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<th>An integration theorem for completely integrable systems with singularities</th>
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<td>Author(s)</td>
<td>Matsuda, Michihiko</td>
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Let \( M \) be a \( C^\infty \)-manifold. We denote the Lie algebra of all vector fields on \( M \) of \( C^\infty \)-class by \( L(M) \). For two elements \( u \) and \( v \) of \( L(M) \), defining \((\text{ad} v)^k u\) inductively as \([v, (\text{ad} v)^{k-1} u]\), we consider a power series

\[
g_t(u, v) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\text{ad} v)^k u.
\]

Let \( c(u, v; x) \) be the radius of convergence of \( g_t(u, v) \) at \( x \) on \( M \). We consider a Lie subalgebra \( L \) of \( L(M) \) which satisfies the following convergence condition \((C)\):

\[\begin{align*}
(\text{i}) & \quad c(u, v; x) \leq c(u, v; K) \quad \text{at every } x \quad \text{on } K, \quad \text{and} \\
(\text{ii}) & \quad g_t(u, v) \text{ is continuously differentiable with respect to } (t, x) \quad \text{term by term at every } (t, x) \quad \text{which satisfies } |t| < c(u, v; K) \quad \text{and} \quad x \in K^\circ, \quad \text{the interior of } K.
\end{align*}\]

**Theorem.** If a Lie subalgebra \( L \) satisfies the condition \((C)\), then through every point \( x \) on \( M \) there passes a maximal integral manifold \( N(x) \) of \( L \). Any integral manifold of \( L \) containing \( x \) is an open submanifold of \( N(x) \).

Here an integral manifold \( N \) of \( L \) is a connected submanifold of \( M \) which satisfies \( T_x(N) = L(x) \) at every \( x \) on \( N \), where \( L(x) = \{ u(x); u \in L \} \).

The problem was solved under the following assumptions (i) \(\sim\) (iii) respectively by Chevalley, Hermann and Nagano:

\[\begin{align*}
(\text{i}) & \quad \text{dim } L(x) \text{ is constant on } M \quad \text{(Frobenius' theorem, Chevalley [1])}, \\
(\text{ii}) & \quad \text{dim } L \text{ is finite} \quad \text{(Hermann [2])},
\end{align*}\]

(iii) \( M \) and \( L(M) \) are of \( C^\infty \)-class, but \( L \) is arbitrary (Nagano [3]).

If we assume (ii) or (iii), then \( L \) satisfies our condition \((C)\) (see Remark 1 and Remark 2).

\[\text{(*) This work was partially supported by the Yukawa Fellowship.}\]
Proof of Theorem. We shall prove only the local existence of an integral manifold of $L$ passing through $x$, since the local uniqueness of integral manifolds and the existence of the maximal integral manifold can be proved in the same way as Nagano [3] and Chevalley [1].

Let $U=\{(x^1, \ldots, x^n); |x^i-x^i_0|<a\}$ be a relatively compact cubic neighbourhood of $x_0=(x^i_0)$ such that $\phi_t(v)$ gives a diffeomorphism from $U$ to $\phi_t(v)U$, if $|t|<T(v, U)$. Here $\phi_t(v)$ is a local one-parameter group of diffeomorphisms generated by $v$, and $T(v, U)$ is a positive number. By our assumption $g_t(u, v)$ satisfies a symmetric hyperbolic partial differential equation

$$\frac{\partial h}{\partial t} + v^i \frac{\partial h}{\partial x^i} - h \frac{\partial v}{\partial x^i} = 0$$

at $(t, x)$ which satisfies $|t|<c(u, v; \bar{U})$ and $x\in U$, where $v=v^i \frac{\partial}{\partial x^i}$. Also $\phi_t(v)*u$ satisfies the same partial differential equation at such $(t, x)$ that $|t|<T(v, U)$ and $x\in U$. Since $g_t(u, v)$ and $\phi_t(v)*u$ have the same initial value $u$ at $t=0$, by the uniqueness theorem we obtain

$$\phi_t(v)*u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\text{ad} v)^k u$$

at $(t, x)$ such that $|t|<\min\{c(u, v; \bar{U}), T(v, U)\}$ and

$$A|t| + \sqrt{\sum_{i=1}^{n} (x^i-x^i_0)^2} < a,$$

where $A=\max \sqrt{\sum_{i=1}^{n} v^i(x)^2}$ on $\bar{U}$.

If $v(x_0)\neq 0$, we may assume that $v=\frac{\partial}{\partial x^3}$ in $U$. Then from the identity (1) we get

$$u(x(t-t)) = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} u^{(k)}(t),$$

where $x(t)=(x^1+t, x^2, \ldots, x^n)$ and $u^{(k)}=\frac{\partial^k u}{\partial (x^i)^k}$. This identity (2) holds for $(t, \tau)$ such that

$$|t|+|\tau|<a, \quad |t|<\min\left\{\frac{a}{2}, c(u, v; \bar{U})\right\}.$$ 

As a function of $\tau$, $u(x(t))$ is real analytic in the interval $(-a, +a)$. Hence we have

$$u^{(l)}(x(t-t)) = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} u^{(k+l)}(x(t))$$

for $(t, \tau)$ which satisfies (3) and for every $l\in Z_+$. From this identity we get
at $x(t)$ for every $t$ which satisfies (3) and for every $l \in \mathbb{Z}_+$. 

Let us consider an integral curve $C$ passing through $x_0$. Take a point $y(s) = \exp(sv)x_0$ on $C$. Then there exists such a positive number $\sigma$ that we have

(6) \[ \phi_\sigma(v)_{\#}(\text{ad} v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\text{ad} v)^{l+k} u \]

at every $y(s')$ on $C$ between $x_0$ and $y(s)$ and for every $l \in \mathbb{Z}_+$. We may assume that $s = m\sigma$ for a positive integer $m$.

Operating $\phi_{m\sigma}(v)_{\#}$ on the identity (6) at $y(\sigma)$, we get

\[ \phi_{m\sigma}(v)_{\#}(\text{ad} v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \phi_{m\sigma}(v)_{\#}(\text{ad} v)^{l+k} u \]

at $y(m\sigma)$ for every $l \in \mathbb{Z}_+$. Then operating $\phi_{m\sigma}(v)_{\#}$ on the identity (6) at $y(2\sigma)$, we have

\[ \phi_{m\sigma}(v)_{\#}(\text{ad} v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \phi_{m\sigma}(v)_{\#}(\text{ad} v)^{l+k} u \]

at $y(m\sigma)$ for every $l \in \mathbb{Z}_+$. Thus we obtain

\[ \phi_{m-n\sigma}(v)_{\#}(\text{ad} v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \phi_{m-n\sigma}(v)_{\#}(\text{ad} v)^{l+k} u \]

at $y(m\sigma)$ for such $(n, l)$ that $0 \leq n \leq m-1$ and $l \in \mathbb{Z}_+$. In particular for $n=m-1$, we have

\[ \phi_{\sigma}(v)_{\#}(\text{ad} v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\text{ad} v)^{l+k} u \]

at $y(m\sigma)$ for every $l \in \mathbb{Z}_+$. Hence inductively we obtain

(7) \[ \phi_{m-n\sigma}(v)_{\#}(\text{ad} v)^l u \in L(\exp(sv)x_0), \quad (0 \leq n \leq m-1) \]

for every $l \in \mathbb{Z}_+$. In particular for $n=l=0$, we get

(8) \[ \phi_\sigma(v)_{\#} u \in L(\exp(sv)x_0). \]

Since $u$ is arbitrary in $L$, we have

(9) \[ \dim L(\exp(sv)x_0) = \dim L(x_0), \]

on an integral curve $C$ passing through $x_0$.

For a point $x$ on $M$, we shall take such a system $\{w_1, \ldots, w_r\}$ of vector fields in $L$ that $w_i(x)$, $\ldots$, $w_r(x)$ are independent at $x$ and span $L(x)$. Let $L'$ be the
linear space spanned by these \( w_i \) in \( L \). We imbed some neighbourhood of the zero in \( L' \) into \( M \) by the mapping: \( w \rightarrow \exp (w)x \). Let \( N \) be its image, which is a submanifold of \( M \). Take two elements \( u \) and \( v \) of \( L' \). We put \( f(s, t) = \exp (t(su+v))x \). Then we claim that

\[
\left[ \frac{\partial f(s, t)}{\partial s} \right]_{s=0} = \int_0^t \phi_s(v)_*u \, dt. 
\]

The left hand side of (10) is a vector field on the curve \( f(0, t) \). To prove this identity we show that both sides of (10) satisfy the same ordinary differential equation

\[
\frac{dh}{dt} = u + h^i \frac{\partial v}{\partial x^i} 
\]

along the curve \( f(0, t) \). We take such a local coordinate system \( (x^i) \) around \( f(0, t) \) that we have \( v = \frac{\partial}{\partial x^i} \) with respect to this coordinate system. Then for sufficiently small \( \Delta t \), we get

\[
\phi_{t+\Delta t}(v)_*u(f(0, t+\Delta t)) = \phi_t(v)_*u(f(0, t)). 
\]

Hence we have

\[
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_0^{t+\Delta t} \phi_s(v)_*u(f(0, t+\Delta t)) \, dt - \int_0^t \phi_s(v)_*u(f(0, t)) \, dt \right\} 
= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{t-\Delta t}^t \phi_s(v)_*u(f(0, t+\Delta t)) \, dt = u(f(0, t)). 
\]

With respect to an arbitrary coordinate system \( (x^i) \), the right hand side of (10) satisfies the equation (11). The left hand side of (10) satisfies (11) along the curve \( f(0, t) \), because we have

\[
\frac{\partial}{\partial t} \left[ \frac{\partial f}{\partial s} \right]_{s=0} = \left[ \frac{\partial}{\partial s} \left( \frac{\partial f}{\partial t} \right) \right]_{s=0} = \left[ \frac{\partial}{\partial s} (su+v) \right]_{s=0}. 
\]

Since both sides of (10) have the same initial value 0 at \( t=0 \), we obtain the identity (10).

By the identity (10) we have

\[
\left[ \frac{\partial f(s, t)}{\partial s} \right]_{s=0} \in L(f(0, t)). 
\]

The tangent space of \( N \) at \( f(0, t) \) is spanned by the left hand side of (12), if \( u \) varies over all elements of \( L' \). Hence, by (9), we see that \( N \) is an integral manifold of \( L \).
Remark 1. Suppose $\dim L$ be finite. We shall show that $c(u, v; x) = \infty$ for every pair of $u$ and $v$ in $L$ and for every $x$ on $M$. Take a basis \( \{u_1, \ldots, u_r\} \) of $L$. We get
\[
u = c^i u_i \quad \text{and} \quad (\text{ad} v) u_i = c_i^j u_j,
\]
where $c^i$ and $c_i^j$ are real constants. We have
\[
(\text{ad} v)^k u = c^{i_0} c^{i_1}_{i_0} \cdots c^{i_k}_{i_{k-1}} u_{i_k},
\]
for $k = 0, 1, 2, \ldots$. Let $c$ be the maximum of $|c^i|$ and $|c_i^j|$, $(1 \leq h, i, j \leq r)$. We obtain the inequality
\[
|c^{i_0} c^{i_1}_{i_0} \cdots c^{i_k}_{i_{k-1}}| \leq c^{k+1} r^k.
\]
Hence $g_i(u, v)$ is expressed in the form $\sum_{i=1}^{r} a^i(t) u_i$, where $a^i(t)$ $(1 \leq i \leq r)$ is an entire function having a majorant series of the form $\sum_{k=0}^{\infty} c^{k+1} r^k t^k(k!)^{-1}$.

Our condition (C) is satisfied by $L$ in this case.

Remark 2. Let $M$ and $L(M)$ be of $C^\omega$-class. Then we have the identity (1) as a direct consequence of the fact that $\phi_t(v) u$ is real analytic with respect to $(t, x)$ at $(0, x)$. Our condition (C) is satisfied by every Lie subalgebra of $L(M)$.

Remark 3. We shall give an example of $L$ which is neither finite dimensional or real analytic, but satisfies (C). Let $M$ be $S^1 \times S^1$. Take a function $f(x)$ on $S^1$ which vanishes at infinitely many points, but does not vanish identically. We define $L$ as the Lie subalgebra generated by $f(x) \frac{\partial}{\partial x} + g(y) \frac{\partial}{\partial y}$, where $g$ varies over all real analytic functions on $S^1$. Then $L$ is neither finite dimensional or real analytic, but satisfies (C).

Remark 4. There exists a Lie subalgebra $L$ which does not satisfy our condition (C). Nagano [3] gave an example of $L$ to which our theorem can not be applied.

Osaka University

Bibliography
