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ON SYMMETRIC SETS OF UNIMODULAR SYMMETRIC MATRICES

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1. Introduction

A binary system A is called a symmetric set if (1) $a\circ a=a$, (2) $(a\circ b)\circ b=a$ and (3) $(a\circ b)\circ c=(a\circ c)\circ(b\circ c)$ for elements a, b and c in A. Define a mapping S_a of A for an element a in A by $S_a(x)=x\circ a$. As in [2], [3] and [4], we denote $S_a(x)$ by xS_a. S_a is a homomorphism of A due to (3), and is an automorphism of A due to (2). Every group is a symmetric set by a definition: $a\circ b=ba^{-1}b$. A subset of a group which is closed under this operation is also a symmetric set. In this paper, we consider a symmetric set which is a subset of the group $SL_n(K)$ consisting of all unimodular symmetric matrices. We denote it by $SM_n(K)$. For a symmetric set A, we consider a subgroup of the group of automorphisms of A generated by all S_aS_b (a and b in A), and call it the group of displacements of A. We can show that the group of displacements of $SM_n(K)$ is isomorphic to $SL_n(K)/\{\pm 1\}$ if $n\geq 3$ or $n\geq 2$ when $K\neq F_3$ (Theorem 5). Also we can show that $PSM_n(K)$, which is defined in a similar way that $PSL(_nK)$ is defined, has its group of displacements isomorphic to $PSL_n(K)$ under the above condition (Theorem 6). A symmetric set A is called transitive if $A=aH$, where a is an element of A and H is the group of displacements. A subset B of A is called an ideal if $BS_a\subseteq B$ for every element a in A. For an element a in A, aH is an ideal since $aHS_x=aS_xH=aS_aS_xH=aH$ for every element x in A. Therefore, A is transitive if and only if A has no ideal other than itself. Let F_q be a finite field of q elements ($q=p^m$). We can show that $SM_n(F_q)$ is transitive if $p\neq 2$ or if n is odd, and that $SM_n(F_q)$ consists of two disjoint ideals both of which are transitive if n is even and $p=2$ (Theorem 7).

A symmetric subset B of A is called quasi-normal if $BT\cap B=B$ or ϕ for every element T of the group of displacements. When A has no proper quasi-normal symmetric subset, we say that A is simple. In [4], it was shown that if A is simple (in this case, A is transitive as noted above) then the group of displacements is either a simple group or a direct product of two isomorphic simple groups. In **4**, we show some examples of $PSM_n(F_q)$. The first example is $PSM_3(F_2)$, which is shown to be a simple symmetric set of 28 elements.

The second example is $PSM_2(F_7)$, which we show consists of 21 elements and is not simple. We analize the structure of it and show that $PSL_2(F_7)$ (which is isomorphic to $PSL_3(F_2)$ and is simple) is a subgroup of A_7. The third example is one of ideals of $PSM_4(F_2)$ which consists of unimodular symmetric matrices with zero diagonal. It has 28 elements and we can show that it is isomorphic to a symmetric set of all transpositions in S_8. This reestablishes the well known theorem that $PSL_4(F_2)$ is isomorphic to A_8.

2. Unimodular symmetric matrices

Theorem 1. *$SL_n(K)$ is generated by unimodular symmetric matrices if $n \geq 3$ or $n \geq 2$ when $K \neq F_3$.*

Proof. Consider a subgroup of $SL_n(K)$ generated by all unimodular symmetric matrices. It is a normal subgroup because if s is a symmetric matrix and u is a non singular matrix then $u^{-1}su=(u^tu)^{-1}(u^tsu)$ which is a product of symmetric matrices. The subgroup clearly contains the center of $SL_n(K)$ properly so that it must coincide with $SL_n(K)$ if $n \geq 3$ or $n \geq 2$ when $K \neq F_2$ or F_3, since $PSL_n(K)$ is simple. If $n=2$ and $K=F_2$, Theorem 1 follows directly from $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$. If $n=2$ and $K=F_3$, Theorem 1 does not hold since $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ is not expressed as a product of unimodular symmetric matrices.

Two matrices a and b are said to be congruent if $b=u^tau$ with a non singular matrix u. Suppose that a is congruent to 1 (the identity matrix) and that $\det a=1$. Then $1=u^tau$, where we may assume that $\det u=1$, because otherwise $\det u=-1$ and then we can replace u by uv with $v=\begin{bmatrix}-1 & & & 0\\ & 1 & & \\ & & \ddots & \\ 0 & & & 1\end{bmatrix}$.

Theorem 2. *Suppose that $n \geq 2$ and $p \neq 2$. Then every unimodular symmetric matrix in $SL_n(F_q)$ is congruent to 1.*

Theorem 2 is known. ([1], p. 16)

Theorem 3. *Suppose that $n \geq 2$ and $q=2^m$. If n is odd, every unimodular symmetric matrix in $SL_n(F_q)$ is congruent to 1. If n is even, every unimodular symmetric matrix in $SL_n(F_q)$ is congruent either to 1 or to $J \oplus J \oplus \cdots \oplus J$, where $J=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. The latter occurs if and only if every diagonal entry of the symmetric matrix is zero.*

Proof. First, we show a lemma.

Lemma. *Suppose that the characteristic of K is 2. If every diagonal entry of a symmetric matrix s over K is zero, then u^tsu has the same property where u is any matrix over K.*

Proof. Let $s=(a_{ij})$, $u=(b_{ij})$ and $u^tsu=(c_{ij})$. Then $a_{ij}=a_{ji}$ and $a_{ii}=0$. We have $c_{ii}=\sum_{k,j} b_{ki}a_{kj}b_{ji}=\sum_{k<j} b_{ki}(a_{kj}+a_{jk})b_{ji}=0$ since $a_{kj}+a_{jk}=2a_{kj}=0$.

Now we return to the proof of Theorem 3. Let $s=(a_{ij})$ be a symmetric matrix in $SL_n(F_q)$. Suppose that $a_{ii}=0$ for all i. Then $a_{1k}\neq 0$ for some k. Taking a product of elementary matrices for u, we have that, in $u^tsu=(b_{ij})$, $b_{12}\neq 0$ and $b_{1j}=0$ for all $j\neq 2$. Since $b_{21}=b_{12}\neq 0$, we can apply the same argument to the second row (and hence to the second column at the same time) to get a matrix (c_{ij}) congruent to s such that $(c_{ij})=\begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}\oplus s'$, where s' is a symmetric matrix of $(n-2)\times(n-2)$. Then take an element d in F_q such that $d^2=c^{-1}$, and let $u=\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}\oplus I_{n-2}$, where I_{n-2} is the identity matrix of $(n-2)\times(n-2)$. Thusfar, we have seen that s is congruent to $J\oplus s'$. By Lemma, s' has the zero diagonal. Proceeding inductively, we can get $J\oplus J\oplus\cdots\oplus J$ which is congruent to s, if s has the zero diagonal. In this case, n must be even. Next, suppose that $a_{ii}\neq 0$ for some i. As in above, we can find u such that $u^tsu=[1]\oplus s'$, where s' is of $(n-1)\times(n-1)$. By induction, s' is congruent either to I_{n-1} or to $J\oplus J\oplus\cdots\oplus J$. In the former case, s is congruent to $1=I$. In the latter case, we just observe that

$$[1]\oplus J=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, we can reduce s to the identity matrix by congruence.

Theorem 4. *Suppose that n is even and $q=2^m$. Then $SL_n(F_q)$ is generated by $a^{-1}b$ where a and b are unimodular symmetric matrices with zero diagonal. Also, $SL_n(F_q)$ is generated by $c^{-1}d$ where c and d are unimodular symmetric matrices which have at least one non zero entry in diagonal.*

Proof. For a and b in Theorem 4, we have $s^{-1}(a^{-1}b)s=(sas)^{-1}(sbs)$, where s is a symmetric matrix in $SL_n(F_q)$. By Lemma, sas and sbs have zero diagonal. Since $SL_n(F_q)$ is generated by symmetric matrices by Theorem 1, the above fact implies that the subgroup of $SL_n(F_q)$ generated by all $a^{-1}b$ is a normal subgroup. On the other hand, the center of $SL_n(F_q)$ consists of zI where z is an element of F_q such that $z^n=1$. Since $zI=a^{-1}(za)$, the center of $SL_n(F_q)$ is contained in the subgroup generated by $a^{-1}b$. It is also easy to see that the subgroup contains an element which is not contained in the center. Again, by the simplicity of PSL_n

(F_q), the subgroup must coincide with the total group. The second part of Theorem 4 is proved in the same way.

3. Symmetric sets of unimodular matrices

Theorem 5. *The group of displacements of $SM_n(K)$ is isomorphic to $SL_n(K)/\{\pm 1\}$ if $n\geq 3$ or $n\geq 2$ when $K\neq F_3$.*

Proof. For $w\in SL_n(K)$ and $a\in SM_n(K)$, we define a mapping T_w of $SM_n(K)$ by $aT_w=w^taw$. T_w is an automorphism of $SM_n(K)$ since $w^t(ba^{-1}b)w=(w^tbw)(w^taw)^{-1}(w^tbw)$. If especially $w=s_1s_2$ with s_1 and s_2 in $SM_n(K)$, then $aT_w=s_2(s_1^{-1}a^{-1}s_1^{-1})^{-1}s_2=aS_{s_1^{-1}}S_{s_2}$, and hence $T_w=S_{s_1^{-1}}S_{s_2}$. By Theorem 1, w is a product (of even number) of s_i in $SM_n(K)$. Thus $w\rightarrow T_w$ gives a homomorphism of $SL_n(K)$ onto the group of displacements of $SM_n(K)$. w is in the kernel of the homomorphism if and only if $w^taw=a$ for every element a in $SM_n(K)$. In this case, especially we have $w^tw=1$ or $w^t=w^{-1}$. Then $w^{-1}aw=a$, or $wa=aw$. Since $SL_n(K)$ is generated by a, the above implies that w must be in the center of $SL_n(K)$. So, $w=zI$ with z in K. Then $w^tw=1$ implies $w^2=1$, or $z=\pm 1$. This completes the proof of Theorem 4.

To define $PSM_n(K)$, we identify elements a and za in $SM_n(K)$, where z is an element in K such that $z^n=1$. The set of all classes defined in this way is a symmetric set in a natural way, and we denote it by $PSM_n(K)$.

Theorem 6. *The group of displacements of $PSM_n(K)$ is isomorphic to $PSL_n(K)$ if $n\geq 3$ or $n\geq 2$ when $K\neq F_3$.*

Proof. Denote by $\bar{a}$ an element of $PSM_n(K)$ represented by a in $SM_n(K)$. For w in $SL_n(K)$, we define T_w: $\bar{a}\rightarrow\overline{w^taw}$. As before, $w\rightarrow T_w$ gives a homomorphism of $SL_n(K)$ onto the group of displacements of $PSM_n(K)$. $T_w=1$ if and only if $\overline{w^taw}=\bar{a}$ for every a. If w is in the center of $SL_n(K)$, then clearly $T_w=1$. So, the kernel of the homomorphism contains the center. On the other hand, we have $\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1&1\\1&2\end{bmatrix}$, which indicates that $w=\begin{bmatrix}1&1\\0&1\end{bmatrix}\oplus I_{n-2}$ is not contained in the kernel. Therefore, the kernel must coincide with the center due to the simplicity of $PSL_n(K)$. This completes the proof of Theorem 6.

Theorem 7. *Suppose that $n\geq 3$ or $n\geq 2$ if $K\neq F_3$. If $p\neq 2$ or if n is odd, then $SM_n(F_q)$ is transitive. If $p=2$ and n is even, then $SM_n(F_q)$ consists of two disjoint ideals, which are transitive.*

Proof. First suppose that $p\neq 2$ or n is odd. Then by Theorems 2 and 3, every unimodular symmetric matrix a is congruent to 1, i.e., $a=u^tu$ with a unimodular matrix u. By Theorem 1, u is a product of even number of unimodular symmetric matrices: $u=s_1\cdots s_{2i}$. Then $T_u=S_{s_1^{-1}}S_{s_2}\cdots S_{s_{2i}}$ as in Theorem 6. Then $a=1T_u\in 1H$, where H is the group of displacements. Thus $SM_n(F_q)$ is transitive in this case. Next suppose that $p=2$ and n is even. Let B_0 be the set of all unimodular symmetric matrices with zero diagonal. Elements of B_0 are congruent to $j=J\oplus J\oplus\cdots\oplus J$. So, for an element a in B_0, there exists u such that $u^t au=j$. Here det $u=1$ since $p=2$. By Theorem 4, u is a product of elements $a^{-1}b$ where a and b are in B_0. For a, b and c in B_0, we have $(b^{-1}c)^t a(b^{-1}c)=aS_bS_c$, from which we can conclude that $aH(B_0)$, where $H(B_0)$ is the group of displacements of B_0, contains j, and hence $a\in jH(B_0)$. Thus, B_0 is transitive. It is also clear that B_0 is an ideal of $SM_n(F_q)$ by Theorems 4 and 5. In the same way, we can show that the complementary set of B_0 in $SM_n(F_q)$ is an ideal of $SM_n(F_q)$ and is transitive as a symmetric set.

4. Examples

First of all, we recall the definition of cycles in a finite symmetric set (see [3]). Let a and b be elements in a finite symmetric set such that $aS_b\neq a$. Then we call a symmetric subset generated by a and b a cycle. To indicate the structure of a cycle, we use an expression: $a_1-a_2-\cdots$, where $a_1=a$, $a_2=b$ and $a_{i+1}=a_{i-1}S_{a_i}$ $(i\geq 2)$. If a symmetric set is effective (i.e. $S_c\neq S_d$ whenever $c\neq d$), the above sequence is repetions of some number of different elements (Theorem 2, [3]). For example, $a_1-a_2-\cdots-a_n-a_1-a_2-\cdots$ where $a_i\neq a_j$ $(1\leq i\neq j\leq n)$. In this case, we denote the cycle by $a_1-a_2-\cdots-a_n$ and call n the length of the cycle.

EXAMPLE 1. $PSM_3(F_2)$ $(=SM_3(F_2))$.
$SM_3(F_2)$ consists of the following 28 elements.

$$a_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ a_2=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\ a_3=\begin{bmatrix}1&1&0\\1&0&0\\0&0&1\end{bmatrix},\ a_4=\begin{bmatrix}0&1&0\\1&1&0\\0&0&1\end{bmatrix},$$

$$a_5=\begin{bmatrix}0&1&1\\1&0&0\\1&0&1\end{bmatrix},\ a_6=\begin{bmatrix}0&1&0\\1&0&1\\0&1&1\end{bmatrix},\ a_7=\begin{bmatrix}1&1&1\\1&0&0\\1&0&1\end{bmatrix},\ a_8=\begin{bmatrix}0&1&0\\1&1&1\\0&1&1\end{bmatrix},$$

$$a_9=\begin{bmatrix}0&1&1\\1&0&1\\1&1&1\end{bmatrix},\ a_{10}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\ a_{11}=\begin{bmatrix}1&0&1\\0&1&0\\1&0&0\end{bmatrix},\ a_{12}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&1\end{bmatrix},$$

$$a_{13}=\begin{bmatrix}0&1&1\\1&1&0\\1&0&0\end{bmatrix},\ a_{14}=\begin{bmatrix}0&0&1\\0&1&1\\1&1&0\end{bmatrix},\ a_{15}=\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix},\ a_{16}=\begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix},$$

$$a_{17}=\begin{bmatrix}0&1&1\\1&1&1\\1&1&0\end{bmatrix},\ a_{18}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\ a_{19}=\begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix},\ a_{20}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&1\end{bmatrix},$$

$$a_{21}=\begin{bmatrix}1&1&0\\1&0&1\\0&1&0\end{bmatrix},\ a_{22}=\begin{bmatrix}1&0&1\\0&0&1\\1&1&0\end{bmatrix},\ a_{23}=\begin{bmatrix}1&1&0\\1&1&1\\0&1&0\end{bmatrix},\ a_{24}=\begin{bmatrix}1&0&1\\0&0&1\\1&1&1\end{bmatrix},$$

$$a_{25}=\begin{bmatrix}1&1&1\\1&0&1\\1&1&0\end{bmatrix},\ a_{26}=\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\end{bmatrix},\ a_{27}=\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix},\ a_{28}=\begin{bmatrix}1&0&1\\0&1&1\\1&1&1\end{bmatrix}.$$

We denote S_{a_i} by S_i, and a transposition (a_i, a_j) by (i, j). Then each S_i is a product of 12 transpositions as follows.

$S_1=(3, 4)(5, 8)(6, 7)(9, 28)(11, 12)(13, 16)(14, 15)(17, 27)(19, 20)(21, 24)(22, 23)(25, 26)$, $S_2=(5, 7)(6, 8)(9, 28)(10, 18)(11, 20)(12, 19)(13, 24)(14, 23)(15, 22)(16, 21)(17, 26)(25, 27)$, $S_3=(1, 4)(5, 7)(6, 28)(8, 9)(10, 22)(11, 24)(12, 17)(13, 20)(15, 18)(16, 25)(19, 26)(21, 27)$, $S_4=(1, 3)(5, 28)(6, 8)(7, 9)(10, 23)(11, 27)(12, 21)(13, 26)(14, 18)(16, 19)(17, 24)(20, 25)$, $S_5=(1, 14)(2, 3)(4, 23)(6, 11)(8, 24)(9, 13)(10, 25)(12, 26)(15, 21)(16, 18)(20, 28)(22, 27)$, $S_6=(1, 22)(2, 4)(3, 15)(5, 19)(7, 16)(9, 21)(10, 24)(12, 28)(13, 23)(14, 26)(17, 18)(20, 27)$, $S_7=(1, 23)(2, 3)(4, 14)(6, 13)(8, 20)(9, 11)(10, 21)(15, 25)(16, 22)(17, 19)(18, 27)(24, 28)$, $S_8=(1, 15)(2, 4)(3, 22)(5, 21)(7, 12)(9, 19)(10, 26)(11, 25)(13, 18)(14, 24)(16, 28)(17, 23)$, $S_9=(1, 2)(3, 10)(4, 18)(5, 17)(6, 25)(7, 27)(8, 26)(11, 14)(12, 23)(15, 20)(16, 24)(19, 22)$, $S_{10}=(2, 18)(3, 19)(4, 20)(5, 23)(6, 24)(7, 21)(8, 22)(9, 26)(13, 15)(14, 16)(17, 27)(25, 28)$, $S_{11}=(1, 12)(2, 21)(3, 23)(4, 9)(5, 19)(7, 18)(8, 25)(13, 15)(14, 27)(16, 17)(20, 26)(22, 28)$, $S_{12}=(1, 11)(2, 24)(3, 28)(4, 22)(5, 26)(6, 18)(8, 20)(9, 23)(13, 27)(14, 16)(15, 17)(19, 25)$, $S_{13}=(1, 6)(2, 25)(3, 14)(4, 26)(5, 17)(7, 22)(8, 18)(10, 11)(12, 24)(16, 23)(19, 27)(21, 28)$, $S_{14}=(1, 21)(2, 23)(4, 27)(5, 24)(6, 26)(7, 11)(8, 15)(9, 18)(10, 12)(13, 20)(17, 22)(19, 28)$, $S_{15}=(1, 24)(2, 22)(3, 17)(5, 14)(6, 12)(7, 25)(8, 21)(9, 20)(10, 11)(16, 19)(18, 28)(23, 27)$, $S_{16}=(1, 7)(2, 26)(3, 25)(4, 15)(5, 18)(6, 23)(8, 27)(9, 24)(10, 12)(11, 21)(13, 22)(17, 20)$, $S_{17}=(1, 10)(2, 11)(3, 6)(4, 24)(7, 19)(8, 23)(9, 13)(12, 18)(14, 25)(15, 28)(16, 26)(20, 21)$, $S_{18}=(2, 10)(3, 12)(4, 11)(5, 16)(6, 15)(7, 14)(8, 13)(9, 27)(17, 28)(21, 23)(22, 24)(25, 26)$, $S_{19}=(1, 20)(2, 13)(3, 9)(4, 15)(6, 11)(7, 17)(8, 10)(12, 27)(14, 28)(21, 23)(22, 26)(24, 25)$, $S_{20}=(1, 19)(2, 16)(3, 14)(4, 28)(5, 10)(6, 27)(7, 12)(9, 15)(11, 17)(21, 26)(22, 24)(23, 25)$, $S_{21}=(1, 5)(2, 17)(3, 27)(4, 22)(6, 25)(7, 10)(8, 14)(11, 26)(13, 28)(15, 24)(16, 20)(18, 19)$, $S_{22}=(1, 13)(2, 15)(3, 26)(5, 27)(6, 16)(7, 23)(8, 19)(9, 10)(11, 28)(12, 21)(14, 25)(18, 20)$, $S_{23}=(1, 16)(2, 14)(4, 25)(5, 20)(6, 22)(7, 13)(8, 17)(9, 12)(10, 28)(11, 24)(15, 26)(18, 19)$, $S_{24}=(1, 8)(2, 27)(3, 23)(4, 17)(5, 15)(6, 10)(7, 26)(9, 16)(12, 25)(13, 19)(14, 21)(18, 20)$, $S_{25}=(1, 18)(2, 19)(3, 16)(4, 5)(7, 15)(8, 11)(9, 21)(10, 20)(12, 13)(17, 22)(23, 28)(24, 27)$, $S_{26}=(1, 18)(2, 20)(3, 8)(4, 13)(5, 12)(6, 14)(9, 22)(10, 19)(11, 16)(17, 21)(23, 27)(24, 28)$, $S_{27}=(1, 10)(2, 12)(3, 21)(4, 7)(5, 22)(6, 20)(9, 14)(11, 18)$

(13, 25) (15, 26) (16, 28) (19, 24), $S_{28}=(1, 2)(3, 18)(4, 10)(5, 25)(6, 17)(7, 26)(8, 27)(11, 22)(12, 15)(13, 21)(14, 19)(20, 23)$.

From the above, we can find that for a fixed element there exist two cycles of length 7, three cycles of length 4 and three cycles of length 3 which contain the given element. Also we can find that there are exactly 8 cycles of length 7 in the set given by C_1: 1—5—14—24—21—15—8, C_2: 1—6—22—16—13—23—7, C_3: 22—19—26—10—9—3—8, C_4: 13—27—25—24—12—2—19, C_5: 23—5—4—28—10—25—20, C_6: 11—26—16—2—21—17—20, C_7: 6—17—3—12—28—15—18 and C_8: 7—18—14—9—11—4—27. By observation we see that every element is contained in exactly two of C_i and that conversely any two of C_i have exactly one element in common. Clearly S_i induces a permutation of C_j, $j=1, 2, \cdots, 8$, and S_i is uniquely determined by its effect on C_j. Now we are going to show that $SM_3(F_2)$ is a simple symmetric set. First, we note that if $t\notin C_i$, then there exists t' in C_i such that $t'S_t=t'$. Let B be a quasi-normal symmetric subset. We may assume that B contains 1 $(=a_1)$. Suppose that B contains one of C_1 or C_2, say, C_1. For $C_i\neq C_1$, let $s_i=C_1\cap C_i$ and let t_i be such that $t_i\in C_i$ and $t_i\notin C_1$. Since there exists t_i' in C_1 such that $t_i'S_{t_i}=t_i'$, we have that $BS_{t_i}=B$ by the definition of quasi-normality of B. Then $s_iS_{t_i}$ is contained in B, which implies that two elements of C_i are contained in B. B is a symmetric subset and the length of C_i is 7 (prime), and hence all of the elements in C_i must be in B. Thus B must coincide with the total symmetric set. To discuss the general case, we consider all cycles of length 4 and 3 containing 1: D_1: 1—9—2—28, D_2: 1—26—18—25, D_3: 1—27—10—17, E_1: 1—3—4, E_2: 1—11—12, E_3: 1—19—20. Clearly, S_2, S_{10} and S_{18} fix the element 1, and we see that $D_1S_{10}=D_2$, $D_1S_{18}=D_3$, $D_2S_2=D_3$, $E_1S_{18}=E_2$, $E_1S_{10}=E_3$ and $E_2S_2=E_3$. Therefore, if B contains one of D_i, it contains all of D_i, and similarly if B contains one of E_i, it contains all of E_i, In this case, we can verify that B contains one of C_i and hence B must coincide with the total set. Lastly suppose that B which contains 1 contains one of 2, 10 and 18, say, 2. Then $B=BS_{10}$ must contain $2S_{10}=18$, and similarly B contains 10. It is concluded that if B contains one of 2, 10 and 18 then B contains all of them. In this case, $2S_4=2$ implies that $BS_4=B$. So, B contains $1S_4=3$. Thus B contains E_1, and then B coincides with the total set. We have completed the proof that $SM_3(F_2)$ is simple.

EXAMPLE 2. $PSM_2(F_7)$ $(=SM_2(F_7)/\{\pm 1\})$.
This symmetric set consists of the following 21 elements (mod $\{\pm 1\}$).

$$a_1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ a_2=\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix},\ a_3=\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix},\ a_4=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

$$a_5=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\ a_6=\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix},\ a_7=\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix},\ a_8=\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix},$$

$$a_9=\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix},\ a_{10}=\begin{bmatrix}1 & 2\\ 2 & -2\end{bmatrix},\ a_{11}=\begin{bmatrix}-2 & 2\\ 2 & 1\end{bmatrix},\ a_{12}=\begin{bmatrix}-1 & 2\\ 2 & 2\end{bmatrix},$$
$$a_{13}=\begin{bmatrix}2 & 2\\ 2 & -1\end{bmatrix},\ a_{14}=\begin{bmatrix}3 & 2\\ 2 & -3\end{bmatrix},\ a_{15}=\begin{bmatrix}-3 & 2\\ 2 & 3\end{bmatrix},\ a_{16}=\begin{bmatrix}1 & 3\\ 3 & 3\end{bmatrix},$$
$$a_{17}=\begin{bmatrix}3 & 3\\ 3 & 1\end{bmatrix},\ a_{18}=\begin{bmatrix}-1 & 3\\ 3 & -3\end{bmatrix},\ a_{19}=\begin{bmatrix}-3 & 3\\ 3 & -1\end{bmatrix},\ a_{20}=\begin{bmatrix}2 & 3\\ 3 & -2\end{bmatrix},$$
$$a_{21}=\begin{bmatrix}-2 & 3\\ 3 & 2\end{bmatrix}.$$

As in Example 1, S_i stands for S_{a_i} and (i, j) for (a_i, a_j) Then we have

$S_1=(2, 3)(4, 9)(5, 8)(6, 7)(10, 13)(11, 12)(16, 19)(17, 18)$, $S_2=(1, 3)(4, 8)(5, 6)(7, 9)(11, 14)(13, 15)(16, 20)(18, 21)$, $S_3=(1, 2)(4, 6)(5, 9)(7, 8)(10, 15)(12, 14)(17, 21)(19, 20)$, $S_4=(1, 20)(2, 8)(3, 18)(5, 10)(7, 12)(13, 17)(14, 19)(16, 21)$, $S_5=(1, 21)(2, 19)(3, 9)(4, 11)(7, 13)(12, 16)(15, 18)(17, 20)$, $S_6=(1, 7)(2, 19)(3, 18)(8, 14)(9, 15)(12, 20)(13, 21)(16, 17)$, $S_7=(1, 6)(2, 17)(3, 16)(4, 15)(5, 14)(10, 21)(11, 20)(18, 19)$, $S_8=(1, 21)(2, 4)(3, 16)(6, 10)(9, 12)(11, 19)(15, 17)(18, 20)$, $S_9=(1, 20)(2, 17)(3, 5)(6, 11)(8, 13)(10, 18)(14, 16)(19, 21)$, $S_{10}=(1, 13)(3, 15)(4, 11)(7, 21)(8, 14)(9, 18)(12, 17)(16, 20)$, $S_{11}=(1, 12)(2, 14)(5, 10)(7, 20)(8, 19)(9, 15)(13, 16)(17, 21)$, $S_{12}=(1, 11)(3, 14)(4, 15)(5, 16)(6, 20)(8, 13)(10, 19)(18, 21)$, $S_{13}=(1, 10)(2, 15)(4, 17)(5, 14)(6, 21)(9, 12)(11, 18)(19, 20)$, $S_{14}=(2, 11)(3, 12)(4, 19)(6, 10)(7, 13)(9, 16)(15, 20)(17, 18)$, $S_{15}=(2, 13)(3, 10)(5, 18)(6, 11)(7, 12)(8, 17)(14, 21)(16, 19)$, $S_{16}=(1, 15)(2, 10)(4, 21)(5, 12)(6, 17)(7, 8)(9, 14)(11, 18)$, $S_{17}=(1, 14)(3, 11)(4, 13)(5, 20)(6, 16)(7, 9)(8, 15)(10, 19)$, $S_{18}=(1, 14)(2, 12)(4, 6)(5, 15)(7, 19)(8, 20)(9, 10)(13, 16)$, $S_{19}=(1, 15)(3, 13)(4, 14)(5, 6)(7, 18)(8, 11)(9, 21)(12, 17)$, $S_{20}=(2, 10)(3, 13)(4, 9)(5, 17)(6, 12)(7, 11)(8, 18)(14, 21)$, $S_{21}=(2, 12)(3, 11)(4, 16)(5, 8)(6, 13)(7, 10)(9, 19)(15, 20)$.

It can be verified that we have the following quasi-normal symmetric subsets B_i which are mapped each other by S_j. $B_1=\{a_1, a_{14}, a_{21}\}$, $B_2=\{a_3, a_{11}, a_{18}\}$, $B_3=\{a_2, a_{12}, a_{17}\}$, $B_4=\{a_{20}, a_{19}, a_{16}\}$, $B_5=\{a_7, a_8, a_{13}\}$, $B_6=\{a_6, a_5, a_{10}\}$, and $B_7=\{a_{15}, a_9, a_4\}$. Then we have a homomorphism ϕ of the group generated by all S_i to the symmetric group of 7 objects B_j $(j=1, 2, \cdots, 7)$. For example, since $B_2S_1=B_3$, $B_5S_1=B_6$ and $B_kS_1=B_k$ $(k\neq 2, 3, 5, 6)$, we have $\phi(S_1)=(B_2, B_3)(B_5, B_6)$. Moreover we can see that the mhoomorphism is into A_7 (the alternating group). Naturally the homomorphism induces a homomorphism of $PSL_2(F_7)$ (=the group of displacements of $PSM_2(F_7)$) into A_7. Since the former is a simple group, it is an isomorphism onto a subgroup of A_7. Thus we have shown that $PSL_2(F_7)$ is a subgroup of A_7.

EXAMPLE 3. An ideal in $SM_4(F_2)$.

We consider the set of all unimodular symmetric matrices of 4×4 over F_2 that have zero diagonal. It is a symmetric set (an ideal of $SM_4(F_2)$) and consists of the following 28 elements. In the following, $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $J=\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

$$a_1=\begin{bmatrix}J&0\\0&J\end{bmatrix},\quad a_2=\begin{bmatrix}J&\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\\\begin{smallmatrix}1&0\\0&0\end{smallmatrix}&J\end{bmatrix},\quad a_3=\begin{bmatrix}J&\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\\\begin{smallmatrix}0&0\\0&1\end{smallmatrix}&J\end{bmatrix},\quad a_4=\begin{bmatrix}J&\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\\\begin{smallmatrix}0&0\\1&0\end{smallmatrix}&J\end{bmatrix},$$

$$a_5=\begin{bmatrix}J&\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\\\begin{smallmatrix}0&1\\0&0\end{smallmatrix}&J\end{bmatrix},\quad a_6=\begin{bmatrix}J&\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\\\begin{smallmatrix}1&0\\1&0\end{smallmatrix}&J\end{bmatrix},\quad a_7=\begin{bmatrix}J&\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\\\begin{smallmatrix}0&1\\0&1\end{smallmatrix}&J\end{bmatrix},\quad a_8=\begin{bmatrix}J&\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\\\begin{smallmatrix}1&1\\0&0\end{smallmatrix}&J\end{bmatrix},$$

$$a_9=\begin{bmatrix}J&\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\\\begin{smallmatrix}0&0\\1&1\end{smallmatrix}&J\end{bmatrix},\quad a_{10}=\begin{bmatrix}J&\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\\\begin{smallmatrix}1&1\\1&1\end{smallmatrix}&J\end{bmatrix},\quad a_{11}=\begin{bmatrix}0&I\\I&0\end{bmatrix},\quad a_{12}=\begin{bmatrix}0&J\\J&0\end{bmatrix},$$

$$a_{13}=\begin{bmatrix}0&\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\\\begin{smallmatrix}1&0\\1&1\end{smallmatrix}&0\end{bmatrix},\quad a_{14}=\begin{bmatrix}0&\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\\\begin{smallmatrix}1&1\\0&1\end{smallmatrix}&0\end{bmatrix},\quad a_{15}=\begin{bmatrix}0&\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\\\begin{smallmatrix}1&1\\1&0\end{smallmatrix}&0\end{bmatrix},\quad a_{16}=\begin{bmatrix}0&\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\\\begin{smallmatrix}0&1\\1&1\end{smallmatrix}&0\end{bmatrix},$$

$$a_{17}=\begin{bmatrix}0&I\\I&J\end{bmatrix},\quad a_{18}=\begin{bmatrix}0&J\\J&J\end{bmatrix},\quad a_{19}=\begin{bmatrix}0&\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\\\begin{smallmatrix}1&0\\1&1\end{smallmatrix}&J\end{bmatrix},\quad a_{20}=\begin{bmatrix}0&\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\\\begin{smallmatrix}1&1\\0&1\end{smallmatrix}&J\end{bmatrix},$$

$$a_{21}=\begin{bmatrix}0&\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\\\begin{smallmatrix}1&1\\1&0\end{smallmatrix}&J\end{bmatrix},\quad a_{22}=\begin{bmatrix}0&\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\\\begin{smallmatrix}0&1\\1&1\end{smallmatrix}&J\end{bmatrix},\quad a_{23}=\begin{bmatrix}J&I\\I&0\end{bmatrix},\quad a_{24}=\begin{bmatrix}J&J\\J&0\end{bmatrix},$$

$$a_{25}=\begin{bmatrix}J&\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\\\begin{smallmatrix}1&0\\1&1\end{smallmatrix}&0\end{bmatrix},\quad a_{26}=\begin{bmatrix}J&\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\\\begin{smallmatrix}1&1\\0&1\end{smallmatrix}&0\end{bmatrix},\quad a_{27}=\begin{bmatrix}J&\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\\\begin{smallmatrix}1&1\\1&0\end{smallmatrix}&0\end{bmatrix},\quad a_{28}=\begin{bmatrix}J&\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\\\begin{smallmatrix}0&1\\1&1\end{smallmatrix}&0\end{bmatrix}.$$

As before, we have

$S_1=(17, 23)(18, 24)(19, 25)(20, 26)(21, 27)(22, 28)$, $S_2=(3, 11)(7, 14)(9, 13)(10, 16)(18, 27)(21, 24)$, $S_3=(2, 11)(6, 13)(8, 14)(10, 15)(18, 28)(22, 24)$, $S_4=(5, 12)(7, 16)(8, 15)(10, 14)(17, 25)(19, 23)$, $S_5=(4, 12)(6, 15)(9, 16)(10, 13)(17, 20)(23, 26)$, $S_6=(3, 13)(5, 15)(8, 12)(9, 11)(20, 28)(22, 26)$, $S_7=(2, 14)(4, 16)(8, 11)(9, 12)(19, 27)(21, 25)$, $S_8=(3, 14)(4, 15)(6, 12)(7, 11)(19, 28)(22, 25)$, $S_9=(2, 13)(5, 16)(6, 11)(7, 12)(20, 27)(21, 26)$, $S_{10}=(2, 16)(3, 15)(4, 14)(5, 13)(17, 24)(18, 23)$, $S_{11}=(2, 3)(6, 9)(7, 8)(15, 16)(21, 22)(27, 28)$, $S_{12}=(4, 5)(6, 8)(7, 9)(13, 14)(19, 20)(25, 26)$, $S_{13}=(2, 9)(3, 6)(5, 10)(12, 14)(18, 20)(24, 26)$, $S_{14}=(2, 7)(3, 8)(4, 10)(12, 13)(18, 19)(24, 25)$, $S_{15}=(3, 10)(4, 8)(5, 6)(11, 16)(17, 22)(23, 28)$, $S_{16}=(2, 10)(4, 7)(5, 9)(11, 15)$

(21, 22) (23, 27), $S_{17}=(1, 23)(4, 25)(5, 26)(10, 24)(15, 22)(16, 21)$, $S_{18}=(1, 24)(2, 27)(3, 28)(10, 23)(13, 20)(14, 19)$, $S_{19}=(1, 25)(4, 23)(7, 27)(8, 28)(12, 20)(14, 18)$, $S_{20}=(1, 26)(5, 23)(6, 28)(9, 27)(12, 19)(13, 18)$, $S_{21}=(1, 27)(2, 24)(7, 25)(9, 26)(11, 22)(16, 17)$, $S_{22}=(1, 28)(3, 24)(6, 26)(8, 25)(11, 21)(15, 17)$, $S_{23}=(1, 17)(4, 19)(5, 20)(10, 18)(15, 28)(16, 27)$, $S_{24}=(1, 18)(2, 21)(3, 22)(10, 17)(13, 26)(14, 25)$, $S_{25}=(1, 19)(4, 17)(7, 21)(8, 22)(12, 26)(14, 24)$, $S_{26}=(1, 20)(5, 17)(6, 22)(9, 21)(12, 25)(13, 24)$, $S_{27}=(1, 21)(2, 18)(7, 19)(9, 20)(11, 28)(16, 23)$, $S_{28}=(1, 22)(3, 18)(6, 20)(8, 19)(11, 27)(15, 23)$.

We can verify that the length of all cycles is three and there exist six cycles which contain a given element. On the other hand, the symmetric set consisting of all transpositions in S_8 satisfies the same property. As a matter of fact, we can find an isomorphism ϕ of our symmetric set to the latter as follows. $\phi(a_1)=(1, 2)$, $\phi(a_2)=(4, 7)$, $\phi(a_3)=(4, 8)$, $\phi(a_4)=(3, 5)$, $\phi(a_5)=(3, 6)$, $\phi(a_6)=(6, 8)$, $\phi(a_7)=(5, 7)$, $\phi(a_8)=(5, 8)$, $\phi(a_9)=(6, 7)$, $\phi(a_{10})=(3, 4)$, $\phi(a_{11})=(7, 8)$, $\phi(a_{12})=(5, 6)$, $\phi(a_{13})=(4, 6)$, $\phi(a_{14})=(4, 5)$, $\phi(a_{15})=(3, 8)$, $\phi(a_{16})=(3, 7)$, $\phi(a_{17})=(1, 3)$, $\phi(a_{18})=(2, 4)$, $\phi(a_{19})=(2, 5)$, $\phi(a_{20})=(2, 6)$, $\phi(a_{21})=(1, 7)$, $\phi(a_{22})=(1, 8)$, $\phi(a_{23})=(2, 3)$, $\phi(a_{24})=(1, 4)$, $\phi(a_{25})=(1, 5)$, $\phi(a_{26})=(1, 6)$, $\phi(a_{27})=(2, 7)$, $\phi(a_{28})=(2, 8)$. Since the group of displacements of the symmetric set of all transpositions in S_8 coincides with A_8, this reestablishes the well known theorem of Dickson that $PSL_4(F_2)$ is isomorphic to A_8.

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