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SOME ESTIMATES OF GREEN’S FUNCTIONS IN THE SHADOW

GEORGI POPOV

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0. Introduction

The purpose of this work is to investigate the asymptotic behaviour of Green’s functions in the so-called shadow for Laplace operator in an exterior domain. As a consequence a field scattered by a non-trapping obstacle will be examined at high frequencies.

These asymptotics have been studied by many authors since Keller’s article [6] appeared. It was shown that for some convex obstacles the scattered field in the shadow should be as small as the exponent $\exp(-A |k|^{1/3})$, $A > 0$, is when the frequency $k$ tends to infinity. Such an estimate is believed to take place for a large class of domains but it has not been proved yet even for strictly convex obstacles except for some special cases. In [12], Ludwig constructed an asymptotic solution $u_N$ for Helmholtz equation in the deep shadow which behaved like $\exp(-A |k|^{1/3})$, $A > 0$, as $k \to \infty$, but he did not show that the difference between $u_N$ and the exact solution could be estimated by the same exponent.

The asymptotics of Green’s functions in the shadow were investigated in [1], [2], [3], [14]. Recently, an asymptotic solution of Green’s functions in the deep shadow was obtained by Zayaev and Philippov [4] for planar strictly convex obstacles. Probably, the technique developed in [8], [9], [11] may be used to obtain the asymptotic expansions of Green’s functions at high frequencies for any strictly convex obstacle in $\mathbb{R}^n$, $n \geq 2$.

Let $K$ be a compact in $\mathbb{R}^n$, $n \geq 2$, with a real analytic boundary $\Gamma$ and let $\Omega = \mathbb{R}^n \setminus K$. The obstacle $K$ is called non-trapping if for any $R > 0$ with $K \subset B_R = \{x \in \mathbb{R}^n; |x| \leq R\}$ there exists $T_R > 0$ such that there are no generalized geodesics, (for definition see [13]), with length $T_R$ within $\Omega \cap B_R$. Denote by $\Delta_0$, respectively by $\Delta_D$, the self-adjoint extension of the Laplace operator in $\mathbb{R}^n$, respectively in $\Omega$ with Dirichlet boundary conditions. Let

$$R_j(k) = (-\Delta_j - k^2)^{-1}$$

be the resolvent of the operator $-\Delta_j$, $j = 0, D$ in $\pm \text{Im}k > 0$. Consider the
cut-off resolvents

\begin{align}
\{ k \in C; \pm \text{Im} k > 0 \} \ni k \rightarrow R_{\beta, x}(k) = \chi R_{\beta, x}^{\pm}(k) \chi \in L'(L^2(\Omega), L^2(\Omega))
\end{align}

where \( \chi \in C^\infty_c(\Omega) = \{ \phi \in C^\infty(\Omega); \text{ supp} \phi \text{ is compact} \} \) and \( \chi(\cdot) = 1 \) in a neighbourhood of \( \Gamma \). Hereafter \( L'(H, H) \) stands for the Banach space of bounded linear operators mapping from the Banach space \( H \) into the Banach space \( H \) and equipped with the usual norm. Obviously the functions (0.1) are analytic with respect to \( k \) in \( \pm \text{Im} k > 0 \).

Our first result is

**Theorem 1.** Suppose \( K \) non-trapping. Then the function (0.1) admits an analytic continuation in the region

\[ U_{\alpha, \beta}^\pm = \{ k \in C; \pm \text{Im} k \leq \alpha |k|^{1/3} - \beta \} \]

for some positive constants \( \alpha \) and \( \beta \).

This theorem was proved for strictly convex obstacles with \( C^\infty \) boundaries and for \( n=3 \) by Babich and Grigorieva [2]. Recently, in [8], [9], Bardos, Lebeau and Rauch showed that the region \( U_{\alpha, \beta}^\pm \) is free of poles of the scattering matrix for any non-trapping obstacle with an analytic boundary, provided \( n \geq 3 \) odd. They investigated the generator \( B \) of the semi-group \( Z(t) \) introduced by Lax and Phillips in [7]. Using the propagation of the Gevrey singularities of the solutions of the mixed problem for the wave equation they proved the estimate \( ||B'Z(t_0)|| \leq AC'(3j)! \) for some \( t_0 \) and for any \( j \in \mathbb{Z}^+ \). Then the region \( U_{\alpha, \beta}^\pm \) does not contain poles of the scattering matrix according to the results in [7], §3. This result can be obtained also from Theorem 1 since the poles of the scattering matrix coincide with the poles of the meromorphic continuation of \( R_{\beta, x}(k) \).

A result close to Theorem 1 was proved by Vainberg [18] and Rauch [16] when \( K \) is non-trapping and \( \Gamma \) is smooth. In this case the functions (0.1) have analytic continuations in \( \{ k \in C; \pm \text{Im} k \leq \alpha \text{ Log} |k| - \beta \} \). It is an open problem if Theorem 1 can be extended to hold for any smooth, non-trapping obstacle.

Let us now consider the distribution kernel \( G^\pm(k, x, y) \) \((G^-(k, x, y))\) of the resolvent \( R_{\beta, x}(k) \) in \( \pm \text{Im} k \geq 0 \) which is usually called outgoing (incoming) Green's function. For any \( k > 0 \) the distribution \( G^\pm(k, x, y) \) solves the problem

\begin{align}
-(\Delta + k^2) G^\pm(k, x, y) &= -\delta(x-y), \quad (x, y) \in \Omega \times \Omega \\
BG^\pm &= 0 \\
G^\pm(k, x, y) &= 0(r^{(3-n)/2}), \quad \frac{dG^\pm}{dr} \mp ik G^\pm = o(r^{(3-n)/2})
\end{align}

as \( r = |x-y| \rightarrow \infty \) and \( k \in R^+_1 = (0, \infty) \).
SOME ESTIMATES OF GREEN'S FUNCTIONS

where \( B u = u \Gamma \).

The point \( x_0 \in \Omega \) belongs to the shadow \( Sh(y_0) \) of \( K \) with respect to a given point \( y_0 \in \Omega \) if there are no generalized geodesics starting at \( y_0 \) and passing through \( x_0 \). Denote by \( d(x, y) \) the distance function in \( \Omega \), i.e.

\[
d(x, y) = \inf \{ \text{length of } \gamma; \gamma \text{ is a path in } \Omega \text{ connecting } x \text{ and } y \}.\]

Denote \( D^*_\gamma = D_1 \cdots D_n^* \), where \( D_j = i^{-1} \partial/\partial x_j \) and \( p = (p_1, \ldots, p_n) \in \mathbb{Z}_+^n, \mathbb{Z}_+ = \{0, 1, \ldots\} \).

**Theorem 2.** Suppose \( K \) non-trapping and \( x_0 \in Sh(y_0) \). Then there exists a neighbourhood \( \mathcal{O} \) of \( (x_0, y_0) \) in \( \Omega \times \Omega \) such that

\[
| D^*_\gamma D^*_\delta D^*_\zeta G^\pm(k, x, y) | \leq C \exp(-A|k|^{1/2} \mp d(x, y) \text{ Im } k) \]

in \( U^\pm \times \mathcal{O} \) for any \( (m, p, q) \in \mathbb{Z}_+^{2n+1} \) and for some positive constants \( \alpha, \beta, A, \) and \( C = C(m, p, q) \).

Now consider the scattering of plane waves by the obstacle \( K \). Let \( \omega \in S^{n-1} = \{ \theta \in \mathbb{R}^n; |\theta| = 1 \} \) and denote \( L_s = \{ x \in \mathbb{R}^n; \langle x, \omega \rangle = s \} \) where \( \langle x, \omega \rangle = \sum_{j=1}^n x_j \omega_j \). Consider the solution \( u_s(k, x) \) of the problem

\[
\begin{align*}
(\Delta + k^2) u_s(k, x) &= 0 \\
u_s \big|_{x \in \Gamma} &= -e^{ik\langle x, \omega \rangle}/x \in \Gamma \\
u_s &= O(r^{(1-s)/2}), \frac{d}{dr} u_s - iku_s = o(r^{(1-s)/2}) \text{ as } r = |x| \to \infty.
\end{align*}
\]

The point \( x_0 \) belongs to the shadow \( Sh(K, \omega) \) of \( K \) with respect to a given direction \( \omega \) if none of the generalized geodesics \( \gamma(t), t > 0 \), starting at \( L_s \) for some \( s < \min \langle y, \omega \rangle \) and having \( \omega \) as an initial direction passes through the point \( x_0 \) (\( t \) is the natural parameter on \( \gamma \)).

**Theorem 3.** Suppose \( K \) non-trapping and \( x_0 \in Sh(K, \omega) \). Then there exists a neighbourhood \( \mathcal{O} \) of \( x_0 \) in \( \Omega \) such that

\[
| D^*_\gamma D^*_\delta (u_s(k, x) + e^{ik\langle x, \omega \rangle}) | \leq C \exp(-A|k|^{1/2})
\]

in \( [k_0, \infty) \times \mathcal{O} \) for some \( A > 0 \) and any \( k_0 > 0, m \geq 0, p \in \mathbb{Z}_+^n \).

An immediate consequence of (0.4) is the Kirchoff approximation of \( \frac{\partial}{\partial \nu} u_s \big|_{\Gamma} \) in the shadow, where \( \nu \) is the outward normal to \( \Gamma \).

An estimate close to (0.3) was obtained for strictly convex obstacles in [2]. Moreover, some asymptotic expansions in the shadow for \( x_0 \) and \( y_0 \) sufficiently close to \( \Gamma \) and \( n=2 \) were recently obtained by Zayaev and Philippov in [4]. Provided \( x_0 \in Sh(y_0) \) and \( \Gamma \) smooth the Green's functions \( G^\pm \) were
estimated in [14] as follows

$$|G^\pm(k, x, y)| \leq C_N k^{-N}$$

for any $N>0$ in $k \geq k_0>0$ and $(x, y)$ in a neighbourhood of $(x_0, y_0)$.

The estimate (0.4) was predicted by Keller's geometrical theory of diffraction [5], [6], see also [12].

The method we use is close to that developed by Vainberg [18] (see also [16]) in order to prove uniform decay of the local energy for hyperbolic equations. The propagation of Gevrey singularities for the mixed problem studied in [10], [11] and the non-trapping condition allow us to compare the solutions of the mixed problem with suitably chosen solutions of the Cauchy problem for the wave equation. This is used in Proposition 1 to prove that the kernels of the cut-off resolvents $R_{\beta, x}(k)$ coincide with the Fourier transforms of some compactly supported distributions modulo exponentially decreasing functions, holomorphic in $U_{\beta}^{+}$. The theorems follow from Proposition 1 by using once more the results on the propagation of Gevrey singularities for the mixed problem.

1. Estimates of Green's functions

In this section we prove theorems 1 and 2. Let us denote by $U_0(t)$ and $U(t)$ the propagators of the Cauchy problem and the mixed problem respectively, i.e.

\[
\begin{cases}
(\partial_t^2 - \Delta) U_0(t)f(x) = 0 & \text{in } (t, x) \in \mathbb{R}^4 \times \mathbb{R}^n \\
U_0(0)f(x) = 0, \quad \partial_t U_0(0)f(x) = f(x), & f \in C_0(\mathbb{R}^n),
\end{cases}
\]

\[
\begin{cases}
(\partial_t^2 - \Delta) U(t)f(x) = 0 & \text{in } (t, x) \in \mathbb{R}^4 \times \Omega \\
B U(t)f(x) = 0 & \text{in } \mathbb{R}^n \\
U(0)f(x) = 0, \quad \partial_t U(0)f(x) = f(x), & f \in C_\beta(\Omega),
\end{cases}
\]

Using standard energy estimates one can extend the operators $U_0(t)$ and $U(t)$ by continuity in $L^2(\mathbb{R}^n)$ and in $L^2(\Omega)$ respectively. Recall that a function $f(x)$ defined in a domain $M \subset \mathbb{R}^n$ belongs to the Gevrey class $G^s(M)$, $s \geq 1$, if for any compact $M_i \subset M$ there exist some constants $A = A(M_i, f), B = B(M_i, f)$ such that

$$\sup_{x \in M_i} |D^\alpha f(x)| \leq A B^{(|\alpha| - s)}$$

for any $\alpha$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = (\alpha_1!) \cdots (\alpha_n)!$.

Let $\chi \in G^\omega(\mathbb{R}^n)$, $\chi(x) = 1$ in a neighbourhood of $B_R = \{ x; x \leq R \}$ and $\chi(x) = 0$ for $x \not\in B_{R_1}$ for some $R_1 > R$. In view of the non-trapping condition there exists $T > R_1$ such that any generalized geodesic starting at $B_{R_1}$ leaves it by
the time $T$. Then from the theorem about the propagation of Gevrey $G^3$
singularities proved by G. Lebeau [10] follows that the distribution kernel
$U(t, x, y)$ of $U(t)$ is a $G^3$ function in
$$Q_0 = [R^3(-T, T)] \times (B_{R_1} \cap \Omega) \times (B_{R_2} \cap \Omega).$$
Therefore the estimate
\begin{equation}
|D^j D^x_s D^y_u U(t, x, y)| \leq A_Q C^{|\alpha|+|\beta|+\left((j+|\alpha|+|\beta|)l\right)^3}
\end{equation}
holds in $(t, x, y) \in Q$ for any compact $Q \subset Q_0$ and any $j, \alpha, \beta$. Moreover the
constants $A_Q$ and $C_Q$ do not depend on $(t, x, y) \in Q$ and on $j, \alpha, \beta$.

Let $\zeta \in G^3(R^{n+1}), \zeta = 1$ in a neighbourhood of the set \{$(t, x) \in R^{n+1}; ||x|-
-t| < T$\} and $\zeta(t, x) = 0$ if $||x| - t| > T+1$. Consider the operators
$$U_x(t) = \chi U(t) \chi, \ U_{0,x}(t) = \chi U_0(t) \chi, \ E(t) = \zeta U(t) \chi.$$ 
Next we write the modified resolvent $R^\delta_{\gamma}(k)$ in the form
\begin{equation}
R^\delta_{\gamma}(k) = \chi \hat{E}(k) + Z_\gamma(k)
\end{equation}
where
$$\chi \hat{E}(k) = \int_0^\infty e^{ikt} \chi \Lambda(t) dt, \ \text{Im} \ k > 0,$$
denotes the Fourier-Laplace transform of $\chi \Lambda \in L^1(R^4, L^s(L^2(\Omega), L^2(\Omega)))$. Note
that the operator-valued function $\chi \Lambda(t)$ has a compact support with respect to $t$ since $\chi(x) \zeta(t, x)$ has. Therefore $\chi \hat{E}(k)$ is an analytic function with values
in the space $L^s(L^2(\Omega), L^s(\Omega))$, while $Z_\gamma(k)$ is analytic in $\{k \in C; \ \text{Im} \ k > 0\}$. Let
$H^s(\Omega), s \geq 0, s \in Z$, be the closure of $C^0(\Omega)$ with respect to the Sobolev norm
$||u||^2 = \sum_{|\alpha| \leq s} ||D^\alpha u||^2_{L^2(\Omega)}$ and let $H^{-s}(\Omega)$ be the dual space of $H^s(\Omega)$. We shall use
also the domain $D^s$ of the operator $(-\Delta)^{s/2}$, $s \geq 0, s \in Z$, equipped with the
graph topology, where the operator $(-\Delta)^{s/2}$ is given by the functional calculas. Denote by $D^{-s}$ the dual space of $D^s$. Theorems 1 and 2 will follow from

**Proposition 1.** The function $Z_\gamma(k)$ can be extended as an analytic function
$$\{k \in C; \ \text{Im} \ k > 0\} \ni k \mapsto Z_\gamma(k) \in L(H^{-s}(\Omega), H^s(\Omega))$$
for any $s \geq 0, s \in Z$. Moreover, there exist some positive constants $\alpha$ and $\beta$ such
that $Z_\gamma(k)$ has an analytic continuation in $U^\ast_{\alpha, \beta}$ and
\begin{equation}
||D^s \gamma Z_\gamma(k)||_{L(H^{-s}, H^s)} \leq C \exp(-A |k|^{s/2} - T \text{Im} K), \ m \geq 0,
\end{equation}
in $k \in U^\ast_{\alpha, \beta}$ for some positive constants $A$ and $C = C(m, s)$.

**Proof.** Let us denote $F(t) = [\partial_t^2 - \Delta, \zeta] U(t) \chi$, where $[F_1, F_2] = F_1 F_2 - F_2 F_1$,
is the commutator of the operators $F_1$ and $F_2$ and $\xi$ stands for the operator of multiplication by the function $\xi(t, x)$. Then $E(t)$ is the propagator of the problem

\[
\begin{cases}
(\partial_t^2 - \Delta) E(t) f(x) &= F(t) f(x) \\
BE(t) f &= 0 \\
E(0) f(x) &= \chi(x) f(x), \quad f \in L^2(\Omega).
\end{cases}
\] (1.6)

The distribution kernel $F(t, x, y)$ of the operator $F(t)$ belongs to the Gevrey class $G^3(\mathbb{R}^1 \times \Omega \times \Omega)$ in view of the propagation of Gevrey singularities of $U(t, x, y)$ and the definition of the functions $\xi(t, x)$ and $\chi(x)$. Moreover

\[
\text{supp } F \subset \{ (t, x, y) \in \mathbb{R}^1 \times \Omega \times \Omega; \\
|t| > T, \ T \leq |x| - t \leq T + 1, \ |y| \leq R\} \tag{1.7}
\]

in view of the finite propagation speed for the wave equation.

Let $\tilde{F}(t, x, y)$ be a $G^3$ continuation of the function $F(t, x, y)$ such that (1.7) continues to hold. Denote by $\tilde{F}(t)$ the operator with a distribution kernel $\tilde{F}(t, x, y)$ and consider the problem

\[
\begin{cases}
(\partial_t^2 - \Delta) W(t) f(x) &= \tilde{F}(t) f(x) \\
W(0) f(x) &= \partial_t W(0) f(x) = 0, \quad f \in C^\infty_0(\mathbb{R}^n) \tag{1.8}
\end{cases}
\]

The distribution kernel $W(t, x, y)$ of $W(t)$ is a $G^3$ function since the function $\tilde{F}(t, x, y)$ is such, $\tilde{F}(t) = 0$ in $|t| < T$ and since

\[
W(t) = \int_0^t U_0(s) \tilde{F}(t-s) ds
\]

Let $\psi \in C^\infty(\mathbb{R}^n)$, $\psi(x) = 0$ in a neighbourhood of $B_R$ and $\chi(x) = 1$ on supp $(1 - \psi)$. Denote

\[
Q(t) f(x) = (\partial_t^2 - \Delta) (E(t) f(x) - \psi W(t) f(x))
= (1 - \psi) F(t) f(x) + [\Delta, \psi] W(t) f(x)
\]

in $x \in \Omega$ for $f \in C^\infty_0(\Omega)$. In view of (1.6), (1.7) and Duhamel's formula we obtain

\[
E(t) f - \psi W(t) f = U(t) \chi f + \int_0^t U(t-s) \chi Q(s) f ds, \quad f \in L^2(\Omega).
\]

Multiplying the last equality by $\chi$ and performing Fourier-Laplace transform with respect to $t$ we obtain

\[
\chi \hat{E}(k) f = R^+_{\partial, \chi}(k) f + R^+_{\partial, \chi}(k) \hat{Q}(k) f + \psi \chi \hat{W}(k) f
\] (1.9)

for $\text{Im } k > 0$. We are going to prove that the functions $\psi \chi \hat{W}(k)$ and $\hat{Q}(k)$ can
be continued analytically for $\text{Im} \, k \leq 0$.

Let $\mathcal{H} \in C^\omega(\mathbb{R}^n)$, $\mathcal{H}(x) = 0$ for $x \in B_T$, $\mathcal{H}(x) = 1$ outside $B_{T+1}$ and set

$$G(t)f(x) = (\partial_t^2 - \Delta) \left( W(t)f(x) - \mathcal{H}(x) E(t)f(x) \right) = (1 - \mathcal{H}) \tilde{G}(t)f(x) - \left[ \Delta, \mathcal{H} \right] E(t)f(x).$$

The function $\mathbb{R}^1 \ni t \mapsto E(t) \in \mathcal{L}(D^{-s}, D^{-s+1})$ is bounded for any $s \in \mathbb{Z}$, $[\Delta, \mathcal{H}] \in L(D^{-s}, H^{-s}(\mathbb{R}^n))$, and $H^{-s}(\Omega) \subset D^{-1}$ for any $s \geq 0$, $s \in \mathbb{Z}$. Then $\mathbb{R}^1 \ni t \mapsto [\Delta, \mathcal{H}] E(t)$ is a bounded function with values in $L(H^{-s}(\Omega), H^{-s}(\mathbb{R}^n))$, $s \geq 0$, $s \in \mathbb{Z}$, and

$$\left\|G(t)\right\|_{L(H^{-s}(\Omega), H^{-s}(\mathbb{R}^n))} \leq C$$

for any $t \in \mathbb{R}^1$.

In view of (1.6), (1.7), (1.10) and Duhamel’s formula we write

$$W(t)f(x) = \mathcal{H}(x) E(t)f(x) + \int_0^t U_0(t-s) G(s)f(x)ds, \quad f \in H^{-s}(\Omega).$$

Note that the support of the distribution kernel of $G(t)$ is contained in $\{(t, x, y); \quad |t| \leq 2T+2, |x| \leq T+1, |y| \leq T+1\}$. Therefore

$$\chi_2 W(t)f = \chi_2 \int_0^T U_0(t-s) \chi_1 G(s)fds, \quad f \in H^{-s}(\Omega),$$

for any $T \geq 2T+2$, where $\chi_1 \in C_0(\mathbb{R}^n)$, $\chi_1(x) = 1$ in $B_{T+1}$ and $\chi_2 \in C_0(B_T)$.

**Lemma 1.** Let $\chi_2 \in C_0(B_T \setminus B_R)$. Then $\chi_2 U_0(t) \chi_1 \in L(H^{-1}(\mathbb{R}^n), H^1(\Omega))$ for any $s \in \mathbb{R}^1$ and any $t \in [2T+3, \infty)$. Moreover the function

$$[2T+3, \infty) \ni t \mapsto \chi_2 U_0(t) \chi_1 \in L(H^{-1}(\mathbb{R}^n), H^1(\Omega))$$

can be continued analytically in $\{t \in \mathbb{C}; \quad |t| > 2T+3\}$ and

$$\left\|D^j\chi_2 U_0(t) \chi_1\right\|_{L(H^{-1}(\mathbb{R}^n), H^1(\Omega))} \leq A(j)|t|^{-2}$$

for any $t \in \mathbb{C}$, $|t| > 2T+3$, for $j \geq \max(0, 3-n)$, and for some $A$ which does not depend on $j$.

**Proof.** The conclusion is obvious when $n$ is odd because of Huyghens principle. Suppose $n \geq 2$ is even, $j \geq 1$, and set $\mathcal{O}_T = \{(t, x, y) \in C^{2n+1}; |t| > 2T+3, |x| \leq T, |y| \leq T+1\}$. Then $U_0(t, x, y) = C_n (t^2 - |y|^2)^{(n-1)/2}$ for any $(t, x, y) \in \mathcal{O}_T$ and for some constant $C_n$. Using Cauchy integral formula we obtain for any $j \geq 1$, $\alpha, \beta$ the estimate

$$|\partial^\alpha_x \partial^\beta_y U_0(t, x, y)| \leq (2\pi)^{-2n-1}(j-1)! (\alpha + \beta)! 2^{\alpha + \beta} \max\{|\partial^\alpha_x U_0(z, \mathfrak{x}, \mathfrak{y})|; |z-t| = 1, |x-\mathfrak{x}| + |y-\mathfrak{y}| = 1/2\} \leq A_{\alpha, \beta}(j)|t|^{-2}$$

in $\mathcal{O}_T$, which yields (1.12).
According to (1.10), (1.11) and lemma 1 the function

\[ [T_2, \infty) \ni t \mapsto \chi_2 W(t) \in \mathcal{L}(H^{-s}(\Omega), H^s(\Omega)), \quad T_2 = 2T_1 + 2, \]

can be continued as an analytical one in \{t \in \mathbb{C}; |t| > T_2\} for any \( t \geq 0 \) and any \( \chi_2 \in C_0^\infty(B_T \setminus B_R) \). Moreover the estimate

(1.13) \[ ||D^j \chi_2 W(t)||_{\mathcal{L}(H^{-s}, H^s)} \leq A(j)! |t|^{-2} \]

is valid in \( |t| > T_2 \) for any \( j \geq \max(0, 3-n) \) and any \( s \geq 0, s \in \mathbb{Z} \) where the constant \( A \) does not depend on \( j \).

Now we can estimate the norm of the Fourier-Laplace transform of \( \chi_2 W(t) \) in \( \mathcal{L}(H^{-s}, H^s) \). Let \( \Re \alpha > \alpha_0 > 0 \) for some \( \alpha_0 > 0 \). Since \( W(t) = 0 \) in \( |t| < T \) we can write

\[ \chi_2 \hat{W}(k) = k^{-1} \int_0^{T_2} e^{ikt} D_i \chi_2 W(t) \, dt + k^{-1} \int_{T_2}^\infty e^{ikt} D_i \chi_2 W(t) \, dt. \]

Using (1.13) we can change the contour of integration in the second integral to obtain

\[ \exp(C |k|^{1/3}) \chi_2 \hat{W}(k) = \sum_{j=0}^{\infty} \frac{C}{j!} |k|^{-3/2-j} \left[ \int_0^{T_2} e^{ikt} D_i \chi_2 W(t) \, dt + e^{ikt} \int_0^\infty e^{-kt} \chi_2 (D_i W)(T_2 + it) \, dt \right]. \]

Integrating \( [j/3] \) times by parts in any member of the last sum we have

\[ \exp(C |k|^{1/3}) \chi_2 \hat{W}(k) = \sum_{j=0}^{\infty} \frac{C}{j!} k^{3/2-j} \left[ \int_0^{T_2} e^{ikt} \chi_2 D_i^{j/3+1} W(t) \, dt + e^{ikt} \int_0^\infty e^{-kt} \chi_2 (D_i^{j/3+1} W)(T_2 + it) \, dt \right]. \]

where \( [m] \) denotes the integer part of \( m \in \mathbb{R} \). Since \( W \in C^3 \) and in view of (1.13) any member of the last sum can be estimated by

\[ A_1 C \beta B_i e^{-\beta |m| k}, \quad B = \begin{cases} T & \text{when } \Im k \geq 0 \\ T_2 & \text{when } \Im k < 0 \end{cases} \]

in \( \{k \in \mathbb{C}; \Re k \geq k_0 > 0\} \), where the constants \( A_1 \) and \( B_i \) do not depend on \( j \in \mathbb{Z} \). Provided that \( C < B_i^{-1/3} \) we obtain

(1.14) \[ ||\chi_2 \hat{W}(k)||_{\mathcal{L}(H^{-s}(\Omega), H^s(\Omega))} \leq C_0 \exp(-C |k|^{1/3} - B \Im k) \]

for \( \Re k \geq k_0 > 0 \), where \( C_0 = A_1(1-CB_i^{-1/3})^{-1} \). Proceeding in the same way when \( \Re k \leq -k_0 < 0 \) we can continue \( \chi_2 \hat{W}(k) \) analytically in \( \mathbb{C} \setminus \{k; \Im k \leq 0, |\Re k| \leq k_0\} \) so that (1.14) holds in this region for any \( k_0 > 0 \). Then the Fourier-Laplace transform \( \hat{Q}(k) \) of \( Q(t) = (1-\psi)F(t) + [\Delta, \psi] W(t) \) can be continued analytically in \( \mathbb{C} \setminus \{0, -i\infty\} \) and...
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(1.15) \[ \|\hat{Q}(k)\|_{L(H^{-1}(\Omega), H^s(\Omega))} \leq C_\nu \exp(-C|k|^{1/\alpha} - B \text{Im} k) \]

is fulfilled in \( C\setminus \{k; \text{Im} k \leq 0, |\text{Re} k| \leq k_0\} \) for any \( k_0 > 0 \).

**Lemma 2.** The function \( C \ni k \mapsto \hat{\chi} \hat{E}(k) \in L(H^s(\Omega), H^s(\Omega)) \) is analytic and

(1.16) \[ \|\hat{\chi} \hat{E}(k)\|_{L(H^s(\Omega), H^s(\Omega))} \leq C(1 + |k|) \max(0, -\text{Im} k) \]

for any \( s \geq 0 \), \( s \in \mathbb{Z} \).

Proof. The assertion is obvious for \( t = 0 \) since \( U(t) \) is a bounded function in \( \mathbb{R}^d \) with values in \( L(L^2(\Omega), L^2(\Omega)) \) and \( \chi \xi(t, x) = 0 \) for any \( t > 2T + 1 \). Suppose \( s \geq 1 \) and consider

(1.17) \[ \begin{align*}
L(k)f &= -\int_0^\infty e^{ikt} \chi F(t)f dt \in H^s(\Omega) \\
\hat{\chi} \hat{E}(k)f|_\Gamma &= 0
\end{align*} \]

for \( f \in H^s(\Omega) \). Here

\[ L(k)f = -\int_0^\infty e^{ikt} \chi F(t)f dt \in H^s(\Omega) \]

and \( L(k) \) satisfies the estimate (1.16) for any \( s \geq 0 \) since the distribution kernel of the operator \( \chi F(t) \) is smooth and \( \text{supp}(\chi F) \subset \{|x| \leq R_1, |y| \leq R_2, |t| < 2T + 1\} \) in view of (1.7). Then

\[ \|\hat{\chi} \hat{E}(k)f\|_{s-1} \leq C((1 + |k|^2)\|\chi \hat{E}(k)f\|_{s-1} + e^{(2T+1)\max(0, -\text{Im} k)}\|f\|_s) \]

for \( f \in H^s(\Omega) \), for some \( \chi_1 \in C_0^\infty(\Omega) \), \( \chi_\xi = 1 \) in a neighbourhood of \( \text{supp}(\chi) \) which proves (1.16) by induction. Differentiating (1.17) with respect to \( \hat{k} \) and using (1.16) it is easy to prove that

\[ \frac{d}{dk} \chi \hat{E}(k) \in L(H^s(\Omega), H^s(\Omega)) \]

for any \( s \geq 0 \), \( s \in \mathbb{Z} \). Thus \( \chi \hat{E}(k) \) is an analytic function.

According to (1.15) the operator \( I + \hat{Q}(k) \colon H^s(\Omega) \to H^s(\Omega) \) is invertible for any \( k \in U_{\alpha, \beta}^+ \) and for some \( \alpha, \beta \). Then \( R_{\alpha, \beta}(k) \) is an analytic function in \( U_{\alpha, \beta}^+ \) with values in \( L(H^s(\Omega), H^s(\Omega)) \) and satisfies (1.16) in view of (1.9) and Lemma 2. Now, (1.5) follows for \( m = 0 \) from (1.9), (1.14) and (1.15), choosing \( \alpha \) and \( \beta \) small enough. Using Cauchy integral formula we obtain (1.5) for any \( m \in \mathbb{Z}_+ \).

To prove theorem 2 we choose some neighbourhoods \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) of \( x_0 \), respectively \( y_0 \), \( \mathcal{O}_1 \subset \overline{\Omega} \), so that none of the generalized geodesics starting at \( \mathcal{O}_2 \) passes through \( \mathcal{O}_1 \). Set \( \mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2 \) and suppose that \( \mathcal{O} \subset B_R \) and \( T > \text{sup} \{\delta(x, y); (x, y) \in \mathcal{O}\} \). According to proposition 1 we have

\[ G^+(k, x, y) = \int_0^\infty e^{ikt} \xi(t, x) U(t, x, y) dt + Z_x(k, x, y) \]

where
\[ |D^n_x D^n_y Z_n(k, x, y)| = |\langle D^n_x \delta_x, D^n_y Z_n(k) D^n_y \delta_y \rangle| \leq ||D^n_x Z_n(k)||_{\mathcal{L}(H^{-\frac{n}{2}}, H^{\frac{n}{2}})}||\delta_x||_{H^{n+\frac{5}{2}}} \]
\[ \leq C \exp(-A |k|^{\frac{\alpha}{2}} - T \text{Im} k) \leq C \exp(-A_0 |k|^{\frac{\alpha}{2}} - d(x, y) \text{Im} k) \]

in \( U_{\alpha, \theta} \times \mathcal{O} \) for some \( \alpha > 0 \) and \( A_0 > 0 \). Here \( \langle \delta_x, \phi \rangle = \phi(x) \) for any \( \phi \in C_0(\Omega) \) and \( s > n + p + q \). On the other hand \( \zeta(t, x) \mathcal{U}(t, x, y) \) is a \( G^3 \) function in \( \mathbb{R}^3 \times \mathcal{O} \) with a compact support with respect to \( t \). Moreover, \( \mathcal{U}(t, x, y) = 0 \) for \( |t| < d(x, y) \) since the propagation speed for the solutions of the mixed problem for the wave equation equals one (see [17]). Now the arguments used in the proof of (1.14) yield (0.3).

Denote by \( e(\lambda, x, y) \) the spectral function of the operator \( -\Delta_D \) given as the distribution kernel of the spectral projector \( E_\lambda \) of \( -\Delta_D \). Since \( E_\lambda \to I \) in \( L^2(\Omega) \) as \( \lambda \to \infty \) and

\[ \frac{d}{d\lambda}(\lambda^2, x, y) = (2\pi)^{-1} \{ G^+(\lambda, x, y) - G^-(\lambda, x, y) \} \quad \text{for} \quad x \neq y, \lambda > 0, \]

it is easy to obtain from theorem 2 the following

**Corollary 1.** Suppose \( K \) non-trapping and \( x_0 \in Sh(y_0) \). Then

\[ |D^n_x D^n_y e(\lambda, x, y)| \leq C \exp(-A \lambda^\alpha), \quad A > 0, \]

in \( [\lambda_0, \infty) \times \mathcal{O} \) for \( (m, p, q) \in \mathbb{Z}_{2+1}^3, \lambda_0 > 0 \).

2. **Asymptotics of the scattered waves**

In this section we prove theorem 3. Translating the origin to a given point \( x_0 \in \mathbb{R}^n \) the function \( u_0(k, x) \) is multiplied by \( \exp(ik \langle x_0, \omega \rangle) \). Thus we can suppose that \( K \subset B_{R_0}(x_0) = \{ x \in \mathbb{R}^n; |x - x_0| \leq R \} \) and \( \langle x, \omega \rangle > 0 \) for any \( x \in B_{R+1}(x_0) \). Consider the function

\[ v(k, x) = u_0(k, x) + \phi(x) e^{ik \langle x, \omega \rangle} \]

where \( \phi \in G^3(B_{R+1}(x_0)) \) and \( \phi(x) = 1 \) on \( B_R(x_0), \text{supp} \phi \subset B_{R+1}(x_0). \) Then

\[ \begin{cases} (\Delta + k^2) v(k, x) = [\Delta, \phi] e^{ik \langle x, \omega \rangle} \\ v(k, x)|_{\partial} = 0 \end{cases} \]

and \( v(k, x) \) satisfies the outgoing Sommerfeld’s condition at infinity. Therefore

\[ v(k, x) = R_{\phi, x}(k) ([\Delta, \phi] e^{ik \langle x, \omega \rangle}) \]
\[ = Z_n(k) ([\Delta, \phi] e^{ik \langle x, \omega \rangle}) + \mathcal{X}(k) ([\Delta, \phi] e^{ik \langle x, \omega \rangle}) \]

for \( x \in B_R(x_0) \) where \( \mathcal{X} \in G^3(\mathbb{R}^n), \mathcal{X} = 1 \) on \( B_{R+1}(x_0) \), \( \text{supp}(\mathcal{X}) \subset B_{R+2}(x_0) \).

The first term of the last equality is estimated by proposition 1. The second one is equal to the Fourier-Laplace transform of the distribution.
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\[ v_1(t, x) = \chi(x) \int_{\Delta} \zeta(t-s, x) U(t-s) \delta(s-\langle x, \omega \rangle) \, ds \]

since \( v_2(s, y) = [\Delta, \varphi] \delta(s-\langle y, \omega \rangle) \) vanishes for \( s<0 \). The distribution \( v_1 \) is well-defined since \( v_2 \) has a compact support, \( v_2 \in D^{-m} \) for \( m>3 \) and \( \zeta(t-s) \) \( U(t-s) \) is a continuous function with valued in \( L(D^{-m}, D^{m}) \).

We are going to prove that there exists a neighbourhood \( \mathcal{O} \) of \( x_0 \) such that \( v_1 \) is a \( G^2 \) function in \( R^4 \times \mathcal{O} \).

Let us write \( v_1 = Q(v_2) \) where the operator \( Q \) has a distribution kernel
\[ Q(t, s, x, y) = \chi(x) \zeta(t-s) H(t-s) U(t, x, y) \chi(y) \text{ and } H(s) = 0 \text{ for } s \leq 0, H(s) = 1 \text{ for } s > 0. \]

We shall evaluate the Gеvrey \( G^3 \) wave front \( SS^3(v_1) \) of \( v_1 \) using the relation \( SS^3(v_1) \subset SS^3(Q) \circ SS^3(v_2) \). We have
\[ SS^3(v_2) \subset \{(s, y; \tau, \eta); s = \langle y, \omega \rangle > 0, y \in B_{R}(x_0), \eta = -\tau \omega, \tau \neq 0 \} \]

Moreover, theorem 1.4 in [10] yields
\[ SS^3(Q) \subset \{(\phi^{t-s}(s, y, \tau, \eta); s, y, \tau, \eta); s \leq t, \tau \neq 0 \} \cup \{(0, y, \tau, \xi; 0, y, \tau, \eta) \}
\]
where \( \phi^{t-s}(s, y, \tau, \eta) = (t-s, x^{t}(s, y, \tau, \eta), \tau, \xi(s, y, \tau, \eta)) \) is the generalized bi-characteristic starting at \( (s, y, \tau, \eta) \) and \( t \) is the natural parameter on it. Thus we have
\[ SS^3(\phi^{t-s}) \subset \{(t, x^{t-s}(s, y, \tau, -\tau \omega), \tau, \xi); \tau \neq 0, 0 < s = \langle y, \omega \rangle \leq t, y \in B_{R}(x_0) \} \]

Note that the initial codirection of the generalized geodesic \( \gamma(t) = x^{t}(s, y, \tau, \eta) \) is \( \frac{d\gamma}{dt}(0) = -\eta/\tau \) for any \( y \in \Omega \). Then
\[ SS^3(\gamma) \subset \{(t, \gamma(t-s), \tau, \xi); \gamma \text{ is a generalized geodesic with } \gamma(0) \in B_{R}(x_0), \frac{d\gamma}{dt}(0) = \omega, 0 < s = \langle \gamma(0), \omega \rangle \leq t \} \]

Moreover \( \gamma(t) \in B_{R}(x_0) \) for any \( t \geq 0 \) when \( \gamma(0) \in B_{R}(x_0) \) and \( \langle \gamma(0), \omega \rangle \geq \langle x_{0}, \omega \rangle \) while \( \gamma(t-s) = \gamma_{l}(t), \gamma_{l}(t) \) is the generalized geodesic with initial data \( \gamma_{l}(0) = \gamma(0)-s \omega \in L_{\omega}, \frac{d\gamma_{l}}{dt}(0) = \omega, \) when \( \gamma(0) \in B_{R}(x_0) \) and \( \langle \gamma(0), \omega \rangle \leq \langle x_{0}, \omega \rangle \). Therefore
\[ (\text{sing supp } c(\gamma_{l})) \cap B_{R}(x_0) \subset \{t = \gamma(t); t > 0 \text{ and } \gamma \text{ is a generalized geodesic with } \gamma(0) \in L_{\omega}, \frac{d\gamma}{dt}(0) = \omega \} \]

Since \( x_{0} \in Sh(K, \omega) \) we can choose a neighbourhood \( \mathcal{O} \) of \( x_{0} \) such that \( (\text{sing supp } c(\gamma_{l})) \cap \mathcal{O} \equiv \phi \) which proves theorem 3 since \( \text{supp}(v_1) \) is compact.
References


