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*On the Infinitesimal Rigidity of Surfaces**)

By T. MINAGAWA and T. RADO

Introduction

The theory of infinitesimal deformations of surfaces has an extensive literature. References actually needed in this paper are collected at the end, and will be quoted by means of numbers in square brackets. Also, Efimov [3], Darboux [2], Blaschke [1] contain extensive bibliographical comments. The starting point of the line of thought studied in this paper is the classical theorem that a closed convex surface is infinitesimally rigid. The method of proof in the literature is based upon the use of an auxiliary vector function $\mathfrak{y}$ which represents the rotational component of the deformation (see for example Blaschke [1]). Let us note, in passing, that this method (which we shall call the $\mathfrak{y}$-*technique* for brevity) makes it necessary to require that the given surface as well as the deformation vectors admitted should be of class C'''. We shall call this briefly the (C''', C''') assumption. Following through this method of proof, one sees readily that a convex surface with a boundary curve is also infinitesimally rigid provided that the boundary curve is kept point-wise fixed. The question occurred to us whether a similar theorem holds for *surfaces of negative Gauss curvature.* Pursuing this question, we obtained the results presented in § II. 2. Also, we found various rigidity theorems concerning surfaces of zero curvature, included in § II. 3. While we used originally the $\mathfrak{y}$-*technique,* we found that a severe and unmotivated restriction had to be placed upon the boundary of the surface in relation to the asymptotic lines. To remove this restriction, we devised a new and very simple approach, presented in section I. 1. 6. Now this approach presents still another advantage: the assumption (C''', C'''), explained above, could be now reduced to the assumption (C'', C'). Explicitly: we assume only that

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the surface is of class C'' and the deformation vector is of class C'. There arose the question whether a similar improvement is possible in the case of convex surfaces. We found that this is indeed the case. These new results for convex surfaces are presented in part I. Among the many problems yet open, we should like to call attention to the following ones. Under the (C''', C''') assumptions, E. Rembs [4] proved the following remarkable theorem (generalizing previous results of Liebmann). Let S be a convex surface with boundary curves $C_1, \cdots, C_m$. Suppose that each C_k is a plane curve, and that S is in contact with the plane of C_k all along C_k. Then S is infinitesimally rigid even if the boundary curves are not kept fixed. We show, in part I, that the theorem remains valid under the (C'', C') assumption, *provided* that S is of class C''' in some vicinity of the boundary curves. The removal of this restriction may lead to interesting questions. Furthermore, for surfaces of negative or zero curvature, we found that infinitesimal rigidity can be achieved even if only a part of the boundary is kept fixed. For convex surfaces, we could verify only that a corresponding property occurs, in very strong form, for those surfaces of the second order that are projectively equivalent to the sphere (see I. 7. 2). Adequate extension of this special result is the second problem to which we wish to call attention. So far our comments were restricted to surfaces whose Gauss curvature has constant sign. As regards the case of surfaces whose curvature is of variable sign, we present only a special result. In § II. 4, we show that our methods yield a proof of the infinitesimal rigidity of the torus.*)

Since we use a new approach, and also because we operate under the reduced assumption (C'', C'), direct comparison with previous methods is not feasible. For example, for the case of convex surfaces, one feature of our method is to throw a point of the surface to infinity by means of a projection transformation, and then to resolve our problems by a study of the singularity so created. The explicit formulas, due to Darboux [2], for the projective transformation (acting simultaneously upon the surface and the deformation vector) are fundamental in this discussion. We note that the rigidity of surfaces extending to infinity has been studied previously; in particular, an interesting paper of J. J. Stoker [5] proved stimulating for our own work. However, Stoker makes the (C''', C''') assumption, and his methods are very different from ours.

Our assumption (C'', C') may be considered as natural. Indeed, the theorems depend upon the sign of the Gauss curvature which involves the second derivatives of the surface, and the definition of an infinitesi-

*) For the case of analytic deformation, this has been proved already by Liebmann [7], [8].

mal deformation involves the first derivatives of the deformation vector. However, the reader conversant with the modern theory of Real Variables will easily discover further plausible extensions, not involving any new ideas. Accordingly, we restricted ourselves to a presentation in the classical spirit.

PART I. CONVEX SURFACES

§ I. 1. Infinitesimal deformations.

I. 1. 1. We shall consider surfaces in Euclidean xyz space of the following two types.

(i) Smooth closed regular surfaces of the type of the sphere. Such a surface S is topologically equivalent to the sphere, and in the vicinity of any one of its points it is required to admit of a parametric representation of the form

$$\mathfrak{x} = \mathfrak{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in D,$$

where $\mathfrak{x}$ denotes *the position vector* which joins the origin (0, 0, 0) to a point $(x(u, v), y(u, v), z(u, v))$ of S. The coordinate functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are assumed throughout to be of class C'' in the domain D. On setting, as usual,

$$E = \mathfrak{x}_u^2, \quad F = \mathfrak{x}_u \mathfrak{x}_v, \quad G = \mathfrak{x}_v^2, \quad W = (EG - F^2)^{\frac{1}{2}},$$

the term *regular surface* refers to the further condition $W > 0$ in D.

(ii) Smooth regular surfaces of the type of a finitely connected plane Jordan region. Such a surface is required to have, as a whole, a parametric representation

$$S: \quad \mathfrak{x} = \mathfrak{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in R,$$

where R is a finitely connected Jordan region in the parameter plane uv. The representation is required to be a 1-1 correspondence between R and S. The coordinate functions are required to be of class C'' in the interior of R, and it is further required that their partial derivatives of the first two order remain continuous on the boundary of R. The condition $W > 0$ is required to hold in the interior and also on the boundary of R.

(iii) In the course of the proofs, we shall consider further restricted portions of surfaces of the preceding two types. Furthermore, we shall consider unbounded surfaces which arise by subjecting the previously described types of surfaces to a projective transformation.

I. 1. 2. Let S be a surface of one of these types, and let

$\mathfrak{z}=(X, Y, Z)$ be a vector function of class C' defined on S. By this we mean the following set of conditions.

(i) If $\mathfrak{x}=\mathfrak{x}(u, v)$, $(u, v)\in D$, is an admissible representation (as specified in I. 1. 1) of a portion of S, then X, Y, Z as functions of u, v are of class C' in D.

(ii) If $S:\mathfrak{x}=\mathfrak{x}(u, v)$, $(u, v)\in R$, is an admissible representation of S (as specified in I. 1. 1, (ii)), then X, Y, Z and their first partial derivatives remain continuous on the boundary of R.

Now denote by ε a real parameter. If $\mathfrak{x}$ is the position vector of S, then the vector function $\mathfrak{x}+\varepsilon\mathfrak{z}$ determines (as position vector) for each fixed value of ε a surface S_ε. If C is an arc on S and C_ε is the corresponding arc on S_ε, then the length $l(C_\varepsilon)$ of C_ε is a function of ε. If $dl(C_\varepsilon)/d\varepsilon=0$ for $\varepsilon=0$ and for every choice of C on S, then $\mathfrak{z}$ is said to induce an *infinitesimal deformation* of S. In terms of the parameters u, v, one finds readily that the necessary and sufficient condition for this is represented by the differential equations

$$\mathfrak{z}_u\mathfrak{x}_u=0\,,\quad \mathfrak{z}_u\mathfrak{x}_v+\mathfrak{z}_v\mathfrak{x}_u=0\,,\quad \mathfrak{z}_v\mathfrak{x}_v=0\,. \tag{1}$$

I. 1. 3. If $\mathfrak{z}\equiv 0$ on S, then the equations (1) surely hold. Another obvious case arises if $\mathfrak{z}$ can be represented in the special form

$$\mathfrak{z}=(\mathfrak{u}\times\mathfrak{x})+\mathfrak{v}\,, \tag{2}$$

where $\mathfrak{u}, \mathfrak{v}$ are constant vectors. One finds by direct substitution that the equations (1) hold. In fact, this conclusion is evident *a priori* since a vector function $\mathfrak{z}$ of the special form (2) coincides (on S) with the velocity field of a rigid motion. In the special case (2), we shall say that the infinitesimal deformation induced by $\mathfrak{z}$ for S is *trivial*.

I. 1. 4. In the preceding definitions, the vector function $\mathfrak{z}$ is required to be of class C' only. In the extensive literature on infinitesimal deformations, the methods used necessitate the requirement that $\mathfrak{z}$ be of class C'''. The reason for this lies in the method generally used. This method starts with the observation that the equations (1) imply the existence of a unique vector function $\mathfrak{y}$ which satisfies the equations

$$\mathfrak{z}_u=\mathfrak{y}\times\mathfrak{x}_u\,,\quad \mathfrak{z}_v=\mathfrak{y}\times\mathfrak{x}_v\,. \tag{3}$$

This vector $\mathfrak{y}$ plays a fundamental role in the literature. For brevity, we shall refer to its use as the $\mathfrak{y}$-*technique*. In this technique, one needs the first and second partial derivatives of $\mathfrak{y}$. One finds readily that for the existence and continuity of these derivatives it is necessary to assume that $\mathfrak{z}$ itself is of class C'''. Since the basic equations (1) contain only the first derivatives of $\mathfrak{z}$, it appears desirable to devise

a technique which requires only that $\mathfrak{z}$ be of class C'. One of the contributions of this paper is the observation that this objective can be achieved in a very simple manner.

I. 1. 5. *Definition.* Let S be a smooth closed regular surface of class C'' (see I. 1. 1). If S admits only of trivial infinitesimal deformations (in the sense of I. 1. 2, I. 1. 3), then S is said to be *IR* (*infinitesimally rigid*).

Definition. Let S be a smooth regular surface of the type of a finitely connected plane Jordan region (see I. 1. 1).

(α) S is said to be *IR* (*infinitesimally rigid*) if it admits only of trivial infinitesimal deformations.

(β) Let B be the entire boundary of S. Then S is said to be *IRB* (*infinitesimally rigid for fixed boundary B*) provided that the following holds: any vector function $\mathfrak{z}$ which induces an infinitesimal deformation of S and vanishes on B is identically zero on S.

(γ) Let b denote a subset of the boundary of S. Then S is said to be *IRb* if the condition stated under (β) holds with B replaced by b.

I. 1. 6. It is our purpose to derive theorems relating to the *IR*, *IRB*, *IRb* properties. It is important to recall that the vector function $\mathfrak{z}$ inducing the infinitesimal deformation is required only to be of class C'. Accordingly, the classical $\mathfrak{y}$-technique (see I. 1. 4) is not applicable. However, we obtain differential equations appropriate for our purposes in the following manner. Returning to I. 1. 2, let

$$\xi = \frac{\mathfrak{x}_u \times \mathfrak{x}_v}{W}$$

be the unit normal vector of S. We introduce auxiliary functions a, b, c of u, v by the formulas

$$a = \mathfrak{z}\mathfrak{x}_u\,, \quad b = \mathfrak{z}\mathfrak{x}_v\,, \quad c = \mathfrak{z}\xi\,. \tag{4}$$

Since $\mathfrak{z}$ is of class C', clearly a, b, c are of class C' (at least). Differentiation of (4) yields, in view of the equation (1),

$$a_u = \mathfrak{z}\mathfrak{x}_{uu}\,, \quad b_v = \mathfrak{z}\mathfrak{x}_{vv}\,, \quad a_v + b_u = 2\,\mathfrak{z}\mathfrak{x}_{uv}\,. \tag{5}$$

Now $\mathfrak{x}_{uu}$, $\mathfrak{x}_{vv}$, $\mathfrak{x}_{uv}$ can be expressed in terms of $\mathfrak{x}_u$, $\mathfrak{x}_v$, ξ (see Blaschke [1]). Substituting these expressions in (5) and using (4), we obtain

$$a_u = \begin{Bmatrix} 1 & 1 \\ & 1 \end{Bmatrix} a + \begin{Bmatrix} 1 & 1 \\ & 2 \end{Bmatrix} b + Lc\,, \tag{6}$$

$$b_v = \begin{Bmatrix} 2 & 2 \\ & 1 \end{Bmatrix} a + \begin{Bmatrix} 2 & 2 \\ & 2 \end{Bmatrix} b + Nc\,, \tag{7}$$

$$(8) \qquad a_v + b_u = 2\begin{Bmatrix}1 & 2\\ & 1\end{Bmatrix} a + 2\begin{Bmatrix}1 & 2\\ & 2\end{Bmatrix} b + 2Mc\,,$$

where the coefficients of a, b are the so-called Christoffel symbols, and L, M, N are the second fundamental quantities for S. These are the fundamental differential equations in our method. *It is important to realize that a and b depend upon the parametric representation chosen for S.*

I. 1. 7. For later reference, we note the form of our equations for the case when $u = x$, $v = y$, and accordingly $z = z(x, y)$, where $z(x, y)$ is single-valued and of class C''. Denoting the partial derivatives of z by p, q, r, s, t as usual, one finds

$$(9) \qquad a = X + pZ\,, \quad b = Y + qZ\,,$$

$$(10) \qquad a_x = rZ\,, \quad b_y = tZ\,, \quad a_y + b_x = 2\,sZ\,.$$

Let us recall that X, Y, Z are of class C' by assumption. Let us insist again that the functions a, b are *not invariant* under a change of parameters.

§ I. 2. The Darboux transformation.

I. 2. 1. Let $S: \mathfrak{x} = \mathfrak{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in D$, be a portion of a regular surface of class C'' (see I. 1. 1), which satisfies the following conditions in the domain D.

(i) $z(u, v) > 0$ in D.

(ii) If O is the origin of the coordinate system and P is any point of S, then the line g passing through O and P intersects S in the single point P.

(iii) If P is any point of S, then the tangent plane of S at P does not pass through O. It is readily seen that this condition is equivalent to the condition

$$(1) \qquad \begin{vmatrix} x & y & z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \neq 0 \quad \text{in } D\,.$$

To this surface S, we apply the projective transformation

$$(2) \qquad x' = \frac{x}{z}\,, \quad y' = \frac{y}{z}\,, \quad z' = \frac{1}{z}\,.$$

In view of condition (i), this transformation is applicable to S, and we obtain from S a new surface S', given by

$$(3) \quad S': \mathfrak{x} = \mathfrak{x}'(u, v) = (x'(u, v), y'(u, v), z'(u, v))\,, \qquad (u, v) \in D\,.$$

From the conditions (ii), (iii) we draw however the further inference that S' *as a whole* can be represented in the non-parametric form

$$S': \quad z' = z'(x', y'), \tag{4}$$

where $z'(x', y')$ is a single-valued function of class C''. Indeed, if a point $(x_0', y_0', 0)$ is assigned, then the vertical line g' through this point corresponds, by means of the transformation (2), to the line g which joins the origin to the point $(x_0', y_0', 1)$. By condition (ii), such a line g intersects S in one point at most. Hence, the vertical line g' intersects S' in one point at most. Thus S' can be represented in the form (4) with $z'(x', y')$ single-valued. We have to verify yet that z' is again of class C''. For this purpose, we note that the functions $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ are clearly of class C'', in view of the assumptions made concerning the original surface S. In passing from the representation (3) to the representation (4), by well-known elementary considerations class properties are preserved provided that the Jacobian $\partial(x', y')/\partial(u, v)$ is different from zero. One finds readily, in view of (2), that

$$\frac{\partial(x', y')}{\partial(u, v)} = \frac{1}{z} \begin{vmatrix} x & y & z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},$$

which is different from zero by condition (iii).

I. 2. 2. If the original surface S itself is given in the non-parametric form $S: z = z(x, y)$, $(x, y) \in D$, where $z(x, y)$ is single-valued and of class C'', and of course the conditions (i), (ii), (iii) of I. 2. 1 are assumed to hold, then the determinant (1) reduces to the simple form

$$\Delta = z - xp - yq, \tag{5}$$

where we use the conventional notation

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}.$$

On setting, for the function $z'(x', y')$ appearing in (4),

$$p' = \frac{\partial z'}{\partial x'}, \quad q' = \frac{\partial z'}{\partial y'},$$

one finds readily that

$$p' = -\frac{p}{\Delta}, \quad q' = -\frac{q}{\Delta}, \tag{6}$$

where $\Delta \neq 0$ by the condition (iii) in I. 2. 1.

I. 2. 3. Returning to the general situation in I. 2. 1, let us consider a vector function

$$\mathfrak{z} = \mathfrak{z}(u, v) = (X, Y, Z), \quad (u, v) \in D,$$

which is of class C' and induces an infinitesimal deformation of the surface S. By a fundamental observation of Darboux [2], for the transform S' introduced in I. 2. 1 one can set up explicitly a vector function

$$z' = (X', Y', Z') \tag{7}$$

which induces an infinitesimal deformation of S', by means of the formulas

$$X' = \frac{X}{z}, \quad Y' = \frac{Y}{z}, \quad Z' = -\frac{xX + yY + zZ}{z}. \tag{8}$$

Observe that in these formulas x, y, z stand for the coordinate functions of the original surface S, and hence x, y, z are of class C''. Since $z > 0$ on S and X, Y, Z are of class C' by assumption, it is clear that $\mathfrak{z}'$ is of class C'. To see that $\mathfrak{z}'$ induces an infinitesimal deformation of S', we have to verify the relationships

$$\mathfrak{z}_u'\mathfrak{x}_u' = 0, \quad \mathfrak{z}_u'\mathfrak{x}_v' + \mathfrak{z}_v'\mathfrak{x}_u' = 0, \quad \mathfrak{z}_v'\mathfrak{x}_v' = 0. \tag{9}$$

One finds by straightforward calculation that (9) follows from the transformation formulas (2) and (8) jointly with the relations

$$\mathfrak{z}_u\mathfrak{x}_u = 0, \quad \mathfrak{z}_u\mathfrak{x}_v + \mathfrak{z}_v\mathfrak{x}_u = 0, \quad \mathfrak{z}_v\mathfrak{x}_v = 0, \tag{10}$$

which express the fact that the original vector function $\mathfrak{z}$ induces an infinitesimal deformation for the original surface S.

The formulas (2) and (8) constitute the Darboux transformation which will be used as an essential tool in the sequel. At present, we call attention to the following simple facts.

(α) If the vector $\mathfrak{z} = (X, Y, Z)$ vanishes at a point P of S, then the vector $\mathfrak{z}' = (X', Y', Z')$ vanishes at the corresponding point P' of S', and *vice versa*. This is evident from the formulas (8).

(β) If the vector function $\mathfrak{z}$ induces a trivial infinitesimal deformation of S, then $\mathfrak{z}'$ induces a trivial infinitesimal deformation of S', and *vice versa*. Indeed (cf. I. 1. 3) if $\mathfrak{z}$ is of the special form

$$\mathfrak{z} = (\mathfrak{x} \times \mathfrak{u}) + \mathfrak{v},$$

where $\mathfrak{u}, \mathfrak{v}$ are constant vectors and $\mathfrak{x} = (x, y, z)$ is the position vector on S, then it follows immediately from (8) that $\mathfrak{z}'$ is of the same special form, and *vice versa*.

(γ) If the Gauss curvature K of S satisfies at a point P of S one of the relations $K>0$, $K=0$, $K<0$, then the Gauss curvature K' of S' satisfies the same relation at the corresponding point P' of S'. This is merely a special case of the well-known general fact that the sign of the Gauss curvature of a surface is invariant under projective transformations.

I. 2. 4. Assume now that S is given by the non-parametric representation $z=z(x, y)$, as in I. 2. 2. The vector $\mathfrak{z}$ gives rise to the functions $a=a(x, y)$, $b=b(x, y)$ given by the formulas (see I. 1. 7)

$$a=X+pZ\,,\quad b=Y+qZ\,. \tag{11}$$

Similarly, the corresponding vector $\mathfrak{z}'$ gives rise to the functions $a'=a'(x', y')$, $b'=b'(x', y')$ given by

$$a'=X'+p'Z'\,,\quad b'=Y'+q'Z'\,, \tag{12}$$

where p', q' have the meaning explained in I. 2. 2. The formulas (5), (6), (8) yield the following expressions for a', b' by direct calculation.

$$a'=\frac{z-yq}{z\Delta}a+\frac{yp}{z\Delta}b\,, \tag{13}$$

$$b'=\frac{xq}{z\Delta}a+\frac{z-xp}{z\Delta}b\,. \tag{14}$$

§ I. 3. Study of the functions a and b.

I. 3. 1. We consider a fixed square

$$Q:\quad -R\leqq x\leqq R\,,\quad -R\leqq y\leqq R$$

in the xy plane. Over Q, let there be given a piece of surface

$$S:\quad z=z(x, y)\,,$$

and a vector function $\mathfrak{z}=\mathfrak{z}(x, y)=(X, Y, Z)$, satisfying the following conditions.

(i) $z(x, y)$ is of class C'' in Q, and the partial derivatives p, q, r, s, t satisfy the relations

$$(1)\qquad p=0\,,\quad q=0\,,\quad r>0\,,\quad t>0\,,\quad rt-s^2>0\quad \text{at } (0, 0)\,.$$

Furthermore

$$(2)\qquad z=0\quad \text{at } (0, 0)\,.$$

(ii) $\mathfrak{z}=(X, Y, Z)$ is of class C' in Q, and

$$(3)\qquad X=0\,,\quad Y=0\,,\quad Z=0\quad \text{at } (0, 0)\,,$$

$$(4) \quad X_x = 0,\ X_y = 0,\ Y_x = 0,\ Y_y = 0,\ Z_x = 0,\ Z_y = 0 \quad \text{at } (0, 0).$$

(iii) $\mathfrak{z}$ induces an infinitesimal deformation for S. By I. 1. 7 this implies that on setting

$$(5) \qquad a = X + pZ, \quad b = Y + qZ,$$

one has the fundamental differential equations

$$(6) \qquad a_x = rZ, \quad a_y + b_x = 2sZ, \quad b_y = tZ.$$

I. 3. 2. For $(x, y) \in Q$, we set (using polar coordinates)

$$(7) \qquad \rho = (x^2 + y^2)^{\frac{1}{2}}, \quad x = \rho \cos\theta, \quad y = \rho \sin\theta.$$

Since we shall study orders of magnitude, it will be convenient to use the familiar symbols $O(\rho^n)$, $o(\rho^n)$, where n is an integer. Let us recall that if F is any quantity depending upon ρ and any other variables, then $F = O(\rho^n)$ means that

$$|F| \leqq C\rho^n$$

uniformly in all the variables that F may contain, where C is an absolute constant. Similarly, $F = o(\rho^n)$ means that

$$|F| \leqq C\rho^n \varepsilon(\rho)$$

uniformly in all the variables, where C is an absolute constant, and $\varepsilon(\rho)$ is a positive monotone function of ρ alone such that $\varepsilon(\rho) \to 0$ for $\rho \to 0$.

With these notations, we wish to establish the estimates

$$(8) \qquad a = o(\rho^2), \quad b = o(\rho^2).$$

Let us observe that these estimates would be obvious under the usual assumption that $z(x, y)$, $\mathfrak{z}(x, y)$ are of class C'''. Since we assume only that z is of class C'' and $\mathfrak{z}$ is of class C', the proof of (8) will be somewhat laborious, yet quite elementary.

I. 3. 3. Let us first note a few immediate facts. From (1) and (2) one infers readily that

$$(9) \qquad z = O(\rho^2), \quad p = O(\rho), \quad q = O(\rho), \quad \frac{1}{z} = O(\rho^{-2}).$$

On setting again

$$(10) \qquad \Delta = z - xp - yq,$$

and using (1) and (2), one obtains similarly

$$(11) \qquad \Delta = O(\rho^2), \quad \frac{1}{\Delta} = O(\rho^{-2}).$$

Also, (3) and (4) yield

$$Z = o(\rho), \quad Z_x = o(1), \quad Z_y = o(1). \tag{12}$$

From (12) and (6) one sees that

$$a_x = o(\rho), \quad a_y + b_x = o(\rho), \quad b_y = o(\rho). \tag{13}$$

From (13), it follows (since $a(0, 0) = 0$, $b(0, 0) = 0$ by (3) and (5)) that

$$a(x, 0) = o(x^2), \quad b(0, y) = o(y^2). \tag{14}$$

Now set

$$\Lambda = a \cos\theta + b \sin\theta, \tag{15}$$

where the arguments in a and b are $x = \rho\cos\theta$, $y = \rho\sin\theta$. For fixed θ, Λ is a function of ρ, and one obtains, in view of (13),

$$\frac{d\Lambda}{d\rho} = a_x \cos^2\theta + (a_y + b_x)\cos\theta\sin\theta + b_y \sin^2\theta = o(\rho). \tag{16}$$

Observe the circumstance that a_y and b_x occur only in the combination $a_y + b_x$ for which we have an estimate in (13). Integrating (16) with respect to ρ and noting that $\Lambda = 0$ for $\rho = 0$ by (3) and (5), we obtain

$$\Lambda = a\cos\theta + b\sin\theta = o(\rho^2). \tag{17}$$

Next introduce

$$\mu = \frac{da}{d\theta}\sin\theta - \frac{db}{d\theta}\cos\theta. \tag{18}$$

Now we have

$$\frac{da}{d\theta} = \rho(-a_x \sin\theta + a_y \cos\theta),$$

$$\frac{db}{d\theta} = \rho(-b_x \sin\theta + b_y \cos\theta).$$

It follows that

$$\mu = \rho(-a_x \sin^2\theta + (a_y + b_x)\sin\theta\cos\theta - b_y \cos^2\theta).$$

Accordingly, by (13),

$$\mu = \frac{da}{d\theta}\sin\theta - \frac{db}{d\theta}\cos\theta = o(\rho^2). \tag{19}$$

I. 3. 4. We need now two simple lemmas, probably well known; for the convenience of the reader, we shall sketch the proofs.

Lemma 1. In a closed interval $0 \leq \lambda \leq A$, let $f(\lambda)$ be a real-valued function, such that

(i) $f(\lambda)/\lambda \to 0$ for $\lambda \to 0$,

(ii) $f(\lambda) - 2f\left(\frac{\lambda}{2}\right) = o(\lambda^2)$.

Then $f(\lambda)=o(\lambda^2)$.

Proof. On setting

$$f(\lambda)-2f\left(\frac{\lambda}{2}\right)=h(\lambda), \tag{20}$$

we have by (ii)

$$|h(\lambda)|\leqq C\lambda^2\varepsilon(\lambda), \quad \varepsilon(\lambda)\to 0 \text{ for } \lambda\to 0, \tag{21}$$

where C is a constant and $\varepsilon(\lambda)$ is monotone. Take a λ in the given interval, and let n be a positive integer. Replacing, in (20), λ by $\lambda/2,\cdots,\lambda/2^{n-1}$, multiplying the resulting equations by $2,\cdots,2^{n-1}$ and adding to (20), one obtains

$$f(\lambda)=2^n f\left(\frac{\lambda}{2^n}\right)+\sum_{k=0}^{n-1}2^k h\left(\frac{\lambda}{2^k}\right). \tag{22}$$

By (21) we have (since $\varepsilon(\lambda)$ is monotone)

$$\left|h\left(\frac{\lambda}{2^k}\right)\right|\leqq C\lambda^2\varepsilon(\lambda)2^{-2k}.$$

Accordingly, the summation in (22) is dominated by the quantity

$$C\lambda^2\varepsilon(\lambda)\sum_{k=0}^{n-1}2^{-k}<2C\lambda^2\varepsilon(\lambda). \tag{23}$$

For $n\to\infty$ the first term on the right in (22) converges to zero by the condition (i) above. Thus (22) yields, for $n\to\infty$, in view of (23) the estimate

$$|f(\lambda)|\leqq 2C\lambda^2\varepsilon(\lambda),$$

showing that $f(\lambda)=o(\lambda^2)$.

Lemma 2. In a closed interval $0\leqq\lambda\leqq A$, let $f(\lambda)$, $g(\lambda)$ be two real-valued functions satisfying the following conditions.

(i) $f(\lambda)/\lambda\to 0$, $g(\lambda)/\lambda\to 0$ for $\lambda\to 0$.

(ii) For $0\leqq\alpha\leqq A$, $0\leqq\beta\leqq A$, the well-defined function

$$H(\alpha,\beta)=\alpha\, f(\beta)+\beta g(\alpha) \tag{24}$$

satisfies the relation

$$H(\alpha,\beta)=o(\sigma^3), \text{ where } \sigma=(\alpha^2+\beta^2)^{\frac{1}{2}}. \tag{25}$$

Then $f(\lambda)=o(\lambda^2)$, $g(\lambda)=o(\lambda^2)$.

Proof. On setting, in (24), first $\alpha=\lambda$ and $\beta=\lambda$, and next $\alpha=\lambda$, $\beta=\dfrac{\lambda}{2}$, one obtains in view of (25) the relations

$$f(\lambda)+g(\lambda)=H(\lambda,\lambda)/\lambda=o(\lambda^2) \tag{26}$$
$$2f\left(\frac{\lambda}{2}\right)+g(\lambda)=H\left(\lambda,\frac{\lambda}{2}\right)/\lambda=o(\lambda^2).$$

Subtraction yields

$$f(\lambda)-2f\left(\frac{\lambda}{2}\right)=o(\lambda^2).$$

By Lemma 1, this relation implies (in view of condition (i)) that $f(\lambda)=o(\lambda^2)$. By (26) it follows then that $g(\lambda)=o(\lambda^2)$.

I. 3. 5. Now return to the situation in I. 3. 1. For $0<\alpha<R$, $0<\beta<R$, $0\leqq\tau\leqq 1$, consider

$$\psi(\tau)=a((1-\tau)\alpha,\tau\beta)\alpha-b((1-\tau)\alpha,\tau\beta)\beta, \tag{27}$$

where α, β are fixed, and τ varies from 0 to 1. Since a, b are of class C' by (5), we can differentiate in (27) with respect to τ, obtaining

$$\psi'(\tau)=-a_x\alpha^2+(a_y+b_x)\alpha\beta-b_y\beta^2,$$

where the arguments in a_x, a_y, b_x, b_y are $x=(1-\tau)\alpha$, $y=\tau\beta$. By (13), we see that

$$\psi'(\tau)=o(\sigma^3),\ \text{where}\ \sigma=(\alpha^2+\beta^2)^{\frac{1}{2}}$$

Integration from 0 to 1 yields

$$\psi(1)-\psi(0)=a(0,\beta)\alpha+b(\alpha,0)\beta-a(\alpha,0)\alpha-b(0,\beta)\beta=o(\sigma^3). \tag{28}$$

Now by (14) we have $a(\alpha,0)\alpha=o(\sigma^3)$, $b(0,\beta)\beta=o(\sigma^3)$. Accordingly, (28) yields

$$a(0,\beta)\alpha+b(\alpha,0)\beta=o(\sigma^3). \tag{29}$$

From (3), (4), and (5) we see that $a(0,0)=0$, $b(0,0)=0$, $a_y(0,0)=0$, $b_x(0,0)=0$, and hence

$$a(0,\beta)/\beta\to 0\quad\text{for}\quad\beta\to 0, \tag{30}$$
$$b(\alpha,0)/\alpha\to 0\quad\text{for}\quad\alpha\to 0. \tag{31}$$

Hence, on setting (for $0\leqq\lambda\leqq A<R$),

$$f(\lambda)=a(0,\lambda),\quad g(\lambda)=b(\lambda,0),$$

we see from (29), (30), (31) that the assumptions of Lemma 2 in I. 3. 4 are satisfied. Hence

$$a(0,\lambda)=o(\lambda^2),\quad b(\lambda,0)=o(\lambda^2),\quad 0\leqq\lambda<R. \tag{32}$$

Now introduce

$$\Gamma=a\sin\theta-b\cos\theta, \tag{33}$$

where the arguments in a, b are $x=\rho\cos\theta$, $y=\rho\sin\theta$. For fixed ρ we obtain, in view of (15), (17) and (19),

$$\frac{d\Gamma}{d\theta}=\mu+\Lambda=o(\rho^2). \tag{34}$$

Integration yields (since for $\theta=0$ we have $\Gamma=-b(\rho,0)$)

$$\Gamma=-b(\rho,0)+\int_0^\theta(\mu+\Lambda)\,d\theta. \tag{35}$$

In view of (34), (32) it follows that

$$\Gamma=a\sin\theta-b\cos\theta=o(\rho^2). \tag{36}$$

By (17) we have also

$$\Lambda=a\cos\theta+b\sin\theta=o(\rho^2).$$

It follows that

$$a=\Lambda\cos\theta+\Gamma\sin\theta=o(\rho^2),$$
$$b=\Lambda\sin\theta-\Gamma\cos\theta=o(\rho^2),$$

and thus the relations (8) are proved.

I. 3. 6. Returning to the situation in I. 3. 1, we observe that in view of the conditions (1) we can take, in the xy plane, a circular disc D around the origin so that

$$r>0,\quad t>0,\quad rt-s^2>0\quad \text{in } D. \tag{37}$$

Let S^* be the portion of S over D. The Darboux transformation, discussed in I. 2. 1 to I. 2. 4, applies then to S^* if the origin is first deleted (observe that S^* is convex by (37), hence the assumptions made in § I. 2 are satisfied). The transformation yields a piece of surface

$$S':\quad z'=z'(x',y'),\quad (x',y')\in R',$$

where R' is an unbounded region consisting of all points (x',y') which are exterior to a certain simple closed curve C', as well of the points on C'. Indeed, since $z=O(\rho^2)$ by I. 3. 3, it is clear that

$$x'^2+y'^2=\frac{x^2+y^2}{z^2}\to\infty\quad\text{for}\quad x^2+y^2\to 0, \tag{38}$$

uniformly in all directions. Let us now denote by C_ρ the circle in the xy plane with center at the origin and of radius ρ, by C_ρ^* the simple closed curve on S^* that lies over C_ρ, by $C_\rho^{*\prime}$ the image of C_ρ^* on S' under the Darboux transformation, and finally by C_ρ' the projection of $C_\rho^{*\prime}$ upon the $x'y'$ plane. From (38) one sees that the following holds.

(i) If $0 < \sigma < \rho$, then C_σ' encloses C_ρ'.

(ii) On assigning any circular disc in the $x'y'$ plane, this disc will be interior to C_ρ' for ρ sufficiently small.

Now let $\mathfrak{z}' = (X', Y', Z')$ be the vector function that corresponds to the vector function $\mathfrak{z}$ of I. 3. 1 by means of the Darboux transformation (see I. 2. 3), and consider the corresponding functions a', b', discussed in I. 2. 4. As noted there, we have

$$a' = A_{11}a + A_{12}b\,, \tag{39}$$

$$b' = A_{21}a + A_{22}b\,, \tag{40}$$

where

$$A_{11} = \frac{z - yq}{z\Delta}\,, \quad A_{12} = \frac{yp}{z\Delta}\,, \tag{41}$$

$$A_{21} = \frac{xq}{z\Delta}\,, \quad A_{22} = \frac{z - xp}{z\Delta}\,. \tag{42}$$

From the estimates (9) and (11), jointly with the formula (10) for Δ (see I. 3. 3), it follows by inspection that

$$A_{ik} = O(\rho^{-2})\,, \quad \frac{\partial A_{ik}}{\partial x} = O(\rho^{-3})\,, \quad \frac{\partial A_{ik}}{\partial y} = O(\rho^{-3})\,, \quad i, k = 1, 2\,. \tag{43}$$

Now since, for fixed $\rho > 0$,

$$\frac{dA_{ik}}{d\theta} = \rho\left(-\frac{\partial A_{ik}}{\partial x}\sin\theta + \frac{\partial A_{ik}}{\partial y}\cos\theta\right),$$

we infer from (43) that

$$\frac{dA_{ik}}{d\theta} = O(\rho^{-2})\,. \tag{44}$$

We proceed to verify the fundamental estimate

$$\int_{C_\rho'} a'\,db' = o(1)\,. \tag{45}$$

Proof. In view of the 1-1 correspondence between C_ρ' and C_ρ, this line integral can be evaluated by using the polar angle θ (in the xy plane) as the variable of integration. Accordingly, we consider the integral

$$\int_{C_\rho} a'\,\frac{db'}{d\theta}\,d\theta\,. \tag{46}$$

In view of (39) and (40), the integrand can be written as the sum of eight terms which we distribute into three groups as follows:

$$(47)\qquad A_{11}\frac{dA_{21}}{d\theta}a^2,\quad A_{11}\frac{dA_{22}}{d\theta}ab,\quad A_{12}\frac{dA_{21}}{d\theta}ab,\quad A_{12}\frac{dA_{22}}{d\theta}b^2,$$

$$(48)\qquad \frac{1}{2}A_{11}A_{21}\frac{d(a^2)}{d\theta},\quad \frac{1}{2}A_{12}A_{22}\frac{d(b^2)}{d\theta},$$

$$(49)\qquad A_{11}A_{22}\,a\frac{db}{d\theta},\quad A_{12}A_{21}\,b\frac{da}{d\theta}.$$

We proceed to estimate the contributions of these terms as follows. For the integrands in (47) we obtain, in view of (43), (44) and (8), directly the estimate

$$O(\rho^{-2})\ O(\rho^{-2})\ o(\rho^4)=o(1),$$

and thus the corresponding integrals are also of the order $o(1)$. The integrals corresponding to the integrands in (48) are transformed, by partial integration, into integrals with the integrands

$$\frac{1}{2}a^2\frac{d(A_{11}A_{21})}{d\theta},\quad \frac{1}{2}b^2\frac{d(A_{12}A_{22})}{d\theta}.$$

For these expressions, we infer from (43), (44) and (8) the estimate $o(1)$ as before, and hence the corresponding integrals are also of the order $o(1)$. The terms in (49) require special treatment. Multiplying the functions Λ, μ defined in (15) and (18) respectively, we obtain after some re-arrangements the formulas

$$(50)\qquad a\frac{db}{d\theta}=\frac{1}{2}\frac{d(a^2)}{d\theta}\sin\theta\cos\theta-\frac{1}{2}\frac{d(b^2)}{d\theta}\sin\theta\cos\theta+\frac{d(ab)}{d\theta}\sin^2\theta-\Lambda\mu,$$

$$(51)\qquad b\frac{da}{d\theta}=-\frac{1}{2}\frac{d(a^2)}{d\theta}\sin\theta\cos\theta+\frac{1}{2}\frac{d(b^2)}{d\theta}\sin\theta\cos\theta+\frac{d(ab)}{d\theta}\cos^2\theta+\Lambda\mu.$$

Now the contribution of the first term in (49) can be estimated as follows. On multiplying (50) by $A_{11}A_{22}$, one obtains four terms on the right. The first term is

$$\frac{1}{2}A_{11}A_{22}\sin\theta\cos\theta\frac{d(a^2)}{d\theta}.$$

The integral of this term can be transformed by partial integration into the integral of the expression

$$\frac{1}{2}a^2\frac{d(A_{11}A_{22}\sin\theta\cos\theta)}{d\theta}.$$

By (43), (44) and (8) this expression is of the order $o(\rho^4)\ O(\rho^{-4}) = o(1)$, hence its integral is also of the order $o(1)$. The next two ones of the four terms involved are estimated in the same manner. The fourth term is

$$\Lambda\mu\, A_{11}A_{22}\,,$$

which is of the order $o(\rho^2)\ o(\rho^2)\ O(\rho^{-2})\ O(\rho^{-2}) = o(1)$ by (17), (19), (43), and hence its integral is of the same order. The contribution of the second term in (49) is estimated in the same manner, using the identity (51), and the proof of (45) is complete.

§ 1. 4. An integral formula.

I. 4. 1. Let R be a finitely connected, bounded Jordan region in the xy plane, bounded by $m+1$ smooth simple closed curves $C_0, C_1, \cdots, C_m$, where $m \geqq 0$. We assume that C_0 is the exterior boundary curve of R which encloses $C_1, \cdots, C_m$. For $m = 0$, these interior boundary curves are missing, and certain portions of the following argument become vacuously trivial. Over R, let there be given a piece of surface

$$S:\quad z = z(x, y),\quad (x, y) \in R\,,$$

with the following properties.

(i) $z(x, y)$ is continuous in R and of class C'' in the interior of R. The partial derivatives p, q, r, s, t of z remain continuous on the boundary curves of R.

(ii) $rt - s^2 > 0$ on a dense set in R (and hence $rt - s^2 \geqq 0$ in all of R).

Furthermore, let $\mathfrak{z} = \mathfrak{z}(x, y) = (X, Y, Z)$ be a vector function defined in R, with the following properties.

(iii) $\mathfrak{z}(x, y)$ is continuous in R, and it is of class C' in the interior of R. Its partial derivatives $\mathfrak{z}_x$, $\mathfrak{z}_y$ remain continuous on the boundary curves of R.

(iv) $\mathfrak{z}$ induces an infinitesimal deformation for S, in the sense of I. 1. 2.

Setting, as in I. 1. 7,

$$a = X + pZ\,,\quad b = Y + qZ\,, \tag{1}$$

clearly a, b are continuous in R, of class C' in the interior of R, and the partial derivatives a_x, a_y, b_x, b_y remain continuous on the boundary of R. By I. 1. 7 we have the fundamental equations

$$a_x = rZ\,, \tag{2}$$

$$b_y = tZ\,, \tag{3}$$

$$a_y + b_x = 2sZ\,. \tag{4}$$

Multiply (2) and (3), square (4) and divide by 4, and subtract. There results the equation

$$(5) \qquad a_x b_y - a_y b_x = (rt - s^2) Z^2 + \frac{1}{4}(a_y - b_x)^2 .$$

Integrate (5) over R, transform the left hand side into a line integral, and set

$$(6) \qquad I_k = \int_{C_k} a\, db, \quad k = 0, 1, \cdots, m,$$

where the integration is taken in the counter-clockwise sense around C_k. There follows the basic integral formula

$$(7) \qquad \iint_R \left[(rt - s^2) Z^2 + \frac{1}{4}(a_y - b_x)^2 \right] dxdy = I_0 - I_1 - \cdots - I_m .$$

I. 4. 2. *Lemma.* Assume that

$$(8) \qquad I_0 \leqq 0, \quad I_k \geqq 0, \quad k = 1, \cdots, m .$$

Then the vector function $\mathfrak{z}$ is constant in R.

Proof. As a consequence of (8), one infers from (7) that the double integral appearing there is $\leqq 0$. Now the integrand is $\geqq 0$ by condition (ii) in I. 4. 1. Hence the integrand must vanish identically in R. Since $rt - s^2 > 0$ on a dense set in R, it follows that

$$(9) \qquad Z = 0, \quad a_y - b_x = 0 \quad \text{in } R .$$

From (9), (2), (3), (4) one sees now that $a_x = 0$, $a_y = 0$, $b_x = 0$, $b_y = 0$ in R. Hence

$$(10) \qquad a = \text{constant}, \quad b = \text{constant} \quad \text{in } R .$$

From (9), (10) and (1) one infers directly that X and Y also reduce to constants in R, and thus $\mathfrak{z}$ is constant in R.

I. 4. 3. Now consider the following situation. In the xy plane, take $m \geqq 0$ mutually exclusive, smooth simple closed curves $C_1, \cdots, C_m$ (if $m = 0$, then certain parts of the following argument become vacuously trivial). Let R^0 denote the set of those (finite) points (x, y) which are exterior to all the curves $C_1, \cdots, C_m$, and set $R = R^0 + C_1 + \cdots + C_m$ (if $m = 0$, then R coincides with the whole xy plane). Let $S: z = z(x, y)$, $(x, y) \in R$, and let $\mathfrak{z} = \mathfrak{z}(x, y)$, $(x, y) \in R$, be a vector function such that the conditions (i), (ii), (iii), (iv) stated in I. 4. 1 hold (for the present unbounded region R). Make the following additional assumptions.

(v) One has

$$(11) \qquad I_k \geqq 0, \quad k = 1, \cdots, m,$$

where I_k is defined as in (6) (thus the integration around C_k is taken in the counter-clockwise sense).

(vi) There exists a sequence C_0^n of smooth simple closed curves, enclosing $C_1, \cdots, C_m$ and converging to infinity, such that

$$I_0^n = \int_{C_0^n} a\, db \to 0 \quad \text{for } n \to \infty. \tag{12}$$

In (12), the integration is taken in the counter-clockwise sense around C_0^n. The statement that C_0^n converges to infinity has the following meaning: C_0^{n+1} encloses C_0^n, and on assigning any circular disc D in the xy plane, D will be interior to C_0^n for n sufficiently large.

Lemma. Under the conditions stated, $\mathfrak{z}$ is constant in R.

Proof. Let A_n be the doubly connected region bounded by C_0^n and C_0^{n+1}. Applying the integral formula (7) to A_n, one obtains

$$\iint_{A_n} \left[(rt - s^2) Z^2 + \frac{1}{4} (a_y - b_x)^2 \right] dxdy = I_0^{n+1} - I_0^n .$$

Since the integral on the left is $\geqq 0$ (see condition (ii) in I. 4. 1), it follows that $I_0^n \leqq I_0^{n+1}$. As $I_0^n \to 0$ by (12), it follows that

$$I_0^n \leqq 0 . \tag{13}$$

Now denote by R_n the region bounded by $C_0^n, C_1, \cdots, C_m$. In R_n, $z(x, y)$ and $\mathfrak{z}(x, y)$ satisfy the assumptions of the lemma in I. 4. 2, in view of (11) and (13). Hence, by that lemma, $\mathfrak{z}$ is constant in R_n. But for $n \to \infty$ (since C_0^n converges to infinity) the region R_n will contain any assigned point of the original region R. Thus $\mathfrak{z}$ is constant in R, as asserted.

§ I. 5. Study of a contour integral.

I. 5. 1. In the xy plane we consider a bounded, *doubly connected* Jordan region R whose boundary consists of two smooth, simple closed curves C and C^*, where C is the interior boundary curve. Over R, we consider a piece of surface

$$S: \quad z = z(x, y), \quad (x, y) \in R,$$

which satisfies the following conditions.

(i) $z(x, y)$ is continuous in R and of class C''' in the interior of R. The partial derivatives of $z(x, y)$ of the first three orders remain continuous on C.

(ii) The first partial derivatives p and q are constant on C.

Now let $\mathfrak{z} = \mathfrak{z}(x, y) = (X, Y, Z)$ be a vector function satisfying the following conditions.

(iii) $\mathfrak{z}$ is continuous in R and of class C' in the interior of R. The partial derivatives $\mathfrak{z}_x$, $\mathfrak{z}_y$ remain continuous on C.

(iv) $\mathfrak{z}$ induces an infinitesimal deformation for S.

For the corresponding functions a, b (see I. 1. 7) we assert the inequality

$$\int_C a\, db \geqq 0, \tag{1}$$

where the integration is taken in the counter-clockwise sense around C. We note that this inequality corresponds, in our approach, to a similar relation discovered by Rembs [4]. However, his inequality is stated in terms of the $\mathfrak{y}$-technique (see I. 1. 4), and accordingly his proof requires stronger smoothness assumptions. Furthermore, our proof of (1) is based on more elementary considerations. For clarity, we subdivide the proof into three parts.

I. 5. 2. We begin with a well-known elementary fact.

Lemma (see for example, Landau [6], p. 213, Satz 299). In a domain D of the xy plane, let $f(x, y)$ be a continuous function such that f_x, f_y, f_{xy} exist and are continuous in D. Then f_{yx} also exists and is continuous in D, and one has $f_{yx} = f_{xy}$.

Now let us consider (see I. 1. 7) the fundamental equations

$$a_x = rZ, \tag{2}$$

$$b_y = tZ, \tag{3}$$

$$a_y + b_x = 2sZ. \tag{4}$$

Since z, $\mathfrak{z}$ are of class C''', C' respectively, we obtain from (2) and (3) the existence and continuity of the derivatives (in the interior of R and on C)

$$a_{xx},\ a_{xy},\ b_{yx},\ b_{yy}. \tag{5}$$

From $a = X + pZ$, $b = Y + qZ$ it is clear that the derivatives

$$a_x,\ a_y,\ b_x,\ b_y \tag{6}$$

exist and are continuous (in the interior of R and on C). By the Lemma, it follows that

$$a_{yx} = a_{xy},\quad b_{xy} = b_{yx} \tag{7}$$

exist and are continuous (in the interior of R and on C). In view of (5), (6), (7), we have to account yet for the derivatives a_{yy}, b_{xx}. Now

if we write (4) in the form $a_y = 2sZ - b_x$, then by (7) the desired conclusion follows from (7). Finally, b_{xx} is treated by writing (4) in the form $b_x = 2sZ - a_y$ and using (7). Summing up: the first and second partial derivatives of a, b exist and are continuous in the interior of R, and remain continuous on C.

I. 5. 3. *Lemma.* The expression

$$\lambda = a_y - sZ \tag{8}$$

is constant on C.

Proof. In view of I. 5. 2, we can differentiate λ with respect to the arc-length σ on C. Using (2), (3), (4), we obtain the formula

$$\frac{d\lambda}{d\sigma} = \left[(r_y - s_x)\frac{dx}{d\sigma} + (s_y - t_x)\frac{dy}{d\sigma}\right]Z - \left(s\frac{dx}{d\sigma} + t\frac{dy}{d\sigma}\right)Z_x + \left(r\frac{dx}{d\sigma} + s\frac{dy}{d\sigma}\right)Z_y . \tag{9}$$

Now $r_y = s_x$, $s_y = t_x$. Furthermore, since $p = \text{constant}$, $q = \text{constant}$ on C by assumption, we have on C:

$$\frac{dp}{d\sigma} = r\frac{dx}{d\sigma} + s\frac{dy}{d\sigma} = 0, \quad \frac{dq}{d\sigma} = s\frac{dx}{d\sigma} + t\frac{dy}{d\sigma} = 0 . \tag{10}$$

Thus, by (9), $d\lambda/d\sigma = 0$, and hence λ is constant on C.

I. 5. 4. We can now prove the inequality (1) as follows. By I. 5. 3 we have

$$a_y - sZ = c , \tag{11}$$

where c is a constant. From (2), (10), (11) we obtain (along C)

$$\begin{aligned}\frac{da}{d\sigma} &= a_x\frac{dx}{d\sigma} + a_y\frac{dy}{d\sigma} = rZ\frac{dx}{d\sigma} + (sZ + c)\frac{dy}{d\sigma} \\ &= \left(r\frac{dx}{d\sigma} + s\frac{dy}{d\sigma}\right)Z + c\frac{dy}{d\sigma} = c\frac{dy}{d\sigma} .\end{aligned}$$

It follows that

$$\frac{d}{d\sigma}(a - cy) = 0 ,$$

and hence

$$a = cy + \alpha , \tag{12}$$

where α is a constant. Using (3), (4), (10), (11), a similar calculation shows that

$$b = -cx + \beta , \tag{13}$$

where β is a constant. Accordingly, from (12) and (13) we calculate

$$\int_C a \frac{db}{d\sigma} d\sigma = -c^2 \int_C y \frac{dx}{d\sigma} d\sigma - \alpha c \int_C \frac{dx}{d\sigma} d\sigma . \tag{14}$$

The second integral on the right vanishes, since C is closed. On the other hand, since we integrate along C in the counter-clockwise sense, we have

$$-\int_C y \frac{dx}{d\sigma} d\sigma = \int_C x \frac{dy}{d\sigma} d\sigma = A > 0 ,$$

where A is the area enclosed by C. Thus we get from (14) the relation

$$\int_C a \frac{db}{d\sigma} d\sigma = c^2 A \geqq 0 ,$$

and thus (1) is proved.

§ I. 6. Infinitesimal rigidity of convex surfaces.

I. 6. 1. **Theorem.** *Let S be a closed convex surface of class C''* (see I. 1. 1). *We assume that the Gauss curvature K of S is positive on a dense subset of S (clearly then $K \geqq 0$ everywhere on S). Then S is infinitesimally rigid* (see I. 1. 5). *Explicitly: if $\mathfrak{z} = (X, Y, Z)$ is any vector function of class C' on S which induces an infinitesimal deformation of S, then $\mathfrak{z}$ is trivial* (see I. 1. 2, I. 1. 3).

In the case when $S, \mathfrak{z}$ are both assumed to be of class C''', this theorem is classical (see Efimov [3] for references to the work of Liebmann, Weyl, Blaschke). To deal with the general situation considered here, we proceed as follows. We select on S a point O at which the Gauss curvature is positive, and make O the origin of the coordinate system xyz. Furthermore, we make the xy plane coincide with the tangent plane of S at O, and for definiteness we assume that S lies above the xy plane, except for the point O of course. If S^* is a sufficiently small portion of S around O, then S^* can be represented, in a vicinity of the origin, in the non-parametric form

$$S^* : \quad z = z(x, y), \quad -R \leqq x \leqq R , \quad -R \leqq y \leqq R ,$$

and in this vicinity the function $z(x, y)$ satisfies the conditions stated in I. 3. 1.

Now let $\mathfrak{z} = (X, Y, Z)$ be a vector function as described in the statement of the theorem. In the vicinity of the origin, X, Y, Z are functions of x, y of class C'. We first verify that we can assume, without loss of generality, that X, Y, Z and their first partial derivatives $X_x, \cdots, Z_y$ vanish at the origin. To see this, denote by $X^0, \cdots, Z_y^0$

the values of these functions at the origin. The differential equations in I. 1. 2, applied to the portion S^* with $u=x$, $v=y$, yield the following information (recall that $p=0$, $q=0$ at the origin):

$$X_x^0=0\,,\quad Y_y^0=0\,,\quad X_y^0+Y_x^0=0\,. \tag{1}$$

Now consider, on the whole surface S, the auxiliary vector function $\tilde{\mathfrak{z}}=(\tilde{X},\tilde{Y},\tilde{Z})$, defined by the formulas

$$\tilde{X}=-Z_x^0z+X_y^0y+X^0\,. \tag{2}$$

$$\tilde{Y}=-X_y^0x-Z_y^0z+Y^0\,, \tag{3}$$

$$\tilde{Z}=\quad Z_y^0y+Z_x^0x+Z^0\,. \tag{4}$$

Observe that $\mathfrak{z}$ *induces a trivial infinitesimal deformation of* S, *in the sense of* I. 1. 3. In the vicinity of the origin, the components of $\tilde{\mathfrak{z}}$ are single-valued functions of class C' of x, y. Recalling that $p=0$, $q=0$ at the origin, and using the relations (1), one finds by direct calculation that $\tilde{X},\cdots,\tilde{Z}_y$ agree with $X,\cdots,Z_y$ at the origin. Accordingly, if we introduce the vector function

$$\bar{\mathfrak{z}}=\mathfrak{z}-\tilde{\mathfrak{z}}$$

on S, then the components of $\bar{\mathfrak{z}}$ as well as their first partial derivatives all vanish at the origin (and obviously $\bar{\mathfrak{z}}$ also induces an infinitesimal deformation of S). Hence, if we can show that $\bar{\mathfrak{z}}$ induces a trivial infinitesimal deformation, then the same result will follow for $\mathfrak{z}=\bar{\mathfrak{z}}+\tilde{\mathfrak{z}}$, since $\tilde{\mathfrak{z}}$ induces a trivial infinitesimal deformation, as observed above.

Summing up: without loss of generality, we can assume that in the vicinity of the origin we have the situation considered in I. 3. 1. In view of the assumption made about S, clearly the portion $S-O$ of S presents the situation considered in I. 2. 1. Accordingly, if we apply to $S-O$ the transformation

$$x'=\frac{x}{z}\,,\quad y'=\frac{y}{z}\,,\quad z'=\frac{1}{z}\,,$$

then we obtain, by the discussion in I. 2. 1, a surface

$$S':\quad z'=z'(x',y')\,,\quad -\infty<x'<\infty\,,\quad -\infty<y'<\infty\,,$$

where $z'(x',y')$ is single-valued and of class C'' over the entire $x'y'$ plane. Furthermore, the Gauss curvature of S' is positive on a dense set in the $x'y'$ plane. Simultaneously, by the discussion in I. 2. 3, we obtain a vector function

$$\mathfrak{z}'=(X',Y',Z')\,,\quad -\infty<x'<\infty\,,\quad -\infty<y'<\infty\,,$$

given by the explicit formulas

$$X' = \frac{X}{z}, \quad Y' = \frac{Y}{z}, \quad Z' = -\frac{xX + yY + zZ}{z},$$

which is of class C' over the entire $x'y'$ plane, and induces an infinitesimal deformation of S'. To S' and $\mathfrak{z}'$ we wish to apply the Lemma in I. 4. 3 (with $m = 0$). To justify this step, we have to verify the condition (vi) stated there. For this purpose, let us return to the portion S^* of S in the vicinity of the origin. We noted above that S^* presents the situation discussed in I. 3. 1. Accordingly, the basic estimate

$$\int a' \, db' = o(1),$$

derived in I. 3. 6, yields directly the condition we have to verify, in view of the comments in I. 3. 6. By I. 4. 3, it follows that $\mathfrak{z}'$ is constant, and hence certainly induces a trivial infinitesimal deformation for S'. Accordingly, by I. 2. 3, the same is true for $\mathfrak{z}$ in relation to $S - O$. Thus $\mathfrak{z}$ can be written, on $S - O$, in the form

$$\mathfrak{z} = (\mathfrak{u} \times \mathfrak{x}) + \mathfrak{v}, \tag{5}$$

where $\mathfrak{u}$, $\mathfrak{v}$ are constant vectors and $\mathfrak{x}$ is the position vector of S. By continuity, it follows that (5) holds at the point O also, and the proof of the theorem is complete.

I. 6. 2. Let us return to the surface S of I. 6. 1. Taking a point P on S, consider a smooth simple closed curve C on S which does not pass through P. Let Γ be the set of those points of S which are connected on S with P, without crossing C (thus Γ is topologically equivalent with an open circular disc). We shall then say that the surface $S^* = S - \Gamma$ is derived from S by removing *the open cap* Γ bounded by C (and containing P). Since by assumption the Gauss curvature K of S is positive on a dense set, it is clearly legitimate to assume that $K > 0$ at P.

Theorem. *Let $P_0, P_1, \cdots, P_n$ be points on S, where $n \geqq 0$. Let S^* be obtained from S by removing open caps $\Gamma_0, \Gamma_1, \cdots, \Gamma_n$, bounded by mutually exclusive curves $C_0, C_1, \cdots, C_n$, and containing $P_0, P_1, \cdots, P_n$ respectively (thus S^* is topologically equivalent to a plane Jordan region bounded by $n + 1$ curves). Then S^* is IRB in the sense of* I. 1. 5. *Explicitly: if $\mathfrak{z}$ is a vector function of class C' on S^* which induces an infinitesimal deformation of S^* and vanishes on the whole boundary of S^*, then $\mathfrak{z} \equiv 0$ on S^*.*

Observe that for $n = 0$ we remove just one cap; in this case, certain portions of the following argument become vacuously trivial.

If S^*, $\mathfrak{z}$ were assumed to be both of class C''', then the theorem would reduce to a well-known fact; thus the main point is again that S^*, $\mathfrak{z}$ are only of class C'' and C' respectively. To proceed with the proof, take the coordinate system xyz so that the origin coincides with the point P_0 and the xy plane is tangent to S at P_0. Since the Gauss curvature is positive at P_0, we can assume that $z > 0$ on $S - P_0$, hence also on S^*. Now clearly S^* presents the situation discussed in I. 2. 1. On applying the Darboux transformation (see I. 2. 1, I. 2. 3), we obtain a surface

$$S' : \quad z' = z'(x', y'), \quad (x', y') \in R',$$

as well a corresponding vector function $\mathfrak{z}'$, where R' is a bounded Jordan region bounded by $n+1$ curves $C_0', C_1', \cdots, C_n'$. Now since $\mathfrak{z}$ vanishes on the whole boundary of S^*, it is clear (see I. 2. 3) that $\mathfrak{z}'$ vanishes on the whole boundary of S'. Thus the functions

$$a' = X' + p'Z', \quad b' = Y' + q'Z',$$

corresponding to z' and $\mathfrak{z}'$ in the sense of I. 1. 7, also vanish on the whole boundary of R'. Thus clearly the line integral of $a'\,db'$, taken around each boundary curve C_k', is equal to zero. Thus the assumptions of the lemma in I. 4. 2 are satisfied, and hence $\mathfrak{z}'$ is constant. But since $\mathfrak{z}'$ vanishes on the boundary, it follows that $\mathfrak{z}' \equiv 0$. By I. 2. 3, it follows that $\mathfrak{z} \equiv 0$, and the proof is complete.

I. 6. 3. In the literature one finds examples to show that a convex surface with holes (like the S^* of I. 6. 2) is generally not infinitesimally rigid if the boundary curves are not kept fixed. Yet, a set of beautiful results due to Liebmann [3] and Rembs [4] show that infinitesimal rigidity can be achieved, without keeping the boundary fixed, provided that the boundary curves satisfy certain special conditions. We shall derive presently a strengthened version of the general theorem due to Rembs.

We consider a *convex* piece of surface S with m smooth boundary curves $C_1, \cdots, C_m$, such that the following conditions hold.

(i) S is of class C'' (including the boundary, see I. 1. 1).

(ii) The Gauss curvature of S is positive on a dense subset of R.

(iii) For each $k = 1, \cdots, m$, there is a plane π_k such that C_k lies in π_k and S is in contact with π_k along C_k.

(iv) In the vicinity of each C_k, the surface S is of class C''' (by *vicinity*, we mean here some narrow band on S bordering on C_k).

Remarks. (α) In the work of Rembs, it is required that the contact between S and π_k be of the first order only. One finds comments in the literature to the effect that this requirement can be dispensed with

by a study of sufficiently high derivatives of S, the order of these derivatives depending upon the order of contact permitted. In our method, the order of contact plays no role. (β) Previous work on the problem under consideration is based on the $\mathfrak{y}$-technique, and hence the class condition is C''' for S, and also C''' for the vector functions $\mathfrak{z}$ inducing infinitesimal deformations (see I. 1. 4). In this respect, our method yields again a more general result. (γ) Condition (iv) above requires greater smoothness of S in the vicinity of the boundary curves than elsewhere. While the mechanical interpretation may render such special caution near the boundary plausible, we are not certain that condition (iv) is really necessary for the infinitesimal rigidity of S.

Theorem. *Under the conditions* (i)—(iv) *stated above, the surface S is infinitesimally rigid, without keeping the boundary curves fixed. Explicitly: if $\mathfrak{z}$ is any vector function on S, such that $\mathfrak{z}$ is of class C' and induces an infinitesimal deformation of S, then $\mathfrak{z}$ is trivial.*

Remarks. (α) We require only class C' of $\mathfrak{z}$, as compared with class C''' in the literature. (β) In contrast with condition (iv) above for S, we do not require a higher degree of smoothness of $\mathfrak{z}$ in the vicinity of the boundary.

Proof of the theorem. The proof is quite similar to that presented in I. 6. 1 for the case of a closed convex surface, and hence we shall stress details only at the point where the situation considered here demands additional attention. We choose on S a point O, not on the boundary, such that the Gauss curvature at O is positive. We choose a coordinate system xyz such that O is the origin, the xy plane is tangent to S at O, and S lies above the xy plane (except for the point O). Now consider a vector function $\mathfrak{z}$ on S as described in the statement of the theorem. As in I. 6. 1, we can assume that the components X, Y, Z of $\mathfrak{z}$ as well as their first partial derivatives, with respect to x, y, vanish at the origin. We apply again the Darboux transformation, obtaining

$$S': \quad z' = z'(x', y'), \quad (x', y') \in R',$$
$$\mathfrak{z}' = (X', Y', Z'),$$

same as in I. 6. 1, with the following single difference: R' is now not the whole $x'y'$ plane but an unbounded Jordan region whose boundary consists of m simple closed curves $C_1', \cdots, C_m'$. Proceeding as in I. 6. 1 on the basis of the lemma in I. 4. 2, the new feature is that we have to verify the inequalities

$$(1) \qquad \int_{C_k'} a'\, db' \geqq 0, \quad k = 1, \cdots, m,$$

where the integration is made in the counter-clockwise sense around C_k'. Thus the proof will be complete as soon as (1) is established.

For this purpose, let us return to the original boundary curves $C_1, \cdots, C_m$. Under the Darboux transformation these curves are carried into the boundary curves of S', say $\Gamma_1', \cdots, \Gamma_m'$. Now, by assumption, the curve C_k is a plane curve, and along C_k the surface S is in contact with the plane containing C_k. These features being invariant under a projective transformation, it follows that the corresponding Γ_k' is again a plane curve, and S' is in contact with the plane containing Γ_k'. Since the curve C_k' occurring in (1) is merely the projection of Γ_k', it follows that the partial derivatives p', q' of $z'(x', y')$ are both constant along C_k'. Thus (1) follows directly from I. 5. 1.

I. 6. 4. Observe that the condition (iv) on S, in I. 6. 3, is needed solely to justify the application of the result in I. 5. 1. Hence, if that result can be strengthened in any manner, then the restriction (iv) can be relaxed or altogether eliminated. In this connection it may be of interest to point out that the class C''' requirement is needed only to verify that a certain function $\lambda = \lambda(\sigma)$ is constant (see I. 5. 3). A similar issue arises in Calculus of Variations, when the Euler-Lagrange equation is derived under weaker class assumption than the classical ones, and perhaps this analogy may suggest a feasible approach to the problem of eliminating condition (iv).

I. 6. 5. Various further rigidity theorems will readily occur to the reader if he observes that in the proof of the result in I. 6. 3 the basic issue was to find, after the Darboux transformation, a situation where the assumptions of I. 4. 3 are satisfied. For example: under the circumstances considered in I. 6. 3, we can remove from the surface a finite number of caps (as in I. 6. 2); if the boundary curves of these caps are held fixed (that is, if $\mathfrak{z}$ is required to vanish on these curves), then we still have, after the Darboux transformation a situation where I. 4. 3 applies. Further rigidity theorems are obtained by the observation that the conclusion in I. 4. 3 remains valid even if certain isolated singularities are permitted, provided that we can control the integral of $a\,db$ on small closed curves enclosing the singular points. An elementary discussion reveals that certain types of *corners* can be permitted. However, in the absence of relevant mechanical applications, it is not clear what should be considered as reasonable types of singularities, and so we do not pursue this matter any further at this time.

§ I. 7. The *IRb* property for convex surfaces.

I. 7. 1. As a matter of intuition, under comparable boundary conditions a convex surface should be more rigid than a surface of negative curvature. Accordingly, in view of the theorems in part II, relating to surfaces of negative (and zero) curvature, the following question arises in a natural manner. Let S be a finitely connected convex surface, with boundary curves $C_1, \cdots, C_m$. Let b be a sub-arc of C_1, for example. Is some portion of S infinitesimally rigid if only b is kept fixed? We are unable to answer this question in a general way; on the other hand, we can verify easily that this type of rigidity occurs for those quadratic surfaces which are projectively equivalent to the sphere. These special results will be reviewed presently. *We note that we shall require only class C' of the vector $\mathfrak{z}$ inducing the infinitesimal deformation.*

I. 7. 2. It is convenient to begin with the paraboloid of revolution

$$S^*: \quad z = \frac{1}{2}(x^2 + y^2).$$

Let R be any bounded, finitely connected Jordan region in the xy plane, with boundary curves $C_1, \cdots, C_m$, and let

$$S: \quad z = \frac{1}{2}(x^2 + y^2)$$

be the portion of the paraboloid located over R. Finally, let $\mathfrak{b}$ denote any sub-arc of that boundary curve of S which is located over C_1. *We assert that S is then $IR\mathfrak{b}$ (see* I. 1. 5). *Explicitly: if $\mathfrak{z} = (X, Y, Z)$ is any vector function on S of class C' that induces an infinitesimal deformation of S and vanishes on $\mathfrak{b}$, then $\mathfrak{z} \equiv 0$ on S.*

Proof. By I. 1. 7 we have the fundamental equations

$$a_x = rZ, \quad b_y = tZ, \quad a_y + b_x = 2sZ.$$

In the present case, $r \equiv 1$, $t \equiv 1$, $s \equiv 0$. Hence

$$(1) \qquad a_x = Z, \quad b_y = Z, \quad a_y + b_x = 0.$$

These equations show that

$$(2) \qquad a_x = b_y, \quad a_y = -b_x.$$

Also, since now $p = x$, $q = y$, we have, by I. 1. 7,

$$(3) \qquad a = X + xZ, \quad b = Y + yZ.$$

Since X, Y, Z are of class C', (3) shows that a_x, a_y, b_x, b_y are

continuous, and (2) shows that a, b satisfy the Cauchy-Riemann equations in the interior of R. Thus, on introducing the complex variable $w = x + iy$, the function

$$f(w) = a + ib \tag{4}$$

is an analytic function in the interior of R, by the theorem of Goursat. Furthermore, $f(w)$ is continuous in R, and vanishes on the sub-arc $\mathfrak{b}'$ of C_1 which is the projection of the arc $\mathfrak{b}$. By a classical theorem on analytic functions, it follows that $f(w) \equiv 0$. Thus $a \equiv 0$, $b \equiv 0$ in R, and hence also $a_x \equiv 0$. From (1) it follows that $Z \equiv 0$, and finally (3) yields that $X \equiv 0$, $Y \equiv 0$. Thus $\mathfrak{z} \equiv 0$, as asserted.

I. 7. 3. We observe now that *the IRb property is invariant under projective transformations.* Indeed, a general projective transformation can be decomposed into affine transformations and the transformation $x' = x/z$, $y' = y/z$, $z' = 1/z$. For this last one, the explicit formulas of the Darboux transformation (see I. 2. 3) yield the desired result immediately. An affine transformation can be decomposed into a translation and *shearing transformations* of the type $x' = kx$, $y' = y$, $z' = z$, and the assertion is readily verified again. Thus, since the paraboloid in I. 7. 2 is projectively equivalent to the sphere, it follows that *all surfaces projectively equivalent to the sphere possess the very strong IRb property exhibited by the paraboloid of revolution.*

PART II. NON-CONVEX SURFACES.

§ II. 1. Preliminary comments on differential equations.

II. 1. 1. In a bounded, finitely connected Jordan region R of the uv plane, we consider the system of differential equations

$$a_u = A_1(u, v)\, a + B_1(u, v)\, b\,, \tag{1}$$

$$b_v = A_2(u, v)\, a + B_2(u, v)\, b\,. \tag{2}$$

The following assumptions are made.

(i) The coefficients A_1, B_1, A_2, B_2 are continuous and hence bounded in R. We denote by M a common bound for the maximum of their absolute values in R. We choose a positive number δ, kept fixed in the sequel, such that

$$0 < \delta < \frac{1}{4M}\,. \tag{3}$$

(ii) The functions $a(u, v)$, $b(u, v)$ are continuous in R, and the partial derivatives a_u, b_v exist. The equations (1), (2) show then that

a_u, b_v are continuous in the interior of R and remain continuous on the boundary.

We observe that if these smoothness assumptions were suitably strengthened, then differentiation of (1), (2) would yield a hyperbolic system of the second order. The results to be derived below would then follow from classical theorems. Under the general conditions here considered, we shall rely on the familiar *method of differential inequalities*. In fact, the auxiliary results we need, concerning the system (1), (2), are probably contained in the literature. Accordingly, we shall give only a sketchy presentation.

II. 1. 2. Let $r: \alpha_0 \leqq u \leqq \alpha$, $\beta_0 \leqq v \leqq \beta$ be a rectangle, and let P and Q be opposite vertices of r. Furthermore, let c be a simple continuous arc with end-points P, Q which is contained in the interior of r, except for its end-points, and which is intersected by any horizontal or vertical line in one point at most. Then c subdivides r into two triangle-shaped regions. We assume that one of these, say r^*, is contained in the region R of II. 1. 1.

Lemma. If a and b vanish on the arc c, and if the diameter of r^* is less than the δ occurring in (3), then $a \equiv 0$, $b \equiv 0$ in r^*.

Proof. Let $\mathfrak{A}^*$, $\mathfrak{B}^*$ denote the maximum in r^* of $|a|$, $|b|$ respectively. Then we have a point $(u_0, v_0) \in r^*$ such that

$$|a(u_0, v_0)| = \mathfrak{A}^* \tag{4}$$

Due to the special shape of r^*, we can join (u_0, v_0) to a point (u^*, v_0) on c by a horizontal segment in r^* (this segment may reduce to a point). Since $a = 0$ on c, we have then

$$a(u_0, v_0) = \int_{u^*}^{u_0} a_u(u, v_0)\, du. \tag{5}$$

From (1) we infer that

$$|a_u| \leqq M(\mathfrak{A}^* + \mathfrak{B}^*) \quad \text{in } r^*. \tag{6}$$

From (5) and (6) we see that

$$\mathfrak{A}^* = |a(u_0, v_0)| \leqq \delta M(\mathfrak{A}^* + \mathfrak{B}^*),$$

since the diameter of r^* is less than δ. By (3) it follows that

$$\mathfrak{A}^* \leqq \frac{1}{4}(\mathfrak{A}^* + \mathfrak{B}^*). \tag{7}$$

A similar argument yields the inequality

$$(8) \qquad \mathfrak{B}^* \leq \frac{1}{4}(\mathfrak{A}^* + \mathfrak{B}^*).$$

Addition of (7), (8) yields $(\mathfrak{A}^* + \mathfrak{B}^*) \leq (\mathfrak{A}^* + \mathfrak{B}^*)/2$. Since $\mathfrak{A}^* \geq 0$, $\mathfrak{B}^* \geq 0$, it is now clear that $\mathfrak{A}^* = 0$, $\mathfrak{B}^* = 0$, and the lemma is proved.

II. 1. 3. Let next $r: \alpha_0 \leq u \leq \alpha$, $\beta_0 \leq v \leq \beta$ be a rectangle contained in the region R of II. 1. 1. Let this time c denote the union of two adjacent sides of r.

Lemma. If a and b vanish on c, and if the diameter of r is less than the δ occurring in (3), then $a \equiv 0$, $b \equiv 0$ in r.

The proof is the same as in II. 1. 2, the present r, c replacing the r^*, c of that section.

II. 1. 4. *The lemma in* II. 1. 3 *remains valid if the restriction concerning the diameter of r is dropped.*

Proof. We merely sketch the simple proof. For definiteness, let us assume that c is the union of the sides $u = \alpha$, $v = \beta$ of r. If we subdivide r into 2^n congruent small rectangles, then for n sufficiently large each one of the small rectangles so obtained will have a diameter less than the δ in (3). Now consider the small rectangle r' in the upper right corner of r. In r', the assumptions of the lemma of II. 1. 3 are satisfied, and hence $a \equiv 0$, $b \equiv 0$ in r'. But then, the assumptions of the lemma of II. 1. 3 are satisfied in the small rectangle r'' right below r', and hence $a \equiv 0$, $b \equiv 0$ in r'' also. In this manner, one sees that $a \equiv 0$, $b \equiv 0$ in the stack of small rectangles bordering on the side $u = \alpha$ of r. The same process can now be repeated on the next stack on the left, and it follows that $a \equiv 0$, $b \equiv 0$ in r.

II. 1. 5. Again, let $r: \alpha_0 \leq u \leq \alpha$, $\beta_0 \leq v \leq \beta$ be a rectangle, and let P, Q be opposite vertices of r. Join P and Q, without leaving r, by a simple polygonal line c which consists of a finite number of alternating horizontal and vertical segments. Let r^* be one of the two polygonal regions into which c divides r, and suppose that r^* is a sub-region of the region R in II. 1. 1.

Lemma. If a and b vanish on c, then $a \equiv 0$, $b \equiv 0$ in r^*.

Proof. For definiteness, assume that P is the upper left corner of r. On extending the horizontal segments of c across r^*, one subdivides r^* into a stack of rectangles $r_1^*, \cdots, r_m^*$, where these rectangles are numbered from the highest to the lowest. Then in r_1^* the assumptions

of II. 1. 4 hold, and hence $a \equiv 0$, $b \equiv 0$ in r_1^*. One sees that, as a consequence, the assumptions of II. 1. 4 hold in r_2^*, hence $a \equiv 0$, $b \equiv 0$ in r_2^*, and so forth down to r_m^*.

II. 1. 6. Returning to the situation considered in II. 1. 2, let us make the additional assumption that c is a *convex* arc. Again, we assume that one of the triangle-shaped regions, say r^*, into which c subdivides r, is a sub-region of the region R of II. 1. 1.

Lemma. If a and b vanish on c, then $a \equiv 0$, $b \equiv 0$ in r^* (observe: there is no restriction upon the diameter of r).

Proof. If c' is any sub-arc of c, then because of the convexity of c we have a triangle-shaped subregion of r^*, say $\bar{r}^*$, bounded by c' and two line segments, one horizontal and one vertical. If the diameter of c' is sufficiently small, then the diameter of $\bar{r}^*$ is less than the δ in (3). Accordingly, by II. 1. 2, we have then $a \equiv 0$, $b \equiv 0$ in $\bar{r}^*$. Now, using this remark, the pattern of the proof is clear. First, we subdivide c' into sufficiently small sub-arcs $c_1', \cdots, c_m'$. Then $a \equiv 0$, $b \equiv 0$ in the union of the corresponding regions $\bar{r}_1^*, \cdots, \bar{r}_m^*$. The rest of r^* is then subject to the lemma of II. 1. 5.

II. 1. 7. Returning to the region of II. 1. 1, let us say that R is *admissible* if its boundary can be subdivided into a finite number of arcs $\gamma_1, \cdots, \gamma_m$, such that each γ_k satisfies one or the other of the following two conditions.

(i) γ_k is a horizontal or vertical segment.

(ii) γ_k is a convex arc which can be represented by an equation of the form $v = f(u)$, $\alpha \leq u \leq \beta$, where $f(u)$ is strictly monotone, and $\alpha < \beta$. Furthermore, on denoting by r_k the rectangle (with sides parallel to the axes) with opposite vertices at the end-points of γ_k, one of the two triangle-shaped regions, into which γ_k divides r_k, is a subregion of R.

One sees that an admissible region R can be subdivided into a finite number of subregions $R_1, \cdots, R_n$, each of which satisfies one of the following two conditions.

(i*) R_j is a rectangle with sides parallel to the axes.

(ii*) R_j is a triangle-shaped region, bounded by two line segments (one horizontal, one vertical), and by a sub-arc c_j of an arc γ_k satisfying condition (ii).

II. 1. 8. It is now clear that the results stated in II. 1. 4 and II. 1. 6 yield theorems of the following general type: if the solutions a and b

of the differential equations (1), (2) (see II. 1. 1) vanish on a sufficiently comprehensive subset B^* of the boundary B of R, then a, b vanish identically in a corresponding subregion of R, or even in all of R (assuming that R is admissible in the sense of II. 1. 7). By drawing a diagram of an admissible (simply or multiply connected) region R, the reader will discover how the vanishing of a and b on parts of B is propagated into the interior of R by means of the results in II. 1. 4 and II. 1. 6. We state merely a few representative results, with brief hints concerning procedure. The region R is assumed to be admissible throughout.

Lemma 1. If a and b vanish on the whole boundary of R, then $a \equiv 0$, $b \equiv 0$ in R.

To see this, consider a subdivision of R as described in II. 1. 7. The result in II. 1. 6 shows that $a \equiv 0$, $b \equiv 0$ in each one of the regions R_j satisfying the condition (ii*) in II. 1. 7. The remaining regions R_j are then rectangles, and the vanishing of a, b in these rectangles, arranged in an appropriate order, is then inferred by successive applications of the result in II. 1. 4. In the special case when no regions of type (ii*) are present, only II. 1. 4 is needed of course.

Lemma 2. Suppose that the admissible region R is bounded by two simple closed curves, where the exterior boundary curve is strictly convex, and the interior boundary curve is the perimeter of a rectangle, with sides parallel to the axes. If a and b vanish on the exterior curve, then $a \equiv 0$, $b \equiv 0$ in R.

To see this, one subdivides again R as indicated in II. 1. 7. Then $a \equiv 0$, $b \equiv 0$ in the union of the regions R_j of type (ii*). The remaining regions R_j are then rectangles, and on properly arranging these rectangles the vanishing of a and b can be inferred again by successive applications of II. 1. 4, without requiring *a priori* the vanishing of a and b on the interior boundary curve. Actually, the reader will readily perceive that Lemma 2 could be substantially generalized.

Lemma 3. Suppose that a and b vanish on the boundary arc c_j of one of the regions R_j of type (ii*) (see II. 1. 7). Then $a \equiv 0$, $b \equiv 0$ in R_j.

Of course, this is merely a restatement of II. 1. 6.

§ II. 2. Surfaces of negative curvature.

II. 2. 1. The general concept of an infinitesimal deformation, dis-

cussed in § I. 1, applies to non-convex surfaces as well as to convex surfaces. The reader will observe that in § I. 1 no restriction was placed upon the sign of the Gauss curvature. Accordingly, we can utilize the concepts and definitions presented there, and in particular the fundamental differential equations (6), (7), (8) in I. 1. 6 are at our disposal.

II. 2. 2. Given a surface S of negative Gauss curvature, through each individual point of S there pass precisely two asymptotic lines of S. Accordingly, one can use the asymptotic lines as parameter curves *in the small*. If, however, a given surface S is to be represented *in the large* in this manner, then there arise various interesting questions. In order to emphasize the essential issues from the point of view of infinitesimal rigidity, we assume *a priori* that we deal with a surface S, of negative curvature, that can be represented *in the large* in terms of asymptotic parameters. Accordingly, we make the following assumptions. The surface S is given by a parametric representation (in vector notation):

$$S: \quad \mathfrak{x} = \mathfrak{x}(u, v), \quad (u, v) \in R, \tag{1}$$

such that the following holds.

(i) R is a bounded, finitely connected Jordan region which is *admissible* (in the sense of II. 1. 7).

(ii) The position vector $\mathfrak{x}(u, v)$ is continuous in R, of class C'' in the interior of R, and the first and second partial derivatives of $\mathfrak{x}(u, v)$ remain continuous on the boundary of R.

(iii) On denoting by L, M, N, as usual, the second fundamental quantities for the representation (1), we have

$$L \equiv 0, \quad N \equiv 0 \text{ in } R, \tag{2}$$

$$M \neq 0 \text{ on a dense set in } R. \tag{3}$$

(iv) $W = (EG - F^2)^{\frac{1}{2}} > 0$, where E, F, G denote the first fundamental quantities, as usual.

The following comments are in order. Since the lines $u =$ constant, $v =$ constant correspond to the asymptotic lines of S, the requirement that the parameter region R be admissible has a simple geometrical interpretation: the boundary of S consists of a finite number of arcs, each of which is either an asympotic arc or else *definitely not* an asymptotic arc. Next, since by assumption the Gauss curvature is negative and hence different from zero, in view of (2) we have actually $M \neq 0$ throughout, while we state only in (3) that $M \neq 0$ on a dense set in R. As a consequence, the rigidity theorems to be derived below

have a greater scope than what one may expect: the Gauss curvature may vanish on a substantial subset of S, as long as (3) holds. Neither do we require that the representation (1) be 1-1 *in the large*. Note also the class requirement C'' upon S, in contrast with the usual C'''.

II. 2. 3. **Theorem.** *The surface S, described in* II. 2. 2, *is IRB in the sense of* I. 1. 5. *Explicitly: if $\mathfrak{z} = \mathfrak{z}(u, v)$, $(u, v) \in R$, is a vector function of class C' which induces an infinitesimal deformation of S and vanishes on the whole boundary B of S, then $\mathfrak{z} \equiv 0$.*

Proof. In view of II. 2. 2 (2), the fundamental differential equations (6), (7), (8) of I. 1. 6 appear now in the form

$$a_u = \begin{Bmatrix} 1 & 1 \\ & 1 \end{Bmatrix} a + \begin{Bmatrix} 1 & 1 \\ & 2 \end{Bmatrix} b, \tag{4}$$

$$b_v = \begin{Bmatrix} 2 & 2 \\ & 1 \end{Bmatrix} a + \begin{Bmatrix} 2 & 2 \\ & 2 \end{Bmatrix} b, \tag{5}$$

$$a_v + b_u = 2 \begin{Bmatrix} 1 & 2 \\ & 1 \end{Bmatrix} a + 2 \begin{Bmatrix} 1 & 2 \\ & 2 \end{Bmatrix} b + 2Mc. \tag{6}$$

Since $\mathfrak{z} = 0$ on the boundary, we have (see I. 1. 6 (4))

$$a = 0, \quad b = 0, \quad c = 0 \quad \text{on } B. \tag{7}$$

The equations (4), (5) are of the type discussed in § II. 1. Accordingly, by the lemma 1 in II. 1. 8 it follows that

$$a \equiv 0, \quad b \equiv 0 \quad \text{in } R. \tag{8}$$

From (6) it follows that $Mc \equiv 0$ in R. Hence, by (3), $c = 0$ on a dense set in R. By continuity, it follows that

$$c \equiv 0 \quad \text{in } R. \tag{9}$$

By I. 1. 6 (4) we infer from (8), (9) that

$$\mathfrak{z}\mathfrak{x}_u \equiv 0, \quad \mathfrak{z}\mathfrak{x}_v \equiv 0, \quad \mathfrak{z}\xi \equiv 0. \tag{10}$$

Since the vectors $\mathfrak{x}_u$, $\mathfrak{x}_v$, ξ are linearly independent by the condition (iv) in II. 2. 2, (10) implies that $\mathfrak{z} \equiv 0$.

Remark. Two special cases may be mentioned here to illustrate the scope of this result. First, consider the case when the boundary of R consists solely of horizontal and vertical line segments. Then the surface S may be termed an *asymptotic-polygonal surface*. Next, consider the case when R is bounded by a single strictly convex closed curve. The surface S then may be termed *asymptotic-convex*. In either case, S is *IRB*.

II. 2. 4. It is now clear that a variety of theorems of the IRb type (see I. 1. 5) will follow if one applies the results in § II. 1 to the situation considered in II. 2. 2. We state explicitly, in the way of illustration, only two such theorems.

Theorem. *Returning to the surface S of* II. 2. 2, *given by* (1) *with the properties* (i)—(iv) *stated there, assume that the parameter region R is doubly connected. Suppose that the exterior boundary curve C is strictly convex, while the interior boundary curve is the perimeter of a rectangle with sides parallel to the axes. Let $\overline{C}$ correspond to C on S. Then S is $IR\overline{C}$ in the sense of* I. 1. 5. In other words, S is infinitesimally rigid even if only the exterior boundary curve is kept fixed, provided that the interior boundary curve is an *asymptotic quadrilateral.*

The proof is the same as in II. 2. 3, except that one will now use the lemma 2 in II. 1. 8.

Theorem. *Returning to the surface S given by* (1) *in* II. 2. 2, *with the properties* (i)—(iv) *stated there, consider a sub-region R^* of R of the following character: R^* is bounded by two line segments (one horizontal, one vertical), and by a strictly convex sub-arc γ of the boundary of R. Let γ^* correspond to γ on S. Then the corresponding portion S^* of S is $IR\gamma^*$ in the sense of* I. 1. 5. In other words: if one keeps fixed only a sub-arc γ^* of the boundary of S, then a certain portion of S becomes infinitesimally rigid.

The proof is the same as in II. 2. 3, except that the lemma 3 in II. 1. 8 should now be used.

§ II. 3. Surfaces of zero curvature.

II. 3. 1. For general preliminary comments, the reader is referred to II. 2. 1, II. 2. 2. An important special comment is the following one. Let S be a surface whose Gauss curvature K vanishes identically. If P is a particular point of S, then two cases may occur. Either one has precisely one asymptotic direction at P, or else every direction at P is asymptotic. In the latter case, P will be termed a *flat point.* If no flat points are present, then S is a developable surface in the sense of classical Differential Geometry, and if further S is not excessively large, then it admits of a particularly convenient representation in terms of its rectilinear generators and their orthogonal trajectories. For such a representation, one will have (with the usual notations) $G \equiv 1$, $F \equiv 0$, $N \equiv 0$, $M \equiv 0$. However, simple examples show that such a representation may be available even though a substantial set

of flat points is present. To achieve adequate generality, we consider the following situation. We are given a surface

$$S: \quad \mathfrak{x} = \mathfrak{x}(u, v), \quad (u, v) \in R,$$

such that the following holds. Conditions (i), (ii), (iv) in II. 2. 2 are taken over *verbatim*. Condition (iii) is modified as follows.

(iii*) $G \equiv 1$, $F \equiv 0$, $N \equiv 0$, $M \equiv 0$ in R, and $L \neq 0$ on a dense set in R.

Under these conditions, the fundamental differential equations (6), (7), (8) of I. 1. 6 reduce, after a simple calculation, to the following form

$$a_u = \frac{E_u}{2E} a - \frac{E_v}{2E} b + Lc, \tag{1}$$

$$b_v = 0, \tag{2}$$

$$a_v + b_u = \frac{E_v}{2E} a. \tag{3}$$

The discussion now becomes altogether elementary, and the conclusions become very strong. For convenience, we introduce the following terminology. Let γ be a sub-arc of the boundary of R which can be represented in the form

$$\gamma: \quad v = f(u), \quad u_0 \leqq u \leqq u_1, \tag{4}$$

where $f(u)$ is single-valued and continuous in the indicated interval. The corresponding arc γ^* on the boundary of S will be termed a *strong sub-arc*. Geometrically, a strong sub-arc is one that crosses the rectilinear generators of S (or rather a family of them) only once. Returning to the γ in (4), a point $(\bar{u}, \bar{v}) \in R$ will be termed *vertically accessible* from γ if $(\bar{u}, \bar{v})$ can be joined to some point of γ by a vertical segment in R. The set of these vertically accessible points is denoted by R_γ. The corresponding portion of S will be called the *shadow* of the strong arc γ^*.

II. 3. 2. **Theorem.** *The shadow of a strong arc γ^* is $IR\gamma^*$ in the sense of* I. 1. 5. *Explicitly: if $\mathfrak{z} = \mathfrak{z}(u, v)$ is a vector function of class C' which induces an infinitesimal deformation of the shadow and vanishes on γ^*, then $\mathfrak{z} \equiv 0$.*

The proof is altogether elementary. The assumptions imply that (with the notations of II. 3. 1)

$$a = 0, \quad b = 0, \quad c = 0 \quad \text{on } \gamma. \tag{5}$$

By (2), b is constant along vertical segments. By (5) it follows that

$$b \equiv 0 \quad \text{in } R_\gamma. \tag{6}$$

Equation (3) can now be written in the form

$$\frac{\partial}{\partial v}\frac{a}{E}=0\,.$$

Thus a/E is constant along vertical segments. By (5) it follows that

$$(7)\qquad a\equiv 0 \quad \text{in } R_\gamma\,.$$

Now (1) shows that $Lc=0$. Since $L\neq 0$ on a dense set, by continuity one sees that

$$(8)\qquad c\equiv 0 \quad \text{in } R_\gamma\,.$$

From (6), (7), (8) one infers, as in II. 2. 3, that $\mathfrak{z}\equiv 0$ in R_γ.

II. 3. 3. It is now clear that the theorems for surfaces of negative curvature, discussed in II. 2. 3 and II. 2. 4, remain valid for the surfaces of zero curvature we are now considering. In fact, the theorem in II. 3. 2 shows that these surfaces show greater rigidity of the IRb type. Accordingly, we restrict ourselves to an illustrative example concerned with the *flat points* defined in II. 3. 1. In the xy plane, consider the rectangle

$$R:\ 0\leqq x\leqq 1,\ 0\leqq y\leqq k,\quad k>0\,.$$

Let $f(x)$, $0\leqq x\leqq 1$, be a continuous function with continuous first and second derivatives. Consider the cylindrical surface

$$S:\ z=z(x,y)=f(x)\,,\quad (x,y)\in R\,.$$

In terms of x, y as parameters, one finds readily that $G\equiv 1$, $F\equiv 0$, $N\equiv 0$, $M\equiv 0$. As regards L, one finds

$$L=\frac{f''(x)}{[1+f'(x)^2]^{\frac{1}{2}}}$$

Thus all the assumptions of II. 3. 1 are satisfied if we add the condition that $f''(x)\neq 0$ on a dense set in $0\leqq x\leqq 1$. Now clearly all of S is the shadow of the boundary arc γ^* that lies over $0\leqq x\leqq 1$. Thus S is $IR\gamma^*$. Observe that on properly choosing $f(x)$, we obtain an example where the flat points of S form a set whose measure (in terms of area on S) is as close to the total area of S as we please.

§ II. 4. Study of the torus.

II. 4. 1. We represent the torus T in the form

$$(1)\qquad T:\ \mathfrak{x}=\mathfrak{x}(u,v)=(x(u,v)\,,\ y(u,v)\,,\ z(u,v))\,,$$

where

$$x=(1-r\sin u)\cos v\,,\quad y=(1-r\sin u)\sin v\,,\quad z=r\cos u$$

In these formulas, r is a constant such that $0 < r < 1$, and the parameters u, v are restricted by the conditions $-\pi \leqq u < \pi$, $0 \leqq v < 2\pi$. The *outer belt* B_o and the *inner belt* B_i of the torus correspond to $-\pi \leqq u \leqq 0$ and $0 \leqq u \leqq \pi$ respectively. For $u = 0$ and $u = \pi$ we obtain the *upper circle* C_u and the *lower circle* C_l of the torus respectively. The Gauss curvature K of T is zero on C_u and C_l. Otherwise, $K > 0$ on B_o and $K < 0$ on B_i.

Theorem. *The torus is infinitesimally rigid. Explicitly: if* $\mathfrak{z}$ *is a vector function of class* C' *on the torus which induces an infinitesimal deformation, then* $\mathfrak{z}$ *is trivial.*

In preparation for the proof, let us first note that the outer belt B_o of T satisfies the assumptions of the theorem in I. 6. 3, and hence $\mathfrak{z}$ is trivial on B_o. Accordingly, there exist two constant vectors $\mathfrak{u}$ and $\mathfrak{v}$ such that

$$\mathfrak{z} = (\mathfrak{u} \times \mathfrak{x}) + \mathfrak{v} \quad \text{on } B_o .$$

Consider then, on T, the vector function

$$\mathfrak{z}^* = \mathfrak{z} - [(\mathfrak{u} \times \mathfrak{x}) + \mathfrak{v}] .$$

Clearly, $\mathfrak{z}^*$ induces again an infinitesimal deformation of T, and $\mathfrak{z}^* \equiv 0$ on B_o. If $\mathfrak{z}^*$ can be shown to vanish on B_i also, then it follows that $\mathfrak{z}$ is trivial. In other words, we can assume *a priori* that the vector function $\mathfrak{z}$ itself vanishes on B_o, without loss of generality. Accordingly, we add the assumption

$$(2) \qquad \mathfrak{z} \equiv 0 \quad \text{on } B_o .$$

Consider $\mathfrak{z}$ on B_i. We have $K < 0$ on B_i (except for the boundary circles C_u and C_l), and $\mathfrak{z} = 0$ on the boundary of B_i, by (2). Accordingly, if it were possible to use a regular representation of B_i *in the large* in terms of its asymptotic lines, then the theorem in II. 2. 3 would yield directly the conclusion that $\mathfrak{z} \equiv 0$ on B_i also, and the proof of the theorem would be complete. Actually, we shall find that the circles C_u and C_l represent singularities from this point of view. However, the method used in § II. 2 will be found to apply with certain modifications which we shall discuss presently.

II. 4. 2. Returning to the representation (1), it will be convenient to let v range from $-\infty$ to $+\infty$. Then $\mathfrak{x}, \mathfrak{z}$ are defined (and periodic in v with period 2π) in the infinite strip

$$S^* : \quad -\pi \leqq u < \pi , \quad -\infty < v < +\infty ,$$

and $\mathfrak{z}$ is of class C' in S^* (of course, $\mathfrak{x}$ is analytic). On introducing again the auxiliary functions

$$(3) \qquad a = \mathfrak{z}\mathfrak{x}_u\,, \quad b = \mathfrak{z}\mathfrak{x}_v\,,$$

we have (see I. 1. 6) the differential equations

$$(4) \qquad a_u = \mathfrak{z}\mathfrak{x}_{uu}\,, \quad b_v = \mathfrak{z}\mathfrak{x}_{vv}\,, \quad a_v + b_u = 2\,\mathfrak{z}\mathfrak{x}_{uv}\,.$$

Since $\mathfrak{z}$ is of class C', we see from (3) and (4) that a_u, a_v, a_{uv} exist and are continuous in S^*. By the lemma in I. 5. 2, it follows that a_{vu} also exists and $a_{vu} = a_{uv}$. Similarly one finds that b_u, b_v, $b_{vu} = b_{uv}$ are available and continuous in S^*. From (4) if follows then directly that a_{uu}, a_{vv}, b_{vv}, b_{uu} also exist and are continuous in S^*. Briefly a and b are of class C'' in S^*. Now, by (2), we infer that $\mathfrak{z}$, $\mathfrak{z}_u$, $\mathfrak{z}_v$ vanish on the line $u = 0$. From (3) we see then that a, a_u, b, b_u vanish for $u = 0$. Since a, b are of class C'', it follows that

$$(5) \qquad a = O(u^2)\,, \quad b = O(u^2) \quad \text{in } S^*.$$

In fact, a simple discussion would show that one has even the estimate $o(u^2)$, but we do not need this fact.

II. 4. 3. We restrict ourselves from now on to the strip

$$(6) \qquad S: \ 0 \leqq u < \pi\,, \quad -\infty < v < +\infty\,,$$

which corresponds to the inner belt B_i of the torus. For the first and second fundamental quantities we find the expressions

$$E = r^2\,, \quad F = 0\,, \quad G = (1 - r\sin u)^2\,,$$
$$L = r\,, \quad M = 0\,, \quad N = -(1 - r\sin u)\sin u\,.$$

On setting

$$(7) \qquad \lambda(u) = \left[\frac{r}{(1 - r\sin u)\sin u}\right]^{\frac{1}{2}}, \quad 0 < u < \pi\,,$$

the differential equation of the asymptotic lines on B_i appears in the form

$$(8) \qquad \frac{dv}{du} = \pm\,\lambda(u)\,, \quad 0 < u < \pi\,.$$

Now observe that for $u = 0$ and $u = \pi$ the function $\lambda(u)$ is of the order of $u^{-\frac{1}{2}}$, $(\pi - u)^{-\frac{1}{2}}$ respectively. Accordingly, $|\lambda(u)|$ is integrable in the closed interval $0 \leqq u \leqq \pi$, and thus we can introduce the auxiliary function

$$(9) \qquad \varphi(u) = \int_0^u \lambda(u)\,du\,, \quad 0 \leqq u \leqq \pi\,.$$

Then $\varphi(u)$ is strictly increasing, and we have

$$\varphi(0)=0, \quad \varphi'(u)=\lambda(u), \quad \varphi(\pi)=k, \tag{10}$$

where $0<k<+\infty$. An entirely elementary argument yields the relations

$$u^{\frac{1}{2}}\lambda(u)\to r^{\frac{1}{2}} \quad \text{for} \quad u\to 0^+, \tag{11}$$

$$\varphi(u)/u^{\frac{1}{2}}\to 2r^{\frac{1}{2}} \quad \text{for} \quad u\to 0^+. \tag{12}$$

II. 4. 4. In view of (8), one has $v-\varphi(u)=\text{constant}$, $v+\varphi(u)=\text{constant}$ along the two families of asymptotic lines of B_i. Accordingly, we introduce the asymptotic lines as parameter curves by changing to new parameters

$$\alpha=v-\varphi(u), \tag{13}$$

$$\beta=v+\varphi(u). \tag{14}$$

In the (α, β) plane, let $\bar{S}$ denote the infinite strip bounded by the lines $\beta=\alpha$ and $\beta=\alpha+2k$ respectively (where $k=\varphi(\pi)$ as in (10)). Since $\varphi(u)$ is strictly increasing, one sees that the correspondence between the strips S and $\bar{S}$ is $1-1$. The lines $u=\text{constant}$ in S correspond to the lines $\beta=\alpha+\text{constant}$ in $\bar{S}$. From (10), (13), (14) one finds readily that

$$u_\alpha=-\frac{1}{2\lambda}, \quad u_\beta=\frac{1}{2\lambda}, \quad v_\alpha=\frac{1}{2}, \quad v_\beta=\frac{1}{2}, \tag{15}$$

and hence

$$\frac{\partial(u, v)}{\partial(\alpha, \beta)}=-\frac{1}{2\lambda}.$$

Thus the Jacobian of the transformation is different from zero in the interior of S, but it vanishes on the boundary of $\bar{S}$. This makes it necessary to proceed with some caution. In any case, due to the $1-1$ character of the transformation, $\mathfrak{x}$ and $\mathfrak{z}$ are single-valued and continuous on the closed strip $\bar{S}$, and $\mathfrak{x}$ is analytic and $\mathfrak{z}$ is of class C' in the interior of S.

II. 4. 5. For clarity, we first study the situation in the interior $\bar{S}^0$ of the strip $\bar{S}$. We denote by $\bar{E}, \bar{F}\ \bar{G}, \bar{L}, \bar{M}, \bar{N}$ the first and second fundamental quantities relative to the (α, β) representation. We set

$$\bar{W}=(\bar{E}\bar{G}-\bar{F}^2)^{\frac{1}{2}}.$$

We have

$$\mathfrak{x}_\alpha=\mathfrak{x}_u u_\alpha+\mathfrak{x}_v v_\alpha,$$

$$\mathfrak{x}_\beta=\mathfrak{x}_u u_\beta+\mathfrak{x}_v v_\beta.$$

In view of the formulas in II. 4. 3, II. 4. 4 we obtain readily:

$$\bar{E} = \bar{G} = \frac{1 - r \sin u}{4}, \tag{16}$$

$$\bar{F} = \frac{(1 - r \sin u)(1 - 2r \sin u)}{4}, \tag{17}$$

$$\bar{W}^2 = \frac{r(1 - r \sin u)^3 \sin u}{4}. \tag{18}$$

Thus $\bar{W} \neq 0$ in $\bar{S}^0$, but $\bar{W}$ vanishes on the boundary of $\bar{S}^0$. *This is the fact that necessitates a special study of the boundary.* In the interior $\bar{S}^0$ of $\bar{S}$, the (α, β) representation is however entirely regular. Since $\alpha =$ constant, $\beta =$ constant correspond to the asymptotic lines, and since the Gauss curvature is negative in $\bar{S}^0$, we have

$$\bar{L} = 0, \quad \bar{N} = 0, \quad \bar{M} \neq 0 \quad \text{in } \bar{S}^0. \tag{19}$$

Accordingly, in $\bar{S}^0$ we have the situation described and studied in § II. 2 (with u, v replaced by α, β). On introducing the auxiliary functions

$$\bar{a} = \mathfrak{z}\mathfrak{x}_\alpha, \quad \bar{b} = \mathfrak{z}\mathfrak{x}_\beta, \quad \bar{c} = \mathfrak{z}\xi, \tag{20}$$

we have by II. 2. 3 the differential equations

$$\bar{a}_\alpha = \begin{Bmatrix} 1 & 1 \\ & 1 \end{Bmatrix} \bar{a} + \begin{Bmatrix} 1 & 1 \\ & 2 \end{Bmatrix} \bar{b}, \tag{21}$$

$$\bar{b}_\beta = \begin{Bmatrix} 2 & 2 \\ & 1 \end{Bmatrix} \bar{a} + \begin{Bmatrix} 2 & 2 \\ & 2 \end{Bmatrix} \bar{b}, \tag{22}$$

$$\bar{a}_\beta + \bar{b}_\alpha = 2 \begin{Bmatrix} 1 & 2 \\ & 1 \end{Bmatrix} \bar{a} + 2 \begin{Bmatrix} 1 & 2 \\ & 2 \end{Bmatrix} \bar{b} + 2\bar{M}\bar{c}, \tag{23}$$

where the Christoffel symbols are relative to the (α, β) representation. Using (16), (17), (18), (15), we find by straightforward calculation the formulas

$$\begin{Bmatrix} 1 & 1 \\ & 1 \end{Bmatrix} = - \begin{Bmatrix} 2 & 2 \\ & 2 \end{Bmatrix} = - \frac{(3 - 11 r \sin u + 8 r^2 \sin^2 u) \cos u}{8 [r(1 - r \sin u)^3 \sin u]^{\frac{1}{2}}}, \tag{24}$$

$$\begin{Bmatrix} 1 & 1 \\ & 2 \end{Bmatrix} = - \begin{Bmatrix} 2 & 2 \\ & 1 \end{Bmatrix} = \frac{3 \cos u}{8 [r(1 - r \sin u) \sin u]^{\frac{1}{2}}}. \tag{25}$$

Observe that these expressions approach infinity if the point (α, β) approaches the boundary of $\bar{S}^0$. Observe also that these expressions depend upon u alone. Accordingly, they are constant along each line $\beta = \alpha +$ constant in $\bar{S}^0$.

II. 4. 6. We now turn to the study of the situation in the vicinity of the boundary line $g: \beta = \alpha$ of S (which corresponds to the line $u = 0$ in the (u, v) representation). Let $(\alpha, \beta) \in \bar{S}^0$, and let l be the distance of (α, β) from g. Then $l = (\beta - \alpha)/\sqrt{2}$, and $\beta - \alpha = 2\varphi(u)$ by (13) and (14). Hence $l = \sqrt{2}\,\varphi(u)$. By (12) we see that

$$l/u^{\frac{1}{2}} \to 2\sqrt{2}\, r^{\frac{1}{2}} \quad \text{for} \quad u \to 0^+ . \tag{26}$$

Let $\{\ \}$ denote any one of the Christoffel symbols occurring in (24), (25). Since $\sin u/u \to 1$ for $u \to 0$, it is clear that

$$u^{\frac{1}{2}} \left| \left\{ \quad \right\} \right| \to \frac{3}{8\, r^{\frac{1}{2}}} \quad \text{for} \quad u \to 0^+ .$$

In view of (26) if follows that

$$l \left| \left\{ \quad \right\} \right| \to \frac{3\sqrt{2}}{4} \quad \text{for} \quad u \to 0^+ . \tag{27}$$

From (27) we draw the following conclusion: if N is a positive number greater than $3\sqrt{2}/4$, then there exists a positive number $\delta > 0$ such that

$$|\{\ \}| < N/l \quad \text{for} \quad (\alpha, \beta) \in \bar{S}_\delta{}^0 ,$$

where $\bar{S}_\delta{}^0$ is the infinite strip bounded by the lines $\beta = \alpha$ and $\beta = \alpha + \delta$. Since $3 < 7/2$, we can choose $N = 7\sqrt{2}/8$. We have then (for an appropriate constant $\delta > 0$):

$$\left| \left\{ \quad \right\} \right| < \frac{7\sqrt{2}}{8\, l} \quad \text{for} \quad (\alpha, \beta) \in \bar{S}_\delta{}^0 . \tag{28}$$

We derive next an estimate for $\bar{a}, \bar{b}$ (see (20)). Using (3), (5), (15), (11), (26) we obtain:

$$\begin{aligned} \bar{a} = \mathfrak{z}\mathfrak{x}_\alpha &= \mathfrak{z}\,(\mathfrak{x}_u u_\alpha + \mathfrak{x}_v v_\alpha) = a u_\alpha + b v_\alpha = -\frac{1}{2\lambda}\, a + \frac{1}{2}\, b \\ &= O\,(u^{\frac{1}{2}})\, O\,(u^2) + O\,(u^2) = O\,(u^2) = O\,(l^4) . \end{aligned}$$

A similar argument applies to $\bar{b}$. Thus we have $\bar{a} = O\,(l^4)$, $\bar{b} = O\,(l^4)$ in $\bar{S}_\delta{}^0$. Accordingly, $|\bar{a}|/l^4$, $|\bar{b}|/l^4$ are bounded in $\bar{S}_\delta{}^0$. Thus we can define

$$\mathfrak{A} = l.\,u.\,b.\, \frac{|\bar{a}|}{l^4} , \quad \mathfrak{B} = l.\,u.\,b.\, \frac{|\bar{b}|}{l^4} \quad \text{in } \bar{S}_\delta{}^0 . \tag{29}$$

Now we return to the equations (21), (22), which we write, for convenience, in the form

$$\bar{a}_\alpha = A_{11}\bar{a} + A_{12}\bar{b} , \tag{30}$$

$$\bar{b}_\beta = A_{21}\bar{a} + A_{22}\bar{b} . \tag{31}$$

By (28) we have

$$|A_{ij}| < \frac{7\sqrt{2}}{8\, l} \quad \text{in } \bar{S}_\delta{}^0 . \tag{32}$$

Now take any point (α_0, β_0) in $\bar{S}_\delta{}^0$. The horizontal ray, from (α_0, β_0) to the right, intersects the boundary line $g: \beta = \alpha$ in the point (β_0, β_0). Since $\bar{a}\,(\beta_0, \beta_0) = 0$, we obtain from (29), (30), (32) the relations

$$|\bar{a}(\alpha_0, \beta_0)| = \left| \int_{\alpha_0}^{\beta_0} \bar{a}_\alpha(\alpha, \beta_0)\, d\alpha \right| \leqq \int_{\alpha_0}^{\beta_0} \left| A_{11} l \right| \left| \frac{\bar{a}}{l^4} \right| l^3\, d\alpha$$

$$+ \int_{\alpha_0}^{\beta_0} \left| A_{12} l \right| \left| \frac{\bar{b}}{l^4} \right| l^3\, d\alpha \leqq \frac{7\sqrt{2}}{8} (\mathfrak{A} + \mathfrak{B}) \int_{\alpha_0}^{\beta_0} l^3\, d\alpha,$$

where l is again the distance of (α, β_0) from the boundary line $g: \beta = \alpha$. Since clearly $l = (\beta_0 - \alpha)/\sqrt{2}$, one finds readily that

$$\int_{\alpha_0}^{\beta_0} l^3\, d\alpha = \frac{\sqrt{2}}{4} l(\alpha_0, \beta_0)^4.$$

Thus we obtain the inequality

$$\frac{|\bar{a}(\alpha_0, \beta_0)|}{l(\alpha_0, \beta_0)^4} \leqq \frac{7}{16} (\mathfrak{A} + \mathfrak{B}).$$

Since (α_0, β_0) was an arbitrary point in $\bar{S}_\delta^0$, it follows that

$$\mathfrak{A} \leqq \frac{7}{16} (\mathfrak{A} + \mathfrak{B}).$$

A similar argument yields the inequality

$$\mathfrak{B} \leqq \frac{7}{16} (\mathfrak{A} + \mathfrak{B}).$$

Hence finally

$$\mathfrak{A} + \mathfrak{B} \leqq \frac{7}{8} (\mathfrak{A} + \mathfrak{B}),$$

and consequently $\mathfrak{A} = 0$, $\mathfrak{B} = 0$. In view of (29), it follows that $\bar{a} \equiv 0$, $\bar{b} \equiv 0$ in $\bar{S}_\delta^0$. Since $\bar{M} \neq 0$ in $\bar{S}_\delta^0$, from (23) it follows now that $\bar{c} \equiv 0$ in $\bar{S}_\delta^0$, and finally from (20) we infer that

$$\mathfrak{z} \equiv 0 \quad \text{in } \bar{S}_\delta^0. \tag{33}$$

By continuity, we have $\mathfrak{z} = 0$ also on the boundary line $g_\delta: \beta = \alpha + \delta$ of $\bar{S}_\delta^0$. Now take any point P_0 in $\bar{S}^0 - \bar{S}_\delta^0$, not on g_δ. Then P_0 is the vertex of a triangle Δ with vertices P_0, P_1, P_2, where the sides P_0P_1, P_0P_2 are horizontal and vertical respectively and the side P_1P_2 lies on g_δ, and $\Delta \subset \bar{S}^0$. On the side P_1P_2, $\mathfrak{z}$ vanishes. In Δ, the system (21), (22) satisfies all the assumptions of the lemma in II. 1. 6 (with a, b, u, v replaced by $\bar{a}, \bar{b}, \alpha, \beta$). Accordingly, by that lemma, $\bar{a} \equiv 0$, $\bar{b} \equiv 0$ in Δ. Since P_0 was arbitrary, it follows that $\bar{a} \equiv 0$, $\bar{b} \equiv 0$ in all of $\bar{S}^0$, and hence, in view of (23), (19), (20), we see finally that

$\mathfrak{z} \equiv 0$ in $\bar{S}^0$. Thus $\mathfrak{z}$ vanishes on the inner belt of the torus also, and the proof is complete.

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Bibliography

[1] W. Blaschke, Vorlesungen über Differentialgeometrie, vol. 1, J. Springer, Berlin, 1930.

[2] G. Darboux, Leçons sur la théorie générale des surfaces, vol. IV, Gauthier-Villars, Paris, 1896.

[3] N. V. Efimov, Qualitative problems of the theory of deformation of surfaces (Russian), American Mathematical Society, Translation no. 37.

[4] E. Rembs, Unverbiegbare offene Flächen, Sitzungsberichte Preuss. Akad., Phys.-Math. Klasse, 1930, pp. 123-133.

[5] J. J. Stoker, Open convex surfaces which are rigid, Studies and Essays presented to R. Courant, Interscience Publishers, New York, 1948.

[6] E. Landau, Einführung in die Differential-und Integralrechnungen, p. 213, Satz 299.

[7] H. Liebmann, Über die Verbiegung der geschlossenen Ringfläche, Göttinger Nachrichten, 1901, pp. 39-53.

[8] H. Liebmann, Über die Verbiegung von Rotationsflächen, Leipziger Berichte, vol. 53, 1901, pp. 215-234.