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# *Note on Locally Compact Groups*

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§ 1. The purpose of this note is to study the problem proposed by C. Chevalley: Is it true that a locally compact group which has no arbitrarily small[1] subgroup is a Lie group?

Concerning the above problem two theorems will be proved in this note. One of them is:

**Theorem 1.** *A locally euclidean group $G$ which has a neighbourhood of the identity containing no non-trivial subgroup, has a neighbourhood $\tilde{U}$ of the identity, through any point of which there exists one and only one one-parameter subgroup*[2].

The other is:

**Theorem 2.** *If $(U_n)^n$ is contained in $\tilde{U}$, then $G$ is a Lie group, where $U_n$ denotes the aggregate of the $n$-th roots of elements in a neighbourhood $U$.*

§ 2. For an element $x$ of a neighbourhood $U$ of the identity $e$ we denote by $\delta_U(x)$ the smallest number $n$ such that $x^{2^n} \in U$. The group $G$ is said to have the property $(S)$ if there exists a neighbourhood $U$ of $e$ such that $\delta_U(x) < \infty$ for every $x$ in $U - \{e\}$. According to Kuranishi[3] a locally euclidean group $G$ which has the property $(S)$, has a neighbourhood of the identity $e$, through any point of which one can draw one and only one one-parameter subgroup. Therefore we have only to show that a locally compact group which has no arbitrarily small subgroup has the property $(S)$.

In order to prove this we shall need the following Lemma.

**Lemma 1.** Let $W$ be a neighbourhood of the identity $e$ which contains no non-trivial subgroup in it. For an arbitrarily small neigh-

1) A small subgroup means a subgroup contained in a sufficiently small neighbourhood of the identity.

2) This theorem was proved with the co-operation of Dr. Gotô. Cf. the forthcoming Nagoya Math. Journal.

3) See Kuranishi: *Differentiability of locally compact groups*, Nagoya Math. Journal Vol. 1, 1950, 71-81.

bourhood $U$ of $e$ there exists a neighbourhood $U^*$ of $e$ such that if $x, x^k \in U^*$ and $x^i \in W$ for all $i \leqq k$, then $x^i \in U$ for all $i \leqq k$.

Proof.

Put

$$X = \left\{x^k ;\ x^k \in U,\ x^i \in W \ \textit{for all}\ i \leqq k,\ x^j \bar{\in} U \ \textit{for some}\ j \leqq k.\right\}$$

If $U-X$ does not contain any neighbourhood of $e$, then there exists a sequence $\{a_n\}$ such that

$$a_n^{k_n} \rightarrow e$$

and

$$a_n^{j_n} \bar{\in} U,$$

with
$$j_n \leqq k_n .$$

We may suppose that $a_n^{j_n}$ converges to $\bar{a} \in \overline{W}$.

For an integer $r$ we can easily find integers $r_n$ such that

$$r_n \equiv r j_n \quad (\mathit{mod}\ k_n)$$
$$0 < r_n \leqq k_n .$$

Then $\{a_n^{r_n}\}$ converges to $\bar{a}^r$ because

$$a_n^{r_n} = a_n^{r j_n} \cdot a_n^{p_n k_n},$$

where $p_n$ are integers whose absolute values are less than $r$.

Let us denote by $A$ the aggregate of limit points $\bar{a}^r$. $A$ is clearly a non-trivial subgroup of $G$ contained in $W$, because $A \ni \bar{a} \neq e$. This contradicts the hypothesis and whence we complete the proof of Lemma 1.

Now we shall have the

**Theorem 1'.** *A locally compact group which has no arbitrarily small subgroup, has the property* $(S)$.

Proof. Let us take a neighbourhood $U$ of $e$ such that $U^2 \subset W$. Let $V$ be a neighbourhood of $e$ contained in $U^*$. If $x, x^2, x^3, \ldots x^{2^j} \in U$ and $x^{2^{j+1}} \in V$, then clearly $x^i \in W$ for all $i \leqq 2^{j+1}$ and by Lemma 1 $x^i \in U$ for all $i \leqq 2^{j+1}$. Therefore for a large $j$, $x^{2^j} \bar{\in} V$, which shows that $G$ has the property $(S)$.

§ 3. Concerning Theorem 2 we shall also use the Kuranishi's[4] results. He proved that if $(x^{1/n} y^{1/n})^n$ converge uniformly to a continuous function over $U \times U$, then $G$ is a Lie group.

We shall need some preparatory lemmas.

4) l. c. 3).

**Lemma 2.** Under the assumption of Theorem 2 a metric $\rho(x, y)$ can be defined so that $\rho(x^\lambda, e)$ may be differentiable at $\lambda = 0$.

Proof. At first we must define the metric. Without loss of generality $\tilde{U}$ may be taken as a symmetric one.

Let $y$ be an element of the boundary $Bd(\tilde{U})$ of $\tilde{U}$[5], and let $U$ be a neighbourhood of $e$ such that

$$U^2 \subset \tilde{U}.$$

Let us define a metric $\rho(x, y)$ in $U$ such that

$$\rho(x, e) = \inf_{(y)} \sum |\lambda_i|$$
$$\rho(x, z) = \rho(z^{-1}x, e)$$

where inf $(y)$ means the infimum of $\sum |\lambda_i|$ when we take an arbitrary decomposition

$$x = y_1^{\lambda_1} y_2^{\lambda_2} \dots y_p^{\lambda_p}$$

for a suitable $y_i$'s $\in Bd(\tilde{U})$ and real $\lambda_i$'s.

It is clear that

$$\rho(x, y) = \rho(y, x)$$
$$\rho(x, y) + \rho(y, z) \geqq \rho(x, z).$$

This metric $\rho(x, y)$ satisfies the metric conditions when we prove that $\rho(x, y) = 0$ implies $x = y$. We may suppose that $y = e$ and $G$ is connected.

For a sufficiently large $s$

$$x^s \bar{\in} \tilde{U}.$$

We denote by $t(n)$ the smallest integer such that

$$(U_n)^{t(n)} \ni x.$$

Then

$$st(n) > n,$$
$$t(n)/n > 1/s.$$

From this inequality we shall have easily

$$\rho(x, e) > 0.$$

Hence we have proved that $\rho(x, y)$ satisfies the metric conditions. From the definition we can easily see that

5) We may assume that $Bd(\tilde{U})$ intersects in only one point with any one-parameter semi-group. See H. Whitney: *On regular family of curves*, Bull. Amer. Math. Soc. 47, 145-147 (1941).

$$\rho(x^\lambda, e) \leqq \lambda$$

for $x \in U$.

Then $\rho(x^\lambda, e)$ is differentiable because $\rho(x^\lambda, e)$ is subadditive and $\rho(x^\lambda, e)/\lambda \leqq 1$ [6].

§ 4. Now let us study some properties of this metric $\rho(x, y)$.

**Lemma 3.** The metric $\rho(x, y)$ has the following properties:

i) $\rho(x, y)$ is left invariant,

ii) $K_2 \rho(y, e) \leqq \rho(yx, x) \leqq K_1 \rho(y, e)$,[7]

iii) if $\rho(x, e) = O(\mu)$, then

$$\frac{\rho(x^{-1}yx, e)}{\rho(y, e)} = 1 + O(\mu).$$

Proof. i) is evident. From the definition of this metric we have for some $y_i$'s $\in Bd(\tilde{U})$, real $\lambda_i$'s and an arbitrarily small $\varepsilon$,

$$\left|\rho(y, e) - \sum|\lambda_i|\right| \leqq \varepsilon$$

and

$$y = y_1^{\lambda_1} y_2^{\lambda_2} \dots y_p^{\lambda_p}.$$

Let us consider a real number $s(y)$ with

$$x^{-1} y^{s(y)} x \in Bd(\tilde{U}).$$

Then for every $i$

$$0 < K_2 \leqq s(y_i) \leqq K_1 < \infty.$$

Hence by simple calculations we obtain

$$K_2 \rho(y, e) \leqq \rho(x^{-1}yx, e) \leqq K_1 \rho(y, e).$$

Thus the proposition ii) is proved.

In case $\rho(x, e) = O(\mu)$,

$$s(y) = 1 + O(\mu),$$

because

$$\frac{\rho(x^{-1}yx, e)}{\rho(y, e)} - 1 \leqq \frac{\rho(x, e) + \rho(y^{-1}xy, e)}{\rho(y, e)} \leqq \frac{O(\mu)}{\rho(y, e)}.$$

Hence the proposition iii) is proved and we complete the proof of Lemma 3.

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6) See, Einar Hille: *Functional Analysis and Semigroups*, Amer. Math. Soc. Coll. Pub. p. 143.

7) In this note $K_i$'s are all absolute constants. They could be taken near to 1 except $K_4$.

§5. From the following relations

$$x(xaxa^{-1})\,x^{-1}(xaxa^{-1}) = x^2ax^2a^{-1},$$

we have

$$\rho(xaxa^{-1}, e)\,(1+C(x)) \geqq \rho(x^2ax^2a^{-1}, e),$$

where $C(x)$ denotes

$$\sup_y \frac{\rho(x^{-1}yx, e)}{\rho(y, e)}$$

If $\delta_{\tilde{U}}(a)$ and $\delta_{\tilde{U}}(x)$ are $\leqq n$,
then

$$\begin{aligned}&\rho(xaxa^{-1}, e)\prod_{i=1}^{n}(1+C(x^{2^i}))\\ &\geqq \rho(x^{2^n}ax^{2^n}a^{-1}, e)\\ &\geqq \rho(x^{2^{n+1}}, e) + O(1/2^n)\\ &\geqq 2^{n+1}K_3\rho(x, e)\,.\end{aligned}$$

It is possible to take our neighbourhood $\tilde{U}$ so small as to make $K_3$ sufficiently near to 1.

On the other hand

$$\prod^{n}(1 + C(x^{2^i}))/2^n$$

can be taken sufficiently near to 1 too.

Then we have

$$\rho(xaxa^{-1}, e) \geqq K_4\rho(x, e)$$

with $K_4 > 1$.

Put

$$\begin{aligned}x_1 &= x & a_1 &= a\\ &\vdots & &\vdots\\ x_i &= x_{i-1}a_{i-1}x_{i-1}a_{i-1}^{-1}, & a_i &= a^{2^i}\\ &\vdots & &\vdots\end{aligned}$$

Then by simple calculations

$$\rho(x_p, e) \geqq K_4{}^p\rho(x, e)$$

or

$$\rho((xa)^{2^p}a^{-2^p}, e) \geqq K_4{}^p\rho(x, e)\,.$$

Now take $x^{1/2^q}y^{1/2^p}x^{-1/2^q}y^{-1/2^p}$ for $x$ and $y^{1/2^p}$ for $a$ in the above inequality.

Then

$$\begin{aligned}&\rho(x^{1/2^q}yx^{-1/2^q}y^{-1}, e)\\ &\geqq K_4{}^p\rho(x^{1/2^q}y^{1/2^p}x^{-1/2^q}y^{-1/2^p}, e)\,.\end{aligned}$$

This means that

$$2^q\rho\left(x^{1/2^q}y^{1/2^p}x^{-1/2^q}y^{-1/2^p},\ e\right)$$

converge to zero when $p$ and $q$ increase to $\infty$.

Consequently

$$\rho\left(x^{\mu}y^{\lambda}x^{-\mu}y^{-\lambda},\ e\right)/\ \mu$$

converge to zero when $\lambda$ and $\mu$ decrease to zero.

§ 6. Proof of Theorem 2.

Put

$$F_n(x,\ y)\ =\ (x^{1/n}y^{1/n})^n\ .$$

Then

$$\begin{aligned}&\rho\,(F_n(x,\ y),\quad F_{np}(x,\ y))\\ &=\rho\left((x^{1/n}y^{1/n})^n,\quad (x^{1/np}y^{1/np})^{np}\right)\\ &\leqq K_1 n\rho\,(x^{1/n}y^{1/n},\quad (x^{1/np}y^{1/np})^p)\\ &\leqq K_1 n\sum_{i=0}^{p-1}\rho\,(y^{1/np},\ x^{-i/np}y^{1/np}x^{i/np})\end{aligned}$$

By the inequality obtained in the last chapter

$$\begin{aligned}&\leqq K_1\ np/np\cdot\varepsilon_n\\ &=K_1\varepsilon_n\ ,\end{aligned}$$

where $\varepsilon_n$ converges to zero when $n\to\infty$.

Hence we proved that $F_n(x,\ y)$ converge uniformly in $U\times U$, which completes the proof of Theorem 2.

To conclude this note, the author wishes to thank Dr. Gotô and Dr. Kuranishi for their valuable suggestions and advices.

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Added in proof. Theorem 1 is true when the group is locally connected. The proof will be given in the next number of this journal.