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AN INTEGRATION THEOREM FOR COMPLETELY INTEGRABLE SYSTEMS WITH SINGULARITIES

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Let M be a C^∞-manifold. We denote the Lie algebra of all vector fields on M of C^∞-class by $L(M)$. For two elements u and v of $L(M)$, defining $(\mathrm{ad}\, v)^k u$ inductively as $[v, (\mathrm{ad}\, v)^{k-1} u]$, we consider a power series

$$g_t(u, v) = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\mathrm{ad}\, v)^k u .$$

Let $c(u, v; x)$ be the radius of convergence of $g_t(u, v)$ at x on M. We consider a Lie subalgebra L of $L(M)$ which satisfies the following convergence condition (C):

(C) For any pair of u and v in L and for any compact set K in M, there exists a positive number $c(u, v; K)$ such that

(i) we have $c(u, v; x) \geqq c(u, v; K)$ at every x on K, and

(ii) $g_t(u, v)$ is continuously differentiable with respect to (t, x) term by term at every (t, x) which satisfies $|t| < c(u, v; K)$ and $x \in K^i$, the interior of K.

Theorem. *If a Lie subalgebra L satisfies the condition (C), then through every point x on M there passes a maximal integral manifold $N(x)$ of L. Any integral manifold of L containing x is an open snbmanifold of $N(x)$.*

Here an integral manifold N of L is a connected submanifold of M which satisfies $T_x(N) = L(x)$ at every x on N, where $L(x) = \{u(x); u \in L\}$.

The problem was solved under the following assumptions (i)~(iii) respectively by Chevalley, Hermann and Nagano:

(i) dim $L(x)$ is constant on M (Frobenius' theorem, Chevalley [1]),

(ii) dim L is finite (Hermann [2]),

(iii) M and $L(M)$ are of C^ω-class, but L is arbitrary (Nagano [3]).

If we assume (ii) or (iii), then L satisfies our condition (C) (see Remark 1 and Remark 2).

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Proof of Theorem. We shall prove only the local existence of an integral manifold of L passing through x, since the local uniqueness of integral manifolds and the existence of the maximal integral manifold can be proved in the same way as Nagano [3] and Chevalley [1].

Let $U=\{(x^1, \cdots, x^n); |x^i-x_0^i|<a\}$ be a relatively compact cubic neighbourhood of $x_0=(x_0^i)$ such that $\phi_t(v)$ gives a diffeomorphism from U to $\phi_t(v)U$, if $|t|<T(v, U)$. Here $\phi_t(v)$ is a local one-parameter group of diffeomorphisms generated by v, and $T(v, U)$ is a positive number. By our assumption $g_t(u, v)$ satisfies a symmetric hyperbolic partial differential equation

$$\frac{\partial h}{\partial t}+v^i\frac{\partial h}{\partial x^i}-h^i\frac{\partial v}{\partial x^i}=0$$

at (t, x) which satisfies $|t|<c(u, v; \bar{U})$ and $x\in U$, where $v=v^i\frac{\partial}{\partial x^i}$. Also $\phi_t(v)_*u$ satisfies the same partial differential equation at such (t, x) that $|t|<T(v, U)$ and $x\in U$. Since $g_t(u, v)$ and $\phi_t(v)_*u$ have the same initial value u at $t=0$, by the uniqueness theorem we obtain

$$(1) \qquad \phi_t(v)_*u=\sum_{k=0}^{\infty}(-1)^k\frac{t^k}{k!}(\mathrm{ad}\, v)^k u$$

at (t, x) such that $|t|<\min.\{c(u, v; \bar{U}), T(v, U)\}$ and

$$A|t|+\sqrt{\sum_{i=1}^{\infty}(x^i-x_0^i)^2}<a,$$

where $A=\max.\sqrt{\sum_{i=1}^{\infty}v^i(x)^2}$ on $\bar{U}$.

If $v(x_0)\neq 0$, we may assume that $v=\frac{\partial}{\partial x^1}$ in U. Then from the identity (1) we get

$$(2) \qquad u(x(\tau-t))=\sum_{k=0}^{\infty}(-1)^k\frac{t^k}{k!}u^{(k)}(x(\tau)),$$

where $x(\tau)=(x_0^1+\tau, x_0^2, \cdots, x_0^n)$ and $u^{(k)}=\frac{\partial^k u}{\partial(x^1)^k}$. This identity (2) holds for (t, τ) such that

$$(3) \qquad |t|+|\tau|<a, \quad |t|<\min.\left\{\frac{a}{2}, c(u, v; \bar{U})\right\}.$$

As a function of τ, $u(x(\tau))$ is real analytic in the interval $(-a, +a)$. Hence we have

$$(4) \qquad u^{(l)}(x(\tau-t))=\sum_{k=0}^{\infty}(-1)^k\frac{t^k}{k!}u^{(k+l)}(x(\tau))$$

for (t, τ) which satisfies (3) and for every $l\in Z_+$. From this identity we get

$$(5) \qquad \phi_t(v)_*(\mathrm{ad}\, v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\mathrm{ad}\, v)^{l+k} u$$

at $x(\tau)$ for every t which satisfies (3) and for every $l \in Z_+$.

Let us consider an integral curve C passing through x_0. Take a point $y(s) = \exp(sv)x_0$ on C. Then there exists such a positive number σ that we have

$$(6) \qquad \phi_\sigma(v)_*(\mathrm{ad}\, v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\mathrm{ad}\, v)^{l+k} u$$

at every $y(s')$ on C between x_0 and $y(s)$ and for every $l \in Z_+$. We may assume that $s = m\sigma$ for a positive integer m.

Operating $\phi_{(m-1)\sigma}(v)_*$ on the identity (6) at $y(\sigma)$, we get

$$\phi_{m\sigma}(v)_*(\mathrm{ad}\, v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \phi_{(m-1)\sigma}(v)_*(\mathrm{ad}\, v)^{l+k} u$$

at $y(m\sigma)$ for every $l \in Z_+$. Then operating $\phi_{(m-2)\sigma}(v)_*$ on the identity (6) at $y(2\sigma)$, we have

$$\phi_{(m-1)\sigma}(v)_*(\mathrm{ad}\, v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \phi_{(m-2)\sigma}(v)_*(\mathrm{ad}\, v)^{l+k} u$$

at $y(m\sigma)$ for every $l \in Z_+$. Thus we obtain

$$\phi_{(m-n)\sigma}(v)_*(\mathrm{ad}\, v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \phi_{(m-n-1)\sigma}(v)_*(\mathrm{ad}\, v)^{l+k} u$$

at $y(m\sigma)$ for such (n, l) that $0 \leqq n \leqq m-1$ and $l \in Z_+$. In particular for $n = m-1$, we have

$$\phi_\sigma(v)_*(\mathrm{ad}\, v)^l u = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (\mathrm{ad}\, v)^{l+k} u$$

at $y(m\sigma)$ for every $l \in Z_+$. Hence inductively we obtain

$$(7) \qquad \phi_{(m-n)\sigma}(v)_*(\mathrm{ad}\, v)^l u \in L(\exp(sv)x_0)\,, \qquad (0 \leqq n \leqq m-1)$$

for every $l \in Z_+$. In particular for $n = l = 0$, we get

$$(8) \qquad \phi_s(v)_* u \in L(\exp(sv)x_0)\,.$$

Since u is arbitrary in L, we have

$$(9) \qquad \dim L(\exp(sv)x_0) = \dim L(x_0)\,,$$

on an integral curve C passing through x_0.

For a point x on M, we shall take such a system $\{w_1, \cdots, w_r\}$ of vector fields in L that $w_1(x), \cdots, w_r(x)$ are independent at x and span $L(x)$. Let L' be the linear space spanned by these w_i in L. We imbed some neighbourhood of the zero in L' into M by the mapping: $w \rightarrow \exp(w)x$. Let N be its image, which is a submanifold of M. Take two elements u and v of L'. We put $f(s, t) = \exp(t(su+v))x$. Then we claim that

$$\left[\frac{\partial f(s, t)}{\partial s}\right]_{s=0} = \int_0^t \phi_\tau(v)_* u \, d\tau \,. \tag{10}$$

The left hand side of (10) is a vector field on the curve $f(0, t)$. To prove this identity we show that both sides of (10) satisfy the same ordinary differential equation

$$\frac{dh}{dt} = u + h^i \frac{\partial v}{\partial x^i} \tag{11}$$

along the curve $f(0, t)$. We take such a local coordinate system (z^i) around $f(0, t)$ that we have $v = \frac{\partial}{\partial z^1}$ with respect to this coordinate system. Then for sufficiently small Δt, we get

$$\phi_{t+\Delta t}(v)_* u(f(0, t+\Delta t)) = \phi_t(v)_* u(f(0, t)) \,.$$

Hence we have

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_0^{t+\Delta t} \phi_\tau(v)_* u(f(0, t+\Delta t)) \, d\tau - \int_0^t \phi_\tau(v)_* u(f(0, t)) \, d\tau \right\}$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\Delta t}^0 \phi_\tau(v)_* u(f(0, t+\Delta t)) \, d\tau = u(f(0, t)) \,.$$

With respect to an arbitrary coordinate system (x^i), the right hand side of (10) satisfies the equation (11). The left hand side of (10) satisfies (11) along the curve $f(0, t)$, because we have

$$\frac{\partial}{\partial t}\left[\frac{\partial f}{\partial s}\right]_{s=0} = \left[\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right)\right]_{s=0} = \left[\frac{\partial}{\partial s}(su+v)\right]_{s=0} .$$

Since both sides of (10) have the same initial value 0 at $t=0$, we obtain the identity (10).

By the identity (10) we have

$$\left[\frac{\partial f(s, t)}{\partial s}\right]_{s=0} \in L(f(0, t)) \,. \tag{12}$$

The tangent space of N at $f(0, t)$ is spanned by the left hand side of (12), if u varies over all elements of L'. Hence, by (9), we see that N is an integral manifold of L.

REMARK 1. Suppose $\dim L$ be finite. We shall show that $c(u, v; x)=\infty$ for every pair of u and v in L and for every x on M. Take a basis $\{u_1, \cdots, u_r\}$ of L. We get

$$u = c^i u_i \quad \text{and} \quad (\operatorname{ad} v) u_i = c_i^j u_j ,$$

where c^i and $c_j{}^i$ are real constants. We have

$$(\operatorname{ad} v)^k u = c^{i_0} c_{i_0}^{i_1} \cdots c_{i_{k-1}}^{i_k} u_{i_k} ,$$

for $k=0, 1, 2, \cdots$. Let c be the maximum of $|c^h|$ and $|c_i^j|$, $(1 \leqq h, i, j \leqq r)$. We obtain the inequality

$$\left| c^{i_0} c_{i_0}^{i_1} \cdots c_{i_{k-1}}^{i_k} \right| \leqq c^{k+1} r^k .$$

Hence $g_t(u, v)$ is expressed in the form $\sum_{i=1}^{r} a^i(t) u_i$, where $a^i(t)$ $(1 \leqq i \leqq r)$ is an entire function having a majorant series of the form $\sum_{k=0}^{\infty} c^{k+1} r^k t^k (k!)^{-1}$.

Our condition (C) is satisfied by L in this case.

REMARK 2. Let M and $L(M)$ be of C^ω-class. Then we have the identity (1) as a direct consequence of the fact that $\phi_t(v)_* u$ is real analytic with respect to (t, x) at $(0, x)$. Our condition (C) is satisfied by every Lie subalgebra of $L(M)$.

REMARK 3. We shall give an example of L which is neither finite dimensional or real analytic, but satisfies (C). Let M be $S^1 \times S^1$. Take a function $f(x)$ on S^1 which vanishes at infinitely many points, but does not vanish identically. We define L as the Lie subalgebra generated by $f(x)\frac{\partial}{\partial x} + g(y)\frac{\partial}{\partial y}$, where g varies over all real analytic functions on S^1. Then L is neither finite dimensional or real analytic, but satisfies (C).

REMARK 4. There exists a Lie subalgebra L which does not satisfy our condition (C). Nagano [3] gave an example of L to which our theorem can not be applied.

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Bibliography

[1] C. Chevalley: Theory of Lie groups I, Chap. III, Princeton Univ. Press, 1946.

[2] R. Hermann: *The differential geometry of foliations* II, J. Math. Mech. **11** (1962), 303–315.

[3] T. Nagano: *Linear differential systems with singularities and an application to transitive Lie algebras*, J. Math. Soc. Japan **18** (1966), 398–404.