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Osaka University
THE STRUCTURE
OF ALGEBRAIC EMBEDDINGS OF $\mathbb{C}^2$ INTO $\mathbb{C}^3$
(THE NORMAL QUARTIC HYPERSURFACE CASE. II)

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(Received March 14, 2008)

Abstract
We obtain the affirmative answer for a special case of the linearization problem for algebraic embeddings of $\mathbb{C}^2$ into $\mathbb{C}^3$. Indeed, we determine all the compactifications $(X, Y)$ of $\mathbb{C}^2$ such that $X$ are normal quartic hypersurfaces in $\mathbb{P}^3$ without triple points and $Y$ are hyperplane sections of $X$. Moreover, for each $(X, Y)$, we construct a tame automorphism of $\mathbb{C}^3$ which transforms the hypersurface $X \setminus Y$ onto a coordinate hyperplane.

1. Introduction
A polynomial mapping $f: \mathbb{C}^n \to \mathbb{C}^m$ is called an algebraic embedding of $\mathbb{C}^n$ into $\mathbb{C}^m$ for $m > n \geq 1$ if $f$ is injective and if the image of $f$ is a smooth algebraic subvariety of $\mathbb{C}^m$. Let Aut($\mathbb{C}^n$) be the group of algebraic automorphisms of $\mathbb{C}^n$. Here we consider the following conjecture:

Conjecture. Let $f: \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ be an algebraic embedding. Then $f$ is equivalent to a linear embedding up to Aut($\mathbb{C}^n$) and Aut($\mathbb{C}^{n+1}$), equivalently to say, there exists an algebraic automorphism of $\mathbb{C}^{n+1}$ which transforms the image $f(\mathbb{C}^n)$ onto a coordinate hyperplane.

For the case $n = 1$, Abhyankar-Moh [1] and Suzuki [14] showed that the conjecture is true. For the cases $n \geq 2$, the conjecture is still unsolved. In this paper, we will consider the case $n = 2$ only. Our approach is geometric and our main tool is a method of compactifications of $\mathbb{C}^2$. Let $f: \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$ be an algebraic embedding. We identify $\mathbb{C}^3$ with an affine part of the complex projective space $\mathbb{P}^3$ in the standard way. We denote by $X_f$ the closure of the image of $f$ in $\mathbb{P}^3$ and put $Y_f := X_f \setminus f(\mathbb{C}^2)$. By construction, we see that $Y_f$ is a hyperplane section of $X_f$ and that $X_f \setminus Y_f$ is bi-regular to $\mathbb{C}^2$, that is $(X_f, Y_f)$ is a compactification of $\mathbb{C}^2$. We call $Y_f$ the boundary of the compactification. Our main purpose is to write down explicitly a defining equation of the image of $f$ up to affine transformations of $\mathbb{C}^3$ and to construct explicitly

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an algebraic automorphism of $\mathbb{C}^3$ linearizing the defining equation, when the image of $f$ is of low degree. This explicit way is very important for us not only to obtain examples but also to find geometric invariants and inductive methods. In this direction, in Ohta [10], we showed that the conjecture is true when the degree of the image of $f$ is less than or equal to three. For the case of degree three, we needed a so-called Nagata automorphism (cf. [10]) to linearize some embeddings.

Next we consider the case of degree four. Then we have the following three possibilities: (1) $X_f$ is normal and it has at least a triple point; (2) $X_f$ is normal and it has no triple points; (3) $X_f$ is non-normal. For the case (1), in Ohta [11], we showed that the conjecture is true, and we needed a generalization and an analogue of a Nagata automorphism to linearize some embeddings. In this paper, we will treat the case (2) only. The case (3) will be dealt with elsewhere. Thus it suffices to consider a compactification $(X, Y)$ of $\mathbb{C}^2$ such that $X$ is a normal quartic hypersurface in $\mathbb{P}^3$ without triple points and $Y$ is a hyperplane section of $X$. First we will determine the defining equations of such compactifications $(X, Y)$ by using the classification of minimal normal compactifications of $\mathbb{C}^2$ due to Morrow [9] and the structure theorem of minimally elliptic singularities due to Laufer [8]. Finally, for each $(X, Y)$, we will construct a tame automorphism of $\mathbb{C}^3$ explicitly which linearizes the defining equation of $X \setminus Y$.

From now on to the end of this paper, we assume the following:

ASSUMPTION. Let $X$ be a normal quartic hypersurface in $\mathbb{P}^3$ without triple points and $Y$ a hyperplane section of $X$ such that $X \setminus Y$ is biholomorphic to $\mathbb{C}^2$. Denote by $H$ the hyperplane in $\mathbb{P}^3$ with $Y = X \cap H$.

We define some notations as follows. Let $Y = \bigcup_{i=1}^f Y_i$ be the irreducible decomposition of $Y$. We put $\mathcal{Y} := H|_X$. We note that $\text{Supp } \mathcal{Y} = Y$ and $\mathcal{O}_H(X|_H) \cong \mathcal{O}_\mathcal{Y}(4)$. We put $x := \text{Sing } X = \{x_1, \ldots, x_m\}$. Let $\pi : M \to X$ be the minimal resolution of $X$ with exceptional set $E = \bigcup_{i=1}^r E_i := \pi^{-1}(x)$, where each $E_i$ is irreducible. We denote by $\hat{\mathcal{C}}$ the proper transform of a curve $C$ in $X$ by $\pi$. In §2, we shall see that $X$ has a unique minimally elliptic double point, which is denoted by $x_1$, and that $Z^2 = -1, -2$ for the fundamental cycle $Z$ of $\pi^{-1}(x_1)$. Then our main results are the following:

Theorem 1. Let $(X, Y)$ be a pair satisfying Assumption. Then the weighted dual graph of $\hat{Y} \cup E$ is one of Fig. 1, where the notations $\bullet$, $\circ$, $\triangle$ mean smooth rational curves with self-intersection numbers $-1$, $-2$, $-3$ respectively and all $\circ$, $\triangle$ are irreducible components of $E$.

Theorem 2. For each weighted dual graph of $\hat{Y} \cup E$ in Theorem 1, the defining equation of $(X, Y)$ is one of the following up to $\text{Aut}(\mathbb{P}^3)$:

(XV) $X : (z_2z_3 + \alpha z_0^2 + \beta z_0z_1 + \gamma z_1^2)^2 + z_0z_3^3 + z_1^3z_3 = 0$, $\beta^2 - 4\alpha\gamma = 0$,

(XVI) $X : (z_2z_3 + \alpha z_0^2 + \beta z_0z_1 + \gamma z_1^2)^2 + z_0z_3^3 + z_1^3z_3 = 0$, $\beta^2 - 4\alpha\gamma \neq 0$. 


the previous paper [11] and we obtain some invariants as follows:

\begin{align*}
\text{(XV)} & \quad Z = (X_0^2 + \beta X_1 + \gamma X_1^2) - (X_0 X_1 + X_1^2) = 0, \quad [\beta^2 - 4\alpha(\gamma - 1)] [\beta^2 - 4\alpha(\gamma + 1)] = 0, \\
\text{(XVI)} & \quad Z = (X_0^2 + \beta X_1 + \gamma X_1^2) - (X_0 X_1 + X_1^2) = 0, \quad [\beta^2 - 4\alpha(\gamma - 1)] [\beta^2 - 4\alpha(\gamma + 1)] = 0.
\end{align*}

\begin{align*}
\text{(XVII)} & \quad Z = (X_0^2 + \beta X_1 + \gamma X_1^2) - (X_0 X_1 + X_1^2) = 0, \quad [\beta^2 - 4\alpha(\gamma - 1)] [\beta^2 - 4\alpha(\gamma + 1)] = 0, \\
\text{(XVIII)} & \quad Z = (X_0^2 + \beta X_1 + \gamma X_1^2) - (X_0 X_1 + X_1^2) = 0, \quad [\beta^2 - 4\alpha(\gamma - 1)] [\beta^2 - 4\alpha(\gamma + 1)] = 0, \\
\text{(XIX)} & \quad Z = (X_0^2 + \beta X_1 + \gamma X_1^2) - (X_0 X_1 + X_1^2) = 0, \quad [\beta^2 - 4\alpha(\gamma - 1)] [\beta^2 - 4\alpha(\gamma + 1)] = 0, \\
\text{(XX)} & \quad Z = (X_0^2 + \beta X_1 + \gamma X_1^2) - (X_0 X_1 + X_1^2) = 0, \quad [\beta^2 - 4\alpha(\gamma - 1)] [\beta^2 - 4\alpha(\gamma + 1)] = 0, \\
\text{(XI)} & \quad Z = (X_0^2 + \beta X_1 + \gamma X_1^2) - (X_0 X_1 + X_1^2) = 0, \quad [\beta^2 - 4\alpha(\gamma - 1)] [\beta^2 - 4\alpha(\gamma + 1)] = 0.
\end{align*}

where $z = (z_0 : z_1 : z_2 : z_3)$ is a homogeneous coordinate of $\mathbb{P}^3$, $H = [z_3 = 0]$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\alpha \neq 0$.

**Remark.** In Theorems 1 and 2, we continue to number the types of $(X, Y)$ from the previous paper [11] and we obtain some invariants as follows:

\begin{align*}
\text{(XV)} & \quad Z^2 = -2, \quad \mathcal{Y} = 4Y_1 (Y_1 : \text{line}), \quad x = [x_1]. \\
\text{(XVI)} & \quad Z^2 = -2, \quad \mathcal{Y} = 2Y_1 + 2Y_2 (Y_1 : \text{line}), \quad x = [x_1]. \\
\text{(XVII)} & \quad Z^2 = -2, \quad \mathcal{Y} = 2Y_1 + Y_2 + Y_3 (Y_1, Y_2, Y_3 : \text{line}), \quad x = [x_1, x_2]. \\
\text{(XVIII)} & \quad Z^2 = -2, \quad \mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4 (Y_1, Y_2, Y_3 : \text{line}), \quad x = [x_1]. \\
\text{(XIX)} & \quad Z^2 = -2, \quad \mathcal{Y} = 2Y_1 + Y_2 + Y_3 (Y_1 : \text{line}), \quad x = [x_1, x_2]. \\
\text{(XX) } & \quad Z^2 = -2, \quad \mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4 (Y_1 : \text{line}), \quad x = [x_1]. \\
\text{(XI)} & \quad Z^2 = -1, \quad \mathcal{Y} = 2Y_1 + Y_2 (Y_1 : \text{line}, Y_2 : \text{conic}), \quad x = [x_1].
\end{align*}

For each type, $x_1 = (0 : 0 : 1 : 0)$ is the unique minimally elliptic double point and $x_2$...
is a rational double point of type $A_1$, where

$$x_2 = \begin{cases} 
(\beta : -2\alpha : \beta : 0) & \text{for the type (XVII) and } \beta^2 - 4\alpha(\gamma - 1) = 0. \\
(\beta : -2\alpha : -\beta : 0) & \text{for the type (XVII) and } \beta^2 - 4\alpha(\gamma + 1) = 0. \\
(\beta : -2\alpha : 0 : 0) & \text{for the type (XIX)}. 
\end{cases}$$

Moreover, every line in $X$ through $x_1$ is an irreducible component of $Y$ (see Lemma 3.2 (ii) and Lemma 4.1 (v)).

Here we recall some special subgroups of $\text{Aut}(\mathbb{C}^3)$. Let $A(3, \mathbb{C})$ and $J(3, \mathbb{C})$ be the subgroups of all affine transformations and de Jonquières automorphisms respectively. Let us denote by $T(3, \mathbb{C})$ the subgroup generated by $A(3, \mathbb{C})$ and $J(3, \mathbb{C})$. An algebraic automorphism of $\mathbb{C}^3$ is said to be tame if it is an element of $T(3, \mathbb{C})$ (cf. [11]).

**Theorem 3.** For each defining equation of $(X, Y)$ in Theorem 2, there exists a tame automorphism of $\mathbb{C}^3$ which transforms $X \setminus Y$ onto a coordinate hyperplane.

As a consequence of Theorems 2 and 3, we obtain the following:

**Theorem 4.** Let $f: \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$ be an algebraic embedding. Assume that $X_f$ is a normal quartic hypersurface in $\mathbb{P}^3$ without triple points. Then $f$ is equivalent to a linear embedding up to $\text{Aut}(\mathbb{C}^2)$ and $T(3, \mathbb{C})$.

Indeed, if one has such an algebraic embedding $f$, then $(X_f, Y_f)$ has one of the defining equations of the types (XV) through (XXI) up to $\text{Aut}(\mathbb{P}^3)$ by Theorem 2 and there exists a tame automorphism of $\mathbb{C}^3$ transforming $f(\mathbb{C}^2) = X_f \setminus Y_f$ onto a coordinate hyperplane by Theorem 3. Thus we obtain Theorem 4.

**Notation.** $b_i(V) = \dim_{\mathbb{R}} H^i(V, \mathbb{R})$: $i$-th Betti number of $V$.  
Exc $\varphi$: exceptional set of birational morphism $\varphi: V \to W$.  
Pic($V$): Picard group of $V$.  
$K_V$: canonical divisor of $V$.  
$\omega_V$: dualizing sheaf of $V$.  
m$_{V,v}$: maximal ideal of $O_{V,v}$.  
mult$_W V$: multiplicity of $V$ at general point of $W$.  
$D|_V$: restriction of Cartier divisor $D$ to $V$.  
$(D \cdot C)_{V,v}$: local intersection number of $D$ and $C$ at $v \in V$.  
$D_1 \sim D_2$: $D_1$ and $D_2$ are linearly equivalent.  
$(V, v)$: normal two-dimensional singularity.  
$p_g(v)$: geometric genus of $(V, v)$.  
$p_g(v_1, \ldots, v_n) = \sum_{i=1}^n p_g(v_i)$. 
\( \mathbb{N} = \{1, 2, 3, \ldots \} \): set of all positive integers.

\((-n)\)-curve: smooth rational curve with self-intersection number \(-n\).

\(\circ\): \((-n)\)-curve.

\(\odot\): 0-curve.

\(\bullet\): \((-1)\)-curve.

\(\circ\): \((-2)\)-curve.

\(\Delta\): \((-3)\)-curve.

2. Preliminaries

In this section, we shall describe the fundamental properties of a pair \((X, Y)\) satisfying Assumption in §1. We use the same notation as that in §1. Let \(Y = \bigcup_{i=1}^{n} Y_i\) be the irreducible decomposition of \(Y\). We denote by \(\deg Y_i\) the degree of \(Y_i\) as a plane curve of \(H \cong \mathbb{P}^2\). We set \(Y' := H|_X = \sum_{i=1}^{n} k_i Y_i\), where \(k_i \in \mathbb{N}\) and \(\sum_{i=1}^{n} k_i \deg Y_i = 4\).

We put \(x := \text{Sing} X = \{x_1, \ldots, x_m\}\). Let \(\pi: M \to X\) be the minimal resolution of \(X\) with exceptional set \(E = \bigcup_{i=1}^{r} E_i := \pi^{-1}(x)\), where each \(E_i\) is irreducible. We may assume that \(\pi^{-1}(x_i) = \bigcup_{j=1}^{s_i} E_{ij}\) for \(1 \leq i \leq m\), where \(0 =: s_0 \leq s_1 \leq \cdots \leq s_m := s\). Let \(Z = \sum_{j=1}^{s_0} \sum_{i=1}^{s_j} a_{ij} E_{ij}\) be the fundamental cycle of \(\pi^{-1}(x_i)\) with \(a_{ij} \in \mathbb{N}\). We denote by \(\hat{C}\) the proper transform of a curve \(C\) in \(X\) by \(\pi\). Let \(\Gamma\) be a general smooth hyperplane section of \(X\) with \(\Gamma \cap x = \emptyset\). We have the relations \((\hat{\Gamma} \cdot \hat{Y}_i)_M = (\Gamma \cdot Y_i)_X = \deg Y_i\) and \(\hat{\Gamma} \sim \sum_{i=1}^{s} k_i \hat{Y}_i + \sum_{j=1}^{s} b_j E_j\) with \(b_j \in \mathbb{N}\). We note that \(\omega_X = \mathcal{O}_X(K_X) \cong \mathcal{O}_X\) and \(x \subset Y\) and that \(M \setminus (\hat{Y} \cup E)\) is biholomorphic to \(\mathbb{C}^2\). By Kodaira [6] and Ramanujam [12], we see that \(X \setminus Y\) and \(M \setminus (\hat{Y} \cup E)\) are birational to \(\mathbb{C}^2\). In particular, \(X\) and \(M\) are rational surfaces. Then we have the next proposition.

**Proposition 2.1** (Ohta [10]). One obtains the following:

(i) \(H_0(X, \mathbb{Z}) \cong H_0(Y, \mathbb{Z}) = \mathbb{Z}\).

(ii) \(H_1(X, \mathbb{Z}) \cong H_1(Y, \mathbb{Z}) = 0\).

(iii) \(H_2(X, \mathbb{Z}) \cong H_2(Y, \mathbb{Z}) = \bigoplus_{i=1}^{n} \mathbb{Z} \cdot Y_i\).

(iv) \(H_3(X, \mathbb{Z}) \cong H_3(Y, \mathbb{Z}) = 0\).

(v) \(H^1(X, \mathcal{O}_X) = 0\).

(vi) \(p_g(x) = 1\).

(vii) \(X\) is not a cone.

(viii) \(\gcd(\deg Y_1, \ldots, \deg Y_n) = 1\).

(ix) \(\text{mult}_p X \leq \sum_{i=1}^{n} k_i \text{mult}_p Y_i\) \((\forall p \in Y = X \cap H)\).

**Remark.** (1) By (i) and (ii), \(Y\) is a connected divisor without cycles. In particular, each \(Y_i\) is a rational curve without nodes. If \(Y\) contains at least two lines, then \(Y\) consists of lines which meet at only one point. Indeed, this follows since \(Y\) has no cycles and each \(Y_i\) is a plane curve.

(2) By (vi), (vii) and Assumption in §1, we may assume that \(X_1\) is a minimally elliptic double point and \(X \setminus \{x_1\}\) consists of at most rational double points. For simplic-
ity, we put $Z := Z^{(i)}$. By Artin [2] and Laufer [8], we see that $K_M \sim -Z$, $Z^2 = -1, -2$ and $Z^{(i)} = -2$ for $2 \leq i \leq m$.

(3) Since $(M, \tilde{Y} \cup E)$ also satisfies the assertions (i) through (v), $\tilde{Y} \cup E$ is a connected divisor without cycles (cf. [10]). By Noether’s formula, we obtain $b_2(\tilde{Y}) + b_2(E) = b_2(M) = 10 - Z^2$. Thus $\tilde{Y} \cup E$ consists of $10 - Z^2$ rational curves.

For the divisor $\mathcal{Y}$, we obtain the following classification. In the last part of this section, we will make this classification to be detailed.

**Lemma 2.2.** There exist the following seven possibilities for the divisor $\mathcal{Y}$:

(i) $\mathcal{Y} = 4Y_1 \ (Y_1$: line$)$ with $x \subseteq Y_1$.

(ii) $\mathcal{Y} = 3Y_1 + Y_2 \ (Y_1$: line$)$ with $x \subseteq Y_1$.

(iii) $\mathcal{Y} = 2Y_1 + 2Y_2 \ (Y_1$: line$)$ with $x \subseteq Y_2$.

(iv) $\mathcal{Y} = 2Y_1 + Y_2 + Y_3 \ (Y_1$: line$)$ with $x \subseteq Y_1$ and $Y_1 \cap Y_2 \cap Y_3 = \{\text{one point}\}$.

(v) $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4 \ (Y_1$: line$)$ with $x = \{x_1\} = Y_1 \cap Y_2 \cap Y_3 \cap Y_4$.

(vi) $\mathcal{Y} = 2Y_1 + Y_2 \ (Y_1$: line, $Y_2$: conic$)$ with $x \subseteq Y_1$ and $Y_1 \cap Y_2 = \{\text{one point}\}$.

(vii) $\mathcal{Y} = Y_1 + Y_2 \ (Y_1$: line, $Y_2$: cuspidal cubic$)$ with $x \subseteq \text{Sing} \ Y$ and $Y_1 \cap Y_2 = \{\text{one point}\}$.

Proof. By Proposition 2.1 (ii), (viii) and (ix), we obtain the assertions. $\square$

For the fundamental cycles $Z$ and $Z^{(i)}$, we shall prove some lemmas with strong effect to the structure of $(X, Y)$.

**Lemma 2.3** ([10]). One obtains the following:

(1) Assume that $Z^2 = -2$. Then

$$\pi^*m_{X,x_1} \cong \mathcal{O}_M(-Z).$$

Moreover, the blowing-up morphism at $x_1$ of $X$ factors $\pi$ and $(\tilde{C} \cdot Z) = \text{mult}_{x_1} C$ for any curve $C$ in $X$ through $x_1$.

(2) Assume that $Z^2 = -1$. Denote by $E_1$ a unique irreducible component $E_i$ of $Z$ with $(E_i \cdot Z) = -1$ and $a_i = 1$. Then there exists a unique point $p_0$ of $E_1 \setminus \text{Sing}(\text{Supp} \ Z)$ such that

$$(\pi \circ \pi_0)^*m_{X,x_1} \cong \mathcal{O}_{M'}(-Z' - 2E_0'),$$

where $\pi_0: M' \to M$ is a blowing-up at $p_0$ with exceptional curve $E'_0$ and $Z'$ is the proper transform of $Z$ in $M'$. Moreover, the blowing-up morphism at $x_1$ of $X$ factors $\pi \circ \pi_0$ and $(\tilde{C}' \cdot Z' + 2E'_0) = \text{mult}_{x_1} C$ for any curve $C$ in $X$ through $x_1$, where $\tilde{C}'$ is the proper transform of $C$ in $M'$.

Proof. First we use Theorem 3.13 in [8] and the universal property of blowing-up (cf. Proposition II.7.14 in [5]). By applying the same argument as in the proof of Lemma 3 in [10], we obtain the assertions. $\square$
LEMMA 2.4 ([2], [10]). Assume that $x$ contains at least two points. Then

$$
\pi^* m_{X,x_i} \cong \mathcal{O}_M(-Z^{(i)})
$$

for $2 \leq i \leq m$. Moreover, the blowing-up morphism at $x_i$ of $X$ factors $\pi$ and $(\hat{C} \cdot Z^{(i)}) = \text{mult}_{x_i} C$ for any curve $C$ in $X$ through $x_i$.

Proof. First we use Theorem 4 in [2] and the universal property of blowing-up (cf. Proposition II.7.14 in [5]). By applying the same argument as in the proof of Lemma 3 in [10], we obtain the assertions.

LEMMA 2.5. One obtains the following:

(i) Assume that $C$ is a line or a conic in $X$. Then $\hat{C} \cong \mathbb{P}^1$ and $(\hat{C} \cdot Z^{(i)}) = 1$ when $x_i \in C$ ($1 \leq i \leq m$). If $x_i \in C$, then $\hat{C}$ is a $(−1)$-curve with $(\hat{C} \cdot Z) = 1$. If $x_i \notin C$, then $\hat{C}$ is a $(−2)$-curve with $(\hat{C} \cdot Z) = 0$.

(ii) Assume that $C$ is a plane cuspidal cubic in $X$ and $\text{Sing} C = \{x_1\}$. Then $\hat{C} \cong \mathbb{P}^1$ and $(\hat{C} \cdot Z^{(i)}) = 1$ when $x_i \in C$ ($2 \leq i \leq m$). If $Z^2 = −2$, then $\hat{C}$ is a 0-curve with $(\hat{C} \cdot Z) = 2$. If $Z^2 = −1$, then $\hat{C}$ is a 0-curve with $(\hat{C} \cdot Z) = 2$ or a $(−1)$-curve with $(\hat{C} \cdot Z) = 1$.

Proof. By Lemma 2.3, we note that the blowing-up morphism at $x_1$ of $X$ factors $\pi$ or $\pi \circ \pi_0$ if $Z^2 = −2$ or −1 respectively. By Lemmas 2.3, 2.4 and the adjunction formula, we obtain the assertions.

REMARK. In (i) and (ii), we note that $\hat{C} \cup \pi^{-1}((x \setminus \{x_1\}) \cap C)$ is a simple normal crossing divisor of smooth rational curves. In (i), we see that $\hat{C}$ meets $\pi^{-1}(x_1)$ transversally at only one point if $x_1 \in C$.

LEMMA 2.6 (Reid [13]). One obtains the following:

(i) $Z$ is a numerically 2-connected divisor of $M$ with $\omega_Z \cong \mathcal{O}_Z$ and $p_a(Z) = h^0(\mathcal{O}_Z) = h^1(\mathcal{O}_Z) = 1$. Here an effective divisor $D$ of smooth projective surface is said to be numerically $n$-connected for $n \geq 0$ if it satisfies the condition $(D_1 \cdot D_2) \geq n$ for every effective decomposition $D = D_1 + D_2$ with $D_1, D_2 > 0$.

(ii) There exists an exact sequence

$$
0 \to \mathbb{C} \to \text{Pic}(Z) \xrightarrow{\text{deg}} \mathbb{Z}^\oplus s_1 \to 0
$$

of abelian groups, where the homomorphism $\text{deg}$ is given for $L \in \text{Pic}(Z)$ by

$$
\text{deg } L = (\deg_{E_1} L|_{E_1}, \ldots, \deg_{E_n} L|_{E_i}).
$$
(iii) If $L$ is a nef line bundle on $Z$ with $\deg_Z L := \sum_{i=1}^{s_1} a_i \deg E_i \rvert E_i = 1$, then there exists a unique smooth point $P$ of $Z$ such that $L_{\vert P} \cong \mathcal{O}_Z(P)$.

(iv) If $P$ and $Q$ are smooth points of $Z$, then $P = Q$ if and only if $\mathcal{O}_Z(n(P - Q)) \cong \mathcal{O}_Z$ for some integer $n \geq 1$.

Proof. (i) By the adjunction formula, Lemma 3.11 and Theorem 4.21 in [13], we obtain the assertions.

(ii) Note that $H^1(Z, \mathbb{Z}) = 0$ and $H^2(Z, \mathbb{Z}) \cong \mathbb{Z}^{s_1}$ since $\hat{Y} \cup E$ has no cycles and $\text{Supp} Z$ consists of $s_1$ rational curves. Note that $H^1(\mathcal{O}_Z) \cong \mathbb{C}$ and $H^2(\mathcal{O}_Z) = 0$ by (i) and $\dim_{\mathbb{C}} Z = 1$. By applying the exponential cohomology sequence of sheaves on $Z$, we obtain the assertion.

(iii) By Lemma 4.23 in [13], we obtain the assertion.

(iv) By noting (ii) and (iii), we obtain the assertion. □

We shall prove some useful lemmas for smooth compactifications of $\mathbb{C}^2$. It is well-known that the weighted dual graph of a boundary of minimal normal compactification of $\mathbb{C}^2$ is a linear tree of smooth rational curves by Ramanujam [12] and these graphs are classified by Morrow [9] (cf. Proposition 2 in [10]). Here a smooth compactification $(S, C)$ of $\mathbb{C}^2$ is said to be minimally normal if it satisfies the following two conditions: (1) $C$ is a simple normal crossing divisor; (2) any $(-1)$-curve in $C$ meets at least three other irreducible components of $C$.

**Lemma 2.7.** There exists no boundary $C$ of smooth compactification of $\mathbb{C}^2$ satisfying the following conditions:

(i) $C$ contains a smooth rational curve $C_0$ with $C_0^2 \geq -1$.

(ii) $C \setminus C_0$ consists of at least three connected components, which are denoted by $C_1$, $C_2$, $\ldots$, $C_n$ with $n \geq 3$.

(iii) $C_i$ meets $C_0$ transversally at only one point for any $1 \leq i \leq n$.

(iv) $C_i$ is a simple normal crossing divisor of smooth rational curves whose self-intersection numbers are less than or equal to $-2$ for any $i = 1, 2$.

Proof. Assume that there exists such a smooth compactification $(S, C)$ of $\mathbb{C}^2$. By applying some blowing-ups on $(C_3 \cup \cdots \cup C_n) \setminus C_0$, we obtain a smooth compactification $(S', C')$ of $\mathbb{C}^2$ with simple normal crossing boundary, where $C'$ is the total transform of $C$ in $S'$. Let $C'_i$ be the total transform of $C_i$ in $S'$ for $0 \leq i \leq n$. Then $C'$ satisfies the following conditions:

(1) $C'_0$ is a smooth rational curve in $C'$ with $(C'_0)^2 \geq -1$;

(2) $C' \setminus C'_0$ consists of the $n$ connected components $C'_1$, $C'_2$, $\ldots$, $C'_n$ with $n \geq 3$;

(3) $C'_i$ meets $C'_0$ transversally at only one point for any $1 \leq i \leq n$;

(4) $C'_i$ is a simple normal crossing divisor of smooth rational curves whose self-intersection numbers are less than or equal to $-2$ for any $i = 1, 2$;

(5) $C'_i$ is a simple normal crossing divisor of smooth rational curves for any $3 \leq i \leq n$. 

By [12], we obtain a linear tree in Fig. 2 as a boundary of minimal normal compactification of $\mathbb{C}^2$ by applying some blowing-downs in $\mathbb{C}^3\setminus BC$, where $\mathbb{C}^3\setminus BC/BC_0$, $\mathbb{C}^3\setminus BC/BC_1$, and $\mathbb{C}^3\setminus BC/BC_2$ are the proper transforms of $C_0$, $C_1$, and $C_2$ respectively. However this dual graph is not found in Morrow’s classification. This is a contradiction.

**Lemma 2.8.** Assume that $C$ is a boundary of smooth compactification of $\mathbb{C}^2$ satisfying the following conditions:

(i) $C$ contains a $(-1)$-curve $C_0$.

(ii) $C_1$ consists of exactly two connected components $C_1$ and $C_2$.

(iii) $C_i$ meets $C_0$ transversally at only one point for any $i = 1, 2$.

(iv) $C_1$ is a simple normal crossing divisor of $(-2)$-curves.

Then the weighted dual graph of $C$ is a linear tree.

**Proof.** Assume that the weighted dual graph of $C_0 \cup C_1$ is not such a linear tree. Then there exists an irreducible component $C_{1,1}$ of $C_1$ such that $(C_0 \cup C_1) \setminus C_{1,1}$ consists of at least three connected components and such that the weighted dual graph of the connected component of $(C_0 \cup C_1) \setminus C_{1,1}$ containing $C_0$ is a linear tree. By contracting the connected component of $(C_0 \cup C_1) \setminus C_{1,1}$ containing $C_0$, we obtain a boundary of smooth compactification of $\mathbb{C}^2$ satisfying the conditions in Lemma 2.7. This is a contradiction.

**Lemma 2.9.** Assume that $C$ is a simple normal crossing boundary of smooth compactification of $\mathbb{C}^2$ which is a union of only one $(-1)$-curve and some $(-2)$-curves. Then the weighted dual graph of $C$ is a linear tree or a tree as in Fig. 3.

**Proof.** Let $C_0$ be the unique $(-1)$-curve in $C$ and $C \setminus C_0 = \bigcup_{i=1}^{n} C_i$ the decomposition into connected components with $n \geq 1$. By Lemma 2.7, we obtain $n = 1, 2$. Then we consider the following cases (1) and (2).
(1) Assume that \( n = 2 \). By Lemma 2.8, the weighted dual graphs of \( C_0 \cup C_1 \) and \( C_0 \cup C_2 \) are linear trees \( \cdots \bullet \circ \circ \circ \cdots \o \). By using Morrow’s classification after the contraction of \( C_0 \cup C_1 \) or \( C_0 \cup C_2 \), we see that the weighted dual graph of \( C \) is a linear tree \( \circ \o \bullet \circ \circ \o \).

(2) Assume that \( n = 1 \). Note that the weighted dual graph of \( C \) is not a linear tree by the assumption. Then there exists an irreducible component \( C_{1,1} \) of \( C_1 \) such that \( C \setminus C_{1,1} \) consists of at least three connected components and such that the weighted dual graph of the connected component of \( C \setminus C_{1,1} \) containing \( C_0 \) is a linear tree. By contracting the connected component of \( C \setminus C_{1,1} \) containing \( C_0 \), we obtain a simple normal crossing boundary \( D \) of smooth compactification of \( C_2 \) which is a union of only one \((1)\)-curve \( D_0 \) and some \((2)\)-curves and such that \( D \setminus D_0 \) consists of at least two connected components. By Lemma 2.7, \( D \setminus D_0 \) consists of exactly two connected components. By the same argument as that in (1), the weighted dual graph of \( D \) is a linear tree \( \circ \o \bullet \circ \circ \o \). Hence the weighted dual graph of \( C \) is obtained as in Fig. 3.

From now on to the end of this section, we shall show that some cases do not occur for the classification of the divisor \( Y \) in Lemma 2.2. In the proofs of the following lemmas, we mainly use Lemmas 2.5, 2.7 and 2.8. Especially, we always note Remark of Lemma 2.5.

**Lemma 2.10.** It does not occur the case where

\[ Y = 2Y_1 + 2Y_2 \quad (Y_i: \text{line}) \text{ with } x_1 \notin Y_1 \cap Y_2. \]

Proof. Assume that this case occurs. We may assume that \( x_1 \in Y_1 \setminus Y_2 \). By Lemma 2.5, \( \hat{Y}_1 \) is a \((-1)\)-curve and \( \hat{Y}_2 \) is a \((-2)\)-curve in \( M \). Note the linear equivalence

\[ \hat{\Gamma} \sim 2\hat{Y}_1 + 2\hat{Y}_2 + \sum b_i E_i \]

with \( b_i \in \mathbb{N} \). Then we consider the following cases (1) and (2).

(1) Assume that \( Y_1 \cap Y_2 \) is a smooth point of \( X \). Note that \( (\hat{Y}_1 \cdot \hat{Y}_2) = 1 \) by computing the intersection number of the above linear equivalence and \( \hat{Y}_1 \). By Lemma 2.7, \( x \cap (Y_1 \setminus Y_2) \) consists of only one point \( x_1 \). Since \( (\sum b_i E_i \cdot \hat{Y}_2) = 3 > 0 \), \( x \cap (Y_2 \setminus Y_1) \) contains at least one point. By Lemma 2.8, \( x \cap (Y_2 \setminus Y_1) \) consists of exactly one point, which is denoted by \( x_2 \), and the weighted dual graph of \( \hat{Y}_1 \cup \hat{Y}_2 \cup \pi^{-1}(x_2) \) is a linear tree as in Fig. 4 (1), where \( E_{s_1+1}, \ldots, E_{s_2} \) are the irreducible components of \( \pi^{-1}(x_2) \) \((s_2 - s_1 \geq 1)\). By computing the intersection number of the above linear equivalence and \( \hat{Y}_2 + Z^{(2)} \), we obtain \( b_{s_2} = -1 \). This is a contradiction.

(2) Assume that \( Y_1 \cap Y_2 \) is a singular point of \( X \), which is denoted by \( x_2 \). By Lemma 2.4, the blowing-up morphism at \( x_2 \) of \( X \) factors \( \pi \). Thus we obtain \( (\hat{Y}_1 \cdot \hat{Y}_2) = 0 \).
By Lemma 2.7, \( x \cap (Y_1 \setminus Y_2) \) consists of only one point \( x_1 \). By Lemma 2.8, \( x \cap (Y_2 \setminus Y_1) \) consists of no points or exactly one point, which is denoted by \( x_3 \), and the weighted dual graph of \( \hat{Y}_1 \cup \hat{Y}_2 \cup \pi^{-1}(x) \) is a linear tree as in Fig. 4 (2), where \( E_{s_1+1}, \ldots, E_{s_2} \) are the irreducible components of \( \pi^{-1}(x_2) \) and \( E_{s_3+1}, \ldots, E_{s_3} \) are the irreducible components of \( \pi^{-1}(x_3) \) (\( s_2 - s_1 \geq 1, s_3 - s_2 \geq 0 \)). If \( x = \{x_1, x_2\} \), then we have \( b_{s_1+1} = -1 \) by computing the intersection number of the above linear equivalence and \( \hat{Y}_2 + Z^{(2)} \). This is a contradiction. If \( x = \{x_1, x_2, x_3\} \), then we have \( b_{s_1+1} + b_{s_3} = 1 \) by computing the intersection number of the above linear equivalence and \( \hat{Y}_2 + Z^{(2)} + Z^{(3)} \). This is a contradiction.

**Lemma 2.11.** It does not occur the case where

\[
\mathcal{V} = 2Y_1 + Y_2 + Y_3 \quad (Y_i: \text{line}) \quad \text{with} \quad x_1 \in Y_1 \setminus (Y_2 \cup Y_3).
\]

**Proof.** Assume that this case occurs. Note that \( x \subset Y_1 \) and that \( \hat{Y}_1, \hat{Y}_2, \hat{Y}_3 \) are a \((-1)\)-curve and two \((-2)\)-curves in \( M \) respectively by Lemma 2.5. If \( Y_1 \cap Y_2 \cap Y_3 \) is a smooth point of \( X \), then we have \( (\hat{Y}_1 \cdot \hat{Y}_2) = (\hat{Y}_2 \cdot \hat{Y}_3) = (\hat{Y}_3 \cdot \hat{Y}_1) = 1 \) by computing the intersection number of each \( \hat{Y}_i \) and \( \hat{F} \sim 2\hat{Y}_1 + \hat{Y}_2 + \hat{Y}_3 + \sum b_i E_i \) \( (b_i \in \mathbb{N}) \). By applying the blowing-up on \( \hat{Y}_1 \cap \hat{Y}_2 \cap \hat{Y}_3 \), we obtain a smooth compactification of \( \mathbb{C}^2 \) with the conditions in Lemma 2.7. This is a contradiction. Thus \( Y_1 \cap Y_2 \cap Y_3 \) is a singular point of \( X \), which is denoted by \( x_2 \). By Lemma 2.4, the blowing-up morphism at \( x_2 \) of \( X \) factors \( \pi \). Thus we have \( (\hat{Y}_i \cdot \hat{Y}_j) = 0 \) for \( i \neq j \). Hence the weighted dual graph of \( \hat{Y}_1 \cup \hat{Y}_2 \cup \hat{Y}_3 \cup \pi^{-1}(x) \) is not a linear tree and \( \bar{x} = \{x_1, x_2\} \) by Lemma 2.7. On the other hand, the weighted dual graph of \( \hat{Y}_1 \cup \hat{Y}_2 \cup \hat{Y}_3 \cup \pi^{-1}(x) \) is a linear tree by Lemma 2.8. This is a contradiction.

**Lemma 2.12.** It does not occur the case where

\[
\mathcal{V} = 2Y_1 + Y_2 \quad (Y_1: \text{line}, \ Y_2: \text{conic}) \quad \text{with} \quad x_1 \in Y_1 \setminus Y_2.
\]
Proof. Assume that this case occurs. Note that $x \subset Y_1$ and that $\hat{Y}_1$ and $\hat{Y}_2$ are a $(-1)$-curve and a $(-2)$-curve in $M$ respectively by Lemma 2.5. Note the linear equivalence

$$\hat{\Gamma} \sim 2\hat{Y}_1 + \hat{Y}_2 + \sum b_i E_i$$

with $b_i \in \mathbb{N}$. If $Y_1 \cap Y_2$ is a smooth point of $X$, then we have $(\hat{Y}_1 \cdot \hat{Y}_2) = 2$ by computing the intersection number of the above linear equivalence and $\hat{Y}_2$. By applying the blowing-ups twice on $\hat{Y}_1 \cap \hat{Y}_2$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction. Thus $Y_1 \cap Y_2$ is a singular point of $X$, which is denoted by $x_2$. By computing the intersection number of the above linear equivalence and $\hat{Y}_2$, we have $(\hat{Y}_1 \cdot \hat{Y}_2) = 0, 1$. If $(\hat{Y}_1 \cdot \hat{Y}_2) = 1$, then each pair of $\hat{Y}_1$, $\hat{Y}_2$ and $\pi^{-1}(x_2)$ meets transversally at only one point since $(\hat{Y}_1 \cdot Z(2)) = (\hat{Y}_2 \cdot Z(2)) = 1$. By applying the blowing-up on $\hat{Y}_1 \cap \hat{Y}_2 \cap \pi^{-1}(x_2)$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction. Thus we have $(\hat{Y}_1 \cdot \hat{Y}_2) = 0$. Hence $x = \{x_1, x_2\}$ by Lemma 2.7 and the weighted dual graph of $\hat{Y}_1 \cup \hat{Y}_2 \cup \pi^{-1}(x_2)$ is a linear tree as in Fig. 5 by Lemma 2.8, where $E_{s_1+1}, \ldots, E_{s_2}$ are the irreducible components of $\pi^{-1}(x_2)$ ($s_2 - s_1 \geq 1$). By computing the intersection number of the above linear equivalence and $\hat{Y}_2 + Z(2)$, we obtain $b_{s_1+1} = -1$. This is a contradiction.

\[\text{Lemma 2.13.} \quad \text{It does not occur the case where}\]

$$\mathcal{Y} = Y_1 + Y_2 \quad (Y_1: \text{line}, \ Y_2: \text{cuspidal cubic}) \quad \text{with} \quad Y_1 \cap Y_2 = \{x_1\} \neq \text{Sing} \ Y_2.$$

Proof. Assume that this case occurs. In $H \cong \mathbb{P}^2$, $Y_1$ and $Y_2$ meet tangentially to the third order at $x_1$ which is a smooth point of $Y_2$. By Lemmas 2.3 and 2.5, $\hat{Y}_1$ is a $(-1)$-curve in $M$ and $(\hat{Y}_1 \cdot Z) = (\hat{Y}_2 \cdot Z) = 1$. Note that $(\hat{Y}_1 \cdot \hat{Y}_2) = 0, 1$ by computing the intersection number of $\hat{Y}_1$ and $\hat{\Gamma} \sim \hat{Y}_1 + \hat{Y}_2 + \sum b_i E_i$ ($b_i \in \mathbb{N}$). If Sing $Y_2$ is a smooth point of $X$, then we see that $\hat{Y}_2 \cong Y_2$ and $\hat{Y}_2^2 = 1$ by the adjunction formula. By applying the blowing-ups three times on Sing $\hat{Y}_2$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction. Thus Sing $Y_2$ is a singular point of $X$, which is denoted by $x_2$, and in particular $x = \{x_1, x_2\}$. By Lemma 2.4, the blowing-up morphism at $x_2$ of $X$ factors $\pi$. Thus $\hat{Y}_2$ is a smooth curve and in partic-
ular a \((-1)\)-curve in \(M\) since \((\hat{Y}_2 \cdot Z) = 1\). Note that \((\hat{Y}_2 \cdot Z^{(2)}) = 2\) by Lemma 2.4 and 
\[Z^{(2)} = \sum_{i=0}^{n} a_i E_i \quad (a_i \in \mathbb{N})\]
Then we consider the following cases (1), (2) and (3).

(1) Assume that there exists an irreducible component \(E_{i_1}\) of \(\pi^{-1}(x_2)\) such that \((\hat{Y}_2 \cdot E_{i_1}) = 2\) and \(a_{i_1} = 1\). Note that \((\hat{Y}_2 \cdot Z^{(2)} - E_{i_1}) = 0\). By applying the blowing-ups twice on \(\hat{Y}_2 \cap E_{i_1}\), we obtain a smooth compactification of \(C^2\) with the conditions in Lemma 2.7. This is a contradiction.

(2) Assume that there exist two irreducible components \(E_{i_1}\) and \(E_{i_2}\) of \(\pi^{-1}(x_2)\) such \((\hat{Y}_2 \cdot E_{i_1}) = (\hat{Y}_2 \cdot E_{i_2}) = 1\) and \(a_{i_1} = a_{i_2} = 1\). Note that \((\hat{Y}_2 \cdot Z^{(2)} - E_{i_1} - E_{i_2}) = 0\).

By applying the blowing-up on \(\hat{Y}_2 \cap E_{i_1} \cap E_{i_2}\), we obtain a smooth compactification of \(C^2\) with the conditions in Lemma 2.7. This is a contradiction.

(3) Assume that there exists an irreducible component \(E_{i_1}\) of \(\pi^{-1}(x_2)\) such that \((\hat{Y}_2 \cdot E_{i_1}) = 1\) and \(a_{i_1} = 2\). Note that \((\hat{Y}_2 \cdot Z^{(2)} - 2E_{i_1}) = 0\). Note that \(\hat{Y}_2 \cup \pi^{-1}(x_2)\) is of simple normal crossing and that \(x_2\) is a rational double point not of type \(A\). If \((\hat{Y}_1 \cdot \hat{Y}_2) = 0\), then \(x_2\) is a rational double point of type \(A\) by Lemma 2.8. This is a contradiction.

Thus we have \((\hat{Y}_1 \cdot \hat{Y}_2) = 1\). Since \((\hat{Y}_1 \cdot \hat{Y}_2) = (\hat{Y}_1 \cdot Z) = (\hat{Y}_2 \cdot Z) = 1\), each pair of \(\hat{Y}_1, \hat{Y}_2\) and \(\pi^{-1}(x_1)\) meets transversally at only one point. By applying the blowing-up on \(\hat{Y}_1 \cap \hat{Y}_2 \cap \pi^{-1}(x_1)\), we obtain a smooth compactification of \(C^2\) with the conditions in Lemma 2.7. This is a contradiction.

\[\textbf{Lemma 2.14.} \text{ It does not occur the case where} \]

\[\mathcal{Y} = Y_1 + Y_2 \quad (Y_1: \text{ line}, \ Y_2: \text{ cuspidal cubic}) \text{ with } Y_1 \cap Y_2 \neq \{x_1\} = \text{Sing} \ Y_2.\]

\[\textbf{Proof.} \text{ Assume that this case occurs. Note that } Y_1 \text{ and } Y_2 \text{ meet in } H \cong \mathbb{P}^2 \text{ tangentially to the third order at a smooth point of } Y_2 \text{ and that } x \setminus \{x_1\} \text{ is contained in } Y_1 \cap Y_2 \text{ and } \{x_1\} = \text{Sing} \ Y_2. \]

By Lemma 2.5, \(\hat{Y}_1\) is a \((-2)\)-curve in \(M\) with \((\hat{Y}_1 \cdot Z) = 0\) and \(\hat{Y}_2\) is a \((-1)\)-curve with \((\hat{Y}_2 \cdot Z) = 1\) or a \(0\)-curve with \((\hat{Y}_2 \cdot Z) = 2\). Note the linear equivalence 

\[\hat{\Gamma} \sim \hat{Y}_1 + \hat{Y}_2 + \sum b_i E_i\]

with \(b_i \in \mathbb{N}\). If \(Y_1 \cap Y_2\) is a smooth point of \(X\), then we have \((\hat{Y}_1 \cdot \hat{Y}_2) = 3\) by computing the intersection number of the above linear equivalence and \(\hat{Y}_1\). By applying the blowing-ups three times on \(\hat{Y}_1 \cap \hat{Y}_2\), we obtain a smooth compactification of \(C^2\) with the conditions in Lemma 2.7. This is a contradiction. Thus \(Y_1 \cap Y_2\) is a singular point of \(X\), which is denoted by \(x_2\), and in particular \(x = \{x_1, x_2\}\). Note that \((\hat{Y}_1 \cdot Z^{(2)}) = (\hat{Y}_2 \cdot Z^{(2)}) = 1\) by Lemma 2.4. By computing the intersection number of the above linear equivalence and \(\hat{Y}_1\), we have \((\hat{Y}_1 \cdot \hat{Y}_2) = 0, 1, 2\). If \((\hat{Y}_1 \cdot \hat{Y}_2) = 1, 2\), then there exists a unique irreducible component \(E_{i_1}\) of \(\pi^{-1}(x_2)\) such that \(\hat{Y}_1, \hat{Y}_2\) and \(E_{i_1}\) meet at only one point since \((\hat{Y}_1 \cdot Z^{(2)}) = (\hat{Y}_2 \cdot Z^{(2)}) = 1\). By applying the blowing-ups \((\hat{Y}_1 \cdot \hat{Y}_2)\) times on \(\hat{Y}_1 \cap \hat{Y}_2 \cap E_{i_1}\), we obtain a smooth compactification of \(C^2\) with the
conditions in Lemma 2.7. This is a contradiction. Hence we have \((\hat{Y}_1 : \hat{Y}_2) = 0\). Then we consider the following cases (1) and (2).

1. Assume that \(\hat{Y}_2\) is a \((-1)\)-curve in \(M\) with \((\hat{Y}_2 : Z) = 1\). By Lemma 2.8, the weighted dual graph of \(\hat{Y}_1 \cup \hat{Y}_2 \cup \pi^{-1}(x_2)\) is a linear tree as in Fig. 6 (1), where \(E_{s_1+1}, \ldots, E_{s_2}\) are the irreducible components of \(\pi^{-1}(x_2)\) \((s_2 - s_1 \geq 1)\). By computing the intersection number of the above linear equivalence and \(\hat{Y}_1 + Z^{(2)}\), we have \(b_{s_1+1} = -1\). This is a contradiction.

2. Assume that \(\hat{Y}_2\) is a 0-curve in \(M\) with \((\hat{Y}_2 : Z) = 2\). Since \((\hat{Y}_2 : Z) = 2\), we have \(\operatorname{mult}_p Z = 1, 2\), where \(\{p\} := \hat{Y}_2 \cap \operatorname{Supp} Z\). If \(\operatorname{mult}_p Z = 1\), then we obtain a smooth compactification of \(\mathbb{C}^2\) with the conditions in Lemma 2.7 by applying the blowing-ups twice on \(p\). This is a contradiction. If \(\operatorname{mult}_p Z = 2\), then after applying the blowing-up on \(p\), by Lemma 2.8, we have the weighted dual graph of \(\hat{Y}_1 \cup \hat{Y}_2 \cup \pi^{-1}(x_2)\) as in Fig. 6 (2), where \(E_{s_1+1}, \ldots, E_{s_2}\) are the irreducible components of \(\pi^{-1}(x_2)\) \((s_2 - s_1 \geq 1)\). By computing the intersection number of the above linear equivalence and \(\hat{Y}_1 + Z^{(2)}\), we obtain \(b_{s_1+1} = -1\). This is a contradiction.

As a consequence, we obtain the following refined classification of the divisor \(\mathcal{Y}\).

**Proposition 2.15.** There exist the following seven possibilities for the divisor \(\mathcal{Y}\):

1. \(\mathcal{Y} = 4Y_1 \ (Y_1: \text{ line})\) with \(x \subset Y_1\).
2. \(\mathcal{Y} = 3Y_1 + Y_2 \ (Y_i: \text{ line})\) with \(x \subset Y_1\).
3. \(\mathcal{Y} = 2Y_1 + 2Y_2 \ (Y_i: \text{ line})\) with \(x \subset Y\) and \(Y_1 \cap Y_2 = \{x_1\}\).
4. \(\mathcal{Y} = 2Y_1 + Y_2 + Y_3 \ (Y_i: \text{ line})\) with \(x \subset Y_1\) and \(Y_1 \cap Y_2 \cap Y_3 = \{x_1\}\).
5. \(\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4 \ (Y_i: \text{ line})\) with \(x = \{x_1\} = Y_1 \cap Y_2 \cap Y_3 \cap Y_4\).
6. \(\mathcal{Y} = 2Y_1 + Y_2 \ (Y_1: \text{ line, } Y_2: \text{ conic})\) with \(x \subset Y_1\) and \(Y_1 \cap Y_2 = \{x_1\}\).
7. \(\mathcal{Y} = Y_1 + Y_2 \ (Y_1: \text{ line, } Y_2: \text{ cuspidal cubic})\) with \(x = \{x_1\} = Y_1 \cap Y_2 = \text{Sing } Y_2\).

In particular, for each case, \(Y\) contains at least one line through \(x_1\).

### 3. Proof of Theorem 1 for \(Z^2 = -2\)

In this section, we shall prove Theorem 1 for the case \(Z^2 = -2\). Let \((X, Y)\) be a pair satisfying Assumption in §1 and \(Z^2 = -2\). We use the same notation as that in §1 and §2. We mainly consider a projection from \(x_1\) and a blowing-up at \(x_1\) to investigate the pair \((X, Y)\). First we note that \(\hat{Y} \cup E\) is a connected divisor without cycles which
consists of twelve rational curves. For each irreducible component \( E_i \) of \( Z \), we may assume that \( (E_i \cdot Z) < 0 \) for \( 1 \leq i \leq s_{0,1} \) and \( (E_i \cdot Z) = 0 \) for \( s_{0,1} < i \leq s_1 \), where \( s_{0,1} \) is an integer with \( 1 \leq s_{0,1} \leq s_1 \). We put 
\[ Z_1 := \sum_{i=1}^{s_{0,1}} a_i E_i \] 
and 
\[ Z_2 := \sum_{i=s_{0,1}+1}^{s_1} a_i E_i, \]
where \( Z_2 = 0 \) is allowed. Thus we obtain an effective decomposition \( Z = Z_1 + Z_2 \). Since \( Z_2^2 = -2 \), we note that \( s_{0,1} = 1, 2 \). Let \( \sigma: \mathbb{P}^3 \to \mathbb{P}^3 \) be the blowing-up at \( x_1 \) with exceptional divisor \( \Delta \), which is isomorphic to \( \mathbb{P}^2 \). Let \( \overline{T} \) be the proper transform of a closed algebraic subset \( T \) of \( \mathbb{P}^3 \) by \( \sigma \). We have that \( \sigma|_{\overline{T}}: \mathbb{P}^3 \setminus \Delta \cong \mathbb{P}^3 \setminus \{ x_1 \} \) and \( O_{\overline{T}}(\Delta) \cong O_{\mathbb{P}^2}(-1) \). We set \( \overline{E} := \overline{X} \cap \Delta \). We have that \( \sigma|_{\overline{X}\setminus T}: \overline{X}\setminus \overline{E} \cong \overline{X}\setminus \{ x_1 \} \) and that \( (\overline{X}, \overline{Y} \cup \overline{E}) \) is a compactification of \( \mathbb{C}^2 \) with dualizing sheaf \( \omega_{\overline{X}} \cong O_{\overline{X}} \). Let \( v: \overline{X} \to \overline{X} \) be the normalization of \( \overline{X} \). Let \( \overline{C} \) be the proper transform of a curve in \( \overline{X} \) by \( \sigma|_{\overline{X}} \circ v \). We have that \( v_1|_{\overline{X}\setminus v^{-1}(\overline{E})}: \overline{X}\setminus v^{-1}(\overline{E}) \cong \overline{X}\setminus \overline{E} \) and that \( (\overline{X}, \overline{Y} \cup v^{-1}(\overline{E})) \) is a compactification of \( \mathbb{C}^2 \). Let \( \psi: \mathbb{P}^3 \to \mathbb{P}^2 \) be the projection from \( x_1 \) and \( \overline{v}: \mathbb{P}^3 \to \mathbb{P}^2 \) the resolution of indeterminacy of \( \psi \). We have that \( \overline{v}|_{\Delta}: \Delta \to \mathbb{P}^2 \) is an isomorphism and \( \overline{v}\overline{\sigma}|_{\overline{X}}: \overline{X} \to \mathbb{P}^2 \) is a generically finite morphism of degree two.

We note that \( \overline{\Gamma} \sim \overline{H}|_{\overline{X}} + \Delta |_{\overline{X}}, \overline{\Gamma}' \sim v^*(\overline{H}|_{\overline{X}}) + v^*(\Delta |_{\overline{X}}) \) and \( \overline{\Gamma} \sim \sum_{i=1}^{s_1} k_i \overline{Y}_i + \sum_{j=1}^{s_i} b_j E_j \) with \( b_j \in \mathbb{N} \). Then we have some fundamental lemmas.

**Lemma 3.1.** One obtains the following:

(i) \( \overline{X} \) is non-normal. Moreover, \( \overline{X}|_{\Delta} = 2\overline{E} = 2\text{line} \) and \( \Delta \cap \text{Sing} \overline{X} = \overline{E} \).

(ii) Sing \( \overline{X} \) consists of at most rational double points.

(iii) There exists a birational morphism \( \overline{\pi}: M \to \overline{X} \) satisfying \( \sigma|_{\overline{X}} \circ \overline{\pi} = \pi \). Then \( \overline{\pi^*}(\Delta|_{\overline{X}}) = Z \) and \( \overline{\pi^*}(\psi|_{\overline{X}})|_{\mathbb{P}^3} \cong O_{\mathbb{P}^3}(\overline{\Gamma} - Z) \). Moreover, \( \deg(\overline{\pi}|_{E_i}) = -(E_i \cdot Z) \) for each irreducible component \( E_i \) of \( Z_1 \) and \( \pi(\text{Supp} Z_2) \) is a finite set. In particular, \( \overline{\pi}|_{M|E}: M \setminus E \cong \overline{X} \setminus (\overline{E} \cup (\sigma|_{\overline{X}})^{-1}(x \setminus \{ x_1 \})) \).

(iv) There exists a birational morphism \( \overline{\pi}' = M \to \overline{X} \) satisfying \( v \circ \overline{\pi}' = \pi \). Then \( \deg(\overline{\pi}'|_{E_i}) = 1 \) and \( \deg(\psi|_{\overline{X}}) = -(E_1 \cdot Z) \) with \( \overline{E}_i := \overline{\pi}'(E_i) \) for each irreducible component \( E_i \) of \( Z_1 \), and \( \overline{\pi}'(\text{Supp} Z_2) \) is a finite set. Moreover, \( \overline{\pi}' \) is a minimal resolution of \( \overline{X} \) with \( \text{Exc} \overline{\pi}' = \text{Supp} Z_2 \cup \pi^{-1}(x \setminus \{ x_1 \}) \). Here one puts the push-forward Weil divisor \( \overline{\sigma}' := (\overline{\pi}'|_Z)_{x}(Z) \) of \( \overline{X} \).

(v) Let \( \overline{X} \xrightarrow{\sigma} V \xrightarrow{h} \mathbb{P}^2 \) be the Stein factorization of \( \overline{\psi}|_{\overline{X}} \circ v \). Then \( V \) is normal, \( g \) is a birational morphism and \( h \) is a finite morphism of degree two. In particular, \( g|_{\overline{X}\setminus \text{Exc} g}: \overline{X}\setminus \text{Exc} g \cong V \setminus g(\text{Exc} g) \) and \( \text{Exc} g \) is the proper transform of the union of all lines in \( X \) through \( x_1 \) by \( \sigma|_{\overline{X}} \circ v \). Thus one obtains the commutative diagram as in Fig. 7.

(vi) Assume that \( l \) is a line in \( X \) through \( x_1 \). Then \( \overline{l} \setminus \text{Sing} \overline{X} \) consists of at most one rational double point of type \( A \) and the weighted dual graph of \( (\overline{\pi}^*)^{-1}(\overline{l}) = \overline{l} \cup (\overline{\pi}^*)^{-1}(\overline{\Gamma} \setminus \text{Sing} \overline{X}) \) is a linear tree \( \bullet \rightarrow \circ \rightarrow \cdots \rightarrow \circ \). In particular, \( \overline{g}(\overline{l}) \) is a smooth point of \( V \) and \( x \cap l = \{ x_1 \}, \{ x_1, A_n \} \) for some \( n \geq 1 \).

(vii) Sing \( V \) consists of at most rational double points.
Proof. (i), (ii) The assertions are the general properties of the minimally elliptic singularity \((X, x_1)\) with \(Z^2 = -2\). Indeed, we can check the assertions by applying a blowing-up of the local analytic defining equation of \((X, x_1)\) in Theorem 4.57 (2) of [7] (cf. [8]).

(iii) There exists a birational morphism \(\pi: M \to X\) satisfying \(\sigma \circ \pi = \pi\). In particular, we have \((\pi)^{-1}(E) = \text{Supp} \ Z\). By the isomorphisms

\[
\pi^*(O_X(-\Delta|_X)) \cong \pi^*(\mathbb{P}^2) \cong \pi^* \mathbb{P}^2 = \mathcal{O}_M(-Z),
\]

we obtain \(\pi^*(\Delta|_X) \sim Z\). Since \(\pi^*(\Delta|_X)\) is an effective divisor of \(M\) whose support equals to \(\text{Supp} \ Z\) and the intersection matrix of \(\text{Supp} \ Z\) is negative definite, we obtain \(\pi^*(\Delta|_X) = Z\). In particular, we have

\[
\pi^*(\psi|_X|_\mathbb{P}^2(1) \cong \pi^*(\mathbb{P}|_X) \cong \pi^* \mathbb{P} = \mathcal{O}_M(-Z) = \mathcal{O}_M(\hat{Y} - Z).
\]

Let \(E_i\) be any irreducible component of \(Z\). Since \(\pi|_{E_i}\) is identified with \(\psi|_{E_i} \circ \pi|_{E_i}\), we obtain \(\text{deg}(\pi|_{E_i}) = -(E_i \cdot Z)\). By noting that \((\sigma|_X)^{-1}(x) = \hat{E} \cup (\sigma|_X)^{-1}(x \setminus \{x_1\}))\), we have \(\pi|_{M \setminus E}: M \setminus E \cong X \setminus (E \cup (\sigma|_X)^{-1}(x \setminus \{x_1\}))\).

(iv) There exists a birational morphism \(\pi^*: M \to \bar{X}\) satisfying \(\nu \circ \pi^* = \pi\) by the lifting property of normalization (cf. Proposition 8.4.3 in [4]). By noting (iii) and that \(\nu\) is a finite morphism, we obtain the assertions.

(v) Note the general properties of Stein factorization (cf. Corollary III.11.5 in [5]).

(vi) First we note (ii) and that \(\hat{I}\) is a \((-1)\)-curve in \(M\) by Lemma 2.5 (i). By applying Lemma 4 in [10] for the morphisms \(\pi^*: (M, (\pi^*)^{-1}(\hat{I})) \to (\bar{X}, \hat{I})\) and \(g: (\bar{X}, \hat{I}) \to (V, g(\hat{I}))\), we obtain the assertions.

(vii) By using (ii), (v) and (vi), we obtain the assertion. \(\square\)

**Lemma 3.2.** One obtains the following:

(i) \(\hat{H} \cap \Delta = \hat{E}\). In particular, \(Y\) is a union of lines through \(x_1\).

(ii) Every line in \(X\) through \(x_1\) is contained in \(Y\).

(iii) \(\hat{Y}_i\) is a \((-1)\)-curve in \(M\) with \((\hat{I} \cdot \hat{Y}_i) = (\hat{Y}_1 \cdot Z) = 1\) \((1 \leq i \leq t)\).
(iv) $Y_i \cap Y_j = \mathcal{Y}_i \cap \mathcal{Y}_j = \emptyset \ (i \neq j)$.
(v) $x \setminus \{x_1\}$ consists of at most rational double points of type A.
(vi) $\text{Exc } g = \overline{Y}$ and $g|_{\mathcal{X}} : \mathcal{X} \setminus \overline{Y} \cong V \setminus g(\overline{Y})$.
(vii) $(\text{Sing } \mathcal{X}^u) \setminus \overline{Y} = g^{-1}(\text{Sing } V)$.
(viii) $(\mathcal{X}^u, \overline{Y} \cup \text{Supp } \mathcal{Z}^u)$ and $(V, g(\text{Supp } \mathcal{Z}^u))$ are compactifications of $\mathbb{C}^2$.

Proof. (i) Assume that $\overline{H} \cap \Delta \neq \overline{E}$. Let $l$ be any line in $H$ such that $x_1 \in l \not\subset X$ and $l \cap \mathcal{X} = \emptyset$. Since $\mathcal{X} \cdot l = 2$ and $(l \cap \Delta) \cap \mathcal{X} = \emptyset$, we have

$$
\sum_{p \in \mathcal{X} \setminus \{x_1\}} \left( \sum_{i=1}^t k_i Y_i \cdot l \right)_{H,p} = \sum_{p \in \mathcal{X} \setminus \{x_1\}} (X \cdot l)_{\mathcal{X},p} = 2.
$$

On the other hand, by Proposition 2.15, we have $\sum_{p \in \mathcal{X} \setminus \{x_1\}} (\sum_{i=1}^t k_i Y_i \cdot l)_{H,p} = 0, 1$ for a general line $l$ in $H$ through $x_1$. This is a contradiction. Thus we obtain $\overline{H} \cap \Delta = \overline{E}$. Since $E \subset \text{Sing } \mathcal{X}$ and $(\mathcal{X} \cdot \overline{l}_{\mathcal{X}} = 2$ for a general line $l$ in $H$ through $x_1$, we see that $Y$ is a union of lines through $x_1$.

(ii) Let $l$ be any line in $X$ through $x_1$. Then we have $l \cap \Delta \subset \mathcal{X} \cap \Delta = \mathcal{E} = \overline{H} \cap \Delta$. Hence we obtain $l \subset H$ and thus $l \subset X \cap H = \mathcal{Y}$.

(iii) Note that each $Y_i$ is a line in $X$ through $x_1$ by (i).

(iv) Since each $\mathcal{Y}_i$ is the proper transform of a line through $x_1$ by the blowing-up at $x_1$, we see that $\mathcal{Y}_i \cap \mathcal{Y}_j = \emptyset \ (i \neq j)$. By noting this, we obtain the assertions.

(v) By using (i), (ii) and Lemma 3.1 (vi), we obtain the assertion.

(vi) By using (i), (ii) and Lemma 3.1 (v), we obtain the assertions.

(vii) Note (vi) and that $g(\mathcal{Y}^u)$ consists of smooth points of $V$.

(viii) Note (vi) and that $v^{-1}(\mathcal{E}) = \text{Supp } \mathcal{Z}^u$. $\square$

**Lemma 3.3.** One obtains the following:

(i) $\mathcal{Z}^u$ is the Cartier divisor $v^*(\Delta|_\mathcal{X})$ of $\mathcal{X}^u$. In particular, $(\mathcal{X}^u)^*(\mathcal{Z}^u) = Z$.

(ii) $g_s(\mathcal{Z}^u) = h^*(\mathcal{Y}(E))$, $g^*g_s(\mathcal{Z}^u) = \mathcal{Z}^u + \sum_{i=1}^t k_i \mathcal{Y}_i = v^*(\mathcal{H}|_\mathcal{X})$.

(iii) $k_i \mathcal{Y}_i$ is a Cartier divisor of $\mathcal{X}^u$ $(1 \leq i \leq t)$.

(iv) $\mathcal{Y}^u = \sum_{i=1}^t k_i \mathcal{Y}_i + 2\mathcal{Z}^u$, $\mathcal{Y} = \sum_{i=1}^t (\mathcal{X}^u)^*(k_i \mathcal{Y}_i) + 2\mathcal{Z}$.

(v) $g_s(\mathcal{Y})$ is a smooth Cartier divisor of $V$ with $g_s(\mathcal{Y}) \cap \text{Sing } V = \emptyset$.

(vi) $g_s(\mathcal{Y}) \sim 2g_s(\mathcal{Z}^u)$, $g^*g_s(\mathcal{Y}) = \mathcal{G}^u + \sum_{i=1}^t k_i \mathcal{Y}_i$.

(vii) $(g_s(\mathcal{Y}) \cdot g_s(\mathcal{Y}))_{V} = 8$, $(g_s(\mathcal{Y}) \cdot g_s(\mathcal{Z}^u))_{V} = 4$, $(g_s(\mathcal{Z}^u) \cdot g_s(\mathcal{Z}^u))_{V} = 2$.

(viii) $(g_s(\mathcal{Y}) \cdot g_s(\mathcal{Z}^u))_{V,\mathcal{Z}^u} = (\mathcal{Y}(\mathcal{E}) \cdot \mathcal{Y}(\mathcal{E}))_{\mathcal{Z}^u} = k_i$ $(1 \leq i \leq t)$.

(ix) $k_\mathcal{X} \sim -\mathcal{Z}^u$, $k_V \sim -g_s(\mathcal{Z}^u) = -h^*(\mathcal{Y}(E))$.

Proof. (i) By Lemma 3.1 (iii) and (iv), we have $(\mathcal{X}^u)^* v^*(\Delta|_\mathcal{X}) = Z$. By pushing this forward, we obtain $v^*(\Delta|_\mathcal{X}) = (\mathcal{X}^u)_s(\mathcal{Z}) = \mathcal{Z}^u$. 

(ii) By noting that \( v^*(\Delta |_{\mathbb{T}^r}) = \mathbb{Z}^r \) and \( v^*|_{\mathbb{T}^r}: \mathbb{E} \cong \mathbb{E} \), we have \( g^*(\mathbb{Z}^r) = h^*(\mathbb{Z}^r) \). Thus we obtain \( g^*g_*(\mathbb{Z}^r) = v^*(\mathbb{Z}^r)(\mathbb{E}) = v^*(\mathbb{H}|_{\mathbb{T}^r}) = \mathbb{Z}^r + \sum_{i=1}^t k_i \mathbb{Y}_i \).

(iii) By (i) and (ii), we see that \( \sum_{i=1}^t k_i \mathbb{Y}_i \) is a Cartier divisor of \( \mathbb{X}^r \). By Lemma 3.2 (iv), we obtain the assertion.

(iv) Note (i), (ii), (iii) and (iv), we have \( (v) \hat{i} k \) is a Cartier divisor of \( \mathbb{X}^r \). By Lemma 3.2 (iv), we obtain the assertion.

(v) First we note that \( \mathbb{T}^r \) is a smooth Cartier divisor of \( \mathbb{X}^r \) with \( \mathbb{T}^r \cap \text{Sing} \mathbb{X}^r = \emptyset \) which intersects each \( \mathbb{Y}_i \) transversally at only one point by \((\Gamma \cdot Y_i)_X = 1\). Since each \( g(\mathbb{Y}_i) \) is a smooth point of \( V \), we obtain the assertion.

(vi) By pushing the first relation of (iv) forward, we obtain the first assertion. By noting (i), (ii), (iii) and (iv), we have \( (\bar{\mathbb{T}}^r)^* (g^*g_*(\bar{\mathbb{T}}^r)) = \hat{\Gamma}^r + \sum_{i=1}^t (\bar{\mathbb{T}}^r)^*(k_i \mathbb{Y}_i) \). Since \( \text{Supp}(\bar{\mathbb{T}}^r)^*(g^*g_*(\bar{\mathbb{T}}^r)) = \hat{\Gamma}^r \cup (\bar{\mathbb{T}}^r)^{-1}(\bar{\mathbb{Y}}^r) \) and the intersection matrix of \((\bar{\mathbb{T}}^r)^{-1}(\bar{\mathbb{Y}}^r) = (g \circ \bar{\mathbb{T}}^r)^{-1}(g(\bar{\mathbb{Y}}^r)) \) is negative definite, we have \( (\bar{\mathbb{T}}^r)^*(g^*g_*(\bar{\mathbb{T}}^r)) = \hat{\Gamma}^r + \sum_{i=1}^t (\bar{\mathbb{T}}^r)^*(k_i \mathbb{Y}_i) \). By pushing this forward, we obtain the second assertion.

(vii) Note (vi) and that \( \text{deg} h = 2 \) and \( \text{Supp}(g(\mathbb{Z}^r)) = h^*(\mathbb{T}^2) \).

(viii) By noting that \( \text{Supp}(\mathbb{Z}^r)^*(g^*g_*(\mathbb{Z}^r)) = \mathbb{Z}^r \), we obtain the first equality of the assertion. By the same argument as in the proof of Proposition 2.2 (vi) in [11], we obtain the second equality of the assertion.

(ix) Since both of \( \text{Sing} \mathbb{X}^r \) and \( \text{Sing} V \) consist of at most rational double points, we see that \( K_{\mathbb{X}^r} \) and \( K_V \) are Cartier divisors. Since \( \bar{\mathbb{T}}^r \) is the minimal resolution of \( \mathbb{X}^r \), we have \( (\bar{\mathbb{T}}^r)^*k_{\mathbb{X}^r} \sim K_M \sim -\mathbb{Z}^r \). By pushing this forward, we also have \( K_{\mathbb{X}^r} \sim -\mathbb{Z}^r \). By noting Lemma 3.2 (vi), we obtain \( K_V \sim -g_*(\mathbb{Z}^r) \).

**Remark.** The branch locus \( B \) of \( h \) is a reduced plane quartic curve. Indeed, this is showed as follows. First we note that \( \text{Pic}(V) \) is torsion-free by Lemma 3.1 (vii), Lemma 3.2 (viii), and Proposition 1 in [10]. Thus we obtain the injectivity of \( h^* : \text{Pic}(\mathbb{P}^2) \cong \mathbb{Z} \to \text{Pic}(V) \). Let \( B \) be the ramification divisor of \( h \). Since \( \text{deg} h = 2 \), we have \( K_V \sim h^*K_{\mathbb{P}^2} + R \) and \( h^*B = 2R \). By noting (ix), we obtain \( h^*(B - 4L) \sim 0 \) and hence \( B - 4L \sim 0 \), where \( L \) is a line in \( \mathbb{P}^2 \). In the following, we omit the investigation of a detailed structure of \( B \) since there is no necessity in our arguments.

**Lemma 3.4.** One obtains the following:

(i) The weighted dual graph of \( (\bar{\mathbb{T}}^r)^{-1}(\bar{\mathbb{Y}}^r) \) is given as in Fig. 8(a), (b), (c), (d) for \( k_i = 1, 2, 3, 4 \) respectively, where the integers adjacent to vertices are coefficients in the divisor \( (\bar{\mathbb{T}}^r)^*(k_i \mathbb{Y}_i) \).

(ii) \( b_2((\bar{\mathbb{T}}^r)^{-1}(\bar{\mathbb{Y}}^r)) = 4 \). In particular, \( 8 \leq b_2((\bar{\mathbb{T}}^r)^{-1}(x_i)) \leq 11 \).

(iii) \( (\text{Sing} \mathbb{X}^r) \setminus \mathbb{Y}^r \neq \emptyset \). In particular, \( \text{Sing} \mathbb{X}^r \neq \emptyset \) and \( \text{Sing} V \neq \emptyset \).

(iv) If \( x \cap Y_i \) contains more than one point for some \( i \), then \( 2 \leq k_i \leq 4 \) and \( (x \cap Y_i) \setminus \{x_i\} \) consists of only one rational double point of type \( A_{k_i - 1} \).

(v) \( \hat{\mathbb{Y}} \cup B \) is a simple normal crossing divisor of twelve smooth rational curves whose weighted dual graph is not a linear tree. In particular, each irreducible component \( E_i \) of \( E \) is a smooth rational curve with \( E_i^2 = (E_i \cdot Z) - 2 \).
Proof. (i) By Lemma 3.3 (vi), we have $(\pi^\ast)^p(g^\ast g_\ast(\overline{\Gamma}^\ast)) = \tilde{\Gamma} + \sum_{i=1}^s (\pi^\ast)^p(k_i \overline{Y}_i^\ast)$.

By noting that $(g \circ \pi^\ast)|_{M \setminus (g \circ \pi^\ast)^{-1}(\text{Sing} V)} : M \setminus (g \circ \pi^\ast)^{-1}(\text{Sing} V) \to V \setminus \text{Sing} V$ is a composite of blowing-ups whose exceptional set is $(\pi^\ast)^{-1}(\overline{Y}^\ast) = (g \circ \pi^\ast)^{-1}(g(\overline{Y}^\ast))$ and that $(\pi^\ast)^p(g^\ast g_\ast(\overline{\Gamma}^\ast))$ is the total transform of $g_\ast(\overline{\Gamma}^\ast)$ by $g \circ \pi^\ast$, we obtain the assertion.

(ii) By noting (i) and that $\tilde{\Gamma} \cup E = (\pi^\ast)^{-1}(\overline{Y}^\ast) \cup \pi^{-1}(x_1)$, we obtain the assertions.

(iii) Assume that $\text{Sing} \overline{X}^\ast \subset \overline{Y}^\ast$. Then we have $\tilde{\Gamma} \cup E = (\pi^\ast)^{-1}(\overline{Y}^\ast) \cup \text{Supp} Z_1$. Thus we obtain $b_2(\tilde{\Gamma} \cup E) \leq 4 + 2 = 6$. This is a contradiction.

(iv) By noting (i) and that $\overline{X}^\ast \setminus \nu^{-1}(\overline{E}) \cong X \setminus \{x_1\}$, we obtain the assertion.

(v) By using $b_2(\pi^{-1}(x_1)) \geq 4$ and Proposition 3.5 in [8], we have that $\pi^{-1}(x_1)$ is a simple normal crossing divisor of smooth rational curves. By using $p_e(x_1) \neq 0$ and Satz 2.10 in Brieskorn [3], we see that the weighted dual graph of $\pi^{-1}(x_1)$ is not a linear tree. By Lemma 3.1 (vi) and $(\tilde{\Gamma} \cdot Z) = 1 (1 \leq i \leq t)$, we obtain the assertions.

Since $Z^2 = -2$, there exist the following three possibilities for the divisor $Z_1 = \sum_{i=1}^{n_1} a_i E_i$. From now on, we shall consider these cases separately:

1. $Z_1 = E_1$ with $(E_1 \cdot Z) = -2$.
2. $Z_1 = 2E_1$ with $(E_1 \cdot Z) = -1$.
3. $Z_1 = E_1 + E_2$ with $(E_1 \cdot Z) = (E_2 \cdot Z) = -1$.

**3.1. The case $Z_1 = E_1$ with $(E_1 \cdot Z) = -2$.**

**Proposition 3.5.** *This case does not occur.*

Proof. Assume that this case occurs. By Lemma 3.4 (v), we have that $E_1$ is a $(-4)$-curve in $M$. By Lemma 3.3 (iv) and Lemma 3.4 (i), (v), we see that $(\tilde{\Gamma} \cdot E_1) = 1$ and $((\pi^\ast)^p(k_i \overline{Y}_i^\ast) - k_i \tilde{\Gamma} \cdot E_1) = 0 (1 \leq i \leq t)$. In particular, $\overline{Y}^\ast \setminus \overline{E}_1$ consists of smooth points of $\overline{X}^\ast$. By contracting the curve $(\pi^\ast)^{-1}(\overline{Y}^\ast)$, we obtain a boundary of a minimal normal compactification of $\mathbb{C}^2$ which is a union of seven $(-2)$-curves and one 0-curves. However its weighted dual graph cannot be found in Morrow’s classification. This is a contradiction.

**3.2. The case $Z_1 = 2E_1$ with $(E_1 \cdot Z) = -1$.** By Lemma 3.4 (v), we have that $E_1$ is a $(-3)$-curve in $M$. By Lemma 3.1 (iii) and the isomorphism $\overline{\psi}(\overline{E}^\ast)$, we obtain $E_1 \cong \overline{E}^\ast \cong \overline{\psi}(\overline{E}) \cong \overline{E} \cong \mathbb{P}^1$. By Lemma 3.1 (iv) and Lemma 3.3 (ii), we also obtain $\overline{E}_1^\ast = \overline{\pi}^\ast(\text{Supp} Z)$ and $g(\overline{E}_1^\ast) = h^{-1}(\overline{\psi}(\overline{E}))$. 
**Proposition 3.6.** There exist the following two cases:

(i) \( \mathcal{Y} = 4Y_1 \) (\( Y_1 \): line) with \( x = [x_1] \).

(ii) \( \mathcal{Y} = 2Y_1 + 2Y_2 \) (\( Y_i \): line) with \( x = [x_1] = Y_1 \cap Y_2 \).

Moreover, for the cases (i) and (ii), the weighted dual graphs of \( \hat{Y} \cup E \) are of types (XV) and (XVI) in Theorem 1 respectively.

Proof. First we note that \( \hat{Y} \cup E \) is of simple normal crossing. Since \( a_1 = 2 \) and \( (\hat{Y}_i \cdot Z) = 1 \), we obtain \( (\hat{Y}_i \cdot E_i) = 0 \) (1 \( \leq i \leq t \)). In particular, \( \bar{Y}^* \cap \bar{E}_i^* \) consists of singular points of \( \bar{X}^* \). By Lemma 3.4 (i), we have that \( x = [x_1] \) and \( 2 \leq k_i \leq 4 \) (1 \( \leq i \leq t \)). Thus we obtain \( \mathcal{Y} = 4Y_1 \) or \( \mathcal{Y} = 2Y_1 + 2Y_2 \) (\( Y_i \): line). By Lemma 3.3 (iv), we obtain \( (\sum_{i=1}^{t}(\bar{Y}_i)^*k_i) \cdot E_1 = 2 \). By noting this, we see that the weighted dual graphs of \( (\bar{Y}_i)^{-1}(\bar{Y}^*) \cup E_1 \) are given as in Fig. 9 (a) and (b) for the cases \( \mathcal{Y} = 4Y_1 \) and \( \mathcal{Y} = 2Y_1 + 2Y_2 \) respectively. By contracting the curve \( (\bar{Y}_i)^{-1}(\bar{Y}^*) \), we have a boundary of a smooth compactification of \( \mathbb{C}^2 \) which is a simple normal crossing divisor of seven \((-2)\)-curves and one \((-1)\)-curve. By Lemma 2.9, its weighted dual graph is given as in Fig. 9 (c). Since the \((-1)\)-curve in Fig. 9 (c) is the proper transform of \( E_1 \), we obtain the assertions. \( \square \)

### 3.3. The case \( Z_1 = E_1 + E_2 \) with \( (E_1 \cdot Z) = (E_2 \cdot Z) = -1 \)

By Lemma 3.4 (v), we have that \( E_1 \) and \( E_2 \) are \((-3)\)-curves in \( M \). By Lemma 3.1 (iv), (v) and Lemma 3.3 (ii), we also have that \( \bar{E}_1^* \neq \bar{E}_2^* \), \( g(\bar{E}_1^*) \neq g(\bar{E}_2^*) \), \( \bar{E}_1^* \cup \bar{E}_2^* = \bar{X}^*(\text{Supp} Z) \) and \( g(\bar{E}_1^*) \cup g(\bar{E}_2^*) = h^{-1}(\bar{\psi}(\bar{E})) \). Since \( \text{Supp} Z \) is connected, both of \( \bar{E}_1^* \cup \bar{E}_2^* \) and \( g(\bar{E}_1^*) \cup g(\bar{E}_2^*) \) are also connected.

**Lemma 3.7.** One obtains the following:

(i) \( E_i \cong \bar{E}_i \cong g(\bar{E}_i^*) \cong \bar{\psi}(\bar{E}) \cong \bar{E} \cong \mathbb{P}^1 \) (\( i = 1, 2 \)).

(ii) Both of \( \bar{E}_1^* \cap \bar{E}_2^* \) and \( g(\bar{E}_1^*) \cap g(\bar{E}_2^*) \) consist of only one point.

(iii) \( \text{Sing } V = g(\bar{E}_1^*) \cap g(\bar{E}_2^*) \), \( \text{Sing } \bar{X}^* \setminus \bar{Y} = \bar{E}_1^* \cap \bar{E}_2^* \).

(iv) \( (\hat{Y}_i \cdot E_1 + E_2) = 1 \), \( ((\bar{Y}_i)^*k_1) - k_1\hat{Y}_i \cdot E_1 + E_2 = 0 \) (1 \( \leq i \leq t \)).

(v) \( \text{Sing } \bar{X}^* \cap (\bar{E}_1^* \cup \bar{E}_2^*) = \bar{E}_1^* \cap \bar{E}_2^* \).

(vi) \( (\bar{Y}_i)^{-1}(\bar{E}_1^* \cup \bar{E}_2^*) = \text{Supp}(Z - E_1 - E_2) \), \( b_2(\text{Supp}(Z - E_1 - E_2)) = 6 \).
(vii) There exist the following two cases:

(a) \( \mathcal{Y} = 2Y_1 + Y_2 + Y_3 \) \((Y_i:\ \text{line})\), \(x = x \cap Y_1 = \{x_1, A_1\}, \{x_1\} \cap Y_2 \cap Y_3\).

(b) \( \mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4 \) \((Y_i:\ \text{line})\), \(x = \{x_1\} \cap Y_2 \cap Y_3 \cap Y_4\).

Moreover, for the cases (a) and (b), the weighted dual graphs of \((\pi^r)^{-1}(\mathcal{Y}) \cup E_1 \cup E_2\) are given as in Fig. 10 (a) and (b) respectively.

(viii) \(E_1 \cap E_2 = \emptyset\). Moreover, there exists no irreducible component of \(\text{Supp}(Z - E_1 - E_2)\) intersecting both of \(E_1\) and \(E_2\).

Proof. (i) By Lemma 3.1 (iii) and the isomorphism \(\overline{\psi}|_{\overline{\mathcal{X}}}\), we obtain the assertion.

(ii) First we note that \((\overline{\mathcal{X}}, \overline{\mathcal{Y}} \cup E_1 \cup E_2)\) and \((V, g(\overline{E_1}) \cup g(\overline{E_2}))\) are compactifications of \(\mathbb{C}^2\) and that \(\overline{\mathcal{X}}\) and \(V\) are normal. By Proposition 1 (ii) in [10], we obtain the assertion.

(iii) By Lemma 3.4 (iii), we have that \(\text{Sing} V \neq \emptyset\). Since \(h^*(\overline{\psi}(\overline{E})) = g(\overline{E_1}) + g(\overline{E_2})\) and \(\sum_{q \in h^{-1}(p)} \text{mult}_q V \leq \deg h = 2\) for any point \(p\) of \(\mathbb{P}^2\), we obtain the first assertion. By Lemma 3.2 (vii), we also obtain the second assertion.

(iv) By noting (iii), Lemma 3.3 (iv) and Lemma 3.4 (i), (v), we obtain the assertions.

(v) By using (iii) and (iv), we obtain the assertion.

(vi) By using (v) and \(b_3((\pi^r)^{-1}(\mathcal{Y})) = 4\), we obtain the assertions.

(vii) By Lemma 3.3 (iv), we obtain \(\left(\sum_{j=1}^{2} (\pi^r)^j(k_j \overline{Y_j}) \cdot E_j\right) = 2 \ (j = 1, 2)\). By noting this and (iv), we have that \(\mathcal{Y} = 2Y_1 + 2Y_2, 2Y_1 + Y_2 + Y_3\) or \(Y_1 + Y_2 + Y_3 + Y_4\) \((Y_i:\ \text{line})\). Now we may assume that \(Y_i\) and \(E_i \setminus \text{Sing}(\text{Supp} Z)\) meet transversally at only one point, which is denoted by \(p_i\), for each \(i = 1, 2\). Let \(L\) be a line in \(\mathbb{P}^2\) such that \((h \circ g \circ \pi^r)(p_1) \in L \neq \overline{\psi}(\overline{E})\).

Since \(h^*(\overline{\psi}(\overline{E})) = g(\overline{E_1}) + g(\overline{E_2})\), we see that the divisor \((\pi^r)^*g^*h^*L\) intersects \(Z\) transversally at only two points \(p_1\) and \(q_2\), where \(q_2\) is a point of \(E_2 \setminus \text{Sing}(\text{Supp} Z)\). By noting this and Lemma 3.1 (iii), we obtain

\[ O_Z(Z) \cong O_Z((-\tilde{\Gamma} - Z)) \cong O_Z(-((\pi^r)^*g^*h^*L)) \cong O_Z(-p_1 - q_2). \]

By Lemma 3.3 (iv), we obtain \(O_Z(2Z) \cong O_Z(-2p_1 - 2p_2)\). Thus we obtain \(O_Z(2(q_2 - p_2)) \cong O_Z\). By Lemma 2.6 (iv), we see that \(q_2 = p_2\). Hence we obtain

\[ (h \circ g \circ \pi^r)(p_1) = (h \circ g \circ \pi^r)(q_2) = (h \circ g \circ \pi^r)(p_2). \]
By noting that \((h \circ g \circ \pi^\omega)(p_i) = \overline{\psi}(Y_i)\) \((i = 1, 2)\), we have \(\overline{\psi}(Y_1) = \overline{\psi}(Y_2)\). On the other hand, we have \(\overline{\psi}(Y_1) \neq \overline{\psi}(Y_2)\) since \(Y_1 \) and \(Y_2\) are distinct two lines through \(x_1\). This is a contradiction. Thus we obtain the assertions.

(viii) Since \(E_1^\omega \cap E_2^\omega\) consists of only one point and \(\text{Supp} \ Z\) is of simple normal crossing, we have \(E_1 \cap E_2 = \emptyset\). Next we assume that there exists an irreducible component of \(\text{Supp}(Z - E_1 - E_2)\) intersecting both of \(E_1\) and \(E_2\). By contracting the curve \((\pi^\omega)^{-1}(\overline{Y}) \cup E_1 \cup E_2\), we obtain a boundary of a minimal normal compactification of \(\mathbb{C}^2\) which is a union of five \((-2)\)-curves and one \(0\)-curve. However its weighted dual graph cannot be found in Morrow’s classification. This is a contradiction. Thus we obtain the assertions.

Now we have that \(E_1^\omega \cap E_2^\omega\) is a rational double point of \(\overline{X}\) of type \(A_6, D_6\) or \(E_6\) by Lemma 3.7 (vi). Let \(W\) be the fundamental cycle of \((\pi^\omega)^{-1}(E_1^\omega \cap E_2^\omega)\). Then we note that \((\pi^\omega)^{-1}(E_1^\omega \cap E_2^\omega) = \text{Supp}(Z - E_1 - E_2) = \text{Supp} \ W = \bigcup_{i=3}^{8} \text{Supp}(E_i)\) and \(Z = E_1 + E_2 + \sum_{i=3}^{8} a_i E_i\) with \(a_3, \ldots, a_8 \in \mathbb{N}\). By Lemmas 2 and 3 in [10], we also note that \((E_1 \cdot W) = (E_2 \cdot W) = 1\). Since \((Z \cdot E_i) = 0 \ (3 \leq i \leq 8)\), we obtain \((Z \cdot W) = 0\).

**Proposition 3.8.** Assume that \(E_1^\omega \cap E_2^\omega\) is of type \(A_6\). Then the weighted dual graph of \(\hat{Y} \cup E\) is of type (XVII) or (XVIII) in Theorem 1, for the case \(Y = 2Y_1 + Y_2 + Y_3\) or \(Y = Y_1 + Y_2 + Y_3 + Y_4\) (\(Y_i\): line) respectively.

**Proof.** Assume that \(E_1^\omega \cap E_2^\omega\) is of type \(A_6\). Then the weighted dual graph of \(\text{Supp} \ W\) is given as in Fig. 11 (a), where the integers adjacent to vertices are coefficients in \(W\). We note that \((E_3 \cdot W) = (E_8 \cdot W) = -1\) and \((E_i \cdot W) = 0 \ (4 \leq i \leq 7)\). By computing the intersection number \((Z \cdot W)\), we have

\[ 0 = (Z \cdot W) = (E_1 \cdot W) + (E_2 \cdot W) + \sum_{i=3}^{8} a_i (E_i \cdot W) = 2 - a_3 - a_8. \]

Thus we obtain \(a_3 = a_8 = 1\). Moreover, by computing the intersection numbers \((Z \cdot E_1)\) and \((Z \cdot E_2)\), we obtain \((\sum_{i=3}^{8} a_i E_i \cdot E_1) = (\sum_{i=3}^{8} a_i E_i \cdot E_2) = 2\). By noting that \(\text{Supp} \ Z\) is of simple normal crossing, we see that \(E_1 \cap (E_3 \cup E_8) = E_2 \cap (E_3 \cup E_8) = \emptyset\). Here we note Lemma 3.7 (viii). By contracting the curve \((\pi^\omega)^{-1}(\overline{Y}) \cup E_1 \cup E_2\) and suitable curves in \(\hat{Y} \cup E\), we have a boundary of a minimal normal compactification of \(\mathbb{C}^2\). Since its weighted dual graph must be found in Morrow’s classification, the weighted dual graph of \(\text{Supp} \ Z\) is uniquely determined as in Fig. 11 (b). By Lemma 3.7 (vii), we obtain the assertion.

**Proposition 3.9.** It does not occur the case where \(E_1^\omega \cap E_2^\omega\) is of type \(D_6\).

**Proof.** Assume that \(E_1^\omega \cap E_2^\omega\) is of type \(D_6\). Then the weighted dual graph of \(\text{Supp} \ W\) is given as in Fig. 12 (a), where the integers adjacent to vertices are co-
coefficients in $W$. We note that $(E_1 \cdot W) = (E_2 \cdot W) = 1$. By contracting the curve $(\overline{\pi^*})^{-1}(\overline{Y^*}) \cup E_1 \cup E_2$ and suitable curves in $\hat{Y} \cup E$, we have a boundary of a minimal normal compactification of $\mathbb{C}^2$. Since its weighted dual graph must be found in Morrow’s classification, the weighted dual graph of $\text{Supp} \ Z$ is uniquely determined as in Fig. 12 (b). We note that $(E_6 \cdot W) = -1$ and $(E_i \cdot W) = 0$ ($i = 3, 4, 5, 7, 8$). By computing the intersection numbers $(Z \cdot W)$ and $(Z \cdot E_7)$, we have $0 = (Z \cdot W) = 2 - a_6$ and $0 = (Z \cdot E_7) = a_6 - 2a_7 + 1$. Thus we obtain $a_7 = 3/2 \not\in \mathbb{N}$. This is a contradiction.

**Proposition 3.10.** Assume that $\overline{E_1^w} \cap \overline{E_2^w}$ is of type $E_6$. Then the weighted dual graph of $\hat{Y} \cup E$ is of type (XIX) or (XX) in Theorem 1, for the case $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ or $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ ($Y_i$: line) respectively.

Proof. Assume that $\overline{E_1^w} \cap \overline{E_2^w}$ is of type $E_6$. Then the weighted dual graph of $\text{Supp} \ W$ is given as in Fig. 13 (a), where the integers adjacent to vertices are coefficients in $W$. By Lemma 3.7 (viii) and $(E_1 \cdot W) = (E_2 \cdot W) = 1$, the weighted dual graph of $\text{Supp} \ Z$ is uniquely determined as in Fig. 13 (b). By Lemma 3.7 (vii), we obtain the assertion.

Thus we complete the proof of Theorem 1 for the case $Z^2 = -2$.

**4. Proof of Theorem 1 for $Z^2 = -1$**

In this section, we shall prove Theorem 1 for the case $Z^2 = -1$. Let $(X, Y)$ be a pair satisfying Assumption in §1 and $Z^2 = -1$. We use the same notation as that in §1 and §2. We mainly consider a projection from $x_1$ and a blowing-up at $x_1$ to investigate the pair $(X, Y)$. First we note that $\hat{Y} \cup E$ is a connected divisor without
cycles which consists of eleven rational curves. Since $Z^2 = -1$, we may assume that $E_1$ is a unique irreducible component of $Z = \sum_{i=1}^{11} a_i E_i$ such that $(E_1 \cdot Z) = -1$ and $a_1 = 1$. Then we have that $(E_i \cdot Z) = 0$ for any irreducible component $E_i$ of $Z - E_1$. By Lemma 2.3 (ii), there exists a unique point $p_0$ of $E_1 \setminus \text{Sing}(\text{Supp } Z)$ such that $(\pi \circ \pi_0)^* m_{X,x_1} \cong O_M(-Z' - 2E'_0)$, where $\pi_0: M' \to M$ is a blowing-up at $p_0$ with exceptional curve $E'_0$ and $Z'$ is the proper transform of $Z$ by $\pi_0$. We put $\pi' := \pi \circ \pi_0$.

Let $\hat{C}'$ be the proper transform of a curve $C$ in $X$ by $\pi'$. Let $E'_{i}$ and $E_{0}'$ be the proper transforms of $E_i$ and $E_0$ respectively for $i \geq 1$. We note that $\pi'_0 Z = Z' + E'_{0}$. Supp $\pi'_0 Z = (\pi')^{-1}(x_1)$, $\text{Exc } \pi' = Z' \cup E'_0$, $K_{M'} \sim -Z'$ and $Z'^2 = -2$. Let $\sigma: \mathbb{P}^3 \to \mathbb{P}^3$ be the blowing-up at $x_1$ with exceptional divisor $\Delta$, which is isomorphic to $\mathbb{P}^2$.

Let $\bar{T}$ be the proper transform of a closed algebraic subset $T$ of $\mathbb{P}^3$ by $\sigma$. We have that $\sigma|_{\mathbb{P}^3 \setminus \Delta}: \mathbb{P}^3 \setminus \Delta \cong \mathbb{P}^3 \setminus \{x_1\}$ and $O_{\mathbb{P}^3}(\Delta)|_{\Delta} \cong O_{\mathbb{P}^2}(-1)$. We set $\bar{E} := \bar{X} \cap \Delta$. We have that $\sigma|_{\bar{X} \setminus \bar{E}}: \bar{X} \setminus \bar{E} \cong X \setminus \{x_1\}$ and that $(\bar{X}, \bar{Y}, \bar{E})$ is a compactification of $C^2$ with dualizing sheaf $\omega_{\bar{X}} \cong O_{\bar{X}}$. Let $\psi: \mathbb{P}^3 \to \mathbb{P}^2$ be the projection from $x_1$ and $\bar{\psi}: \mathbb{P}^3 \to \mathbb{P}^2$ the resolution of indeterminacy of $\psi$. We have that $\bar{\psi}|_{\Delta}: \Delta \to \mathbb{P}^2$ is an isomorphism and $\bar{\psi}|_{\bar{X}}: \bar{X} \to \mathbb{P}^2$ is a generically finite morphism of degree two. We note that $\bar{Y} \sim \bar{H}|_{\bar{X}} + \Delta|_{\bar{X}}$ and $\hat{f} \sim \sum_{i=1} \hat{k}_i E_i + \sum_{j=1} b_j E_j$ with $b_j \in \mathbb{N}$. Then we have some fundamental lemmas.

**Lemma 4.1.** One obtains the following:

(i) $\bar{X}$ is normal. Moreover, $\bar{X}|_{\Delta} = 2E = 2\text{line}$ and $E \cap \text{Sing } \bar{X}$ consists of only one point, which is denoted by $\bar{x}_1$.

(ii) $\text{Sing } \bar{X}$ consists of exactly one minimally elliptic singular point $\bar{x}_1$ and at most rational double points.

(iii) There exists a birational morphism $\bar{\pi} : M' \to \bar{X}$ satisfying $(\sigma|_{\bar{X}}) \circ \bar{\pi} = \pi'$. Then $(\bar{\pi}')^*(\Delta|_{\bar{X}}) = Z' + 2E'_0$, $(\bar{\pi}')^*(\bar{\psi}|_{\bar{X}}) O_{\mathbb{P}^2}(1) \cong O_{M'}(\hat{f}' - Z' - 2E'_0)$, $\bar{\pi}'|_{E'_0}: E'_0 \cong \bar{E}$ and $\bar{\pi}'(\text{Supp } Z') = \{\bar{x}_1\}$. Moreover, $\bar{\pi}'$ is a minimal resolution of $\bar{X}$ with $\text{Exc } \bar{\pi}' = (\bar{\pi}')^{-1}(\bar{x}_1) \cup (\bar{\pi}')^{-1}(x \setminus \{x_1\})$.

(iv) $Z'$ is the fundamental cycle of $(\bar{\pi}')^{-1}(\bar{x}_1)$ with $Z'^2 = -2$. Moreover, $\bar{x}_1$ is a minimally elliptic double point of $\bar{X}$ and $(\bar{\pi}')^* m_{\bar{X}, \bar{x}_1} \cong O_M(-Z')$. 

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Fig. 13.
There exists exactly one line $l_1$ in $X$ through $x_1$. Then $x_1 \in l_1 \subset Y$ and $\overline{\mathbb{X}}_1 \in \overline{\mathbb{Y}}_1 \subset \overline{\mathbb{Y}}$. Moreover, $\hat{l}_1$ is a $(−1)$-curve in $M'$ with $(\hat{l}_1 \cdot Z') = 1$ and $(\hat{l}_1 \cdot E'_0) = 0$ and $\hat{l}_1$ is a $(−1)$-curve in $M$ with $(\hat{l}_1 \cdot Z) = 1$ and $p_0 \notin \hat{l}_1$.

Let $\overline{\mathbb{X}} \to V \to \mathbb{P}^2$ be the Stein factorization of $\overline{\mathbb{Y}}|_{\overline{\mathbb{X}}}$. Then $V$ is normal, $g$ is a birational morphism and $h$ is a finite morphism of degree two. In particular, $g|_{\overline{\mathbb{X}}, \hat{l}_1} \colon \overline{\mathbb{X}} \setminus \hat{l}_1 \cong V \setminus g(\hat{l}_1)$ with $\text{Exc } g = \overline{\mathbb{I}}_1 = g^{-1}(g(\overline{\mathbb{X}}_1))$ and $(V, g(\overline{\mathbb{Y}} \cup \overline{\mathbb{E}}))$ is a compactification of $\mathbb{C}^2$. Thus one obtains the commutative diagram as in Fig. 14.

$V$ is a projective normal Gorenstein surface with dualizing sheaf $\omega_V \cong \mathcal{O}_V$. Moreover, $\text{Sing } V$ consists of exactly one minimally elliptic double point $g(\overline{\mathbb{X}}_1)$ and at most rational double points.

**Proof.** (i) The assertions are the general properties of the minimally elliptic singularity $(X, x_1)$ with $Z^2 = −1$. Indeed, we can check the assertions by applying a blowing-up of the local analytic defining equation of $(X, x_1)$ in Theorem 4.57 (3) of [7] (cf. [8]).

(ii) By using $K_{\overline{\mathbb{X}}} \sim 0$ and Proposition 1 (vi) in [10], we obtain $p_\phi(\text{Sing } \overline{\mathbb{X}}) = 1$. Since $\sigma|_{\overline{\mathbb{X}}, \hat{l}_1} : \overline{\mathbb{X}} \setminus \hat{l}_1 \cong X \setminus \{x_1\}$ and $\mathbb{E} \cap \text{Sing } \overline{\mathbb{X}} = \{\overline{\mathbb{X}}_1\}$, we obtain the assertion.

(iii) There exists a birational morphism $\overline{\mathbb{I}}' : M' \to \overline{\mathbb{X}}$ satisfying $(\sigma|_{\overline{\mathbb{X}}}) \circ \overline{\mathbb{I}}' = \pi'$ by Lemma 2.3 (ii). In particular, we obtain $(\overline{\mathbb{I}}')^{-1}(\mathbb{E}) = \text{Supp}(Z' + 2E'_0)$. By the isomorphisms

$$(\overline{\mathbb{I}}')^*(\mathcal{O}_{\overline{\mathbb{X}}}(-\Delta|_{\overline{\mathbb{X}}}) \cong (\overline{\mathbb{I}}')^*(\sigma|_{\overline{\mathbb{X}}})^*\mathcal{O}_{X, x_1} \cong \mathcal{O}_{M'}(-Z' - 2E'_0),$$

we obtain $(\overline{\mathbb{I}}')^*(\Delta|_{\overline{\mathbb{X}}}) \sim Z' + 2E'_0$. Since $(\overline{\mathbb{I}}')^*(\Delta|_{\overline{\mathbb{X}}})$ is an effective divisor of $M'$ whose support equals to $\text{Supp}(Z' + 2E'_0)$ and the intersection matrix of $\text{Supp}(Z' + 2E'_0)$ is negative definite, we obtain $(\overline{\mathbb{I}}')^*(\Delta|_{\overline{\mathbb{X}}}) = Z' + 2E'_0$. In particular, we have

$$(\overline{\mathbb{I}}')^*(\overline{\psi}|_{\overline{\mathbb{X}}})^*\mathcal{O}_{\mathbb{E}^c}(1) \cong (\overline{\mathbb{I}}')^*\mathcal{O}_{\overline{\mathbb{X}}}(\overline{\mathbb{I}}|_{\overline{\mathbb{X}}}) \cong (\overline{\mathbb{I}}')^*\mathcal{O}_{\overline{\mathbb{X}}}((\mathbb{I}')^* - \Delta|_{\overline{\mathbb{X}}}) \cong \mathcal{O}_{\overline{\mathbb{I}}'}((\mathbb{I}')^* - Z' - 2E'_0).$$

Let $E'_i$ be any irreducible component of $(\mathbb{I}')^{-1}(x_1)$ for $i \geq 0$. Since $(\overline{\mathbb{I}}')^*|_{E'_i}$ is identified with $(\overline{\psi}|_{\overline{\mathbb{X}}}) \circ (\overline{\mathbb{I}}')^*|_{E'_i}$, we obtain $\deg(\overline{\mathbb{I}}')^*|_{E'_i} = -(Z' + 2E'_0 \cdot E'_i)_{\mathbb{M}'}$. Thus we see that $\overline{\mathbb{I}}' : E'_0 \cong \mathbb{E}$ and $\overline{\mathbb{I}}'(\text{Supp } Z') = \{\overline{\mathbb{X}}_1\}$. Here we note that $\text{Exc } \overline{\mathbb{I}}' = (\overline{\mathbb{I}}')^{-1}(\mathbb{X} \setminus \{x_1\})$. Since $\text{Supp } Z = \pi^{-1}(x_1)$ and $p_0 \in E_1 \setminus \text{Sing}(\text{Supp } Z)$, there exist no $(-1)$-curves in $\text{Supp } Z' = (\overline{\mathbb{I}}')^{-1}(\mathbb{X}_1)$. Thus $\overline{\mathbb{I}}'$ is a minimal resolution of $\overline{\mathbb{X}}$. 

![Diagram](https://via.placeholder.com/150)
(iv) Note that $\pi'$ gives a minimal resolution of the minimally elliptic singularity $(\overline{X}, \overline{\tau}_1)$ and $K_{M'} \sim -Z'$. By using Theorem 3.4 in [8], we have that $Z'$ is the fundamental cycle of $(\pi')^{-1}(\overline{\tau}_1)$. By using $Z'^2 = -2$ and Theorem 3.13 in [8], we see that $\text{mult}_{\overline{\pi}' \overline{X}} = 2$ and $(\pi')^*m_{\overline{X}, \overline{\tau}_1} \cong \mathcal{O}_{M'}(-Z')$.

(v) There exists at least a line $l$ in $X$ through $x_1$ by Proposition 2.15. By Lemma 2.3 (ii), we obtain $(\hat{l} \cdot Z + 2E_0) = 1$. Hence $\hat{l}$ is a $(-1)$-curve in $M'$ with $(\hat{l} \cdot Z') = 1$ and $(\hat{l} \cdot E'_0) = 0$. In particular, $\hat{l}$ is a $(-1)$-curve in $M$ with $(\hat{l} \cdot Z) = 1$ and $p_0 \notin \hat{l}$. Since $(\hat{l} \cdot Z') = 1$, we see that $\hat{l}$ passes through $\overline{x}_1$ necessarily. From this, the existence of $l$ is unique. Thus we put $l_1 := l$. By Proposition 2.15 again, we obtain $l_1 \subset Y$ and in particular $\overline{l}_1 \subset \overline{Y}$.

(vi) Note the general properties of Stein factorization (cf. Corollary III.11.5 in [5]).

(vii) Since $\deg g = 2$, $\text{Sing} V$ consists of at most double points. Note that any double point is a hypersurface singularity and in particular a Gorenstein singularity. Thus $V$ is Gorenstein. Since $g|_{\overline{X} \backslash \overline{l}_1}: \overline{X} \backslash \overline{l}_1 \cong V \backslash g(\overline{\tau}_1)$ and $K_{\overline{X}} \sim 0$, we obtain $K_V \sim 0$. By applying Proposition 1 (vi) in [10], we obtain $p_g(\text{Sing} V) = 1$. Hence we obtain the assertions. □

**Remark.** The branch locus $B$ of $h$ is a reduced plane sextic curve. Indeed, this is showed as follows. First we note that $\text{Pic}(V)$ is torsion-free by Lemma 4.1, and Proposition 1 in [10]. Thus we obtain the injectivity of $h^*: \text{Pic}(\mathbb{P}^2) \cong \mathbb{Z} \rightarrow \text{Pic}(V)$. Let $R$ be the ramification divisor of $h$. Since $\deg h = 2$, we have $K_V \sim h^*K_{\mathbb{P}^2} + R$ and $h^*B = 2R$. By noting that $K_V \sim 0$, we obtain $h^*(B - 6L) \sim 0$ and hence $B - 6L \sim 0$, where $L$ is a line in $\mathbb{P}^2$. In the following, we omit the investigation of a detailed structure of $B$ since there is no necessity in our arguments.

**Lemma 4.2.** One obtains the following:

(i) $x = [x_1]$, $\text{Sing} \overline{X} = [\overline{x}_1]$, $1 \leq b_2(Y) \leq 2$, $b_2(E) = 11 - b_2(Y)$. In particular, $\text{Sing} V = [g(\overline{x}_1)]$.

(ii) $E$ is a simple normal crossing divisor of one $(-3)$-curve $E_1$ and some $(-2)$-curves whose weighted dual graph is not a linear tree.

(iii) $\hat{l}_1$ is a $(-1)$-curve in $M$ with $(\hat{l}_1 \cdot E_1) = 1$, $(\hat{l}_1 \cdot Z - E_1) = 0$ and $p_0 \notin \hat{l}_1$.

(iv) $\hat{l}_1 \cap \text{Supp} Z$ is a smooth point of $Z$, which is denoted by $p_1$. In particular, $p_0 \neq p_1 \in E_1 \setminus \text{Sing}(\text{Supp} Z)$.

(v) $\mathcal{O}_{\mathbb{Z}}(Z) \cong \mathcal{O}_{\mathbb{Z}}(-p_0)$.

**Proof.** (i) First we show that $(\hat{l}_1 \cap \text{Sing} \overline{X}) \setminus [\overline{x}_1] = \emptyset$. Assume that $(\hat{l}_1 \cap \text{Sing} \overline{X}) \setminus [\overline{x}_1] \neq \emptyset$. By Lemma 2.5 (i) and Lemma 4.1 (ii), there exists an irreducible component $E'_1$ of $(\pi')^{-1}(\hat{l}_1 \cap \text{Sing} \overline{X}) \setminus [\overline{x}_1]$ which is a $(-2)$-curve in $M'$ with $(\hat{l}' \cdot E'_1) = 1$. By using $Z'^2 = -2$ and Lemma 4.1 (v), we obtain $(Z' + E'_1 + 2\hat{l}')^2 = 0$ directly. On the other hand, the intersection matrix of $(g \circ \pi')^{-1}(g(\overline{l}_1)) = \hat{l}_1 \cup \text{Supp} Z' \cup (\pi')^{-1}(\hat{l}_1 \cap \text{Sing} \overline{X}) \setminus [\overline{x}_1]$ is negative definite. This is a contradiction. Thus we have that $\hat{l}_1 \cap \text{Sing} \overline{X} = [\overline{x}_1]$ and
in particular \( x \cap l = \{ x_1 \} \). By noting Proposition 2.15 and Lemma 4.1 (v), we obtain \( 1 \leq b_2(Y) \leq 2 \) and \( x = \{ x_1 \} \). Immediately, we also obtain the other assertions.

(ii) By using \( b_2(E) \geq 4 \) and Proposition 3.5 in [8], we have that \( E \) is a simple normal crossing divisor of smooth rational curves. By using \( p_\xi(x_1) \neq 0 \) and Satz 2.10 in [3], we see that the weighted dual graph of \( E \) is not a linear tree. By the adjunction formula, we obtain the assertion.

(iii) Note that \( \hat{I}_1 \) is a \((-1)\)-curve in \( M' \) with \((\hat{I}_1 \cdot Z') = 1 \) and \((\hat{I}_1 \cdot E'_0) = 0 \) by Lemma 4.1 (v). Since \((\hat{I}_1 \cdot Z') = 1 \), there exists a unique irreducible component \( E'_i \) of \( Z' \) such that \((\hat{I}_1 \cdot E'_i) = 1 \) and \((\hat{I}_1 \cdot Z' - E'_i) = 0 \). Since the intersection matrix of \((g \circ \pi')^{-1}(g(\hat{l}_1)) = \hat{I}_1 \cup \text{Supp} Z' \) is negative definite, we obtain \( 3(E'_i)^2 + 6 = (Z' + E'_i + 2\hat{I}_1)^2 < 0 \) and thus \((E'_i)^2 \leq -3 \). Since \( p_0 \in E_1 \setminus \text{Sing}(\text{Supp } Z) \), we have \( E'_i = E'_1 \). By noting that \((\hat{I}_1 \cdot E'_0) = 0 \), we obtain the assertion.

(iv) By noting (iii), we obtain the assertions.

(v) Let \( L \) be a line in \( \mathbb{P}^2 \) such that \((h \circ g)(x_1) = \overline{\psi}(x_1) \neq L \). Then we have \( \text{Supp}(h \circ g \circ \pi')^*L \cap \text{Supp } Z' = \emptyset \). By the projection formula and \((h \circ g \circ \pi')|_{E'_0} : E'_0 \cong \overline{\psi}(E) \), we also have \((h \circ g \circ \pi')^*L \cdot E'_0 = (L \cdot \overline{\psi}(E))_{\mathbb{P}^2} = 1 \). Hence \((\pi_0)_*(h \circ g \circ \pi')^*L \) intersects \( Z \) transversally at only one point \( p_0 \), which is a smooth point of \( Z \). By Lemma 4.1 (iii), we obtain \( \hat{I} - Z \sim (\pi_0)_*(h \circ g \circ \pi')^*L \). By restricting this relation to \( Z \), we obtain the assertion.

\[ \mathcal{Y} = 2Y_1 + Y_2 \text{ (\( Y_1 \): line, \( Y_2 \): conic) with } x = Y_1 \cap Y_2 = \{ x_1 \}. \]

In this case, one has that \( Y_1 = l_1, \overline{\psi}(H) = \overline{\psi}(E) \) and \( H|_{X} = 2Y_1 + Y_2 + E \). Moreover, one has that \( g(Y_2) \neq g(E), g(Y_2) + g(E) = h^*(\overline{\psi}(E)) \) and \( Y_2 \cong g(Y_2) \cong \overline{\psi}(E) \cong g(E) \cong E \cong \mathbb{P}^1 \).

Proof. By Proposition 2.15, Lemma 4.1 (v) and Lemma 4.2 (i), there exist the following four possibilities:

1. \( \mathcal{Y} = 4Y_1 \text{ (\( Y_1 \): line) with } x = x \cap Y_1 = \{ x_1 \}. \)
2. \( \mathcal{Y} = 3Y_1 + Y_2 \text{ (\( Y_1 \), \( Y_2 \): line) with } x = x \cap (Y_1 \setminus Y_2) = \{ x_1 \}. \)
3. \( \mathcal{Y} = 2Y_1 + Y_2 \text{ (\( Y_1 \): line, \( Y_2 \): conic) with } x = Y_1 \cap Y_2 = \{ x_1 \}. \)
4. \( \mathcal{Y} = Y_1 + Y_2 \text{ (\( Y_1 \): line, \( Y_2 \): cusploid cubic) with } x = Y_1 \cap Y_2 = \text{Sing } Y_2 = \{ x_1 \}. \)

For each case, we note that \( Y_1 = l_1, x_i \in Y_1 \) and \( Y_i \cong \mathbb{P}^1 \) for any \( i \). By noting the position of \( x_1 \) in \( Y \) and that \( \overline{\psi}|_{X} \) is a generically finite morphism of degree two, we obtain \( \overline{\psi}(H) = \overline{\psi}(E) \). First we consider the case (3). In this case, we obtain \( H|_{X} = 2Y_1 + Y_2 + E \) since \( \overline{\psi}(H) = \overline{\psi}(E) \). By noting Lemma 4.1 (vi) and that \( \overline{\psi}|_{E} : E \cong \overline{\psi}(E) \), we have that \( g(Y_2) \neq g(E), g(Y_2) + g(E) = h^*(\overline{\psi}(E)) \) and \( Y_2 \cong g(Y_2) \cong \overline{\psi}(E) \cong g(E) \cong E \cong \mathbb{P}^1 \). Next we show that the cases (1), (2) and (4) do not occur.

1. Assume that the case (1) occurs. Since \( \overline{\psi}(H) = \overline{\psi}(E) \), we obtain \( H|_{X} = 4Y_1 + 2E \). By Lemma 4.1 (iii), there exists an effective divisor \( D \) of \( M \) such that \( \text{Supp } D = \)
Supp $Z$ and $\hat{\Gamma} - Z \sim 4\hat{Y}_1 + D$. From this, we obtain $(D \cdot E_i) = (3Z \cdot E_i)$ for each irreducible component $E_i$ of $Z$. Since the intersection matrix of Supp $Z = \pi^{-1}(x_1)$ is negative definite, we have $D = 3Z$ and thus $\hat{\Gamma} \sim 4\hat{Y}_1 + 4Z$. In particular, we have $O_Z(4Z) \cong O_Z(-4p_1)$. By Lemma 4.2 (v), we also have $O_Z(4(p_0 - p_1)) \cong O_Z$. By Lemma 2.6 (iv), we obtain $p_0 = p_1$. This is a contradiction.

(2) Assume that the case (2) occurs. Since $\overline{\psi(\hat{H})} = \overline{\psi(\hat{E})}$, we obtain $\overline{H}_{\overline{\pi}} = 3\overline{Y}_1 + \overline{Y}_2 + \overline{E}$. By Lemma 4.1 (iii), there exists an effective divisor $D$ of $M$ such that Supp $D = \text{Supp} Z$ and $\hat{\Gamma} - Z \sim 3\hat{Y}_1 + \hat{Y}_2 + D$. From this, we obtain $(D \cdot E_i) = (2Z \cdot E_i)$ for each irreducible component $E_i$ of $Z$. Since the intersection matrix of Supp $Z = \pi^{-1}(x_1)$ is negative definite, we have $D = 2Z$ and thus $\hat{\Gamma} \sim 3\hat{Y}_1 + \hat{Y}_2 + 3Z$. In particular, we have $O_Z(3Z) \cong O_Z(3(p_0 - p_1)) \cong O_Z$. By Lemma 4.2 (v), we have $O_Z(3(p_0 - p_1)) \cong O_Z$. By Lemma 2.6 (iv), we obtain $p_0 = p_1$. This is a contradiction.

(4) Assume that the case (4) occurs. Note that $\overline{Y}_1$ and $\overline{Y}_2$ meet at only one point $\overline{x}_i$ transversally. In particular, $\overline{Y}_2$ is smooth. By using $(\overline{\pi})^*m_{Y_2,E} \cong O_{M'}(-Z')$ and Lemma 3 in [10], we obtain $(\hat{Y}_2 \cdot Z') = 1$. On the other hand, we obtain $(\hat{Y}_2 \cdot Z') = 0, 2$ by Lemma 2.3 (ii). This is a contradiction.

**Lemma 4.4.** One obtains the following:

(i) $\hat{Y}_2$ is a $(-1)$-curve in $M$ with $(\hat{Y}_2 \cdot Z) = 1$, $(\hat{Y}_1 \cdot \hat{Y}_2) = 0$ and $p_0 \notin \hat{Y}_2$.

(ii) $\hat{Y}_2 \cap \text{Supp} Z$ is a smooth point of $Z$, which is denoted by $p_2$, and $p_2 \neq p_0, p_1$.

(iii) $\hat{Y} \cup E$ is a simple normal crossing divisor of two $(-1)$-curves $\hat{Y}_1$, $\hat{Y}_2$, one $(-3)$-curve $E_1$ and eight $(-2)$-curves whose weighted dual graph is not a linear tree.

(iv) $\hat{\Gamma} \sim 2\hat{Y}_1 + \hat{Y}_2 + 3Z$. In particular, $O_Z(3Z) \cong O_Z(2p_1 - p_2)$. $O_Z(3p_0 - 2p_1 - p_2) \cong O_Z$. In particular, $p_2 \in E_1 \setminus \text{Sing(Supp} Z)$.

Proof. (i) Note that each pair of $\overline{Y}_1$, $\overline{Y}_2$ and $\overline{E}$ meet transversally at only one point $\overline{x}_i$ and that the blowing-up morphism at $\overline{x}_i$ of $\overline{X}$ factors $\overline{\pi}'$ by Lemma 4.1 (iv) and Proposition II.7.14 in [5]. From these, we have $(\hat{Y}_1 \cdot \hat{Y}_2) = (\hat{Y}_1 \cdot E_0') = (\hat{Y}_2 \cdot E_0') = 0$. Thus we obtain the assertion.

(ii) By using (i), we obtain the assertion.

(iii) By (i), (ii) and Lemma 4.2 (ii), (iii), we obtain the assertion.

(iv) Note that $\overline{H}_{\overline{\pi}} = 2\overline{Y}_1 + \overline{Y}_2 + \overline{E}$. By Lemma 4.1 (iii), there exists an effective divisor $D$ of $M$ such that Supp $D = \text{Supp} Z$ and $\hat{\Gamma} - Z \sim 2\hat{Y}_1 + \hat{Y}_2 + D$. From this, we have $(D \cdot E_i) = (2Z \cdot E_i)$ for each irreducible component $E_i$ of $Z$. Since the intersection matrix of Supp $Z = \pi^{-1}(x_1)$ is negative definite, we obtain $D = 2Z$ and thus $\hat{\Gamma} \sim 2\hat{Y}_1 + \hat{Y}_2 + 3Z$. In particular, we obtain $O_Z(3Z) \cong O_Z(2p_1 - p_2)$. 

(v) By (iv) and Lemma 4.2 (v), we have $O_Z(3p_0 - 2p_1 - p_2) \cong O_Z$ and in particular $\deg O_Z(3p_0 - 2p_1 - p_2) = \deg O_Z = (0, \ldots, 0)$. By noting (ii) and that $p_0, p_1 \in E_1$, we obtain $p_2 \in E_1 \setminus \text{Sing(Supp} Z)$. 

\qed
Proposition 4.5. There exists only the case where
\[
\mathcal{Y} = 2Y_1 + Y_2 \ (Y_1:\ line, \ Y_2:\ conic) \text{ with } x = Y_1 \cap Y_2 = \{x_1\}.
\]
Moreover, the weighted dual graph of \( \hat{\mathcal{Y}} \cup E \) is of type (XXI) in Theorem 1.

Proof. We have already obtained the first assertion. Now we prove the second assertion. Let \( \hat{\mathcal{Y}}_2^* \), \( E_i^* \) and \( E^* \) be the proper transforms of \( \hat{\mathcal{Y}}_2 \), \( E_i \) and \( E \) by the contraction morphism of \( \hat{\mathcal{Y}}_1 \) respectively. By Lemma 4.2 (iv) and Lemma 4.4 (ii), (iii), \( \hat{\mathcal{Y}}_2^* \cup E^* \) is a boundary of a smooth compactification of \( \mathbb{C}^2 \) which is a simple normal crossing divisor of one \((-1)\)-curve \( \hat{\mathcal{Y}}_2^* \) and nine \((-2)\)-curves. By Lemma 2.9 and \( p_2 \in E_1 \setminus \text{Sing}(\text{Supp } Z) \), the weighted dual graph of \( \hat{\mathcal{Y}}_2^* \cup E^* \) is given as in Fig. 15. Since \( p_1 \in E_1 \setminus \text{Sing}(\text{Supp } Z) \) and \( p_1 \neq p_2 \), we obtain the assertion.

Thus we complete the proof of Theorem 1 for the case \( Z^2 = -1 \).

5. Proof of Theorems 2 and 3

In this section, we shall prove Theorems 2 and 3. We use the same notation as that in the previous sections. For each weighted dual graph of \( \hat{\mathcal{Y}} \cup E \) of type (XV) through (XXI) in Theorem 1, we know the shape of the divisor \( \hat{\mathcal{Y}} \cup (\hat{\mathcal{Y}} \cup E) \) by noting that \( (\hat{\mathcal{Y}} \cdot \hat{\mathcal{Y}})_M = (\hat{\mathcal{Y}} \cdot Y)_X = \deg Y_i \) and \( (\hat{\mathcal{Y}} \cdot E)_M = 0 \). By contracting suitable \((-1)\)-curves in \( \hat{\mathcal{Y}} \cup E \) repeatedly, we can obtain \( \mathbb{P}^2 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \) as a compactification of \( \mathbb{C}^2 \). Let \( \tau = \tau_1 \circ \cdots \circ \tau_N : M_N := M \to \cdots \to M_0 \) be the composite of blowing-downs to \( M_0 := \mathbb{P}^2 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \), where \( 10 \leq N \leq 11 \). Conversely, we obtain \( M \) by applying blowing-ups of \( M_0 \) on \( \tau(\hat{\mathcal{Y}} \cup E) \) repeatedly. We denote by \( P_i \) the center of the blowing-up \( \tau_i \) and by \( E_i \) the proper transform of \( \text{Exc } \tau_i = \tau_i^{-1}(P_{i-1}) \) in \( M \) for \( 1 \leq i \leq N \). The birational map \( \phi := \pi \circ \tau^{-1} : M_0 \to X \), which has points of indeterminacy at \( \tau(\text{Exc } \tau) \), gives an isomorphism \( M_0 \setminus \tau(\hat{\mathcal{Y}} \cup E) \cong X \setminus Y \). The commutative diagram in Fig. 16 gives a resolution of indeterminacy of \( \phi \). The image \( G := \tau_2 \hat{\mathcal{Y}} \) is an irreducible curve on \( M_0 \) with \( \text{Sing } G = \tau(\text{Exc } \tau) \). Since \( \pi \) is determined by the linear system \( |\hat{\mathcal{Y}}| \) on \( M \), the map \( \phi \) is determined by the linear system \( \tau_2^*|\hat{\mathcal{Y}}| = |G - m_0 P_0 - m_1 P_1 - \cdots - m_{N-1} P_{N-1}| \) on \( M_0 \) with \( m_i \geq 1 \). By chasing the process of the resolution of indeterminacy of \( \phi \), we can determine a basis of the four-dimensional \( \mathbb{C} \)-vector space associated with \( \tau_2^*|\hat{\mathcal{Y}}| \). Thus we write down the map \( \phi \) and the pair \( (X, Y) \) as the image of \( \phi \) concretely. Finally we construct a tame automorphism of \( \mathbb{C}^3 \) explicitly which linearizes the hypersurface...
The map \(\pi\) with \(\kappa\) for the type whose weighted dual graph is given as in Fig. 17(XV) or (XVI). By the shape of \(\hat{\phi}\), we see that \(\hat{\phi}\) is determined by the linear system \(\ell(x, y) = (x_0 : x_1), (y_0 : y_1)\) be a bihomogeneous coordinate of \(\mathbb{P}^1 \times \mathbb{P}^1\).

5.1. The types (XV) and (XVI). For each type, there exists a composite \(\tau = \tau_1 \circ \cdots \circ \tau_{11}: M_{11} = M \rightarrow \cdots \rightarrow M_0 = \mathbb{P}^2\) of blowing-downs to \(\mathbb{P}^2\) such that \(\text{Exc} \tau\) is contained in \(\hat{\mathcal{Y}} \cup E\). Let \(L\) be the image \(\tau(\hat{\mathcal{Y}} \cup E)\), which is a line in \(\mathbb{P}^2\), and \(\mathcal{L}\) the proper transform of \(L\) in \(\mathcal{M}\). We note that \(\tau(\text{Exc} \tau) = \{P_0\}\) and \(\hat{\mathcal{Y}} \cup E = \overline{\mathcal{L}} \cup (\bigcup_{i=1}^{11} F_i)\), whose weighted dual graph is given as in Fig. 17(XV) or (XVI). By the shape of \(\hat{\mathcal{Y}} \cup (\hat{\mathcal{Y}} \cup E)\) and \(\hat{\mathcal{Y}}^2 = 4\), we see that \(G\) is a plane sextic curve with \(\text{Sing} G = G \cap L = \{P_0\}\) and that \(\phi\) is determined by the linear system \(6L - 2P_0 - 2P_1 - 2P_2 - 2P_3 - 2P_4 - 2P_5 - 2P_6 - P_7 - P_8 - P_9 - P_{10}\), whose base locus consists of only one point \(P_0\). We may assume that \(L = \{w_2 = 0\}\) and \(P_0 = (0 : 1 : 0)\). Then we have the next proposition by computing directly.

**Proposition 5.1.** One obtains the following:

(i) The map \(\phi\) is given, up to automorphisms of \(\mathbb{P}^2\) and \(\mathbb{P}^3\), as follows:

\[
\phi:\begin{cases}
z_0 = w_0 w_2^5 \\
z_1 = f_1(w) w_2^3 \\
z_2 = w_1 w_2^2 + [f_1(w) + \lambda_1 w_0 w_2^2] f_1(w) + \lambda_2 w_0 w_2^2 \\
z_3 = w_2^6
\end{cases}
\]

with \(f_1(w) = f_1(w_0, w_1, w_2) = w_0^3 + w_1^2 w_2\) and \(\lambda_1, \lambda_2 \in \mathbb{C}\), where \(\lambda_1 = \lambda_2\) for the type (XV) and \(\lambda_1 \neq \lambda_2\) for the type (XVI).

(ii) The pair \((X, Y)\) is given, up to automorphisms of \(\mathbb{P}^3\), as follows:

\[
\begin{align*}
X & : (z_2 z_3 + \alpha z_0^2 + \beta z_0 z_1 + \gamma z_1^2)^2 + z_0 z_3^2 + z_1^3 z_3 = 0 \\
Y & : z_3 = (\alpha z_0^2 + \beta z_0 z_1 + \gamma z_1^2)^2 = 0
\end{align*}
\]

with \(\alpha, \beta, \gamma \in \mathbb{C}\) and \(\alpha \neq 0\), where \(\beta^2 - 4\alpha \gamma = 0\) for the type (XV) and \(\beta^2 - 4\alpha \gamma \neq 0\) for the type (XVI).
The map computing directly.

For each type \( AB \) or \( AS \), where \( f \), \( AL \), \( 4 \), \( AS \) is contained in \( AS \), the pair \( (ii) \) consists of two points of \( \hat{X} \) with \( f \), \( AL \), 1, 4, \( CV \), \( BI \), \( AC \), \( 3 \) is contained in \( AS \), \( CV \), \( BI \), \( AC \), \( 3 \). We note that \( (0 : 1), (1 : 0) \) and \( \phi \) is determined by the linear system \( \{4C_1 + 4C_2\} - (2P_0 + 2P_1 + 2P_2 + P_3 + P_4) - (2P_5 + 2P_6 + 2P_7 + P_8 + P_9) \), whose base locus consists of two points \( P_0 \) and \( P_5 \). We may assume that \( C_1 = \{y_1 = 0\}, C_2 = \{x_1 = 0\}, P_0 = ((0 : 1), (1 : 0)) \) and \( P_5 = ((1 : 0), (0 : 1)) \). Then we have the next proposition by computing directly.

**Proposition 5.2.** One obtains the following:

(i) The map \( \phi \) is given, up to automorphisms of \( P^1 \times P^1 \) and \( P^3 \), as follows:

\[
\begin{align*}
\phi &: \begin{cases}
z_0 &= x_0 x_1^3 y_0 y_1^3 \\
z_1 &= f_2(x, y) x_1^2 y_1^2 \\
z_2 &= \lambda_3 x_1^4 y_0 y_1 + \{f_2(x, y) + \lambda_1 x_0 x_1 y_0 y_1\} [f_2(x, y) + \lambda_2 x_0 x_1 y_0 y_1] \\
z_3 &= \lambda_1^4 y_1^4
\end{cases}
\end{align*}
\]

with \( f_2(x, y) = f_2(x_0, x_1, y_0, y_1) = x_0^2 y_0^2 + x_0 x_1 y_0^2 + x_1^2 y_0 y_1 \), \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \) and \( \lambda_3 \neq 0 \), where \( (\lambda_1 - \lambda_2)^2 \{\lambda_1 - \lambda_2\}^2 + 4\lambda_3 = 0 \) for the type (XVII) and \( (\lambda_1 - \lambda_2)^2 \{\lambda_1 - \lambda_2\}^2 + 4\lambda_3 \neq 0 \) for the type (XVIII).

(ii) The pair \( (X, Y) \) is given, up to automorphisms of \( P^3 \), as follows:

\[
\begin{align*}
X &: \begin{cases}
z_2 z_3 + \alpha z_0^2 + \beta z_0 z_1 + \gamma z_1^2 - (z_0 z_3 + z_1^2) = 0
\end{cases}
\end{align*}
\]

\[
\begin{align*}
Y &: \begin{cases}
z_3 = (\alpha z_0^2 + \beta z_0 z_1 + \gamma z_1^2) - z_1^3 = 0
\end{cases}
\end{align*}
\]

with \( \alpha, \beta, \gamma \in \mathbb{C} \) and \( \alpha \neq 0 \), where \( \{\beta^2 - 4\alpha(\gamma - 1)\} \{\beta^2 - 4\alpha(\gamma + 1)\} = 0 \) for the type (XVII) and \( \{\beta^2 - 4\alpha(\gamma - 1)\} \{\beta^2 - 4\alpha(\gamma + 1)\} \neq 0 \) for the type (XVIII).

(iii) For each type, there exists a tame automorphism of \( \mathbb{C}^3 \) which transforms the hypersurface \( X \setminus Y \) onto a coordinate hyperplane.

**5.3. The types (XIX) and (XX).** For each type, there exists a composite \( \tau = \tau_1 \circ \cdots \circ \tau_{10} : M_{10} = M \rightarrow \cdots \rightarrow M_0 = \mathbb{P}^2 \) of blowing-downs to \( \mathbb{P}^2 \) such that \( \text{Exc} \tau \) is contained in \( \hat{Y} \cup E \). Let \( L \) be the image \( \tau(\hat{Y} \cup E) \), which is a line in \( \mathbb{P}^2 \), and \( \mathcal{L} \) the proper transform of \( L \) in \( M \). We note that \( \tau(\text{Exc} \tau) = \{P_0, P_7\} \) and \( \hat{Y} \cup E = \mathcal{L} \).
\( L \cup (\bigcup_{i=1}^{14} F_i) \), whose weighted dual graph is given as in Fig. 17 (XIX) or (XX). By the shape of \( \hat{\Gamma} \cup (\hat{Y} \cup E) \) and \( \hat{\Gamma}^2 = 4 \), we see that \( G \) is a plane sextic curve with

\[ \text{Sing } G = G \cap L = \{ P_0, P_7 \} \]

and that \( \phi \) is determined by the linear system \( |6L - (2P_0 + 2P_1 + 2P_2 + 2P_3 + 2P_4 + P_5 + P_6) - (2P_7 + 2P_8 + P_9 + P_{10})| \), whose base locus consists of two points \( P_0 \) and \( P_7 \). We may assume that \( L = \{ w_2 = 0 \} \), \( P_0 = (0 : 1 : 0) \) and \( P_7 = (1 : 0 : 0) \). Then we have the next proposition by computing directly.

**Proposition 5.3.** One obtains the following:

(i) The map \( \phi \) is given, up to automorphisms of \( \mathbb{P}^2 \) and \( \mathbb{P}^3 \), as follows:

\[
\phi: \begin{cases} 
  z_0 = w_0w_2^3 \\
  z_1 = f_3(w)w_2^3 \\
  z_2 = w_0^3w_2^4 + w_1w_2^3 + \{ f_3(w) + \lambda_1w_0w_2^3 \}[f_3(w) + \lambda_2w_0w_2^3] \\
  z_3 = w_2^6 
\end{cases}
\]

with \( f_3(w) = f_3(w_0, w_1, w_2) = w_0^3w_1 + w_1^2w_2 + \lambda_3w_1w_2^2 \) and \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \), where \( (\lambda_1 - \lambda_2)^2[(\lambda_1 - \lambda_2)^2 - 4] = 0 \) for the type (XIX) and \( (\lambda_1 - \lambda_2)^2[(\lambda_1 - \lambda_2)^2 - 4] \neq 0 \) for the type (XX).

(ii) The pair \((X, Y)\) is given, up to automorphisms of \( \mathbb{P}^3 \), as follows:

\[
\begin{align*}
  X: (z_2z_3 + \alpha z_0^2 + \beta z_0z_1 + \gamma z_1^2)^2 - z_1^4 + z_0z_3^2 + \delta z_1^2z_3^2 &= 0 \\
  Y: z_3 = (\alpha z_0^2 + \beta z_0z_1 + \gamma z_1^2)^2 - z_1^4 &= 0
\end{align*}
\]

with \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \) and \( \alpha \neq 0 \), where \( [\beta^2 - 4\alpha(\gamma - 1)][\beta^2 - 4\alpha(\gamma + 1)] = 0 \) for the type (XIX) and \( [\beta^2 - 4\alpha(\gamma - 1)][\beta^2 - 4\alpha(\gamma + 1)] \neq 0 \) for the type (XX).

(iii) For each type, there exists a tame automorphism of \( \mathbb{C}^3 \) which transforms the hypersurface \( X \setminus Y \) onto a coordinate hyperplane.

**5.4. The type (XXI).** For this type, there exists a composite \( \tau = \tau_1 \circ \cdots \circ \tau_{10} \): \( M_{10} = M \rightarrow \cdots \rightarrow M_0 = \mathbb{P}^2 \) of blowing-downs to \( \mathbb{P}^2 \) such that \( \text{Exc } \tau \) is contained in \( \hat{Y} \cup E \). Let \( L \) be the image \( \tau(\hat{Y} \cup E) \), which is a line in \( \mathbb{P}^2 \), and \( \hat{L} \) the proper transform of \( L \) in \( M \). We note that \( \tau(\text{Exc } \tau) = \{ P_0 \} \), \( F_0 = \hat{Y}_2, F_{10} = \hat{Y}_1 \) and \( \hat{Y} \cup E = \hat{L} \cup (\bigcup_{i=1}^{14} F_i) \), whose weighted dual graph is given as in Fig. 17 (XXI). By the shape of \( \hat{\Gamma} \cup (\hat{Y} \cup E) \) and \( \hat{\Gamma}^2 = 4 \), we see that \( G \) is a plane curve of degree nine with \( \text{Sing } G = G \cap L = \{ P_0 \} \) and that \( \phi \) is determined by the linear system \( |9L - 3P_0 - 3P_1 - 3P_2 - 3P_3 - 3P_4 - 3P_5 - 3P_6 - 2P_7 - 2P_8 - P_9| \), whose base locus consists of only one point \( P_0 \). We may assume that \( L = \{ w_2 = 0 \} \) and \( P_0 = (0 : 1 : 0) \). Then we have the next proposition by computing directly.
**Proposition 5.4.** One obtains the following:

(i) The map \( \phi \) is given, up to automorphisms of \( \mathbb{P}^2 \) and \( \mathbb{P}^3 \), as follows:

\[
\phi: \begin{cases} 
  z_0 = f_4(w)w_2^6 \\
  z_1 = w_0w_2^8 + f_4(w)^2w_2^3 \\
  z_2 = w_1w_2^8 - f_4(w)^3 + \frac{3}{2} f_4(w)\{w_0w_2^5 + f_4(w)^3\} \\
  z_3 = w_2^9
\end{cases}
\]

with \( f_4(w) = f_4(w_0, w_1, w_2) = w_0^3 + w_1^2w_2 + \lambda w_0w_2^2 \) and \( \lambda \in \mathbb{C} \).

(ii) The pair \((X, Y)\) is given, up to automorphisms of \( \mathbb{P}^3 \), as follows:

\[
\begin{cases} 
  X: z_2^2z_3^2 + (2z_0^3 + 3z_0z_1z_3)z_2 - z_1^3z_3 - \frac{3}{4} z_0^2z_1^2 + z_0z_3^3 + \delta(z_1z_3 + z_0^2)z_3^2 = 0 \\
  Y: z_3 = z_0^2\left(z_0z_2 - \frac{3}{8} z_1^2\right) = 0
\end{cases}
\]

with \( \delta \in \mathbb{C} \).

(iii) There exists a tame automorphism of \( \mathbb{C}^3 \) which transforms the hypersurface \( X \setminus Y \) onto a coordinate hyperplane.

**Remark.** In (ii), the hypersurface \( X \setminus Y \) is expressed as follows:

\[
0 = z_2^2 + (2z_0^3 + 3z_0z_1)z_2 - z_1^3 - \frac{3}{4} z_0^2z_1^2 + z_0 + \delta(z_1 + z_0^2)
\]

\[
= \left(z_2 + z_0^3 + \frac{3}{2} z_0z_1\right)^2 - (z_1 + z_0^2)^3 + z_0 + \delta(z_1 + z_0^2)
\]

where \((z_0, z_1, z_2)\) is a coordinate of \( \mathbb{C}^3 = \mathbb{P}^3 \setminus H \).

Thus we complete the proof of Theorems 2 and 3 for the types (XV) through (XXI) in Theorem 1.

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Fig. 17.
ALGEBRAIC EMBEDDINGS OF $\mathbb{C}^2$ INTO $\mathbb{C}^3$ 597

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