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## AN INTEGRATION THEOREM FOR COMPLETELY INTEGRABLE SYSTEMS WITH SINGULARITIES

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Let  $M$  be a  $C^\infty$ -manifold. We denote the Lie algebra of all vector fields on  $M$  of  $C^\infty$ -class by  $L(M)$ . For two elements  $u$  and  $v$  of  $L(M)$ , defining  $(\text{ad } v)^k u$  inductively as  $[v, (\text{ad } v)^{k-1} u]$ , we consider a power series

$$g_t(u, v) = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\text{ad } v)^k u.$$

Let  $c(u, v; x)$  be the radius of convergence of  $g_t(u, v)$  at  $x$  on  $M$ . We consider a Lie subalgebra  $L$  of  $L(M)$  which satisfies the following convergence condition (C):

(C) For any pair of  $u$  and  $v$  in  $L$  and for any compact set  $K$  in  $M$ , there exists a positive number  $c(u, v; K)$  such that

- (i) we have  $c(u, v; x) \geq c(u, v; K)$  at every  $x$  on  $K$ , and
- (ii)  $g_t(u, v)$  is continuously differentiable with respect to  $(t, x)$  term by term at every  $(t, x)$  which satisfies  $|t| < c(u, v; K)$  and  $x \in K^\circ$ , the interior of  $K$ .

**Theorem.** *If a Lie subalgebra  $L$  satisfies the condition (C), then through every point  $x$  on  $M$  there passes a maximal integral manifold  $N(x)$  of  $L$ . Any integral manifold of  $L$  containing  $x$  is an open submanifold of  $N(x)$ .*

Here an integral manifold  $N$  of  $L$  is a connected submanifold of  $M$  which satisfies  $T_x(N) = L(x)$  at every  $x$  on  $N$ , where  $L(x) = \{u(x); u \in L\}$ .

The problem was solved under the following assumptions (i)~(iii) respectively by Chevalley, Hermann and Nagano:

- (i)  $\dim L(x)$  is constant on  $M$  (Frobenius' theorem, Chevalley [1]),
- (ii)  $\dim L$  is finite (Hermann [2]),
- (iii)  $M$  and  $L(M)$  are of  $C^\omega$ -class, but  $L$  is arbitrary (Nagano [3]).

If we assume (ii) or (iii), then  $L$  satisfies our condition (C) (see Remark 1 and Remark 2).

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**Proof of Theorem.** We shall prove only the local existence of an integral manifold of  $L$  passing through  $x$ , since the local uniqueness of integral manifolds and the existence of the maximal integral manifold can be proved in the same way as Nagano [3] and Chevalley [1].

Let  $U = \{(x^1, \dots, x^n); |x^i - x_0^i| < a\}$  be a relatively compact cubic neighbourhood of  $x_0 = (x_0^i)$  such that  $\phi_t(v)$  gives a diffeomorphism from  $U$  to  $\phi_t(v)U$ , if  $|t| < T(v, U)$ . Here  $\phi_t(v)$  is a local one-parameter group of diffeomorphisms generated by  $v$ , and  $T(v, U)$  is a positive number. By our assumption  $g_t(u, v)$  satisfies a symmetric hyperbolic partial differential equation

$$\frac{\partial h}{\partial t} + v^i \frac{\partial h}{\partial x^i} - h^i \frac{\partial v}{\partial x^i} = 0$$

at  $(t, x)$  which satisfies  $|t| < c(u, v; \bar{U})$  and  $x \in U$ , where  $v = v^i \frac{\partial}{\partial x^i}$ . Also  $\phi_t(v)_* u$  satisfies the same partial differential equation at such  $(t, x)$  that  $|t| < T(v, U)$  and  $x \in U$ . Since  $g_t(u, v)$  and  $\phi_t(v)_* u$  have the same initial value  $u$  at  $t=0$ , by the uniqueness theorem we obtain

$$(1) \quad \phi_t(v)_* u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\text{ad } v)^k u$$

at  $(t, x)$  such that  $|t| < \min. \{c(u, v; \bar{U}), T(v, U)\}$  and

$$A|t| + \sqrt{\sum_{i=1}^{\infty} (x^i - x_0^i)^2} < a,$$

where  $A = \max. \sqrt{\sum_{i=1}^{\infty} v^i(x)^2}$  on  $\bar{U}$ .

If  $v(x_0) \neq 0$ , we may assume that  $v = \frac{\partial}{\partial x^1}$  in  $U$ . Then from the identity (1) we get

$$(2) \quad u(x(\tau - t)) = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} u^{(k)}(x(\tau)),$$

where  $x(\tau) = (x_0^1 + \tau, x_0^2, \dots, x_0^n)$  and  $u^{(k)} = \frac{\partial^k u}{\partial (x^1)^k}$ . This identity (2) holds for  $(t, \tau)$  such that

$$(3) \quad |t| + |\tau| < a, \quad |t| < \min. \left\{ \frac{a}{2}, c(u, v; \bar{U}) \right\}.$$

As a function of  $\tau$ ,  $u(x(\tau))$  is real analytic in the interval  $(-a, +a)$ . Hence we have

$$(4) \quad u^{(l)}(x(\tau - t)) = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} u^{(k+l)}(x(\tau))$$

for  $(t, \tau)$  which satisfies (3) and for every  $l \in \mathbb{Z}_+$ . From this identity we get

$$(5) \quad \phi_t(v)_*(\text{ad } v)^t u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\text{ad } v)^{t+k} u$$

at  $x(\tau)$  for every  $t$  which satisfies (3) and for every  $l \in Z_+$ .

Let us consider an integral curve  $C$  passing through  $x_0$ . Take a point  $y(s) = \exp(sv)x_0$  on  $C$ . Then there exists such a positive number  $\sigma$  that we have

$$(6) \quad \phi_\sigma(v)_*(\text{ad } v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\text{ad } v)^{l+k} u$$

at every  $y(s')$  on  $C$  between  $x_0$  and  $y(s)$  and for every  $l \in Z_+$ . We may assume that  $s = m\sigma$  for a positive integer  $m$ .

Operating  $\phi_{(m-1)\sigma}(v)_*$  on the identity (6) at  $y(\sigma)$ , we get

$$\phi_{m\sigma}(v)_*(\text{ad } v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \phi_{(m-1)\sigma}(v)_*(\text{ad } v)^{l+k} u$$

at  $y(m\sigma)$  for every  $l \in Z_+$ . Then operating  $\phi_{(m-2)\sigma}(v)_*$  on the identity (6) at  $y(2\sigma)$ , we have

$$\phi_{(m-1)\sigma}(v)_*(\text{ad } v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \phi_{(m-2)\sigma}(v)_*(\text{ad } v)^{l+k} u$$

at  $y(m\sigma)$  for every  $l \in Z_+$ . Thus we obtain

$$\phi_{(m-n)\sigma}(v)_*(\text{ad } v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \phi_{(m-n-1)\sigma}(v)_*(\text{ad } v)^{l+k} u$$

at  $y(m\sigma)$  for such  $(n, l)$  that  $0 \leq n \leq m-1$  and  $l \in Z_+$ . In particular for  $n = m-1$ , we have

$$\phi_\sigma(v)_*(\text{ad } v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\text{ad } v)^{l+k} u$$

at  $y(m\sigma)$  for every  $l \in Z_+$ . Hence inductively we obtain

$$(7) \quad \phi_{(m-n)\sigma}(v)_*(\text{ad } v)^l u \in L(\exp(sv)x_0), \quad (0 \leq n \leq m-1)$$

for every  $l \in Z_+$ . In particular for  $n = l = 0$ , we get

$$(8) \quad \phi_s(v)_* u \in L(\exp(sv)x_0).$$

Since  $u$  is arbitrary in  $L$ , we have

$$(9) \quad \dim L(\exp(sv)x_0) = \dim L(x_0),$$

on an integral curve  $C$  passing through  $x_0$ .

For a point  $x$  on  $M$ , we shall take such a system  $\{w_1, \dots, w_r\}$  of vector fields in  $L$  that  $w_1(x), \dots, w_r(x)$  are independent at  $x$  and span  $L(x)$ . Let  $L'$  be the

linear space spanned by these  $w_i$  in  $L$ . We imbed some neighbourhood of the zero in  $L'$  into  $M$  by the mapping:  $w \rightarrow \exp(w)x$ . Let  $N$  be its image, which is a submanifold of  $M$ . Take two elements  $u$  and  $v$  of  $L'$ . We put  $f(s, t) = \exp(t(su + v))x$ . Then we claim that

$$(10) \quad \left[ \frac{\partial f(s, t)}{\partial s} \right]_{s=0} = \int_0^t \phi_\tau(v)_* u d\tau.$$

The left hand side of (10) is a vector field on the curve  $f(0, t)$ . To prove this identity we show that both sides of (10) satisfy the same ordinary differential equation

$$(11) \quad \frac{dh}{dt} = u + h^i \frac{\partial v}{\partial x^i}$$

along the curve  $f(0, t)$ . We take such a local coordinate system  $(z^i)$  around  $f(0, t)$  that we have  $v = \frac{\partial}{\partial z^1}$  with respect to this coordinate system. Then for sufficiently small  $\Delta t$ , we get

$$\phi_{t+\Delta t}(v)_* u(f(0, t+\Delta t)) = \phi_t(v)_* u(f(0, t)).$$

Hence we have

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \int_0^{t+\Delta t} \phi_\tau(v)_* u(f(0, t+\Delta t)) d\tau - \int_0^t \phi_\tau(v)_* u(f(0, t)) d\tau \right\} \\ = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\Delta t}^0 \phi_\tau(v)_* u(f(0, t+\Delta t)) d\tau = u(f(0, t)). \end{aligned}$$

With respect to an arbitrary coordinate system  $(x^i)$ , the right hand side of (10) satisfies the equation (11). The left hand side of (10) satisfies (11) along the curve  $f(0, t)$ , because we have

$$\frac{\partial}{\partial t} \left[ \frac{\partial f}{\partial s} \right]_{s=0} = \left[ \frac{\partial}{\partial s} \left( \frac{\partial f}{\partial t} \right) \right]_{s=0} = \left[ \frac{\partial}{\partial s} (su + v) \right]_{s=0}.$$

Since both sides of (10) have the same initial value 0 at  $t=0$ , we obtain the identity (10).

By the identity (10) we have

$$(12) \quad \left[ \frac{\partial f(s, t)}{\partial s} \right]_{s=0} \in L(f(0, t)).$$

The tangent space of  $N$  at  $f(0, t)$  is spanned by the left hand side of (12), if  $u$  varies over all elements of  $L'$ . Hence, by (9), we see that  $N$  is an integral manifold of  $L$ ,

REMARK 1. Suppose  $\dim L$  be finite. We shall show that  $c(u, v; x) = \infty$  for every pair of  $u$  and  $v$  in  $L$  and for every  $x$  on  $M$ . Take a basis  $\{u_1, \dots, u_r\}$  of  $L$ . We get

$$u = c^i u_i \quad \text{and} \quad (\text{ad } v)u_i = c_j^i u_j,$$

where  $c^i$  and  $c_j^i$  are real constants. We have

$$(\text{ad } v)^k u = c^{i_0} c_{i_0}^{i_1} \dots c_{i_{k-1}}^{i_k} u_{i_k},$$

for  $k=0, 1, 2, \dots$ . Let  $c$  be the maximum of  $|c^h|$  and  $|c_i^j|$ , ( $1 \leq h, i, j \leq r$ ). We obtain the inequality

$$|c^{i_0} c_{i_0}^{i_1} \dots c_{i_{k-1}}^{i_k}| \leq c^{k+1} r^k.$$

Hence  $g_t(u, v)$  is expressed in the form  $\sum_{i=1}^r a^i(t) u_i$ , where  $a^i(t)$  ( $1 \leq i \leq r$ ) is an entire function having a majorant series of the form  $\sum_{k=0}^{\infty} c^{k+1} r^k t^k (k!)^{-1}$ .

Our condition (C) is satisfied by  $L$  in this case.

REMARK 2. Let  $M$  and  $L(M)$  be of  $C^\infty$ -class. Then we have the identity (1) as a direct consequence of the fact that  $\phi_t(v)_* u$  is real analytic with respect to  $(t, x)$  at  $(0, x)$ . Our condition (C) is satisfied by every Lie subalgebra of  $L(M)$ .

REMARK 3. We shall give an example of  $L$  which is neither finite dimensional or real analytic, but satisfies (C). Let  $M$  be  $S^1 \times S^1$ . Take a function  $f(x)$  on  $S^1$  which vanishes at infinitely many points, but does not vanish identically. We define  $L$  as the Lie subalgebra generated by  $f(x) \frac{\partial}{\partial x} + g(y) \frac{\partial}{\partial y}$ , where  $g$  varies over all real analytic functions on  $S^1$ . Then  $L$  is neither finite dimensional or real analytic, but satisfies (C).

REMARK 4. There exists a Lie subalgebra  $L$  which does not satisfy our condition (C). Nagano [3] gave an example of  $L$  to which our theorem can not be applied.

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