is a main tool of the present paper.

Let X be a connected finite CW-complex, H the first homology group $H_1(X; \mathbf{Z})$, and $\mathbf{Z}[H]$ the group ring generated by H over $\mathbf{Z}$. Let $p\colon \hat{X} \to X$ be the maximal abelian covering whose covering transformation group is H. Then $\hat{X}$ is also a connected CW-complex whose CW-structure is naturally induced by X. Let $\mathbf{C}_*\left(\hat{X}\right)$ be the cellular chain complex of $\hat{X}$, where $C_i\left(\hat{X}\right)$ is the set of formal linear combinations of oriented i-cells of $\hat{X}$ with integer coefficients. Since H acts on $\hat{X}$ as the covering transformation group, H also acts on $\mathbf{C}_*\left(\hat{X}\right)$. We may regard $\mathbf{C}_*\left(\hat{X}\right)$ as a module over $\mathbf{Z}[H]$. This module is free. A *fundamental family* of cells is a family of cells in $\hat{X}$ such that over each cell of X lies exactly one cell of this family. We can observe that each ordered fundamental family of ordered cells determines basis in $\mathbf{C}_*\left(\hat{X}\right)$ over $\mathbf{Z}[H]$. Let F be a field, and $\varphi\colon \mathbf{Z}[H] \to F$ a ring homomorphism. We define

$$\mathbf{C}_*^{\varphi}(X) = \mathbf{C}_*\left(\hat{X}\right) \otimes_{\mathbf{Z}[H]} F,$$

and the *Reidemeister torsion* of X, $\tau^{\varphi}(X)$, associated to a ring homomorphism $\varphi\colon \mathbf{Z}[H] \to F$.

$$\tau^{\varphi}(X) = \begin{cases} \tau(\mathbf{C}_*^{\varphi}(X)) \in F - \{0\} & \text{if} \quad H_*(\mathbf{C}_*^{\varphi}(X)) = 0, \\ 0 \in F & \text{if} \quad H_*(\mathbf{C}_*^{\varphi}(X)) \neq 0. \end{cases}$$

The invariant $\tau^{\varphi}(X)$ is a simple-homotopy invariant determined up to a multiplication of an element in $\pm\varphi(H)$. A simple-homotopy invariant is a topological invariant by A. Chapman [4]. For a finite CW-pair (X, Y), we can also define $\tau^{\varphi}(X, Y)$ associated to $C_*^{\varphi}(X, Y) = C_*\big(\hat{X}, p^{-1}(Y)\big)$.

The following theorems are fundamental to compute the Reidemeister torsion. We denote the first homology group $H_1(X;\mathbf{Z})$ of X over $\mathbf{Z}$ by $H_1(X)$ for short.

Theorem 2.1 (Turaev [26]; the excision theorem). *Let X_1 and X_2 be subcomplexes of X whose union is X, and whose intersection is Y. Let $j\colon \mathbf{Z}[H_1(Y)] \to \mathbf{Z}[H_1(X)]$ and $j_i\colon \mathbf{Z}[H_1(X_i)] \to \mathbf{Z}[H_1(X)]$ $(i = 1, 2)$ be homomorphisms induced by the natural inclusions. If $\tau^{\varphi\circ j}(Y) \neq 0$, then*

$$\tau^{\varphi}(X) = \tau^{\varphi\circ j_1}(X_1)\tau^{\varphi\circ j_2}(X_2)[\tau^{\varphi\circ j}(Y)]^{-1}.$$

For example, if t is a generator of $H_1(S^1) \cong \mathbf{Z}$, then $\tau(S^1) = (t-1)^{-1}$ and $\tau(S^1 \times S^1) = 1$. The following theorem is a special case of more general result [26].

Theorem 2.2 (Milnor [16], Turaev [26]). *Let K be a knot in a homology 3-sphere Σ, t a generator of $H_1\big(\overline{\Sigma - N(K)}\big) \cong \mathbf{Z}$ where $N(K)$ is a tubular neighborhood of K, and $\Delta_K(t)$ the Alexander polynomial of K. Then*

$$\tau\big(\overline{\Sigma - N(K)}\big) = \Delta_K(t)(t-1)^{-1}.$$

Since any homology lens space is obtained by a p/q-surgery along a knot K in a homology 3-sphere Σ, where $p \geq 2$ and $q \neq 0$, we can compute the Reidemeister torsion of it by Theorem 2.1 and Theorem 2.2. By $(\mathbf{Z}/n\mathbf{Z})^{\times}$ we denote the multiplicative group of invertible elements in the ring $\mathbf{Z}/n\mathbf{Z}$ with respect to the multiplicity. For an element x of $(\mathbf{Z}/n\mathbf{Z})^{\times}$, we denote the inverse element of x by $\bar{x}$.

Theorem 2.3 (Turaev [25, 26, 27]; Sakai [21]). *Let K be a knot in a homology 3-sphere Σ, $\Delta_K(t)$ the Alexander polynomial of K, and $M = \Sigma(K; p/q)$, where $p \geq 2$ and $q \neq 0$. Let d (≥ 2) be a divisor of p, $\zeta = \zeta_d$ a primitive d-th root of unity, and $\varphi_d\colon \mathbf{Z}[t, t^{-1}]/(t^p - 1) \to \mathbf{Q}(\zeta)$ a homomorphism such that $\varphi_d(t) = \zeta$. Then the Reidemeister torsion of M, $\tau^{\varphi_d}(M)$, associated to φ_d is*

$$\tau^{\varphi_d}(M) = \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}.$$

Theorem 2.4 (Reidemeister [20]; Franz [8]). *Let $L(p, q)$ be the lens space of type (p, q), t a generator of the first homology group $H_1(L(p, q))$, d (≥ 2) a divisor of p, $\zeta = \zeta_d$ a primitive d-th root of unity, and $\varphi_d\colon \mathbf{Z}[t, t^{-1}]/(t^p - 1) \to \mathbf{Q}(\zeta)$ a homomorphism such that $\varphi_d(t) = \zeta$. Then the Reidemeister torsion of $L(p, q)$, $\tau^{\varphi_d}(L(p, q))$, associated to φ_d is*

$$\tau^{\varphi_d}(L(p, q)) = (\zeta - 1)^{-1}\big(\zeta^{\bar{q}} - 1\big)^{-1}.$$

Lens spaces are completely classified by using Theorem 2.4 and Franz's lemma (see [8] and Section 3). We apply Franz's lemma to show Main Theorem 1 in Section 3.

2.3. Closed 3-manifold of lens type. Let M be an oriented closed 3-manifold whose first homology group $H_1(M)$ is a finite cyclic group of order p (i.e., M is a *homology lens space*), and t a generator of $H_1(M)$. Let d (≥ 2) be a divisor of p, ζ a primitive d-th root of unity, and $\varphi_d\colon \mathbf{Z}[H_1(M)] \to \mathbf{Q}(\zeta)$ a ring homomorphism such that $\varphi_d(t) = \zeta$. A homology lens space M is *of lens type* if its Reidemeister torsion $\tau^{\varphi_d}(M)$ has the form $(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1}$ for every d where i and j are coprime to p, and do not depend on d. In particular, a homology lens space M is *of* (p, q)*-lens type* if $\bar{i} j \equiv \pm q$ or $\pm \bar{q}$ (mod p).

It is clear that the lens space $L(p, q)$ is of (p, q)-lens type. If a homology lens space of (p, q)-lens type is a lens space, then it is homeomorphic to $L(p, \pm q)$ or $L(p, \pm \bar{q})$. If a Dehn surgery along a knot in a homology 3-sphere yields a 3-manifold of lens type, then we call it *lens type surgery*. It is clear that a lens surgery is a lens type surgery.

3. Proof of Main Theorem 1

In this section we show Main Theorem 1, which states a necessary and sufficient condition for the Reidemeister torsion of $\Sigma(K; p/q)$ to be of lens type in the case that the Alexander polynomial $\Delta_K(t)$ of K is equal to that of the (r, s)-torus knot.

3.1. Franz's lemma and norm of an algebraic number. We prepare Franz's lemma and some results about algebraic numbers.

Theorem 3.1 (Franz [8]). *Let ζ be a primitive n-th root of unity, and $\{a_i \ (i \in (\mathbf{Z}/n\mathbf{Z})^{\times})\}$ the set of integers satisfying the following conditions*:

(1) $a_{-i} = a_i$,

(2) $\sum_{i \in (\mathbf{Z}/n\mathbf{Z})^{\times}} a_i = 0$,

(3) $\prod_{i \in (\mathbf{Z}/n\mathbf{Z})^{\times}} (\zeta^i - 1)^{a_i} = 1$.

Then $a_i = 0$ for all $i \in (\mathbf{Z}/n\mathbf{Z})^{\times}$.

Let F be a finite Galois extension over $\mathbf{Q}$, and α an element of F. We denote the norm of α over $\mathbf{Q}$ by $N_{F/\mathbf{Q}}(\alpha)$, or simply $N(\alpha)$.

$$N_{F/\mathbf{Q}}(\alpha) = \prod_{\sigma \in \mathrm{Gal}(F/\mathbf{Q})} \sigma(\alpha)$$

The followings are fundamental facts in Number Theory (see [3, p.89], [30]).

Proposition 3.2. *In the situation above, we have the followings.*
(1) *$N(\alpha)$ is a rational number, and $N(\alpha) = 0$ if and only if $\alpha = 0$.*
(2) *If α is an algebraic integer, then $N(\alpha)$ is an integer.*
(3) *An algebraic integer α is a unit in the ring of algebraic integers if and only if $N(\alpha) = \pm 1$.*

3.2. Proof of Lemma 1.1. If $\Sigma(K; p/q)$ is of lens type, then there are integers i, j and m such that

$$\Delta_K(\zeta)(\zeta - 1)^{-1}\left(\zeta^{\bar{q}} - 1\right)^{-1} = \pm\zeta^m(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1},$$

where i and j are coprime to p.

By taking the norms of both sides, we have

$$N(\Delta_K(\zeta)) = N(\pm\zeta^m),$$

because

$$N(\zeta - 1) = N\left(\zeta^{\bar{q}} - 1\right) = N(\zeta^i - 1) = N(\zeta^j - 1) \neq 0.$$

Since

$$N(\pm\zeta^m) = \begin{cases} \pm 1 & (d = 2), \\ 1 & (d \geq 3), \end{cases}$$

we have the result. □

3.3. Proof of Main Theorem 1. Let $\Delta_{r,s}(t)$ be the Alexander polynomial of the (r, s)-torus knot

$$\Delta_{r,s}(t) = \frac{(t^{rs} - 1)(t - 1)}{(t^r - 1)(t^s - 1)},$$

d (≥ 2) a divisor of p, ζ a primitive d-th root of unity, and $\varphi_d\colon \mathbf{Z}[t]/(t^p - 1) \to \mathbf{Q}(\zeta)$ a ring homomorphism such that $\varphi_d(t) = \zeta$. Since $M = \Sigma(K; p/q)$ is the p/q-surgery along a knot K whose Alexander polynomial is $\Delta_{r,s}(t)$, we have

$$\tau^{\varphi_d}(M) = \Delta_{r,s}(\zeta)(\zeta - 1)^{-1}\left(\zeta^{\bar{q}} - 1\right)^{-1}$$

by Theorem 2.3. Suppose M is of lens type, then there are integers i, j and m such that

$$\Delta_{r,s}(\zeta)(\zeta - 1)^{-1}\left(\zeta^{\bar{q}} - 1\right)^{-1} = \pm\zeta^m(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1},$$

where i and j are coprime to p.

Suppose $\gcd(p,r) \geq 2$, we take $d = \gcd(p,r)$. Then $\gcd(d,s) = 1$ because $\gcd(r,s) = 1$. We set $p = p'd$ and $r = r'd$. Then

$$\begin{aligned}\Delta_{r,s}(t) &= \frac{(t^{rs}-1)(t-1)}{(t^r-1)(t^s-1)} = \frac{(t^{r'sd}-1)(t-1)}{(t^{r'd}-1)(t^s-1)} \\ &= \left(t^{(s-1)r'd} + t^{(s-2)r'd} + \cdots + t^{2r'd} + t^{r'd} + 1\right) \cdot \frac{t-1}{t^s-1},\end{aligned}$$

and therefore

$$N(\Delta_{r,s}(\zeta)) = s^{\varphi(d)}.$$

By Lemma 1.1, M is not of lens type. Thus we have the conclusion (1).

We assume $\gcd(p,r) = 1$ and $\gcd(p,s) = 1$, and take any divisor d (≥ 2) of p. Then

$$\Delta_{r,s}(\zeta) = \frac{(\zeta^{rs}-1)(\zeta-1)}{(\zeta^r-1)(\zeta^s-1)}.$$

If M is of lens type, then

$$(\zeta^{rs}-1)(\zeta^i-1)(\zeta^j-1) = \pm\zeta^m\left(\zeta^{\bar{q}}-1\right)(\zeta^r-1)(\zeta^s-1).$$

Multipling the complex conjugates to both sides, we have

$$\begin{aligned}&(\zeta^{rs}-1)(\zeta^i-1)(\zeta^j-1)(\zeta^{-rs}-1)(\zeta^{-i}-1)(\zeta^{-j}-1) \\ &= \left(\zeta^{\bar{q}}-1\right)(\zeta^r-1)(\zeta^s-1)\left(\zeta^{-\bar{q}}-1\right)(\zeta^{-r}-1)(\zeta^{-s}-1).\end{aligned}$$

By the same argument as the proof of the classification of lens spaces (see [6, 8, 20, 27], and Theorem 3.1),

$$\{rs, i, j\} = \{\bar{q}, r, s\} \quad \text{in} \quad (\mathbf{Z}/d\mathbf{Z})/\{\pm 1\}.$$

There are two cases.

(i) $rs \equiv \pm\bar{q} \pmod{d}$.

This is equivalent to $qrs \equiv \pm 1 \pmod{d}$.

(ii) $rs \equiv \pm s \pmod{d}$ or $rs \equiv \pm r \pmod{d}$.

This is equivalent to $r \equiv \pm 1 \pmod{d}$ or $s \equiv \pm 1 \pmod{d}$.

If $qrs \equiv \pm 1$ (resp. $r \equiv \pm 1$, $s \equiv \pm 1$) $\pmod{p}$ holds, then $qrs \equiv \pm 1$ (resp. $r \equiv \pm 1$, $s \equiv \pm 1$) $\pmod{d}$ holds for any d. So we state only the case of $d = p$.

The converse is obvious. This completes the proof. □

4. Proof of Main Theorem 2

In this section we show Main Theorem 2, which states a necessary and sufficient condition for the Reidemeister torsion of $\Sigma(K; p/q)$ to be of lens type in the case that the Alexander polynomial $\Delta_K(t)$ of K is of degree 2.

We prepare some lemmas.

Let n be a positive integer, $\varphi(n)$ the Euler function, ζ a primitive n-th root of unity, and

$$\mathbf{\Phi}_n(x) = \prod_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} (x - \zeta^i)$$

the *n-th cyclotomic polynomial*. Then $\mathbf{\Phi}_n(x)$ is an irreducible monic symmetric polynomial over $\mathbf{Z}$ with degree $\varphi(n)$, and

$$\mathbf{\Phi}_n(1) = \begin{cases} 0 & (n = 1), \\ p & (n = p^r,\ p\text{: prime}), \\ 1 & (\text{otherwise}). \end{cases}$$

The Alexander polynomial of a knot in a homology sphere with degree 2 has the following form for some integer $n \neq 0$:

$$\Delta_n(t) = nt^2 - (2n-1)t + n = t + n(t-1)^2 \quad (n \neq 0) \tag{4.1}$$

Let ζ be a primitive p-th root of unity, and α_1 and α_2 the roots of $\Delta_n(t) = 0$. Then we have

$$N(\Delta_n(\zeta)) = \prod_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} n(\zeta^i - \alpha_1)(\zeta^i - \alpha_2) = n^{\varphi(p)} \mathbf{\Phi}_p(\alpha_1)\mathbf{\Phi}_p(\alpha_2) \tag{4.2}$$

We regard the right-hand side of (4.2) as a polynomial of n over $\mathbf{Z}$ depending on p, denote it by $f_p(n)$ (i.e., $f_p(n) \in \mathbf{Z}[n]$), and call it *the p-th norm polynomial* or simply *the norm polynomial*.

Lemma 4.1. (1) *If $n \leq -1$, then $f_p(n) \neq \pm 1$.*
(2) *If $|n| \geq 2$ and p is a prime number, then $f_p(n) \neq \pm 1$.*

Main Theorem 2 is proved by Lemma 1.1 and Lemma 4.1: By the assumption that $\Sigma(K; p/q)$ is of lens type, by definition, $N(\Delta_K(\zeta)) = \pm 1$ holds not only in the p-th cyclotomic field but also in the d-th cyclotomic field for any divisor d of p. In the case that $n \geq 2$ and p is not prime, we study $f_d(n)$ for a prime divisor d of p in Lemma 4.1 (2).

In the case that $p = 2$, Lemma 4.1 holds because $f_2(n) = 4n - 1$. From now on, we assume $p \geq 3$. To show Lemma 4.1, we study properties of $f_p(n)$.

Proposition 4.2. (1) *The degree of $f_p(n)$ is $\varphi(p)$.*
(2) *If $p \geq 3$, then there exists a polynomial of n, $g_p(n)$, over $\mathbf{Z}$ with degree $\varphi(p)/2$ such that $f_p(n) = \{g_p(n)\}^2$.*

Proof. (1) Since $\Delta_n(\zeta) = (1-\zeta)^2 n + \zeta$, the degree of $f_p(n)$ is $\varphi(p)$.

(2) Firstly we note

$$\Delta_n(\zeta) = \zeta^2 \Delta_n(\zeta^{-1}).$$

From this equation,

$$\delta(\zeta) = \frac{\Delta_n(\zeta)}{\zeta}$$

satisfies $\delta(\zeta^{-1}) = \delta(\zeta)$, and $\delta(\zeta)$ is an element of $\mathbf{Q}(\zeta + \zeta^{-1})$. Since $\zeta \neq \zeta^{-1}$, we have $[\mathbf{Q}(\zeta) : \mathbf{Q}(\zeta + \zeta^{-1})] = 2$ and $[\mathbf{Q}(\zeta + \zeta^{-1}) : \mathbf{Q}] = \varphi(p)/2$. If we set

$$g_p(n) = N_{\mathbf{Q}(\zeta+\zeta^{-1})/\mathbf{Q}}(\delta(\zeta)),$$

then $g_p(n)$ is a polynomial of n over $\mathbf{Z}$ with degree $\varphi(p)/2$ such that $f_p(n) = \{g_p(n)\}^2$. □

We write down $f_p(n)$ and $g_p(n)$ in the following form:

$$f_p(n) = \sum_{i=0}^{\varphi(p)} a_i n^i, \quad g_p(n) = \sum_{j=0}^{\varphi(p)/2} b_j n^j.$$

Let $F(n) = s_0 n^m + s_1 n^{m+1} + s_2 n^{m+2} + \cdots + s_d n^{m+d}$ ($s_0 \neq 0$, $s_d \neq 0$) be a polynomial of n over $\mathbf{R}$. If (i) $d = 0$ or (ii) $d \geq 1$ and $s_{i-1}s_i < 0$ ($i = 1, 2, \ldots, d$), then we say that $F(n)$ is an *alternating polynomial*. We note that if all roots of $F(n) = 0$ are positive real numbers or 0, then $F(n)$ is an alternating polynomial.

Lemma 4.3. (1) *The polynomials $f_p(n)$ and $g_p(n)$ are alternating polynomials.*
(2) $a_{\varphi(p)} = \{\mathbf{\Phi}_p(1)\}^2$ *and* $a_0 = 1$.
(3) $b_{\varphi(p)/2} = (-1)^{\varphi(p)/2}\mathbf{\Phi}_p(1)$ *and* $b_0 = 1$.
(4) $a_1 = 2T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) - 2\varphi(p)$, *where $T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta)$ is the trace of ζ in $\mathbf{Q}(\zeta)/\mathbf{Q}$. In particular, if p is an odd prime number, then $a_1 = -2p$ and $b_1 = -p$.*

Proof. (1) Firstly we note

$$\delta(\zeta) = \frac{\Delta_n(\zeta)}{\zeta} = 1 - \{2 - (\zeta + \zeta^{-1})\}n.$$

Since $2 - (\zeta + \zeta^{-1}) > 0$, the polynomials $f_p(n)$ and $g_p(n)$ are alternating polynomials.
(2) $a_{\varphi(p)} = N_{\mathbf{Q}(\zeta)/\mathbf{Q}}((1-\zeta)^2) = \{\mathbf{\Phi}_p(1)\}^2$ and $a_0 = N_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) = 1$.
(3) It is clear by (1) and (2).

(4) By the definition,

$$a_1 = \sum_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} (1 - \zeta^i)^2 \cdot \frac{N_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta)}{\zeta^i} = \sum_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} (\zeta^i + \zeta^{-i} - 2)$$
$$= 2 \sum_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} \zeta^i - 2\varphi(p) = 2T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) - 2\varphi(p),$$

If p is a prime number, then $T_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta) = -1$ and $\varphi(p) = p - 1$. Therefore $a_1 = -2p$. This completes the proof. □

Proof of Lemma 4.1 (1). Assume $n \leq -1$. From Lemma 4.3 (1) and (2), we see

$$f_p(n) \geq \{\mathbf{\Phi}_p(1)\}^2 + 1 \geq 2.$$

This completes the proof. □

To prove Lemma 4.1 (2), we need the following lemma.

Lemma 4.4. (1) *If p is a prime number, then*

$$f_p(n) = n^p\big(\alpha_1^p - 1\big)\big(\alpha_2^p - 1\big),$$

where α_1 and α_2 are roots of $\Delta_n(t)$ in (4.1).
(2) *If p is an odd prime number, then $b_j \equiv 0$* (mod p) *for $j = 1, 2, \ldots, \varphi(p)/2$.*

Proof. (1) From $\varphi(p) = p - 1$ and $t^p - 1 = (t-1)\mathbf{\Phi}_p(t)$ if p is a prime, we have

$$\frac{n^p(\alpha_1^p - 1)(\alpha_2^p - 1)}{n^{\varphi(p)}\mathbf{\Phi}_p(\alpha_1)\mathbf{\Phi}_p(\alpha_2)} = n(\alpha_1 - 1)(\alpha_2 - 1) = \Delta_n(1) = 1.$$

By (4.2), we have the equality.
(2) Let (p) be an ideal in the polynomial ring $\mathbf{Z}[n]$ generated by p, and α_1 and α_2 roots of the Alexander polynomial $\Delta_n(t)$ with degree 2 in (4.2). Since $\Delta_n(t)$ is a polynomial over $\mathbf{Z}$ and p is an odd prime number, we have

$$n^p(\alpha_1 - 1)^p(\alpha_2 - 1)^p \equiv n^p\big(\alpha_1^p - 1\big)\big(\alpha_2^p - 1\big) \quad (\mathrm{mod}\ (p)).$$

By (1),

$$f_p(n) \equiv n^p(\alpha_1 - 1)^p(\alpha_2 - 1)^p = \{n(\alpha_1 - 1)(\alpha_2 - 1)\}^p = 1 \quad (\mathrm{mod}\ (p)).$$

Since $\mathbf{Z}[n]/(p) = (\mathbf{Z}/p\mathbf{Z})[n]$ is a unique factorization domain,

$$g_p(n) \equiv b_0 = 1 \quad (\mathrm{mod}\ (p)).$$

This means $b_j \equiv 0$ (mod p) for $j = 1, 2, \ldots, \varphi(p)/2$. □

Proof of Lemma 4.1 (2). Let p be an odd prime number, and $h_p(n)$ a polynomial of n satisfying

$$g_p(n) = pnh_p(n) + 1.$$

Then $h_p(n)$ is a polynomial over $\mathbf{Z}$ by Lemma 4.4 (2), and $f_p(n) = 1$ if and only if $h_p(n) = 0$. We write down

$$h_p(n) = \sum_{k=0}^{\varphi(p)/2-1} c_k n^k.$$

By Lemma 4.3 (3) and (4), $c_{\varphi(p)/2-1} = \pm 1$ and $c_0 = -1$. From this, if $h_p(n) = 0$, then $n = \pm 1$. Therefore if $|n| \geq 2$, then $h_p(n) \neq 0$. This completes the proof of Lemma 4.1 (2). $\square$

For example, $h_3(n) = -1$, $h_5(n) = n - 1$, $h_7(n) = -(n-1)^2$, $h_{11}(n) = -(n-1)(n^3 - 4n^2 + 3n - 1)$.

Corollary 4.5. *Let K be a knot in a homology 3-sphere Σ, $\Delta_K(t)$ the Alexander polynomial of K, and $M = \Sigma(K; p/q)$ the p/q-surgery for $p \geq 2$ and $q \neq 0$. If $\Delta_K(t)$ is divisible by $nt^2 - (2n-1)t + n$ ($n \in \mathbf{Z}; n \neq 0, 1$), then M is not of lens type.*

Appendix

We introduce a result in [18] due to Ozsváth and Szabó, which is a necessary condition on the Alexander polynomial of a knot in S^3 which yields a lens space. They show it by using knot Floer homology ([13, 18, 19]).

Theorem [Ozsváth-Szabó [18]]. *Let K be a knot in S^3, and $M = (K; p)$, where p is an integer. If M is a lens space, then the Alexander polynomial of K is of the following form*

$$\Delta_K(t) = (-1)^m + \sum_{j=1}^{m} (-1)^{m-j}(t^{s_j} + t^{-s_j}),$$

where $0 < s_1 < s_2 < \cdots < s_m$.

By Moser's theorem (Theorem 1.2), the Alexander polynomial of a torus knot satisfies the condition above. We can check it easily as follows.

Proposition. *The Alexander polynomial of a torus knot has the form in Ozsváth-Szabó's theorem.*

Proof. Let $\varDelta_{p,q}(t)$ be the Alexander polynomial of (p,q)-torus knot. We may assume $2 \leq q \leq p$. There are integers c_k and d_k $(k = 0, 1, \dots, q-1)$ such that $pk = qc_k + d_k$ $(0 \leq d_k \leq q-1)$. The integers $d_0, \dots, d_{q-1}$ are mutually distinct, because p and q are coprime integers. It is clear that $c_0 = d_0 = 0$.

We list the following formulas which are proved easily.

$$\varDelta_{p,q}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} = \frac{\left(t^{p(q-1)} + t^{p(q-2)} + \cdots + t^p + 1\right)(t-1)}{t^q-1} \tag{1}$$

$$t^{pk} = t^{d_k}(t^{qc_k}-1) + t^{d_k} \tag{2}$$

$$\frac{t^{pk}(t-1)}{t^q-1} = t^{d_k}\left(t^{q(c_k-1)} + t^{q(c_k-2)} + \cdots + t^q + 1\right)(t-1) + \frac{t^{d_k}(t-1)}{t^q-1} \tag{3}$$

$$\sum_{k=0}^{q-1} \frac{t^{d_k}(t-1)}{t^q-1} = \frac{(t^{q-1}+t^{q-2}+\cdots+t+1)(t-1)}{t^q-1} = 1 \tag{4}$$

$$\varDelta_{p,q}(t) = 1 + \sum_{k=1}^{q-1}\sum_{l=0}^{c_k-1}\left(t^{ql+d_k+1} - t^{ql+d_k}\right) \tag{5}$$

Equation (5) is obtained from equations (1), (2), (3) and (4). If two pairs (k,l) and (k',l') are distinct, then two numbers $ql + d_k$ and $ql' + d_{k'}$ are distinct. Therefore $\varDelta_{p,q}(t)$ has the form in Ozsváth-Szabó's theorem. $\square$

In [12], we characterized the Alexander polynomial of a knot in any homology 3-sphere having a lens type surgery. Infinitely many knots in S^3 having lens surgery appear as certain families or sequences (see [1]), and no counterexample is discovered. The author thinks such a deformation of the Alexander polynomial above is related to the structure of each family or sequence not only in S^3 but also in any homology 3-sphere.

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