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HOMOLOGICAL PROPERTIES OF THE ENDOMORPHISM RINGS OF CERTAIN PERMUTATION MODULES

To the memory of Akira Hattori

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Introduction

Let G be a finite group and k be a field of characteristic $p>0$. The purpose of this paper is to study homological properties of the endomorphism ring Λ of the $k[G]$-module $M=\oplus k[G/H]$, summed over all subgroups H of G. Our main results are the following.

Theorem A. *If G is not a p'-group, then the finitistic dimension of the ring Λ, that is, the supremum of finite projective dimensions of finitely generated Λ-modules, is equal to*

$$1+\sup\{\operatorname{rank} H/\Phi(H)\}$$

where H runs over all p-subgroups of G and $\Phi(H)$ denotes the Frattini subgroup of H.

Theorem B. *Let G be an elementary abelian p-group. For a subgroup H of G, let S_H be a simple Λ-module corresponding to the summand $k[G/H]$ of M. Then, for subgroups H and H', the least integer i such that $\operatorname{Ext}^i_\Lambda(S_H, S_{H'})\neq 0$ is equal to*

$$\operatorname{rank}(H+H')/H+\operatorname{rank}(H+H')/H'\,.$$

In Section 4 we compute $\operatorname{Ext}^i_\Lambda(F, F')$ for certain Λ-modules F, F' when G is an elementary abelian p-group. Theorem B is contained in Theorem 4.1 there. In Section 5 we deduce Theorem A from Theorem 4.1. We need some results in Lusztig [5] on the homology of the partially ordered set of subgroups of an elementary abelian p-group, and simplified proofs of them are given in Section 2.

The ring Λ arises in the theory of G-functors initiated by Green [3]. In fact, Λ-modules and cohomological G-functors over k are equivalent notions, as was shown by Yoshida [6]. Since the language of G-functors is useful for our computations in Section 4, we review the correspondence between Λ-modules and cohomological G-functors in Section 1.

All modules are assumed to be left and finitely generated. For a ring Γ the category of Γ-modules is denoted by Γ-$\mathcal{M}od$.

1. Permutation modules and cohomological G-functors

Let G be a finite group and k a field. A G-set is a set on which G acts on the left. A permutation module is a $k[G]$-module of the form $k[S]=\bigoplus_{s\in S} ks$ with S a finite G-set. Denote by $P(G)$ the category formed by permutation modules and $k[G]$-homomorphisms, and by $P(G)^\wedge$ the category of k-linear functors $P(G)^{\mathrm{op}}\to k$-$\mathcal{M}od$. If $\mathcal{F}$ is a family of subgroups of G such that every subgroup is conjugate to a member of $\mathcal{F}$, then $P(G)^\wedge$ is equivalent to the category of Λ-modules, where $\Lambda=\mathrm{End}_G(\bigoplus_{H\in\mathcal{F}} k[G/H])^{\mathrm{op}}$. The category $P(G)$ is self-dual. In fact the functor $X\mapsto X^\vee=\mathrm{Hom}_k(X, k)$ gives an equivalence $P(G)^{\mathrm{op}}\simeq P(G)$. Moreover, since $X^\vee\cong X$ for any permutation module X, Λ is isomorphic to Λ^{op}. We will work with $P(G)^\wedge$ rather than Λ-$\mathcal{M}od$.

We can also interpret $P(G)^\wedge$ as the category of cohomological G-functors. Let us recall Green's definition in [3]. A cohomological G-functor F is a family of k-modules $F(H)$ for all subgroups H of G and k-linear maps ρ_H^K: $F(K)\to F(H)$, $\tau_H^K: F(H)\to F(K)$, $\gamma_g: F(H)\to F(H^g)=F(g^{-1}Hg)$ defined for all pairs of subgroups H, K such that $H\subset K$ and for all $g\in G$, satisfying the following axioms.

(i) If H is a subgroup of G, then $\rho_H^H(x)=\tau_H^H(x)=\gamma_g(x)=x$ for $x\in F(H)$, $g\in H$. If H, K, L are subgroups of G such that $H\subset K\subset L$, then $\rho_H^K\rho_K^L(x)=\rho_H^L(x)$ for $x\in F(L)$ and $\tau_K^L\tau_H^K(y)=\tau_H^L(y)$ for $y\in F(H)$. If H is a subgroup of G and $g, g'\in G$, then $\gamma_{g'}\gamma_g(x)=\gamma_{gg'}(x)$ for $x\in F(H)$.

(ii) If H, K are subgroups of G such that $H\subset K$ and $g\in G$, then $\gamma_g\rho_H^K(x)=\rho_{H^g}^{K^g}\gamma_g(x)$for $x\in F(K)$ and $\gamma_g\tau_H^K(y)=\tau_{H^g}^{K^g}\gamma_g(y)$ for $y\in F(H)$.

(iii) (Mackey axiom) If H, K, L are subgroups of G such that $H\subset L$, $K\subset L$, then

$$\rho_K^L\tau_H^L(x)=\sum_g \tau_{H^g\cap K}^K\rho_{H^g\cap K}^{H^g}\gamma_g(x)$$

for $x\in F(H)$, where g runs over representatives for (H, K)-cosets in L.

(iv) If H, K are subgroups of G such that $H\subset K$, then

$$\tau_H^K\rho_H^K(x)=|K: H|x$$

for $x\in F(K)$.

The maps ρ_H^K, τ_H^K, γ_g are called restriction maps, transfer maps, conjugation maps respectively. A morphism $F\to F'$ of cohomological G-functors is a family of k-linear maps $F(H)\to F'(H)$ which commute with the restriction, transfer and conjugation maps.

An object M of $P(G)^\wedge$ determines a cohomological G-functor F in the following way. For each subgroup H of G, set $F(H)=M(k[G/H])$. For subgroups H, K such that $H\subset K$, define G-homomorphisms $r: k[G/H]\to k[G/K]$ and $t: k[G/K]\to k[G/H]$ by $r(xH)=xK$ for $x\in G$ and $t(yK)=\sum xH$, the sum of H-cosets contained in yK, for $y\in G$. For $g\in G$, define a G-isomorphism c_g: $k[G/H]\to k[G/H^g]$ by $c_g(xH)=xHg=xgH^g$ for $x\in G$. We set $\rho_H^K=M(r)$, $\tau_H^K=M(t)$, $\gamma_g=M(c_g^{-1})$. Then one can verify that $F(H)$, ρ_H^K, τ_H^K, γ_g form a cohomological G-functor F. Furthermore, Yoshida proved in [6] that the assignment $M\mapsto F$ gives an equivalence from $P(G)^\wedge$ to the category of cohomological G-functors. For details about G-functors, see [3], [6]. But we do not use anything other than this correspondence between objects of $P(G)^\wedge$ and cohomological G-functors.

2. Homology of Sub (V)

Throughout this section k is a finite field with q elements. Let V be a k-vector space of dimension n. Denote by Sub (V) the set of subspaces of V. Sub (V) is partially ordered by inclusion and so viewed as a category in the usual way (see Gabriel and Zisman [2]). Denote by $\mathrm{Sub}(V)^\wedge$ the category of functors $\mathrm{Sub}(V)^{\mathrm{op}}\to k\text{-}\mathcal{M}od$. An object of $\mathrm{Sub}(V)^\wedge$ is a family $\{F(U)(U\in\mathrm{Sub}(V)), \varphi_{U,U'}(U, U'\in\mathrm{Sub}(V), U\subset U')\}$ of vector spaces $F(U)$ and linear maps $\varphi_{U,U'}$: $F(U')\to F(U)$, satisfying the conditions that $\varphi_{U,U}=\mathrm{id}$, $\varphi_{U,U'}\circ\varphi_{U',U''}=\varphi_{U,U''}$ whenever $U\subset U'\subset U''$. The maps $\varphi_{U,U'}$ are called restriction maps. Ext groups in the abelian category $\mathrm{Sub}(V)^\wedge$ are written as $\mathrm{Ext}^i_{\mathrm{Sub}(V)}(\ ,\ )$. For generalities about functor categories we refer to Grothendieck and Verdier [4].

For $U\in\mathrm{Sub}(V)$, let S_U be the object of $\mathrm{Sub}(V)^\wedge$ such that $S_U(X)=k$ if $X=U$, $S_U(X)=0$ if $X\neq U$. Results on $\mathrm{Ext}^i_{\mathrm{Sub}(V)}$ which we need later are the following.

Proposition 2.1. *Let F: $k\text{-}\mathcal{M}od^{\mathrm{op}}\to k\text{-}\mathcal{M}od$ be a (not necessarily additive) functor and let V be a k-vector space. Denote by $F(V/-)$ the functor* $\mathrm{Sub}(V)^{\mathrm{op}}\to k\text{-}\mathcal{M}od$ *taking X to $F(V/X)$. Then we have* $\mathrm{Ext}^i_{\mathrm{Sub}(V)}(S_U, F(V/-))=0$ *for* $U\in\mathrm{Sub}(V)$, $i\neq\dim U$.

Proposition 2.2. *Let V be a vector* space *and V' a subspace of V. Let* $C_{V,V'}$: $\mathrm{Sub}(V)^{\mathrm{op}}\to k\text{-}\mathcal{M}od$ *be the functor such that*

$$\begin{aligned} C_{V,V'}(X) &= k \qquad \text{if}\ \ X+V'=V\,, \\ &= 0 \qquad \text{if}\ \ X+V'\neq V\,, \end{aligned}$$

and the restriction maps from k to k are the identity. Then we have for $U\in\mathrm{Sub}(V)$, $\mathrm{Ext}^i_{\mathrm{Sub}(V)}(S_U, C_{V,V'})\neq 0$ *if and only if* $U+V'=V$ *and* $i=\dim(U\cap V')$.

Proposition 2.3. *Let V be a vector space and U, W subspaces of V. Then* $\mathrm{Ext}^i_{\mathrm{Sub}(V)}(S_U, S_W) \neq 0$ *if and only if* $W \subset U$ *and* $i = \dim U/W$.

Proposition 2.3 is the well-known Solomon-Tits theorem (see [5, p. 12]) and Proposition 2.2 is [5, Theorem 1.11] and Proposition 2.1 is a generalization of [5, Theorem 1.12] (see Corollary 2.6 below). In [5] these results are stated in terms of simplicial complexes. A link between geometric language and ours may be found in [2, Appendix]. We will prove these results here for the sake of completeness. Our proofs are simpler than those in [5].

As preparation we define adjoint functors between categories of the form $\mathrm{Sub}(V)^\wedge$. Let $f\colon V' \to V$ be a linear map. There are functors

$$\mathrm{Sub}\,(V') \underset{f^s}{\overset{f_s}{\rightleftarrows}} \mathrm{Sub}\,(V)$$

defined by $f_s(X') = f(X')$, $f^s(X) = f^{-1}(X)$, and f_s is a left adjoint of f^s (in such a case we write $f_s \dashv f^s$). Then we have four functors

$$\mathrm{Sub}\,(V')^\wedge \;\begin{array}{c} \xrightarrow{\;f_!\;} \\ \xleftarrow{\;f^*\;} \\ \xrightarrow{\;f_*\;} \\ \xleftarrow{\;f^!\;} \end{array}\; \mathrm{Sub}\,(V)^\wedge$$

defined by

$$\begin{aligned} (f_!F')(X) &= \varinjlim_{X \subset f(X')} F'(X') \\ (f^*F)(X') &= F(f(X'))\,, && \text{i.e.,}\quad f^*F = F \circ f_s \\ (f_*F')(X) &= F'(f^{-1}(X))\,, && \text{i.e.,}\quad f_*F' = F' \circ f^s \\ (f^!F)(X') &= \varprojlim_{f^{-1}(X) \subset X'} F(X) \end{aligned}$$

for $F \in \mathrm{Sub}(V)^\wedge$, $F' \in \mathrm{Sub}(V')^\wedge$, $X \in \mathrm{Sub}(V)$, $X' \in \mathrm{Sub}(V')$, and these form a sequence of adjoints $f_! \dashv f^* \dashv f_* \dashv f^!$ (see [4, n° 5]). Since f^*, f_* are exact, there are natural isomorphisms

$$\mathrm{Ext}^i_{\mathrm{Sub}(V')}(f^*F, F') \cong \mathrm{Ext}^i_{\mathrm{Sub}(V)}(F, f_*F')\,.$$

When V' is a subspace of V and $f\colon V' \to V$ is the inclusion map, then $f_!$ is given by

$$\begin{aligned} (f_!F')(X) &= F'(X) && \text{if}\quad X \subset V'\,, \\ &= 0 && \text{if}\quad X \not\subset V'\,. \end{aligned}$$

Hence $f_!$ is also exact and

$$\mathrm{Ext}^i_{\mathrm{Sub}(V)}(f_!F', F) \cong \mathrm{Ext}^i_{\mathrm{Sub}(V')}(F', f^*F).$$

When U is a subspace of V' and $f\colon V' \to V'/U = V$ is the projection, then $f^!$ is given by

$$\begin{aligned}(f^!F)(X') &= F(X'/U) && \text{if} \quad U \subset X', \\ &= 0 && \text{if} \quad U \not\subset X',\end{aligned}$$

and hence

$$\mathrm{Ext}^i_{\mathrm{Sub}(V)}(f_*F', F) \cong \mathrm{Ext}^i_{\mathrm{Sub}(V')}(F', f^!F).$$

Proof of Proposition 2.1. Let $j\colon U \to V$ be the inclusion map and let

$$\mathrm{Sub}(U)^\wedge \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathrm{Sub}(V)^\wedge$$

be the adjoint defined above. Define a functor $F_{V/U}\colon k\text{-}\mathcal{M}od^{\mathrm{op}} \to k\text{-}\mathcal{M}od$ by $F_{V/U}(X) = F(V/U \oplus X)$. Taking a complement of U in V, we have an isomorphism $V/- \cong V/U \oplus U/-$ as functors on $\mathrm{Sub}(U)$. Hence $j^*F(V/-) \cong F_{V/U}(U/-)$. Also $j_!S_U = S_U$. It then follows that

$$\mathrm{Ext}^i_{\mathrm{Sub}(V)}(S_U, F(V/-)) \cong \mathrm{Ext}^i_{\mathrm{Sub}(U)}(S_U, F_{V/U}(U/-)).$$

Thus we are reduced to the case when $U = V$.

In this case we proceed by induction on $\dim V$. When $V = 0$, the assertion is trivial. Let $V \neq 0$ and write $V = L \oplus H$ with $\dim L = 1$. We have four functors

$$\mathrm{Sub}(H)^\wedge \underset{f_\sharp}{\overset{f^\sharp}{\rightleftarrows}} \mathrm{Sub}(V)^\wedge$$

$$\mathrm{Sub}(H)^\wedge \underset{g_\sharp}{\overset{g^\sharp}{\rightleftarrows}} \mathrm{Sub}(V)^\wedge$$

defined by

$$\begin{aligned}(f^\sharp N)(X) &= N(X) && \text{if} \quad X \subset H, \\ &= 0 && \text{if} \quad X \not\subset H, \\ (f_\sharp M)(Y) &= M(Y) \\ (g^\sharp N)(X) &= N((L+X) \cap H) \\ (g_\sharp M)(Y) &= M(L+Y)\end{aligned}$$

for $M \in \mathrm{Sub}(V)^\wedge$, $N \in \mathrm{Sub}(H)^\wedge$, $X \in \mathrm{Sub}(V)$, $Y \in \mathrm{Sub}(H)$, and these form adjoints $f^\sharp \dashv f_\sharp$, $g^\sharp \dashv g_\sharp$. The inclusion maps $Y \to L+Y$ induce a morphism α: $g_\sharp M \to f_\sharp M$. Since $g_\sharp$ is exact and $g_\sharp S_V = S_H$, $g_\sharp$ induces a map

$$\mathrm{Ext}^i_{\mathrm{Sub}(V)}(S_V, M) \to \mathrm{Ext}^i_{\mathrm{Sub}(H)}(S_H, g_\sharp M),$$

which is also written by $g_\sharp$. We claim that the composite

$$\operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_V, M) \to \operatorname{Ext}^i_{\mathrm{Sub}(H)}(S_H, g_\sharp M) \to \operatorname{Ext}^i_{\mathrm{Sub}(H)}(S_H, f_\sharp M)$$

of the map $g_\sharp$ and the map induced by α is zero for any $M \in \mathrm{Sub}(V)^\wedge$ and any $i \in \mathbf{N}$. Indeed, by taking an injective resolution of M and using the fact that $f_\sharp$ and $g_\sharp$ preserve injective objects, we are reduced to the case when $i=0$. Then the claim is clear because $\operatorname{Hom}(S_V, M) \cong \operatorname{Ker}(M(V) \to \prod_{X \subsetneqq V} M(X))$.

Now let $M = F(V/-)$. Then α is a split monomorphism, because $\alpha\colon g_\sharp M \cong F(H/-) \to F(L \oplus H/-) \cong f_\sharp M$ is induced by the projections $L \oplus H/Y \to H/Y$ for $Y \in \mathrm{Sub}(H)$. By the above claim, it follows that the map

$$g_\sharp\colon \operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_V, F(V/-)) \to \operatorname{Ext}^i_{\mathrm{Sub}(H)}(S_H, g_\sharp F(V/-))$$

is zero for any i. On the other hand, since

$$\begin{aligned}(g^\sharp S_H)(X) &= k \qquad \text{if } \ L+X=V, \\ &= 0 \qquad \text{otherwise,}\end{aligned}$$

we have an exact sequence

$$0 \to \bigoplus_{K \in R} S_K \to g^\sharp S_H \xrightarrow{\beta} S_V \to 0 \tag{2.4}$$

in $\mathrm{Sub}(V)^\wedge$, where R is the set of hyperplanes in V which do not contain L, and the morphism β corresponds to the isomorphism $S_H \cong g_\sharp S_V$ under the adjoint situation. The map $\operatorname{Ext}^i_{\mathrm{Sub}(V)}(\beta, F(V/-))$ is zero for any i because it equals the composite

$$\begin{aligned}\operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_V, F(V/-)) &\xrightarrow{g_\sharp} \operatorname{Ext}^i_{\mathrm{Sub}(H)}(S_H, g_\sharp F(V/-)) \\ &\xrightarrow{\sim} \operatorname{Ext}^i_{\mathrm{Sub}(V)}(g^\sharp S_H, F(V/-)).\end{aligned}$$

Then, by (2.4), we have an exact sequence

$$\begin{aligned}0 \to \operatorname{Ext}^{i-1}_{\mathrm{Sub}(V)}(g^\sharp S_H, F(V/-)) &\to \bigoplus_{K \in R} \operatorname{Ext}^{i-1}_{\mathrm{Sub}(V)}(S_K, F(V/-)) \\ &\to \operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_V, F(V/-)) \to 0\end{aligned} \tag{2.5}$$

for any i. By the inductive hypothesis, $\operatorname{Ext}^{i-1}_{\mathrm{Sub}(V)}(S_K, F(V/-)) = 0$ if $i-1 \neq \dim V - 1$ and $K \in R$. Hence $\operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_V, F(V/-)) = 0$ if $i \neq \dim V$. Q.E.D.

For a vector space V of dimension n we define

$$\mathcal{L}(V) = \operatorname{Ext}_{\mathrm{Sub}(V)}(S_V, (V/-)^\vee)$$

where $(V/-)^\vee$ is the functor $\mathrm{Sub}(V)^{\mathrm{op}} \to k\text{-}\mathcal{M}od$ taking X to the dual $(V/X)^\vee$ of V/X. The following is equivalent to Lusztig's result [5, Theorem 1.12].

Corollary 2.6. *Let V be a vector space and U a subspace of V. Then*

$$\begin{aligned}\operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_U, (V/-)^\vee) &= 0 && \text{if } i \neq \dim U,\\ &\cong \mathcal{L}(U) \neq 0 && \text{if } i = \dim U > 0,\\ &\cong V^\vee && \text{if } i = \dim U = 0.\end{aligned}$$

Proof. The isomorphism in the third case is clear and the equality in the first case follows from Proposition 2.1 by taking F as the functor $X \mapsto X^\vee$. Let $i = \dim U > 0$. Using the notation in the preceding proof, we have that $j^*(V/-)^\vee \cong (V/U)^\vee \oplus (U/-)^\vee$. Since the constant functor $(V/U)^\vee$ is an injective object of $\mathrm{Sub}(U)^\wedge$ and $i > 0$, we see that

$$\operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_U, (V/-)^\vee) \cong \operatorname{Ext}^i_{\mathrm{Sub}(U)}(S_U, (U/-)^\vee) \cong \mathcal{L}(U).$$

It remains to show that $\mathcal{L}(V) \neq 0$ whenever $V \neq 0$. By (2.5) with $i = \dim V = n$, we have exact sequences

$$\begin{aligned}&0 \to \mathcal{L}(H) \to \bigoplus_{K \in R} \mathcal{L}(K) \to \mathcal{L}(V) \to 0 && \text{if } n > 1,\\ &0 \to 0 \to V \to \mathcal{L}(V) \to 0 && \text{if } n = 1.\end{aligned}$$

Setting $l(n) = \dim \mathcal{L}(V)$, we see that $l(n) = (\#R - 1)l(n-1) = (q^{n-1} - 1)l(n-1)$ if $n > 1$, and $l(1) = 1$. Thus $l(n) = (q^{n-1} - 1)\cdots(q-1) \neq 0$ for $n \geq 1$. Q.E.D.

Proof of Proposition 2.2. We use the notation in the proof of Proposition 2.1. We have

$$\operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_U, C_{V,V'}) \cong \operatorname{Ext}^i_{\mathrm{Sub}(U)}(S_U, j^* C_{V,V'})$$

and

$$\begin{aligned}j^* C_{V,V'} &= C_{U, U \cap V'} && \text{if } U + V' = V,\\ &= 0 && \text{otherwise.}\end{aligned}$$

So it is enough to show that $\operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_V, C_{V,V'}) \neq 0$ if and only if $i = \dim V'$. We use induction on $\dim V'$. When $V' = 0$, this is clear. Let $V' \neq 0$. Write $V = L \oplus H$ with $L \subset V'$ and $\dim L = 1$. Then the morphism $\alpha: g_\sharp C_{V,V'} \to f_\sharp C_{V,V'} = C_{H, H \cap V'}$ is the identity. By the claim in the proof of Proposition 2.1, it follows that the map

$$g_\sharp : \operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_V, C_{V,V'}) \to \operatorname{Ext}^i_{\mathrm{Sub}(H)}(S_H, g_\sharp C_{V,V'})$$

is zero for any i. Then we have by (2.4) an exact sequence

$$\begin{aligned}0 \to \operatorname{Ext}^{i-1}_{\mathrm{Sub}(H)}(S_H, C_{H, H\cap V'}) &\to \bigoplus_{K \in R} \operatorname{Ext}^{i-1}_{\mathrm{Sub}(K)}(S_K, C_{K, K \cap V'})\\ &\to \operatorname{Ext}^i_{\mathrm{Sub}(V)}(S_V, C_{V,V'}) \to 0.\end{aligned}$$

If $K \in R$, then $V' \not\subset K$, so $\dim(K \cap V') = \dim V' - 1$. If $i \neq \dim V'$, then the middle term of the above sequence is zero by the inductive hypothesis, hence also is the right term. Set

$$\mathcal{P}(V, V') = \mathrm{Ext}^{n'}_{\mathrm{Sub}(V)}(S_V, C_{V,V'})$$

and $p(n, n') = \dim \mathcal{P}(V, V')$ with $n = \dim V$, $n' = \dim V'$. Then $p(n, n') = (q^{n-1}-1)p(n-1, n'-1)$ for $n'>0$ and $p(n, 0) = 1$. Hence $p(n, n') = (q^{n-1}-1)\cdots(q^{n-n'}-1) \neq 0$. Q.E.D.

Corollary 2.7. *Let $W \subset V' \subset V$. Let $C_{V,V',W}$: $\mathrm{Sub}(V)^{\mathrm{op}} \to k\text{-}\mathcal{M}od$ be the functor such that*

$$\begin{aligned} C_{V,V',W}(X) &= k \quad \text{if } W \subset X \text{ and } X+V' = V, \\ &= 0 \quad \text{otherwise}, \end{aligned}$$

and the restriction maps from k to k are the identity. Then we have for $U \in \mathrm{Sub}(V)$,

$$\begin{aligned} \mathrm{Ext}^{i}_{\mathrm{Sub}(V)}(S_U, C_{V,V',W}) &\cong \mathcal{P}(U/W, (U \cap V')/W) \\ &\qquad \text{if } W \subset U,\ U+V'=V \text{ and } i = \dim U/W - \dim V/V', \\ &= 0 \qquad \text{otherwise}. \end{aligned}$$

Proof. Let p: $V \to V/W$ be the projection and let

$$\mathrm{Sub}(V)^{\wedge} \underset{p^!}{\overset{p_*}{\rightleftarrows}} \mathrm{Sub}(V/W)^{\wedge}$$

be the adjoint defined before the proof of Proposition 2.1. Then

$$C_{V,V',W} = p^! C_{V/W, V'/W}$$

and

$$\begin{aligned} p_* S_U &= S_{U/W} \quad \text{if } W \subset U, \\ &= 0 \quad \text{if } W \not\subset U. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{Ext}^{i}_{\mathrm{Sub}(V)}(S_U, C_{V,V',W}) &\cong \mathrm{Ext}^{i}_{\mathrm{Sub}(V/W)}(S_{U/W}, C_{V/W,V'/W}) \quad \text{if } W \subset U, \\ &= 0 \quad \text{if } W \not\subset U, \end{aligned}$$

and so the corollary follows from Proposition 2.2. Q.E.D.

For a vector space V of dimension n we define the Steinberg module

$$\mathcal{S}t(V) = \mathrm{Ext}^{n}_{\mathrm{Sub}(V)}(S_V, S_0)$$

where $0=\{0\}\subset V$.

Proof of Proposition 2.3. As before we have that

$$\begin{aligned}\mathrm{Ext}^i_{\mathrm{Sub}(V)}(S_U, S_W) &\cong \mathrm{Ext}^i_{\mathrm{Sub}(U/W)}(S_{U/W}, S_0) && \text{if } W\subset U,\\ &= 0 && \text{if } W\not\subset U,\end{aligned}$$

and so it is enough to consider the case when $U=V$, $W=0$. We argue by induction on $\dim V$. When $V=0$, the assertion is clear. Let $V\neq 0$ and write $V=L\oplus H$ with $\dim L=1$. Since $g_\sharp S_0=0$, we have by (2.4),

$$\begin{aligned}\mathrm{Ext}^i_{\mathrm{Sub}(V)}(S_V, S_0) &\cong \bigoplus_{K\in R} \mathrm{Ext}^{i-1}_{\mathrm{Sub}(V)}(S_K, S_0)\\ &\cong \bigoplus_{K\in R} \mathrm{Ext}^{i-1}_{\mathrm{Sub}(K)}(S_K, S_0)\end{aligned}$$

for any i. If $i\neq\dim V$, then $\mathrm{Ext}^{i-1}_{\mathrm{Sub}(K)}(S_K, S_0)=0$ by the inductive hypothesis, and hence $\mathrm{Ext}^i_{\mathrm{Sub}(V)}(S_V, S_0)=0$. Set $s(n)=\dim \mathcal{S}t(V)$ with $n=\dim V$. Then $s(n)=q^{n-1}s(n-1)$ for $n>0$ and $s(0)=1$. Thus $s(n)=q^{n(n-1)/2}\neq 0$ for any $n\geq 0$.

Q.E.D.

REMARK 2.8. For $U\in\mathrm{Sub}(V)$, let J_U: $\mathrm{Sub}(V)^{\mathrm{op}}\to k\text{-}\mathcal{M}od$ be the functor such that

$$\begin{aligned}J_U(X) &= k && \text{if } U\subset X,\\ &= 0 && \text{if } U\not\subset X,\end{aligned}$$

and the restriction maps from k to k are the identity. Then J_U is an injective hull of the simple object S_U. Proposition 2.3 says that if $S_W\to I^\bullet$ is a minimal injective resolution of S_W, then J_U is a direct summand of I^i if and only if $W\subset U$ and $i=\dim U/W$. Other results of this section are similarly restated.

3. Homology of $S(G)$

Throughout this section k is a prime field of characteristic $p>0$ and G is an elementary abelian p-group. Denote by $S(G)$ the category of finite G-sets and by $S(G)^\wedge$ the category of those functors $S(G)^{\mathrm{op}}\to k\text{-}\mathcal{M}od$ which transform disjoint sums into direct sums. Ext groups in the abelian categroy $S(G)^\wedge$ are written as $\mathrm{Ext}^i_{S(G)}(\ ,\ )$. In this section we will compute $\mathrm{Ext}^i_{S(G)}(F, F')$ for certain $F, F'\in S(G)^\wedge$.

We view G as a k-vector space and use the notations $\mathrm{Sub}(G)$, $\mathrm{Sub}(G)^\wedge$ in Section 2. We will give a spectral sequence which relates Ext groups in $\mathrm{Sub}(G)^\wedge$ to those in $S(G)^\wedge$. Let $S(G)_0$ be the full subcategory of $S(G)$ consisting of the G-sets G/H with H subgroups of G. Clearly $S(G)^\wedge$ is equivalent to the category of functors $S(G)_0^{\mathrm{op}}\to k\text{-}\mathcal{M}od$. Since G is abelian, we have a functor s: $S(G)_0\to\mathrm{Sub}(G)$ taking G/H to H. Composition with s yields a functor

$s^*: \mathrm{Sub}(G)^\wedge \to S(G)^\wedge$ and s^* has a right adjoint $s_*: S(G) \to \mathrm{Sub}(G)^\wedge$. Given $M \in \mathrm{Sub}(G)^\wedge$, the functor $\mathrm{Hom}(s^*M, -): S(G)^\wedge \to k\text{-}\mathcal{M}od$ is isomorphic to the composite $\mathrm{Hom}(M, -)\circ s_*$, therefore we have a spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathrm{Sub}(G)}(M, R^j s_* F) \Rightarrow \mathrm{Ext}^{i+j}_{S(G)}(s^*M, F)$$

for any $F \in S(G)^\wedge$, where $R^j s_*: S(G)^\wedge \to \mathrm{Sub}(G)^\wedge$ are right derived functors of s_*. We need to know $R^j s_* F$. First note that for $U \in \mathrm{Sub}(G)$, the group $G/U \cong \mathrm{Hom}_G(G/U, G/U)$ acts on $F(G/U)$ and $(s_*F)(U)$ is isomorphic to the G/U-fixed subspace $F(G/U)^{G/U}$. If F is an injective object of $S(G)^\wedge$, then $F(G/U)$ is an injective G/U-module, because F is a direct sum of functors of the form $X \mapsto \mathrm{Map}(\mathrm{Hom}_G(G/H, X), k)$. Therefore, for any $F \in S(G)^\wedge$, we have that $(R^j s_* F)(U) \cong H^j(G/U, F(G/U))$ and the restriction maps of $R^j s_* F$ correspond to the inflation maps of group cohomology.

In Section 2 we defined the simple objects S_K and the injective object J_K of $\mathrm{Sub}(G)^\wedge$ for $K \in \mathrm{Sub}(G)$ (Remark 2.8). The objects s^*S_K and s^*J_K of $S(G)^\wedge$ are also denoted by S_K and J_K respectively.

Theorem 3.1. *Let K, K' be subgroups of G. If $K \subset K'$ and $d = \mathrm{rank}\, K'/K$, then there are isomorphisms*

$$\mathrm{Ext}^i_{S(G)}(S_{K'}, J_K) \cong \mathrm{Ext}^d_{\mathrm{Sub}(\bar{G})}(S_{\bar{K}'}, H^{i-d}(\bar{G}/-))$$
$$\mathrm{Ext}^i_{S(G)}(S_{K'}, S_K) \cong \mathcal{St}(\bar{K}') \otimes H^{i-d}(\bar{G})$$

for any i, where $\bar{K}' = K'/K$, $\bar{G} = G/K$, $H^{i-d}(\bar{G}) = H^{i-d}(\bar{G}, k)$, and $H^{i-d}(\bar{G}/-)$ is the functor $\mathrm{Sub}(\bar{G})^{\mathrm{op}} \to k\text{-}\mathcal{M}od$ taking U to $H^{i-d}(\bar{G}/U)$, and $\mathcal{St}(\bar{K}')$ is the Steinberg module of the k-vector space $\bar{K}'$ defined in Section 2. If $K \not\subset K'$, then

$$\mathrm{Ext}^i_{S(G)}(S_{K'}, J_K) = \mathrm{Ext}^i_{S(G)}(S_{K'}, S_K) = 0$$

for any i.

Proof. The case $K \not\subset K'$ is clear. Let $K \subset K'$. For the first isomorphism, consider the spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathrm{Sub}(G)}(S_{K'}, R^j s_* J_K) \Rightarrow \mathrm{Ext}^{i+j}_{S(G)}(S_{K'}, J_K)\,.$$

For $U \in \mathrm{Sub}(G)$, we have

$$\begin{aligned}(R^j s_* J_K)(U) &= H^j(G/U, J_K(G/U)) \\ &= H^j(G/U, k) \qquad \text{if } K \subset U\,, \\ &= 0 \qquad \text{if } K \not\subset U\,.\end{aligned}$$

The projection $\pi: G \to \bar{G}$ induces the adjoint

$$\mathrm{Sub}(G)^\wedge \underset{\pi^!}{\overset{\pi_*}{\rightleftarrows}} \mathrm{Sub}(\bar{G})^\wedge$$

as defined in Section 2, and we have that $R^j s_* J_K = \pi^1 H^j(\bar{G}/-)$ and $\pi_* S_{K'} = S_{\bar{K}'}$, so

$$E_2^{i,j} \cong \mathrm{Ext}^i_{\mathrm{Sub}(\bar{G})}(S_{\bar{K}'}, H^j(\bar{G}/-)) .$$

Applying Proposition 2.1 to the functors $H^j(-)\colon k\text{-}\mathcal{M}od^{\mathrm{op}} \to k\text{-}\mathcal{M}od$, we see that $E_2^{i,j}=0$ for $i \neq \mathrm{rank}\, \bar{K}'=d$, hence the spectral sequence yields the desired isomorphism $E_2^{d,i-d} \cong \mathrm{Ext}^i_{S(G)}(S_{K'}, J_K)$ for any i.

Secondly, we have that $R^j s_* S_K \cong S_K \otimes H^j(G/K)$ and so

$$\begin{aligned} E_2^{i,j} &:= \mathrm{Ext}^i_{\mathrm{Sub}(G)}(S_{K'}, R^j s_* S_K) \\ &\cong \mathrm{Ext}^i_{\mathrm{Sub}(G)}(S_{K'}, S_K) \otimes H^j(G/K) \\ &\cong \mathcal{St}(K'/K) \otimes H^j(G/K) && \text{if } i=d, \\ &= 0 && \text{if } i \neq d, \end{aligned}$$

by Proposition 2.3. Thus $E_2^{d,i-d} \cong \mathrm{Ext}^i_{S(G)}(S_{K'}, S_K)$ for any i. Q.E.D.

Corollary 3.2. *Let $K \subsetneqq K' \subset G$ and $d = \mathrm{rank}\, K'/K$. Then*

$$\begin{aligned} \mathrm{Ext}^i_{S(G)}(S_{K'}, J_K) &= 0 && \text{if } i \leq d, \\ &\cong \mathcal{L}(K'/K) && \text{if } i = d+1, \end{aligned}$$

where $\mathcal{L}(K'/K)$ is as defined in Section 2.

Proof. This follows from Theorem 3.1, Corollary 2.6 and the fact that the constant functor $H^0(\bar{G}/-)$ is an injective object of $\mathrm{Sub}(\bar{G})^\wedge$ and $H^1(X) = \mathrm{Hom}(X, k)$ for elementary abelian p-groups X.

4. Homology of $P(G)$

In the rest of this paper k is a prime field of characteristic $p>0$. For a finite group G, let $P(G)$ and $P(G)^\wedge$ be as defined in Section 1. Ext groups in $P(G)^\wedge$ are written as $\mathrm{Ext}^i_{P(G)}(\ ,\ )$. The purpose of this section is to compute $\mathrm{Ext}^i_{P(G)}(M, M')$ for certain $M, M' \in P(G)^\wedge$ when G is an elementary abelian p-group.

We first define adjoint functors between such categories as $P(G)^\wedge$. Let H be a subgroup of G. There are functors

$$P(H) \underset{r}{\overset{t}{\rightleftarrows}} P(G)$$

defined by

$$t(Y) = k[G] \otimes_H Y \cong \mathrm{Hom}_H(k[G], Y), \qquad r(X) = X$$

and functors

$$P(H)^\wedge \underset{\text{Ind}}{\overset{\text{Res}}{\rightleftarrows}} P(G)^\wedge$$

defined by

$$\operatorname{Res}(M) = M \circ t\,, \quad \operatorname{Ind}(N) = N \circ r\,.$$

Since t is a left and right adjoint of r, Res is a left and right adjoint of Ind. We also write $\text{Res}=\text{Res}^G_H$, $\text{Ind}=\text{Ind}^G_H$.

Suppose next that H is a normal subgroup of G and set $\bar{G}=G/H$. We have functors

$$P(G) \underset{r}{\overset{t}{\rightleftarrows}} P(\bar{G})$$

defined by

$$t(Y) = k[\bar{G}] \otimes_G Y\,, \quad r(X) = X$$

and functors

$$P(G)^\wedge \underset{\text{Def}}{\overset{\text{Inf}}{\rightleftarrows}} P(\bar{G})^\wedge$$

defined by

$$\operatorname{Inf}(M) = M \circ t\,, \quad \operatorname{Def}(N) = N \circ r\,.$$

Since t is a left adjoint of r, Inf is a left adjoint of Def.

We also note that the category $P(G)^\wedge$ is self-dual. Indeed, for an object M of $P(G)^\wedge$, let $D(M)$ be the object of $P(G)^\wedge$ defined by $D(M)(X)=M(X^\vee)^\vee$ for $X\in P(G)$, where $(\ )^\vee=\text{Hom}_k(\ ,k)$. Then the assignment $M\mapsto D(M)$ gives an equivalence $P(G)^{\wedge\text{op}}\simeq P(G)^\wedge$.

From now on we assume that G is an elementary abelian p-group. For a subgroup H of G, let S_H be the object of $P(G)^\wedge$ such that

$$\begin{aligned} S_H(k[G/U]) &= k \quad &&\text{if} \quad U = H\,, \\ &= 0 \quad &&\text{if} \quad U \neq H\,, \end{aligned}$$

and let J_H be the object of $P(G)^\wedge$ such that

$$\begin{aligned} J_H(k[G/U]) &= k \quad &&\text{if} \quad H \subset U\,, \\ &= 0 \quad &&\text{if} \quad H \not\subset U\,, \end{aligned}$$

and that for a G-linear map $f\colon k[G/U]\to k[G/U']$,

$$\begin{aligned} J_H(f) &= \text{id} \quad &&\text{if } H \subset U \text{ and } f \text{ is surjective,} \\ &= 0 \quad &&\text{otherwise.} \end{aligned}$$

Theorem 4.1. *Let H, H' be subgroups of an elementary abelian p-group G*

and let $H''=H+H'$, $m=\operatorname{rank} H''/H$, $m'=\operatorname{rank} H''/H'$, $d=m+m'$. *Then*

(i) $\operatorname{Ext}^i_{P(G)}(S_{H'}, J_H)$

$$\begin{aligned} &= 0 && \text{if } i<d \text{ or if } i=d \text{ and } H'\not\subset H,\\ &\cong \mathcal{S}t(H/H') && \text{if } i=d \text{ and } H'\subset H,\\ &\cong \mathcal{S}t(H''/H')\otimes\mathcal{L}(H''/H) && \text{if } i=d+1 \text{ and } H'\not\subset H.\end{aligned}$$

(ii) $\operatorname{Ext}^i_{P(G)}(S_{H'}, S_H) = 0$ if $i<d$,

$$\cong \mathcal{S}t(H''/H')\otimes\mathcal{S}t(H''/H) \quad \text{if } i=d.$$

These isomorphisms preserve the action of the group $\{f\in\operatorname{Aut}(G); f(H)=H, f(H')=H'\}$.

In the rest of this section we prove this theorem. Let G be an elementary abelian p-group. Let $\mathcal{K}$ be the category of cohomological G-functors F such that the transfer maps $\tau_H^K\colon F(H)\to F(K)$ are zero whenever $H\subsetneqq K\subset G$, and the conjugation maps $\gamma_g\colon F(H)\to F(H^g)$ are the identity maps for all $H\subset G, g\in G$. A cohomological G-functor F in $\mathcal{K}$ is determined by its restriction maps, and it can be verified that the category $\mathcal{K}$ is isomorphic to $\operatorname{Sub}(G)^\wedge$. By the correspondence between cohomological G-functors and objects of $P(G)^\wedge$ stated in Section 1, we can view $\mathcal{K}$ as a full subcategory of $P(G)^\wedge$. Thus we get an imbedding $\operatorname{Sub}(G)^\wedge\to P(G)^\wedge$. By this imbedding the objects S_K, J_K of $\operatorname{Sub}(G)^\wedge$ defined in Section 2 correspond respectively to the objects S_K, J_K of $P(G)^\wedge$ defined just before Theorem 4.1.

Lemma 4.2. *Let* H, G' *be subgroups of* G *such that* $H\subset G'$ *and* $\operatorname{rank} G/G'=1$. *Put* $J'_H=\operatorname{Res}^G_{G'}(J_H)\in P(G')^\wedge$. *Then there are exact sequences*

$$0\to J_H\to \operatorname{Ind}^G_{G'}(J'_H)\to \operatorname{Ind}^G_{G'}(J'_H)\to L\to 0 \tag{4.3}$$

$$0\to L_1\to L\to L_0\to 0 \tag{4.4}$$

$$0\to L_0\to J_H\to L_1\to 0 \tag{4.5}$$

in $P(G)^\wedge$, *where* $L_0, L_1\in\mathcal{K}$ *and* $L_1=C_{G,G',H}$ *as an object of* $\operatorname{Sub}(G)^\wedge$ *with the notation in Corollary* 2.7.

Proof. We regard objects of $P(G)^\wedge$ as cohomological G-functors. Let L be the cohomological G-functor such that

$$\begin{aligned} L(U) &= k && \text{if } H\subset U,\\ &= 0 && \text{if } H\not\subset U,\end{aligned}$$

for $U\in\operatorname{Sub}(G)$, and that the restriction maps $\rho_U^{U'}\colon L(U')\to L(U)$ are given by

$$\begin{aligned} \rho_U^{U'} &= \mathrm{id} && \text{if } H\subset U\subset U' \text{ and } U\not\subset G' \text{ or if } H\subset U\subset U'\subset G',\\ &= 0 && \text{otherwise,}\end{aligned}$$

and the transfer maps $\tau_U^{U'}: L(U) \to L(U')$ with $U \subsetneq U'$ are given by

$$\begin{aligned} \tau_U^{U'} &= \mathrm{id} \qquad \text{if } H \subset U = U' \cap G' \text{ and } U' \not\subset G', \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

and all conjugation maps are identity maps. Let L_0, L_1 be the cohomological G-functors such that

$$\begin{aligned} L_0(U) &= k \qquad \text{if } H \subset U \subset G', \\ &= 0 \qquad \text{otherwise,} \\ L_1(U) &= k \qquad \text{if } H \subset U \not\subset G', \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

for $U \in \mathrm{Sub}(G)$, and that the restriction maps from k to k are the identity, all proper transfer maps are zero and all conjugation maps are identity maps. Then L_0, L_1 belong to $\mathcal{K}$ and $L_1 = C_{G,G',H'}$.

Existence of the last two exact sequences is easily shown. To make the first exact sequence, we describe $\mathrm{Ind}_{G'}^{G}(J'_H)$ as a cohomological G-functor. Since for $U \in \mathrm{Sub}(G)$ the G'-module $k[G/U]$ is isomorphic to $\bigoplus_{g \in G/G'} k[G'/U]$ or $k[G'/(G' \cap U)]$ according as $U \subset G'$ or not, we have

$$\begin{aligned} \mathrm{Ind}_{G'}^{G}(J'_H)(U) &= k[G/G'] \qquad \text{if } H \subset U \subset G', \\ &= k \qquad \text{if } H \subset U \not\subset G', \\ &= 0 \qquad \text{if } H \not\subset U. \end{aligned}$$

Let $\varepsilon: k[G/G'] \to k$ and $\iota: k \to k[G/G']$ be the maps such that $\varepsilon(\sum a_g g) = \sum a_g$ and $\iota(1) = \sum g$, where g runs over G/G'. The restriction maps $\rho_U^{U'}$ of $\mathrm{Ind}_{G'}^{G}(J'_H)$ are as follows.

$$\begin{aligned} \rho_U^{U'} &= \mathrm{id} \quad \text{if } H \subset U \subset U' \text{ and } U \not\subset G' \text{ or if } H \subset U \subset U' \subset G', \\ &= \iota \quad \text{if } H \subset U \subset U',\ U \subset G' \text{ and } U' \not\subset G', \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

The transfer maps $\tau_U^{U'}$ of $\mathrm{Ind}_{G'}^{G}(J'_H)$ with $U \subsetneq U'$ are as follows.

$$\begin{aligned} \tau_U^{U'} &= \varepsilon \quad \text{if } H \subset U = U' \cap G' \text{ and } U' \not\subset G', \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then one can verify that there is an exact sequence

$$0 \to J_H \to \mathrm{Ind}_{G'}^{G}(J'_H) \to \mathrm{Ind}_{G'}^{G}(J'_H) \to L \to 0$$

in $P(G)^\wedge$ such that when evaluated at a subgroup U, it becomes the exact sequence

$$0 \to k \xrightarrow{\iota} k[G/G'] \to k[G/G'] \xrightarrow{\varepsilon} k \to 0$$

of G/G'-modules (note that G/G' is cyclic) or the exact sequence

$$0 \to k \xrightarrow{1} k \xrightarrow{0} k \xrightarrow{1} k \to 0\,,$$

according as $U \subset G'$ or not. Q.E.D.

Proposition 4.6. *We write* $\mathrm{Ext}^i_{P(G)} = \mathrm{Ext}^i$ *simply. Under the same conditions as in Theorem* 4.1, *we have*

$$\begin{aligned}
\text{(i)}\quad \mathrm{Ext}^i(S_{H'}, J_H) &= 0 && \text{if } i<d \text{ or if } i=d \text{ and } H'\not\subset H\,,\\
&\cong \mathrm{Ext}^d(S_{H'}, S_H) && \text{if } i=d \text{ and } H'\subset H\,,\\
&\cong \mathrm{Ext}^{m'}(S_{H'}, S_{H''})\otimes \mathrm{Ext}^{m+1}(S_{H''}, J_H) && \\
& && \text{if } i=d+1 \text{ and } H'\not\subset H\,,
\end{aligned}$$

where the isomorphism in the second case is induced by the inclusion $S_H \to J_H$ *and the one in the third case is induced by the Yoneda product.*

$$\begin{aligned}
\text{(ii)}\quad \mathrm{Ext}^i(S_{H'}, S_H) &= 0 && \text{if } i<d\,,\\
&\cong \mathrm{Ext}^{m'}(S_{H'}, S_{H''})\otimes \mathrm{Ext}^{m}(S_{H''}, S_H) && \text{if } i=d\,,
\end{aligned}$$

where the isomorphism in the latter case is induced by the Yoneda product.

Proof. For nonnegative integers k, k', let $\mathrm{I}(k, k')$ (resp. $\mathrm{II}(k, k')$) denote assertion (i) (resp. (ii)) for all triples (G, H, H') such that $\mathrm{rank}\, G/H \leq k$, $\mathrm{rank}\, G/H' \leq k'$. To prove the proposition, it is enough to show the following statements (a)–(d).

(a) $\mathrm{I}(0, 0)$.
(b) $\mathrm{II}(k, 0)$ implies $\mathrm{I}(0, k)$.
(c) Let $k>0$, $k'\geq 0$. $\mathrm{I}(k-1, k')$ and $\mathrm{II}(k-1, k')$ imply $\mathrm{I}(k, k')$.
(d) Let k, $k'\geq 0$. $\mathrm{I}(k, k')$ and $\mathrm{II}(k-1, k')$ imply $\mathrm{II}(k, k')$.

Proof of (a). Assertion (i) for (G, G, G) is clear because $d=0$ and $J_G = S_G$.

Proof of (b). By the duality of $P(G)^\wedge$ defined at the beginning of this section, $\mathrm{Ext}^i(S_{H'}, S_H) \cong \mathrm{Ext}^i(S_H, S_{H'})$. Thus if (ii) is true for (G, H, G), then (i) is true for (G, G, H).

Proof of (c). For subgroups U, U' of G, set $\delta(U, U') = \mathrm{rank}\,(U+U')/U + \mathrm{rank}\,(U+U')/U'$. This function $\delta(\ ,\ )$ satisfies the triangular innequality. Now suppose given subgroups H, H' of G with $H \neq G$. Assuming $\mathrm{I}(\mathrm{rank}\, G/H-1, \mathrm{rank}\, G/H')$ and $\mathrm{II}(\mathrm{rank}\, G/H-1, \mathrm{rank}\, G/H')$, we will show that (i) for (G, H, H') is true. Take a maximal subgroup G' of G such that $H\subset G'$. In addition, if $H'\not\subset H$, we take G' so that $H'\not\subset G'$. Let L, L_0, L_1 be the objects of $P(G)^\wedge$ associated with the subgroups H, G' in Lemma 4.2, and set $J'_H = \mathrm{Res}^G_{G'}(J_H)$.

If $H'\not\subset H$, then $\mathrm{Res}^G_{G'}(S_{H'})=0$, so

$$\mathrm{Ext}^i_{P(G)}(S_{H'}, \mathrm{Ind}^G_{G'}(J'_H)) \cong \mathrm{Ext}^i_{P(G')}(\mathrm{Res}^G_{G'}(S_{H'}), J'_H) = 0$$

for any i. If $H'\subset H$, then

$$\mathrm{Ext}^i_{P(G)}(\mathrm{S}_{H'}, \mathrm{Ind}^G_{G'}(J'_H)) \cong \mathrm{Ext}^i_{P(G')}(S_{H'}, J'_H)\,.$$

Since $\mathrm{rank}\, G'/H<\mathrm{rank}\, G/H$, (i) for (G', H, H') is true. Therefore $\mathrm{Ext}^i_{P(G')}(S_{H'}, J'_H)=0$ for $i<\mathrm{rank}\, H/H'=d$. From these facts and (4.3) it follows that

$$\mathrm{Ext}^i(S_{H'}, J_H) \cong \mathrm{Ext}^{i-2}(S_{H'}, L) \tag{4.7}$$

if $H'\not\subset H$ or if $H'\subset H$ and $i<d$.

Take a minimal injective resolution $L_1\to I^\bullet$ in $\mathrm{Sub}(G)^\wedge$. We regard this as a resolution in $\mathcal{K}\subset P(G)^\wedge$. We claim that $\mathrm{Ext}^i(S_{H'}, I^j)=0$ if $i<d-j-1$ or if $i=d-j-1$ and $j+1<m$. Indeed, by Corollary 2.7 (and Remark 2.8), I^j is a direct sum of objects J_K with $H\subset K$ and $\mathrm{rank}\, K/H=j+1$. For such $K\in\mathrm{Sub}(G)$, (i) is true for (G, K, H'). Since $\delta(K, H')\geq\delta(H, H')-\delta(H, K)=d-j-1$, we have $\mathrm{Ext}^i(S_{H'}, J_K)=0$ for $i<d-j-1$. If $H'\subset K$, then $H+H'\subset K$, hence $m\leq j+1$. Thus if $j+1<m$, then $H'\not\subset K$ and so $\mathrm{Ext}^i(S_{H'}, J_K)=0$ for $i\leq d-j-1$. This proves the claim. Set $Z^j=\mathrm{Ker}(I^j\to I^{j+1})$. By the claim we see that

$$\mathrm{Ext}^i(S_{H'}, L_1) = 0 \qquad \text{if} \quad i<d-1\,, \tag{4.8}$$

$$\mathrm{Ext}^{d-1}(S_{H'}, L_1)\cong\mathrm{Ext}^{d-m}(S_{H'}, Z^{m-1}) \qquad \text{if} \quad m>0\,. \tag{4.9}$$

Further, in the case when $m>0$, i.e., when $H'\not\subset H$, the functor Z^{m-1} has supports in the set $\{K\in\mathrm{Sub}(G);\, H\subset K,\ \mathrm{rank}\, K/H\geq m\}$, so there is an exact sequence

$$0\to S^{\oplus a}_{H''}\to Z^{m-1}\to T\to 0 \tag{4.10}$$

in $\mathcal{K}$, where $a\in\mathbf{N}$ and $T(H'')=0$. Suppose that $T(K)\neq 0$ for a subgroup K. Then $\mathrm{rank}\, G/K<\mathrm{rank}\, G/H$, so (ii) is true for (G, K, H'). Since $H\subset K$ and $H''\neq K$, we have that $\delta(K, H')=\delta(K, H'')+\delta(H'', H')>\delta(H'', H')=d-m$. Thus $\mathrm{Ext}^i(S_{H'}, S_K)=0$ for $i\leq d-m$. Hence $\mathrm{Ext}^i(S_{H'}, T)=0$ for $i\leq d-m$. By (4.10) and (4.9), it then follows that

$$\mathrm{Ext}^{d-1}(S_{H'}, L_1)\cong\mathrm{Ext}^{d-m}(S_{H'}, S^{\oplus a}_{H''}) \qquad \text{if} \quad H'\not\subset H\,. \tag{4.11}$$

Now by (4.4), (4.5) and (4.8), we have injections

$$\mathrm{Ext}^i(S_{H'}, L) \to \mathrm{Ext}^i(S_{H'}, L_0) \qquad \text{for} \quad i<d-1\,,$$

$$\mathrm{Ext}^i(S_{H'}, L_0) \to \mathrm{Ext}^i(S_{H'}, J_H) \qquad \text{for} \quad i<d\,.$$

Thus, by (4.7), if $H'\not\subset H$ and $i\leq d$ or if $H'\subset H$ and $i<d$, then we have an injection

$$\mathrm{Ext}^i(S_{H'}, J_H) \to \mathrm{Ext}^{i-2}(S_{H'}, J_H),$$

hence $\mathrm{Ext}^i(S_{H'}, J_H)=0$. This proves the equality in the first case of (i).

Suppose that $H' \not\subset H$. We know that $\mathrm{Ext}^i(S_{H'}, L_0)=0$ for $i<d$, hence by (4.4), $\mathrm{Ext}^{d-1}(S_{H'}, L_1) \cong \mathrm{Ext}^{d-1}(S_{H'}, L)$. By this and (4.7) and (4.11), we get an isomorphism α: $\mathrm{Ext}^{d+1}(S_{H'}, J_H) \to \mathrm{Ext}^{d-m}(S_{H'}, S_{H''}^{\oplus a})$. Reasoning as above with (G, H, H') replaced by (G, H, H''), we have a similar isomorphism β: $\mathrm{Ext}^{m+1}(S_{H''}, J_H) \to \mathrm{Ext}^0(S_{H''}, S_{H''}^{\oplus a})$. The diagram

$$\begin{array}{ccc}
\mathrm{Ext}^{d-m}(S_{H'}, S_{H''}) \otimes \mathrm{Ext}^{m+1}(S_{H''}, J_H) & \to & \mathrm{Ext}^{d+1}(S_{H'}, J_H) \\
\downarrow \mathrm{id}\otimes\beta & & \downarrow \alpha \\
\mathrm{Ext}^{d-m}(S_{H'}, S_{H''}) \otimes \mathrm{Ext}^0(S_{H''}, S_{H''}^{\oplus a}) & \to & \mathrm{Ext}^{d-m}(S_{H'}, S_{H''}^{\oplus a})
\end{array}$$

is commutative, where the horizontal maps are given by the Yoneda product. Since the lower horizontal map is an isomorphism, also is the upper one. This proves the isomorphism in the third case of (i).

Finally suppose that $H' \subset H$. There is an exact sequence

$$0 \to S_H \to J_H \to M \to 0.$$

If $K \in \mathrm{Sub}(G)$ and $M(K) \neq 0$, then $H \subsetneqq K$. In particular, (ii) is true for (G, K, H'). Since $\delta(K, H') = \mathrm{rank}\, K/H' > \mathrm{rank}\, H/H'$, we have that $\mathrm{Ext}^i(S_{H'}, S_K) = 0$ for $i \leq d$. Therefore $\mathrm{Ext}^i(S_{H'}, M)=0$ for $i \leq d$, hence we get the desired isomorphism $\mathrm{Ext}^d(S_{H'}, S_H) \to \mathrm{Ext}^d(S_{H'}, J_H)$. Thus (i) is proved.

Proof of (d). Suppose given subgroups H, H' of G. Assuming I(rank G/H, rank G/H') and II(rank $G/H-1$, rank G/H'), we will show that (ii) is true for (G, H, H'). Take a minimal injective resolution $S_H \to I^\bullet$ in $\mathrm{Sub}(G)^\wedge$ and regard this as a resolution in $\mathcal{K} \subset P(G)^\wedge$. By Proposition 2.3, I^j is a direct sum of objects J_K with $H \subset K$ and rank $K/H = j$. As in the claim in the proof of (c), we see that $\mathrm{Ext}^i(S_{H'}, I^j)=0$ if $i<d-j$ or if $i=d-j$ and $j<m$. It then follows that

$$\mathrm{Ext}^i(S_{H'}, S_H) = 0 \quad \text{for} \quad i<d,$$
$$\mathrm{Ext}^d(S_{H'}, S_H) \cong \mathrm{Ext}^{d-m}(S_{H'}, Z^m),$$

where $Z^m = \mathrm{Ker}(I^m \to I^{m+1})$. There is an exact sequence

$$0 \to S_{H''}^{\oplus a} \to Z^m \to T \to 0$$

in $\mathcal{K}$, where $a \in \mathbf{N}$ and T has supports in the set $\{K \in \mathrm{Sub}(G); H \subset K, \mathrm{rank}\, K/H \geq m, K \neq H''\}$. As in the proof of (c), we deduce that $\mathrm{Ext}^i(S_{H'}, T)=0$ for $i \leq d-m$, hence $\mathrm{Ext}^{d-m}(S_{H'}, S_{H''}^{\oplus a}) \cong \mathrm{Ext}^{d-m}(S_{H'}, Z^m)$. Thus we get an isomorphism $\mathrm{Ext}^d(S_{H'}, S_H) \to \mathrm{Ext}^{d-m}(S_{H'}, S_{H''}^{\oplus a})$. Replacing (G, H, H') by (G, H, H''), we have a similar isomorphism $\mathrm{Ext}^m(S_{H''}, S_H) \to \mathrm{Ext}^0(S_{H''}, S_{H''}^{\oplus a})$.

By the same argument as before, we see that the Yoneda product

$$\mathrm{Ext}^{d-m}(S_{H'}, S_{H''})\otimes\mathrm{Ext}^{m}(S_{H''}, S_H)\rightarrow \mathrm{Ext}^{d}(S_{H'}, S_H)$$

is an isomorphism. This proves (ii), and completes the proof of Proposition 4.6.

Ext groups in question in Theorem 4.1 were almost computed in the proof above, but to obtain equivariant isomorphisms, we need the following.

Proposition 4.12. *Let* $H\subset H'\subset G$ *and* $m=\mathrm{rank}\, H'/H$.
(i) *If* $m>0$, *then there are natural isomorphisms*

$$\mathrm{Ext}^{m+1}_{P(G)}(S_{H'}, J_H)\cong\mathrm{Ext}^{m+1}_{S(G)}(S_{H'}, J_H)\cong\mathcal{L}(H'/H)\,.$$

(ii) *There is a natural isomorphism*

$$\mathrm{Ext}^{m}_{P(G)}(S_{H'}, S_H)\cong \mathcal{S}t(H'/H)\,.$$

Proof. (i) The second isomorphism is that of Corollary 3.2. Let $f^*\colon P(G)^\wedge\rightarrow S(G)^\wedge$ be the functor defined by $(f^*M)(X)=M(k[X])$ for G-sets X. This induces maps $\mathrm{Ext}^i_{P(G)}(M, N)\rightarrow \mathrm{Ext}^i_{S(G)}(f^*M, f^*N)$ for $M, N\in P(G)^\wedge$, which are denoted by φ. We will show that the map $\varphi\colon \mathrm{Ext}^{m+1}_{P(G)}(S_{H'}, J_H)\rightarrow \mathrm{Ext}^{m+1}_{S(G)}(S_{H'}, J_H)$ is an isomorphism. Let $G', L, L_0, \cdots$ be as in (c) in the proof of Proposition 4.6. There we constructed the isomorphism $\alpha\colon \mathrm{Ext}^{m+1}_{P(G)}(S_{H'}, J_H)\rightarrow \mathrm{Ext}^{0}_{P(G)}(S_{H'}, S^{\oplus a}_{H'})$. We will show that there is also an isomorphism γ which makes the diagram

$$\begin{array}{ccc}\mathrm{Ext}^{m+1}_{P(G)}(S_{H'}, J_H) & \xrightarrow{\alpha} & \mathrm{Ext}^{0}_{P(G)}(S_{H'}, S^{\oplus a}_{H'})\\ \varphi\downarrow & & \downarrow\varphi\\ \mathrm{Ext}^{m+1}_{S(G)}(S_{H'}, J_H) & \xrightarrow[\gamma]{} & \mathrm{Ext}^{0}_{S(G)}(S_{H'}, S^{\oplus a}_{H'})\end{array}$$

commute. Then the left φ will be an isomorphism, because the right φ is clearly an isomorphism.

First, there is an adjoint $\mathrm{Res}^G_{G'}\dashv\mathrm{Ind}^G_{G'}$:

$$S(G')^\wedge \underset{\mathrm{Ind}^G_{G'}}{\overset{\mathrm{Res}^G_{G'}}{\rightleftarrows}} S(G)^\wedge$$

$$\mathrm{Res}^G_{G'}(M)(Y) = M(G\times^{G'} Y)$$
$$\mathrm{Ind}^G_{G'}(N)(X) = N(X)$$

where $X\in S(G)$, $Y\in S(G')$, and $G\times^{G'}Y$ is a quotient of $G\times Y$ obtained by identifying (g, y) with $(gg', g'^{-1}y)$ for $g'\in G'$. Clearly, $f^*\circ\mathrm{Res}^G_{G'}\cong\mathrm{Res}^G_{G'}\circ f^*$, $f^*\circ\mathrm{Ind}^G_{G'}\cong\mathrm{Ind}^G_{G'}\circ f^*$. Therefore (4.7) holds with $\mathrm{Ext}^i_{P(G)}$ replaced by $\mathrm{Ext}^i_{S(G)}$.

Secondly, reasoning as in the previous proof but using Theorem 3.1 and Corollary 3.2 instead of the inductive hypothesis, we see that (4.8), (4.9), (4.11) hold also for $\mathrm{Ext}^i_{S(G)}$. Then the isomorphism γ is defined similarly to α, and $\varphi\circ\alpha=\gamma\circ\varphi$. This proves (i).

(ii) With the notation in (d) in the proof of Proposition 4.6, we have natural isomorphisms

$$\begin{aligned}\mathrm{Ext}^m_{P(G)}(S_{H'}, S_H) &\cong \mathrm{Ext}^0_{P(G)}(S_{H'}, S_{H'}^{\oplus a})\\ &\cong \mathrm{Ext}^0_{\mathrm{Sub}(G)}(S_{H'}, S_{H'}^{\oplus a})\\ &\cong \mathrm{Ext}^m_{\mathrm{Sub}(G)}(S_{H'}, S_H)\cong \mathcal{St}(H'/H)\,. \qquad \text{Q.E.D.}\end{aligned}$$

Proof of Theorem 4.1. The theorem follows immediately from Propositions 4.6, 4.12 and the duality isomorphisms $\mathrm{Ext}^i(S_H, S_{H'})\cong\mathrm{Ext}^i(S_{H'}, S_H)$.

5. The finitistic dimension of $P(G)^\wedge$

Let G be a finite group and set $\Lambda=\mathrm{End}_G(\oplus k[G/H])^{\mathrm{op}}$, where H runs over all subgroups of G. Then $P(G)^\wedge$ and Λ-$\mathcal{Mod}$ are equivalent and $\Lambda\cong\Lambda^{\mathrm{op}}$. Among homological dimensions of the ring Λ, the global dimension and the injective dimension are rarely finite. Here we will determine the finitistic dimension of Λ (see Bass [1]). Write

$$\mathrm{f.dim}\, P(G)^\wedge = \sup\{\mathrm{pd}\, F;\ F\in P(G)^\wedge,\ \mathrm{pd}\, F<\infty\}$$

where pd F is the projective dimension of F.

Theorem 5.1. *If G is not a p'-group, then*

$$\mathrm{f.dim}\, P(G)^\wedge = 1+\sup\{\mathrm{rank}\, H/\Phi(H)\}$$

where H runs over all p-subgroups of G.

We need some notation. For a $k[G]$-module M, let $M^\sim\in P(G)^\wedge$ be the functor taking $X\in P(G)$ to $\mathrm{Hom}_G(X, M)$. When G is a p-group and H is a subgroup of G, we let $S_H\in P(G)^\wedge$ be a simple functor supported at the indecomposable module $k[G/H]$, and set $P_H=k[G/H]^\sim$, a projective cover of S_H. Ext groups in $P(G)^\wedge$ are written simply as $\mathrm{Ext}^i(\ ,\ )$.

Lemma 5.2. *If G is a nontrivial p-group and $n=\mathrm{rank}\, G/\Phi(G)$, then*

$$\mathrm{Ext}^i(S_G, k^\sim) \begin{cases} = 0 & \text{if } i\le n\,,\\ \neq 0 & \text{if } i=n+1\,.\end{cases}$$

Proof. Set $\bar{G}=G/\Phi(G)$. Consider the functors

$$P(G)^\wedge \underset{\mathrm{Def}}{\overset{\mathrm{Inf}}{\rightleftarrows}} P(\bar{G})^\wedge$$

defined in Section 4. For a subgroup K of G we have that $k[\bar{G}]\otimes_G k[G/K]\cong k$ if and only if $K=G$. Hence $\mathrm{Inf}(S_{\bar{G}})=S_G$. Clearly $\mathrm{Def}(k^\sim)=k^\sim=J_{\{1\}}$. Then by Theorem 4.1 and Corollary 2.6,

$$\begin{aligned}\mathrm{Ext}^i_{P(G)}(S_G, k^\sim)&\cong \mathrm{Ext}^i_{P(\bar{G})}(S_{\bar{G}}, J_{\{1\}})\\ &=0 \qquad \text{if } i\leq n,\\ &\cong \mathcal{L}(\bar{G})\neq 0 \qquad \text{if } i=n+1.\end{aligned}$$

Q.E.D.

Lemma 5.3. *For any $k[G]$-module M,* $\mathrm{Ext}^1(S_G, M^\sim)=0$.

Proof. We may assume that $G\neq 1$. The functor $M\mapsto M^\sim$ preserves injectives because it has the exact left adjoint $F\mapsto F(k[G])$. Take an injective resolution $0\to M\to I_0\to I_1$ in $k[G]$-$\mathcal{M}od$. Then $0\to M^\sim\to I_0^\sim\to I_1^\sim$ is an injective resolution in $P(G)^\wedge$. Since $\mathrm{Hom}(S_G, N^\sim)=0$ for any $k[G]$-module N, we have $\mathrm{Ext}^1(S_G, M^\sim)=0$. Q.E.D.

Lemma 5.4. *Let G be a p-group. Suppose that an exact sequence*

$$E\colon 0\to Q\to P\to M\to 0$$

of $k[G]$-modules satisfies the following conditions.

(i) *$P, Q\in P(G)$.*

(ii) *For any proper subgroup H of G, E splits in $k[H]$-$\mathcal{M}od$.*

(iii) *E does not have a split exact sequence as a direct summand.*

Then G acts on Q trivially and P does not have a trivial module as a direct summand.

Proof. We view Q as a submodule of P. Suppose that $Q=k[G/H]\oplus Q'$ with $H\neq G$. The inclusion map $Q\to P$ splits in $k[H]$-$\mathcal{M}od$ and so the injection $Q/Q'\to P/Q'$ does also. But $Q/Q'=k[G/H]$ is H-relatively injective, hence the map $Q/Q'\to P/Q'$ splits in $k[G]$-$\mathcal{M}od$. Then the inclusion map $k[G/H]\to P$ splits in $k[G]$-$\mathcal{M}od$, which contradicts (iii). Thus Q is a trivial G-module. Next suppose that $P=k\oplus P'$. By (iii), $Q\not\subset P'$. Then a complement of $P'\cap Q$ in Q is also a complement of P' in P, which contradicts (iii). This proves the lemma.

REMARK 5.5. It can be shown that the map $Q\to P$ in E is a direct sum of copies of the map $k\to \bigoplus_{|G:H|=p} k[G/H]$ taking 1 to the sum of all cosets.

The following lemma seems to be well-known.

Lemma 5.6. *Let Γ be a ring and M a nonzero Γ-module, $n\in\mathbf{N}$. If* $\mathrm{Ext}^i_\Gamma(M, \Gamma)=0$ *for all $i\leq n$, then* $\mathrm{f.dim}\,\Gamma^{\mathrm{op}}\geq n+1$.

Proof. Take a projective resolution $P_\bullet\to M$. By hypothesis, we have an exact sequence

$$0 \to P_0^* \to \cdots \to P_n^* \to P_{n+1}^*$$

of Γ^{op}-modules, where $P_i^* = \mathrm{Hom}_\Gamma(P_i, \Gamma)$. Set $N = \mathrm{Cok}(P_n^* \to P_{n+1}^*)$. Then $\mathrm{pd}\, N \leq n+1$. If $\mathrm{pd}\, N < n+1$, then the injection $P_0^* \to P_1^*$ splits, and so the map $P_1 \to P_0$ is surjective, which contradicts that $M \neq 0$. Thus $\mathrm{pd}\, N = n+1 \leq$ f.dim Γ^{op} Q.E.D.

Proof of Theorem 5.1. The proof consists of the four parts (a)–(d).

(a) If G is a nontrivial p-group, then

$$\text{f.dim}\, P(G)^\wedge \geq 1 + \text{rank}\, G/\Phi(G)\,.$$

Proof. Let Λ be the ring defined at the beginning of this section. We apply Lemma 5.6 to the ring Λ^{op} ($\cong \Lambda$). Since projective objects of $P(G)^\wedge$ are direct sums of objects P_H, it is enough to show that $\mathrm{Ext}^i(S_G, P_H) = 0$ for any subgroup H and any $i \leq \text{rank}\, G/\Phi(G)$. If $H = G$, this follows from Lemma 5.2 because $P_G = k^\sim$. If $H \neq G$, then $P_H = \mathrm{Ind}_H^G(k^\sim)$ and $\mathrm{Res}_H^G(S_G) = 0$, so $\mathrm{Ext}^i(S_G, P_H) = 0$ for any i. This proves (a).

(b) If G is a p-group, then

$$\text{f.dim}\, P(G)^\wedge \leq 1 + \sup\{\text{rank}\, H/\Phi(H);\ H \subset G\}\,.$$

Proof. We use induction on $|G|$. Set $m = \sup\{\text{rank}\, H/\Phi(H);\ H \subset G\}$. The case when $G = 1$ is clear. Let $G \neq 1$. Then $m \geq 1$. Assume that f.dim $P(G)^\wedge > m+1$. Then there exists an object F of $P(G)^\wedge$ with $\mathrm{pd}\, F = m+2$. Take a minimal projective resolution

$$0 \to P_{m+2}^\sim \to \cdots \to P_0^\sim \to F \to 0$$

in $P(G)^\wedge$ with $P_i \in P(G)$. Note that the functor $(-)^\sim : k[G]\text{-}\mathcal{M}od \to P(G)^\wedge$ is fully faithful. Set $K_i = \mathrm{Ker}(P_i \to P_{i-1})$ for $i \geq 1$. Then the sequences

$$0 \to K_{i+1}^\sim \to P_{i+1}^\sim \to K_i^\sim \to 0$$

in $P(G)^\wedge$ are exact for $i \geq 1$. For any proper subgroup H of G, $\mathrm{pd}\, \mathrm{Res}_H^G(F) \leq m+2 < \infty$, so $\mathrm{pd}\, \mathrm{Res}_H^G(F) \leq m+1$ by the inductive hypothesis. Therefore the morphism $\mathrm{Res}_H^G(P_{m+2}^\sim) \to \mathrm{Res}_H^G(P_{m+1}^\sim)$ in $P(H)^\wedge$ splits, hence the injection $P_{m+2} \to P_{m+1}$ splits as a map of H-modules. Let E be the exact sequence

$$0 \to P_{m+2} \to P_{m+1} \to K_m \to 0$$

of G-modules. By Lemma 5.4 and the minimality of the resolution, E must be a direct sum of exact sequences

$$\begin{aligned} &0 \to L' \to L \to L'' \to 0 \\ &0 \to 0 \to N \to N \to 0 \end{aligned}$$

where L' is a nonzero trivial G-module and L is a direct sum of modules $k[G/H]$ with $H \neq G$, and $N \in P(G)$. Set $n = \operatorname{rank} G/\Phi(G)$ $(\leq m)$. As was shown in (a), $\operatorname{Ext}^i(S_G, P_H) = 0$ if $H \neq G$ or if $H = G$ and $i \leq n$. It follows that

$$\operatorname{Ext}^{i+n-m}(S_G, K_i^{\sim}) \cong \operatorname{Ext}^{i+n-m+1}(S_G, K_{i+1}^{\sim})$$

for $1 \leq i < m$ and

$$\operatorname{Ext}^n(S_G, K_m^{\sim}) \cong \operatorname{Ext}^n(S_G, L''^{\sim}) \cong \operatorname{Ext}^{n+1}(S_G, L'^{\sim}).$$

Hence $\operatorname{Ext}^{n+1}(S_G, L'^{\sim}) \cong \operatorname{Ext}^1(S_G, K_{m-n+1}^{\sim}) = 0$ by Lemma 5.3. But this contradicts that $\operatorname{Ext}^{n+1}(S_G, k^{\sim}) \neq 0$ in Lemma 5.2. This proves (b).

(c) If H is a subgroup of G, then $\text{f.dim}\, P(G)^{\wedge} \geq \text{f.dim}\, P(H)^{\wedge}$.

Proof. For an H-module M, the map $M \to k[G] \otimes_H M$: $x \mapsto 1 \otimes x$ has a retraction which is natural in M. So for $F \in P(H)^{\wedge}$, the morphism of adjunction $\operatorname{Res}_H^G \operatorname{Ind}_H^G(F) \to F$ has a section. Hence $\operatorname{pd} F = \operatorname{pd} \operatorname{Ind}_H^G(F)$. The assertion follows from this immediately.

(d) If H is a p-Sylow subgroup of G, then $\text{f.dim}\, P(G)^{\wedge} \leq \text{f.dim}\, P(H)^{\wedge}$.

Proof. For a G-module M, the map $k[G] \otimes_H M \to M$: $g \otimes x \mapsto gx$ has the section $x \mapsto \frac{1}{|G:H|} \sum_{gH \in G/H} g \otimes g^{-1}x$. So for $F \in P(G)^{\wedge}$, the morphism of adjunction $F \to \operatorname{Ind}_H^G \operatorname{Res}_H^G(F)$ has a retraction. Thus $\operatorname{pd} F = \operatorname{pd} \operatorname{Res}_H^G(F)$, hence the conclusion follows.

The theorem follows by combining (a)–(d).

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