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NON-LINEARIZABLE REAL ALGEBRAIC ACTIONS OF $O(2, \mathbf{R})$ ON \mathbf{R}^4

HIROYUKI MIKI

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0. Introduction

In algebraic transformation groups, one of the important problems is the following.

Linearization problem ([6]). *Let G be a reductive complex algebraic group. Is any algebraic G action on affine space \mathbf{C}^n linearizable, i.e. isomorphic to some G module as G variety?*

Some positive answers to this problem have been given (see [1] for a survey article) but in 1989, G.W. Schwarz [17] constructed counterexamples for many noncommutative groups with $O(2, \mathbf{C})$ being the most explicit case (in the case that the acting group is commutative, any counterexample have never found, and see [7], [9], [11], [12] for further recent results).

In this paper, we consider the analogous problem in the real algebraic category, which was posed in [15]. Then it would be appropriate to take a compact Lie group as acting group since there is a one-to-one correspondence between the family of compact Lie groups and that of reductive complex algebraic groups through the complexification (see [14] p.247).

Schwarz used the properties of complex algebraic geometry to find the counterexamples, so it is not clear whether his argument works in the real algebraic category because \mathbf{R} is not algebraically closed. We use the methods of Masuda-Petrie [11] to obtain the following result.

Theorem. *There is a continuous family of algebraically inequivalent, non-linearizable real algebraic $O(2, \mathbf{R})$ actions on \mathbf{R}^4 .*

Let G be a compact real algebraic group and $G_{\mathbf{C}}$ be the reductive complex algebraic group obtained from G via the complexification. Let $ACT(G, \mathbf{R}^n)$ (resp. $ACT(G_{\mathbf{C}}, \mathbf{C}^n)$) be the set of equivalence classes of real algebraic G actions on \mathbf{R}^n (resp. complex algebraic $G_{\mathbf{C}}$ actions on \mathbf{C}^n), where the equivalence relation is defined by G variety (resp. $G_{\mathbf{C}}$ variety) isomorphism. Then there is a complexification map

$$c_a : ACT(G, \mathbf{R}^n) \rightarrow ACT(G_{\mathbf{C}}, \mathbf{C}^n).$$

It is natural to ask that c_a is injective, but it turns out that the examples in the theorem above give a negative answer to this question.

Proposition. *The map c_a is not injective.*

This paper is organized as follows. We consider the relation between the linearization problem and algebraic G vector bundles in section 1 and construct non-trivial real (affine) algebraic $O(2, \mathbf{R})$ vector bundles in section 2. In section 3 we consider the complexification of real algebraic G vector bundles and that of algebraic actions. In section 4 we prove the theorem above using vector bundles constructed in section 2, and apply the complexifications to the examples in the theorem. We give an explicit description of a non-linearizable real algebraic $O(2, \mathbf{R})$ action in the appendix. Most of the results in this paper are from the author's master thesis [13].

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1. Algebraic G vector bundles and non-linearizable actions

Let K be the real numbers \mathbf{R} or the complex numbers \mathbf{C} . We say that X ($\subset K^n$) is an *affine variety* if X is the set of the zeros of a map from K^n to some K^m whose coordinate functions are polynomials, and we say that $f: X \rightarrow Y$, where X ($\subset K^n$) and Y ($\subset K^m$) are affine varieties, is an *algebraic map* if f extends to a map from K^n to K^m whose coordinate functions are polynomials. A group G is an *algebraic group* if G is an affine variety and the map $\varphi: G \times G \rightarrow G$ defined by $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is algebraic, X is an *(affine) G variety* if X is an affine variety and the action map $\phi: G \times X \rightarrow X$ is algebraic, and $f: X \rightarrow Y$ is an *algebraic G map* (here X and Y are G varieties) if f is algebraic and G equivariant. An algebraic G map is an *algebraic G isomorphism* if it is bijective and its inverse is also an algebraic G map. Two G varieties are *isomorphic* if there is an algebraic G isomorphism between them.

Let G denote an algebraic group over K and let B, F, S denote G modules over K whose representation maps ($: G \times B \rightarrow B$ etc.) are algebraic.

DEFINITION 1.1. Let $Vec(B, F; S)$ be the set of algebraic G vector bundles E over B such that $E \oplus S$ is isomorphic to $F \oplus S$ as algebraic G vector bundle, where $F = B \times F$ and $S = B \times S$ are product bundles over B . We define $VEC(B, F; S)$ to be the set of isomorphism classes of elements in $Vec(B, F; S)$ as algebraic G vector

bundles.

We recall some results about $Vec(B, F; S)$ from [11]. The following results are established in [11] when $K = \mathbb{C}$. But the same argument works when $K = \mathbb{R}$.

DEFINITION 1.2. Let $sur(F \oplus S, S)$ be the set of algebraic G vector bundle surjections $L: F \oplus S \rightarrow S$ which allow an algebraic G splitting map from S to $F \oplus S$, and let $aut(F \oplus S)$ be the group of algebraic G vector bundle automorphisms τ of $F \oplus S$.

REMARK. In the complex category, any algebraic G vector bundle surjection from $F \oplus S$ to S has a splitting (see [2]). But in the real category, this is not the case. For example, $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ defined by $(a, b) \mapsto (a, (a^2 + 1)b)$ has no splitting, where $\mathbb{R} \times \mathbb{R}$ is viewed as a trivial bundle with the projection on the first factor \mathbb{R} .

The group $aut(F \oplus S)$ acts on $sur(F \oplus S, S)$ by $L \mapsto L \circ \tau$ and $L \in sur(F \oplus S, S)$ defines an element $ker L$ in $Vec(B, F; S)$.

Theorem 1.3 ([11]). *The map sending $L \in sur(F \oplus S, S)$ to $ker L \in Vec(B, F; S)$ induces a bijection*

$$sur(F \oplus S, S) / aut(F \oplus S) \cong VEC(B, F; S).$$

Because of the solution of the Serre conjecture (see [16], [19]), any vector bundle $E \in Vec(B, F; S)$ is trivial if we forget the actions. So E gives an algebraic G action on some K^n . We consider the classification of (the total spaces of) elements in $Vec(B, F; S)$ as G varieties.

DEFINITION 1.4. Let $VAR(B, F; S)$ be the set of isomorphism classes of elements in $Vec(B, F; S)$ as G varieties. Let $Aut(B)^G$ be the group of G variety automorphisms of B .

The group $Aut(B)^G$ acts on $VEC(B, F; S)$ by taking pull back bundles and the trivial element in $VEC(B, F; S)$ is fixed under the action. One easily sees that the natural map from $VEC(B, F; S)$ to $VAR(B, F; S)$ factors through the map

$$VEC(B, F; S) / Aut(B)^G \rightarrow VAR(B, F; S).$$

This map is often (but not always) bijective ([11]). We recall a sufficient condition for the above map to be bijective.

DEFINITION 1.5. Let $E_1, E_2 \in Vec(B, F; S)$ and let $f: E_1 \rightarrow E_2$ be a G variety isomorphism. We say that f maps B as graph if the composition $pfs: B \rightarrow B$ is

in $\text{Aut}(B)^G$, where $p: E_2 \rightarrow B$ is the projection and $s: B \rightarrow E_1$ is the zero-section.

Theorem 1.6 ([11]). *Suppose that any G variety isomorphism between elements in $\text{Vec}(B, F; S)$ maps B as graph. Then the natural map: $\text{VEC}(B, F; S) \rightarrow \text{VAR}(B, F; S)$ induces a bijection*

$$\text{VEC}(B, F; S) / \text{Aut}(B)^G \cong \text{VAR}(B, F; S).$$

In particular, if $E \in \text{Vec}(B, F; S)$ is non-trivial, then the G action on E is non-linearizable.

2. Non-trivial $O(2, \mathbf{R})$ vector bundles

In this section we show that $\text{VEC}(B, F; S)$ can be non-trivial. Let $O(2, \mathbf{R})$ be the real orthogonal group. We identify it with $S^1 \times \mathbf{Z}_2$. Define a two dimensional real $O(2, \mathbf{R})$ module $W_n = \{(a, \bar{a}); a \in \mathbf{C}\}$ ($n \in \mathbf{N}$) as follows (here \bar{a} denotes the complex conjugate of a). For $g \in S^1$ and $1 \neq J \in \mathbf{Z}_2$, the representation map is defined by

$$g \mapsto \begin{pmatrix} g^n & 0 \\ 0 & \bar{g}^n \end{pmatrix}, \quad J \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 2.1. *There exists a bijection: $\text{VEC}(W_1, W_m; \mathbf{R}) \cong \mathbf{R}^{m-1}$.*

In order to prove this theorem, we use Theorem 1.3. We first calculate $\text{sur}(W_m \oplus \mathbf{R}, \mathbf{R})$ and $\text{aut}(W_m \oplus \mathbf{R})$.

Lemma 2.2. (1) *Any surjection $L \in \text{sur}(W_m \oplus \mathbf{R}, \mathbf{R})$ is of the following form on the fiber over $(a, \bar{a}) \in W_1$;*

$$L(a, \bar{a}) = (f\bar{a}^m, fa^m, h),$$

where f, h are relatively prime polynomials of $t = |a|^2$ with real coefficients and $h(0) \neq 0$.

(2) *Any automorphism $\tau \in \text{aut}(W_m \oplus \mathbf{R})$ is of the following form on the fiber over $(a, \bar{a}) \in W_1$;*

$$\tau(a, \bar{a}) = \begin{pmatrix} u & a^{2m}l & a^ms \\ \bar{a}^{2m}l & u & \bar{a}^ms \\ \bar{a}^mr & a^mr & w \end{pmatrix},$$

where u, w, l, r, s are polynomials of $t = |a|^2$ and u, w are congruent to non-zero constants modulo t^m .

Proof. (1) L is linear relative to each coordinate of W_m and \mathbf{R} , so one can write

$$L(a, \bar{a}) = (L_1(a, \bar{a}), L_2(a, \bar{a}), L_3(a, \bar{a})),$$

where L_i is a polynomial for $i=1,2,3$. The S^1 equivariance of L means that

$$L_1(ga, \overline{ga}) = \bar{g}^m L_1(a, \bar{a}), \quad L_2(ga, \overline{ga}) = g^m L_2(a, \bar{a}), \quad L_3(ga, \overline{ga}) = L_3(a, \bar{a}).$$

An elementary computation shows that these imply

$$L_1(a, \bar{a}) = f_1(t) \bar{a}^m, \quad L_2(a, \bar{a}) = f_2(t) a^m, \quad L_3(a, \bar{a}) = h(t)$$

for some polynomials f_1, f_2 and h with real coefficients. The Z_2 equivariance shows that f_1 coincides with f_2 , which we denote by f . The property that f and h are relatively prime follows from the existence of a splitting of L and that $h(0)$ is non-zero follows from the surjectivity of L .

(2) Because of $O(2, \mathbf{R})$ equivariance, one can check that τ is of the form in the statement. Since τ is an automorphism,

$$\det(\tau(a, \bar{a})) = (u - t^m l)(uw - 2t^m rs + t^m lw)$$

must be a unit polynomial, which is a non-zero constant. So each factor at the right hand side is also a non-zero constant. It follows that u and uw are congruent to non-zero constants modulo t^m , hence so is w . \square

NOTATION. Let $L_{f,h}$ denote L in Lemma 2.2 (1) and $E(f, h)$ denote the kernel of $L_{f,h}$. We abbreviate $E(1, h)$ as $E(h)$. Then the vector bundle $E(h)$ (with the obvious projection on W_1) is written as follows;

$$E(h) = \{(a, \bar{a}, x, \bar{x}, z) \in W_1 \times W_m \times \mathbf{R}; \bar{a}^m x + a^m \bar{x} + h(t)z = 0\}.$$

Note that if h is a non-zero constant, $E(h)$ is isomorphic to W_m through the correspondence $(a, \bar{a}, x, \bar{x}, z) \mapsto (a, \bar{a}, x, \bar{x})$.

Lemma 2.3. *There are three vector bundle isomorphisms.*

- (1) $E(f, h) \cong E(f, h/h(0))$.
- (2) $E(f, h) \cong E(h)$.
- (3) $E(h_1) \cong E(h_2)$ if and only if there is a non-zero constant c such that $h_1 \equiv ch_2$ modulo t^m .

Proof. (1) $(x, \bar{x}, z) \mapsto (x, \bar{x}, h(0)z)$ is the required isomorphism.

(2) By Theorem 1.3 and Lemma 2.2 (2), it suffices to show the existence of polynomials u, w, l, r, s such that

$$(\bar{a}^m \ a^m \ h) = (f \bar{a}^m \ f a^m \ h) \begin{pmatrix} u & a^{2m} l & a^m s \\ \bar{a}^{2m} l & u & \bar{a}^m s \\ \bar{a}^m r & a^m r & w \end{pmatrix}$$

and that the determinant of the above 3×3 matrix is a non-zero constant. Choose polynomials ξ and η of t such that $f\xi + h\eta = 1$ (this is possible since f and h are

relatively prime by Lemma 2.2 (1)) and polynomials r' and r'' of t such that $hr' = (1-f) - t^m r''$ (this is possible since $h(0) \neq 0$ by Lemma 2.2 (1)). Then one can check that

$$u = 1 + t^m l, \quad w = 1 - 2t^m f l, \quad s = h l, \quad l = \xi r'' / 2, \quad r = r' + t^m \eta r''$$

satisfies the required conditions.

(3) If $E(h_1) \cong E(h_2)$ there is $\tau \in \text{aut}(W_m \oplus R)$ such that $L_{1,h_1} = L_{1,h_2} \circ \tau$, i.e.

$$(\bar{a}^m \ a^m \ h_1) = (\bar{a}^m \ a^m \ h_2) \begin{pmatrix} u & a^{2m} l & a^m s \\ \bar{a}^{2m} l & u & \bar{a}^m s \\ \bar{a}^m r & a^m r & w \end{pmatrix},$$

where the determinant of the above 3×3 matrix is a non-zero constant. Hence $h_1 = h_2 w + 2t^m s$. Since w is a non-zero constant modulo t^m by Lemma 2.2 (2), the necessity is clear. Conversely if $h_1 = ch_2 + t^m h_0$ for some polynomial h_0 of t , then $\tau \in \text{aut}(W_m \oplus R)$ defined by

$$\tau(a, \bar{a}) = \begin{pmatrix} 1 & 0 & a^m h_0 / 2 \\ 0 & 1 & \bar{a}^m h_0 / 2 \\ 0 & 0 & c \end{pmatrix}$$

is the isomorphism between $E(h_1)$ and $E(h_2)$. □

Proof of Theorem 2.1. By Theorem 1.3 and Lemma 2.2 (1), any element in $\text{VEC}(W_1, W_m; R)$ is of the form $[E(f, h)]$, where $[\]$ denotes the isomorphism class. Then Lemma 2.3 implies that the correspondence

$$R^{m-1} \ni (a_1, \dots, a_{m-1}) \mapsto [(E(h))],$$

where $h(t) = 1 + a_1 t + \dots + a_{m-1} t^{m-1}$, gives the bijection. □

3. Complexification

In this section, we assume that G is a real algebraic group and B, F, S are real G modules. We first define the complexification of real affine varieties and algebraic maps and prove some properties.

DEFINITION 3.1. Let $X(\subset R^n)$ be a real affine variety and let $I(X)$ be the ideal of polynomial maps from R^n to R which vanish on X . We define the complex affine variety X_C to be the common zeros of all the elements in $I(X)$ regarded as maps from C^n to C , and we call X_C the *complexification* of X .

Here are some elementary properties about the complexification.

Proposition 3.2. (1) Let $I(X_C)$ be the ideal of polynomial maps from C^n to C which vanish on X_C . Then $I(X_C) = I(X) \otimes C$.

(2) $(X \times Y)_C = X_C \times Y_C$.

(3) Any algebraic map $f: X \rightarrow Y$ extends to a unique algebraic map $f_C: X_C \rightarrow Y_C$.

Proof. (1) It is clear that $I(X_C) \supset I(X) \otimes C$ by definition. We prove the opposite inclusion. For $f \in I(X_C)$, we express $f = f_1 + if_2$, where f_1 and f_2 are polynomials with real coefficients. Then $f_1|_X + if_2|_X = f|_X = 0$, so f_1 and f_2 are in $I(X)$. This means that $I(X_C) \subset I(X) \otimes C$.

(2) The ideal $I(X \times Y)$ is generated by the elements $f_i h_s$, where $f_i \in I(X)$ and $h_s \in I(Y)$. This together with (1) shows that the ideal $I((X \times Y)_C)$ is generated by the elements $\tilde{f}_i \tilde{h}_s$, where $\tilde{f}_i \in I(X_C)$ and $\tilde{h}_s \in I(Y_C)$. This implies (2).

(3) Suppose $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$ and let $F: \mathbf{R}^n \rightarrow \mathbf{R}^m$ be an extension of f . We regard F as a map from C^n to C^m . One easily checks that F maps X_C to Y_C . Therefore $F|_{X_C}: X_C \rightarrow Y_C$ is an extension of f . Now we prove the uniqueness. Suppose that two maps $f_1, f_2: X_C \rightarrow Y_C$ are extensions of f . Let $F_j: C^n \rightarrow C^m$ be an extension of f_j ($j=1,2$). Then $F_1 - F_2$ is algebraic and vanishes on X . Therefore $F_1 - F_2$ vanishes on X_C by (1). Hence $f_1 - f_2 = (F_1 - F_2)|_{X_C} = 0$, i.e. $f_1 = f_2$. \square

We call f_C the *complexification* of f . By Proposition 3.2, we obtain the following.

Corollary 3.3. (1) The complexification of a real algebraic group is a complex algebraic group.

(2) If G is a real algebraic group and X is a real G variety, X_C is a complex G_C variety.

(3) If X and Y are real G varieties and $f: X \rightarrow Y$ is G equivariant, then $f_C: X_C \rightarrow Y_C$ is G_C equivariant.

(4) If $f: X \rightarrow Y$ and $h: Y \rightarrow Z$ are algebraic G maps between real G varieties, then $(f \circ h)_C = f_C \circ h_C$.

Now we define a complexification of elements in $VEC(B, F; S)$ and an involution on $VEC(B_C, F_C; S_C)$. Note that the usual complexification of vector bundles means to complexify only fibers, but our definition means to complexify also base space. Let L be an element in $sur(F \oplus S, S)$. The map $L_C: (F \oplus S)_C \rightarrow S_C$ is G_C equivariant and has a splitting because if P is an algebraic G splitting of L then P_C is an algebraic G_C splitting of L_C . Hence L_C is in $sur((F \oplus S)_C, S_C)$. Let L' be another element of $sur(F \oplus S, S)$. If $L' = L \circ \tau$ for some $\tau \in aut(F \oplus S)$, then $L'_C = L_C \circ \tau_C$ and $\tau_C \in aut(F \oplus S)_C$. Therefore the following definition makes sense, i.e. it does not depend on the choice of L .

DEFINITION 3.4. Let $[E] \in VEC(B, F; S)$ and let $L \in sur(F \oplus S, S)$ represent E , i.e.

$E = \ker L$. Then we define the *complexification* of $[E]$ by $[\ker L_C] \in \text{VEC}(B_C, F_C; S_C)$.

Let $X(\subset \mathbf{R}^n)$ be a real G variety. For $x \in X_C (\subset \mathbf{C}^n)$, the complex conjugation \bar{x} is also in X_C since $f(\bar{x}) = 0$ for any $f \in I(X)$. Hence X_C has an involution defined by $x \mapsto \bar{x}$. Similarly, G_C has an involution. Since the action map: $G \times X \rightarrow X$ is real algebraic, we have $\overline{g \cdot x} = \bar{g} \cdot \bar{x}$ for any $g \in G_C$ and $x \in X_C$.

DEFINITION 3.5. For $L \in \text{sur}((F \oplus S)_C, S_C)$, we define $\bar{L}: (F \oplus S)_C \rightarrow S_C$ by

$$\bar{L}(b, f, s) = \overline{L(\bar{b}, \bar{f}, \bar{s})}.$$

One can check that \bar{L} is in $\text{sur}((F \oplus S)_C, S_C)$. So the correspondence $L \mapsto \bar{L}$ induces an involution on $\text{VEC}(B_C, F_C; S_C)$. Since $\overline{L_C} = L_C$ for $L \in \text{sur}(F \oplus S, S)$, the complexification in Definition 3.4 induces a map

$$c_b: \text{VEC}(B, F; S) \rightarrow \text{VEC}(B_C, F_C; S_C)^{Z_2}.$$

We ask

Complexification problem (vector bundle case). *Is the above map c_b bijective?*

We turn to the complexification of actions. Let $ACT(G, \mathbf{R}^n)$ (resp. $ACT(G_C, \mathbf{C}^n)$) be the set of the equivalence classes of real algebraic G actions on \mathbf{R}^n (resp. complex algebraic G_C actions on \mathbf{C}^n), where the equivalence relation is defined by G variety (resp. G_C variety) isomorphism. By the complexification of real G varieties, we obtain a map

$$c_a: ACT(G, \mathbf{R}^n) \rightarrow ACT(G_C, \mathbf{C}^n).$$

Complexification problem (action case). *Is the above map injective?*

We deal with these problems in the next section.

4. Non-linearizable actions and the complexification problems

We first classify the elements in $\text{Vec}(W_1, W_m; \mathbf{R})$ as $O(2, \mathbf{R})$ varieties, i.e. we calculate $\text{VAR}(W_1, W_m; \mathbf{R})$. We show that the assumption of Theorem 1.6 is satisfied.

Lemma 4.1. *Any $O(2, \mathbf{R})$ variety isomorphism between elements in $\text{Vec}(W_1, W_m; \mathbf{R})$ maps W_1 as graph.*

Proof. Let E_1, E_2 be elements in $\text{Vec}(W_1, W_m; \mathbf{R})$ and $f: E_1 \rightarrow E_2$ be an $O(2, \mathbf{R})$ variety isomorphism. We show that pfs is in $\text{Aut}(W_1)^{O(2, \mathbf{R})}$, where $p: E_2 \rightarrow W_1$ is the projection and $s: W_1 \rightarrow E_1$ is the zero-section. Take the complexification

$f_C: (E_1)_C \rightarrow (E_2)_C$, which is an $O(2, \mathbb{C})$ variety isomorphism. According to [11], f_C maps $(W_1)_C$ as graph, in fact, $p_C f_C s_C: (W_1)_C \rightarrow (W_1)_C$ is a non-zero scalar multiplication. We recall the proof. The map $f_C s_C$ is $O(2, \mathbb{C})$ equivariant, so it is of the form

$$(W_1)_C \ni (a, b) \mapsto (a f_0, b f_0, a^m h_0, b^m h_0, k_0),$$

where f_0, h_0 and k_0 are polynomials of $t=ab$. If f_0 is not a non-zero constant, f_0 has some zero t_0 . Let ζ be a primitive m -th root of 1. Then $f_C s_C$ maps $(t_0, 1)$ and $(\zeta t_0, \zeta^{-1})$ to the same element $(0, 0, a^m h_0(t_0), b^m h_0(t_0), k_0(t_0))$, which contradicts to the injectivity of $f_C s_C$. Hence f_0 must be a non-zero constant. Finally since $p_C f_C s_C$ is the complexification of $p f s$, it preserves W_1 . This proves that $p f s \in \text{Aut}(W_1)^{O(2, \mathbb{R})}$. \square

We can check $\text{Aut}(W_1)^{O(2, \mathbb{R})} = \mathbb{R}^*$ using the $O(2, \mathbb{R})$ equivariance. Suppose that $E(h_1)$ is isomorphic to $E(h_2)$ as $O(2, \mathbb{R})$ varieties. Then $E(h_1)$ is isomorphic to $c^* E(h_2)$ as $O(2, \mathbb{R})$ vector bundles for some $c \in \text{Aut}(W_1)^{O(2, \mathbb{R})} = \mathbb{R}^*$ by Theorem 1.6 and Lemma 4.1. The fiber of $c^* E(h_2)$ over (a, \bar{a}) is the set of points satisfying the equation; $c^m(\bar{a}^m x + a^m \bar{x}) + h_2(c^2 t)z = 0$. Then

$$\begin{aligned} c^* E(h_2) &= \{(a, \bar{a}, x, \bar{x}, z); c^m(\bar{a}^m x + a^m \bar{x}) + h_2(c^2 t)z = 0\} \\ &\cong \{(a, \bar{a}, x, \bar{x}, z); \bar{a}^m x + a^m \bar{x} + h_2(c^2 t)z = 0\} \end{aligned}$$

by Lemma 2.3 (1). Hence $h_1(t)$ is congruent to $h_2(c^2 t)$ modulo t^m by Lemma 2.3 (3) and we obtain the following bijection.

Theorem 4.2. $\text{VAR}(W_1, W_m; \mathbb{R}) \cong \mathbb{R}^{m-1} / \mathbb{R}^*$, where the \mathbb{R}^* action on \mathbb{R}^{m-1} is defined as follows. For $c \in \mathbb{R}^*$ and $(a_1, \dots, a_{m-1}) \in \mathbb{R}^{m-1}$,

$$(a_1, \dots, a_{m-1}) \xrightarrow{c} (c^2 a_1, c^4 a_2, \dots, c^{2(m-1)} a_{m-1}).$$

Proof of the Theorem (in introduction). By Theorem 1.6, it suffices to show that the set $\text{VAR}(W_1, W_m; \mathbb{R})$ can be continuous density, but Theorem 4.2 says that the case $m \geq 3$ satisfies this condition. \square

Next we apply the complexification defined in section 3 to the $O(2, \mathbb{R})$ case. We recall Schwarz's [17] and Masuda-Petrie's [11] results in the complex category. Here $O(2, \mathbb{C}) = \mathbb{C}^* \ltimes \mathbb{Z}_2$ and its action on $(W_m)_C = \{(a, b) \in \mathbb{C}^2\}$ is defined as follows. For $g \in \mathbb{C}^*$, $1 \neq J \in \mathbb{Z}_2$ and $(a, b) \in (W_m)_C$,

$$(a, b) \xrightarrow{g} (g^m a, g^{-m} b) \quad (a, b) \xrightarrow{J} (b, a).$$

Theorem 4.3 ([11],[17]). $\text{VEC}((W_1)_C, (W_m)_C; \mathbb{C}) \cong \mathbb{C}^{m-1}$, where the correspon-

dence is defined similarly to Theorem 2.1.

Theorem 4.4 ([11]). $VAR((W_1)_C, (W_m)_C; C) \cong C^{m-1} / C^*$, where the C^* action on C^{m-1} is defined similarly to Theorem 4.2.

We study the involution on $VEC((W_1)_C, (W_m)_C; C)$. Any element of $VEC((W_1)_C, (W_m)_C; C)$ is represented by $L \in \text{sur}((W_m \oplus R)_C, C)$ of the form;

$$L(a, b, x, y, z) = b^m x + a^m y + f(t)z,$$

where $t = ab$ and f is a polynomial with real coefficients. Then

$$\bar{L}(a, b, x, y, z) = \overline{L(\bar{a}, \bar{b}, \bar{x}, \bar{y}, \bar{z})} = b^m x + a^m y + \bar{f}(t)z,$$

where \bar{f} is a polynomial whose coefficients are complex conjugate of those of f . So the involution on $VEC((W_1)_C, (W_m)_C; C)$ coincides with the complex conjugate on C^{m-1} through the bijection in Theorem 4.3. This together with Theorem 2.1 shows that the complexification map

$$c_b: VEC(W_1, W_m; R) \rightarrow VEC((W_1)_C, (W_m)_C; C)^{Z_2}$$

is bijective.

Now we turn to the case of actions. Remember that we have the complexification map

$$c_a: ACT(O(2, R), R^4) \rightarrow ACT(O(2, C), C^4).$$

The sets $VAR(W_1, W_m; R)$ and $VAR((W_1)_C, (W_m)_C; C)$ are subsets of $ACT(O(2, R), R^4)$ and $ACT(O(2, C), C^4)$ respectively and c_a maps $VAR(W_1, W_m; R)$ into $VAR((W_1)_C, (W_m)_C; C)$. Through the bijections in Theorems 4.2 and 4.4, one can see that the map c_a restricted to $VAR(W_1, W_m; R)$ is nothing but the map from R^{m-1} / R^* to C^{m-1} / C^* induced from the natural inclusion $R^{m-1} \subset C^{m-1}$. An elementary observation shows that the map from R^{m-1} / R^* to C^{m-1} / C^* is not injective, in fact, the inverse image of an element in C^{m-1} / C^* consists of one or two elements. This gives a negative answer to the complexification problem in the action case. However $c_a^{-1}([0]) = [0]$, where $[0]$ denotes the element in R^{m-1} / R^* or C^{m-1} / C^* represented by 0. Since $[0]$ corresponds to a linear action, we pose

Weak complexification problem. *If the complexification of a real algebraic action on R^n is linearizable, then is the action itself linearizable?*

Appendix

We give an explicit description of a non-linearizable real algebraic $O(2, R)$ action on R^4 obtained from Theorem 4.2. For example, we take $E(1 - t^2) \in \text{Vec}(W_1, W_4; R)$.

The following (nonequivariant) algebraic vector bundle automorphism of $W_4 \oplus R$ gives a trivialization of $E(1-t^2) \cong W_4 \subset W_4 \oplus R$.

$$\tau(a, \bar{a}) = \begin{pmatrix} 1+it & 0 & -a^4/2 \\ 0 & 1-it & -a^{-4}/2 \\ a^{-4} & a^4 & 1-t^2 \end{pmatrix}.$$

We define $\sigma: R^4 \rightarrow W_4$ by $(a, b, x, y) \mapsto (a+ib, a-ib, x+iy, x-iy)$. Then it suffices to calculate the correspondence of the composition map in the following;

$$R^4 \xrightarrow{\sigma} W_4 \xrightarrow{\tau^{-1}} E(1-t^2) \xrightarrow{\text{action}} E(1-t^2) \xrightarrow{\tau} W_4 \xrightarrow{\sigma^{-1}} R^4.$$

It turns out that the actions on R^4 of $g = \cos \theta + i \sin \theta \in S^1$ and $1 \neq J \in Z_2$ ($\subset O(2, R)$) are as follows.

$$\begin{aligned} \left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) &\xrightarrow{g} \left(\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \cos 4\theta & -\sin 4\theta \\ \sin 4\theta & \cos 4\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ \left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) &\xrightarrow{J} \left(\begin{pmatrix} a \\ -b \end{pmatrix}, \begin{pmatrix} -f_2 t + 2t^4 - 2t^2 + 1 & f_1 t + t^5 - 2t^3 + 2t \\ -f_1 t + t^5 - 2t^3 + 2t & -f_2 t - 2t^4 + 2t^2 - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right), \end{aligned}$$

where $t = a^2 + b^2$, and f_1, f_2 are polynomials of a, b with the real coefficients such that $(a+ib)^8 = f_1 + if_2$.

References

- [1] H. Bass: *Algebraic group actions on affine spaces*, Comtemp. Math. **43** (1985), 1–23.
- [2] H. Bass and W. Haboush: *Linearizing certain reductive group actions*, Trans. Amer. Math. Soc. **292** (1984), 463–482.
- [3] H. Bass and W. Haboush: *Some equivariant K-theory of affine algebraic group actions*, Comm. in Alg. **15** (1987), 181–217.
- [4] J. Bochnak, M. Coste and M.F. Roy: *Géométrie algébrique réelle*, Ergebnisse der Math. und ihrer Grenzgebiete, Band 12, Springer-Verlag, 1987.
- [5] J.E. Humphreys: *Linear Algebraic Groups*, Graduate Texts in Math. 21, Springer-Verlag, 1972.
- [6] T. Kambayashi: *Automorphism group of a polynomial ring and algebraic group actions on an affine space*, J. Algebra **60** (1979), 439–451.
- [7] F. Knop: *Nichtlinearisierbare Operationen halbeinfacher Gruppen auf affinen Räumen*, Invent. Math. **105** (1991), 217–220.
- [8] H. Kraft: *G-vector bundles and the linearization problem*, in Group Actions and Invariant Theory, CMS Conf. Proc. **10** (1988), 111–123.
- [9] H. Kraft and G.W. Schwarz: *Reductive group actions with one dimensional quotient*, Inst. Hautes Études Sci. Publ. Math. **76** (1992), 1–97.
- [10] M. Masuda: *Algebraic transformation groups from topological point of view*, Sugaku.
- [11] M. Masuda and T. Petrie: *Equivariant algebraic vector bundles over representations of reductive groups: Theory*, Proc. Nat. Acad. Sci. USA **88** (1991), 9061–9064.
- [12] M. Masuda, L. Moser-Jauslin and T. Petrie: *Equivariant algebraic vector bundles over*

representations of reductive groups: Applications, Proc. Nat. Acad. Sci. USA **88** (1991), 9065–9066.

- [13] H. Miki: *Non-linearizable compact real algebraic group actions on \mathbf{R}^n* , Master Thesis (in Japanese), Osaka Cith Univ. (1991).
- [14] A.L. Onishchik and E.B. Vinberg: *Lie Groups and Algebraic Groups*, Springer-Verlag, 1988.
- [15] T. Petrie and J.D. Randall: *Finite-order algebraic automorphisms of affine varieties*, Comm. Math. Helv. **61** (1986), 203–221.
- [16] D. Quillen: *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.
- [17] G.W. Schwarz: *Exotic algebraic group actions*, C.R. Acad. Sci. **309** (1989), 89–94.
- [18] T.A. Springer: *Aktionen reductiver Gruppen auf Varietäten*, in DMV Seminar Band 13: Algebraische Transformationsgruppen und Invariantentheorie, Birkhäuser-Verlag (1989), 3–39.
- [19] A. Suslin: *Projective modules over a polynomial ring*, Dokl. Akad. Nauk SSSR **26** (1976).

Hiroyuki Miki
Department of Mathematics
Osaka City University
Osaka 558, Japan