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Author(s)	Maruo, Kenji
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EXISTENCE OF SOLUTIONS OF SOME NONLINEAR WAVE EQUATIONS

KENJI MARUO

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0. Introduction and Theorem

Let H be a real Hilbert space and A be a positive self adjoint operator in H . Let ϕ be a lower semi continuous proper convex function from H to $(-\infty, \infty]$ and $\partial\phi$ be the subdifferential of ϕ . Then we shall consider the following equation

$$(0.1) \quad \begin{cases} \frac{d^2}{dt^2}u + Au + \partial\phi u \ni f(\cdot, u) \\ u(0) = a, \quad \frac{d}{dt}u(0) = b \quad \text{on } [0, T] \end{cases}$$

where T is a positive number.

The above equation was studied in Schatzman [3], [4], [5] and Maruo [2]. In this paper we prove the existence of a solution of the problem (0.1) under certain assumptions which are somewhat weaker than those of Schatzman [5] and Maruo [2].

In [5] Schatzman showed the existence and uniqueness of a solution of the following nonlinear wave equation

$$(0.2) \quad \begin{cases} \left(\left(\frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u \right) (u-r) = 0, \quad \frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u \geq 0 \right. \\ \text{in the sense of distributions in } [0, 1] \times [0, T], \\ u(x, t) \geq r(x), \quad u(x, 0) = u_0(x), \quad \text{for } x \in [0, 1], \\ \frac{\partial}{\partial t}u(x, t) = u_1(x) \quad \text{a.e. in } [0, 1], \\ \left. u(0, t) = u(1, t) = 0 \quad \text{for } t \in [0, T], \right\} \end{cases}$$

where r is a continuous given function such that $r(0) < 0$, $r(1) < 0$ and $\frac{d^2}{dx^2}r(x) \geq 0$ (in the distribution sense). Set $K = \{f \in L_2(0, 1); f(x) \geq r(x)\}$. The equation (0.2) is rewritten as the following equation in $L_2(0, 1)$

$$(0.3) \quad \begin{cases} \frac{d^2}{dt^2}u + Au + \partial I_K u \ni 0 \\ u(0) = u_0, \quad \frac{d}{dt}u(0) = u_1 \end{cases}$$

where $I(u) = \begin{cases} 0 & \text{if } u \in K \\ \infty & \text{if } u \notin K \end{cases}$ and

$$A = -\frac{d^2}{dx^2} \quad (\text{Dirichlet problem}).$$

We will show that we can apply our main theorem to this equation if only r is a continuous function satisfying $r(0) < 0$, $r(1) < 0$ to deduce the existence of a solution of (0.3). The solution of (0.3), however, does not always satisfy the locally energy conserving condition (see [5]). Hence we cannot get the uniqueness of a solution.

Let Ω be a domain in R^n with smooth boundary and consider the case $\phi(u) = \int_{\Omega} |u(x)|^{p+1} dx$, $p > 1$. Then the equation (0.1) represents a nonlinear Klein-Gordon equation. It will be shown that if $(n+2) > p(n-2)$, the result of this paper can be applied to (0.1) in this case. Note that when $n=3$ this inequality is satisfied for $p=3$.

Now we state notations which will be used throughout this paper. The inner product of H is denoted by (\cdot, \cdot) . When S is a Banach space, its norm is denoted by $|\cdot|_S$. We denote by V the domain of $A^{1/2}$ endowed with the graph norm. By $\partial\phi_\lambda$ and ϕ_λ we denote the Yosida approximations of $\partial\phi$ and ϕ respectively (i.e. $\partial\phi_\lambda x = \lambda^{-1}(1 - J_\lambda)^{-1}x$ and $\phi_\lambda(x) = (2\lambda)^{-1}|x - J_\lambda x|_H^2 + \phi(J_\lambda x)$ where $J_\lambda = (1 + \lambda\partial\phi)^{-1}$).

Next we shall introduce the assumptions.

Let X_1 and X_2 be real Banach spaces.

ASSUMPTION 1. The following inclusion relations hold:

$$V \subset X_1 \subset H \subset X_2 \quad \text{and} \quad X_1 \subset \{\text{the dual space of } X_2\}$$

where each inclusion mapping is continuous. Moreover X_1 is separable and the inclusion mapping from V to X_1 is compact. H is dense in X_2 .

ASSUMPTION 2. There exists $z \in V$ such that

$$(\partial\phi_\lambda x, x - z) \geq c_1 |\partial\phi_\lambda x|_{X_2} - c_2$$

for $x \in V$, $|x|_V \leq R$ and $|\phi(x)| \leq R$ where c_1 and c_2 are positive constants depending only on R and z .

ASSUMPTION 3. The continuous function f from $[0, T] \times H$ to H satisfies

for any $t \in [0, T]$ and $x, y \in H$

$$\begin{aligned} |f(t, x) - f(t, y)|_H &\leq h(t) |x - y|_H, \\ \left| \frac{d}{dt} f(t, x) \right|_H &\leq h(t) (1 + |x|_H) \end{aligned}$$

where h is a function belonging to $L_1(0, T)$.

ASSUMPTION 4. The closure of $D(\phi) \cap V$ in H is equal to the closure of $D(\phi)$ in H .

Clearly V and X_1 are dense in H . By assumption H is dense in X_2 . We use the same notation (\cdot, \cdot) as the inner product of H to denote the pairings between V , X_1 , X_2 and their corresponding duals.

Now we define the solution of (0.1).

DEFINITION. We say that a function $u \in C([0, T]; X_1) \cap W_\infty^1(0, T; H)$ is a solution of the equation (0.1) when it satisfies the following requirements;

- 1) For any $t \in [0, T]$ $u(t) \in D(\phi) \cap V$.
- 2) There exist weak right and left derivatives $\frac{d^\pm}{dt} u(t) \in H$ for any $t \in [0, T]$.

Moreover for any $t \in [0, T]$

$$\begin{aligned} \left| \frac{d^\pm}{dt} u(t) \right|_H^2 + |u(t)|_V^2 + 2\phi(u(t)) &\leq |b|_H^2 + |a|_V^2 + 2\phi(a) \\ &+ 2 \int_0^t (f(s, u(s)), \frac{d}{ds} u(s)) ds \end{aligned}$$

(with necessary modifications at 0 and T).

- 3) There exists a linear functional F on $C([0, T]; X_1)$ such that

$$F(v - u) \leq \int_0^T \phi(v(s)) ds - \int_0^T \phi(u(s)) ds$$

for any $v \in C([0, T]; X_1)$ and

$$\begin{aligned} \int_0^T \left(\frac{d}{ds} u(s), \frac{d}{ds} v(s) \right) ds + \int_0^T (f(s, u(s)) - Au(s), v(s)) ds \\ + (b, v(0)) - \left(\frac{d^-}{dt} u(T), v(T) \right) = F(v) \end{aligned}$$

for any $v \in C([0, T]; X_1) \cap L_1(0, T; V) \cap W_\infty^1(0, T; H)$.

- 4) The initial conditions are satisfied in the following sense

$$u(0) = a, b - \frac{d}{dt} u(0) \in \partial I_{K_0} a$$

where K_0 is the closure of the domain of ϕ , I_{K_0} is the indicator function of

K_0 and ∂I_{K_0} is the subdifferential of I_{K_0} .

We state the theorem.

Theorem. *Let a and b be given elements satisfying*

$$a \in V \cap D(\phi), \quad b \in H.$$

Then under the assumptions 1, 2, 3 and 4 we have at least one solution of (0.1).

To prove the above theorem we consider the following approximate equations for $\lambda > 0$

$$(0.4) \quad \begin{cases} \frac{d^2}{dt^2} u_\lambda + A u_\lambda + \partial \phi_\lambda u_\lambda = f(\cdot, u_\lambda) \\ u_\lambda(0) = a, \quad \frac{d}{dt} u_\lambda(0) = b. \end{cases}$$

In the next section using a method similar to that of [2] we shall investigate the convergence of the solutions of the approximate equations (0.4). In section 2 we prove the theorem. In section 3 we show some examples.

1. Convergence of approximate solutions

In this section under the assumptions 1, 2, 3 and 4 we shall study the convergence of the solutions of (0.4). In what follows let initial values a and b belong to $V \cap D(\phi)$ and H respectively.

First we show some properties of the approximate solutions.

Lemma 1. *For any $\lambda > 0$ we have solutions of the problem (0.4) such that*

$$u_\lambda \in C([0, T]; H) \cap L_\infty(0, T; V) \cap W_\infty^1(0, T; H) \cap W_\infty^2(0, T; V^*)$$

where V^* is the dual space of V .

Proof. See p. 289 Barbu [1].

Lemma 2. *We hold the following equality and inequality*

$$\begin{aligned} 1) \quad & \left| \frac{d}{dt} u_\lambda(t) \right|_H^2 + |u_\lambda(t)|_V^2 + 2\phi_\lambda(u_\lambda(t)) \\ & = |b|_H^2 + |a|_V^2 + 2\phi_\lambda(a) + 2 \int_0^t (f(s, u_\lambda(s)), \frac{d}{ds} u_\lambda(s)) ds \\ 2) \quad & \left| \frac{d}{dt} u_\lambda(t) \right|_H^2 + |u_\lambda(t)|_V^2 + 2\phi_\lambda(u_\lambda(t)) \\ & \leq C_1(|b|_H^2 + |a|_H^2 + |a|_V^2 + 1) \end{aligned}$$

where C_1 is a constant depending only on h and T .

Proof. See Lemma 2.2 in [2].

Lemma 3. *There exists a constant independent of λ such that*

$$\int_0^T |\partial \phi_\lambda u_\lambda(s)|_{x_2} ds \leq \text{Constant}.$$

Proof. In the inequality of assumption 2 we put $x=u_\lambda(t)$. From Lemma 2 the constants c_1 and c_2 are independent of λ . Replacing $\partial \phi_\lambda u$ by $-(u_\lambda'' + Au_\lambda - f(\cdot, u_\lambda))$, integrating over $[0, T]$, using the integration by parts and noting 2) of Lemma 2 we get the conclusion of the Lemma.

Lemma 4. *We have a continuous function u from $[0, T]$ to H such that a subsequence $\{u_{\lambda_j}\}$ of the sequence $\{u_\lambda\}$ converges uniformly to u in H as $\lambda_j \rightarrow 0$.*

Proof. In view of 2) of Lemma 2 $|u_\lambda(t)|_V$ is uniformly bounded. Hence from the assumption 1 we know that $\{u_\lambda(t)\}$ is a relatively compact subset of H for any $t \in [0, T]$. From 2) of Lemma 2 $\{u_\lambda\}$ is uniformly continuous. Thus using Ascoli-Arzelà's theorem this lemma is proved.

For simplicity we denote this subsequence by $\{u_\lambda\}$.

Lemma 5. *There exists a subsequence $\{\lambda_j\}$ of $\{\lambda\}$ such that $\{u_{\lambda_j}\}$ converges to u in $C([0, T]; X_1)$, $\left\{\frac{d}{dt} u_{\lambda_j}\right\}$ converges to $\frac{d}{dt} u$ in weak*- $L_\infty(0, T; H)$ and $\{u_{\lambda_j}(t)\}$ weakly converges to $u(t)$ in V for any $t \in [0, T]$. Hence we know that*

$$u \in C([0, T]; X_1) \cap W_\infty^1(0, T; H)$$

and

$$u(t) \in D(\phi) \cap V \quad \text{for any } t \in [0, T].$$

Proof. From 2) of Lemma 2 it is easy to prove that some subsequence $\left\{\frac{d}{dt} u_{\lambda_j}\right\}$ converges to $\frac{d}{dt} u$ in weak*- $L_\infty(0, T; H)$. Since $\{u_{\lambda_j}(t)\}$ is bounded in V it follows from Lemma 4 that $\{u_{\lambda_j}(t)\}$ weakly converges to $u(t)$ in V . Next we assume that there exists a sequence $\{t_i\}_{i=1}^\infty$ such that

$$(1.1) \quad \begin{cases} \lim_{i \rightarrow \infty} t_i = t_\infty, \\ |u(t_i) - u(t_\infty)|_{x_1} \geq \delta_0 > 0 \quad \text{and } t_i \in [0, T] \\ \text{for } i = 1, 2, \dots \end{cases}$$

Since $|u(t_i)|_V$ is bounded there exists a subsequence $\{u(t_{i_j})\}$ which converges

to some element w of X_1 . On the other hand u is continuous in H . Hence $w = u(t_\infty)$. The above results contradict (1.1). Thus u is continuous in X_1 . Combining that for any $t \in [0, T]$ $\{u_{\lambda_j}(t)\}$ is a relatively compact set in X_1 , that $\{u_{\lambda_j}\}$ is uniformly convergent to u in H and that u is continuous in X_1 we can prove that $\{u_{\lambda_j}\}$ is uniformly convergent to u in X_1 .

For simplicity we denote the subsequence $\{\lambda_j\}$ by $\{\lambda\}$.

We denote $\int_0^t \partial \phi_\lambda u_\lambda(t) dt$ by $\rho_\lambda(t)$ for any $t \in [0, T]$. Then ρ_λ belongs to $W_\infty^1([0, T]; H)$.

Lemma 6. *There exists a subsequence $\{\lambda_j\} \subset \{\lambda\}$ such that for $\alpha \in X_1$ and $t \in [0, T]$, $\{(\rho_{\lambda_j}(t), \alpha)\}$ converges.*

Proof. From Lemma 3 for any $\alpha \in X_1$ we know that the total variation of the function $\{(\rho_\lambda(t), \alpha)\}$ on $[0, T]$ is uniformly bounded in λ . Noting that X_1 is separable and using Helly's choice theorem and the diagonal method we have a subsequence $\{\lambda_j\}$ such that

$$\lim_{\lambda_j \rightarrow 0} (\rho_{\lambda_j}(t), \alpha) \text{ exists for any } \alpha \in X_1.$$

For simplicity we denote $\{\lambda_j\}$ by $\{\lambda\}$.

Put

$$\int_0^t (\partial \phi_\lambda u_\lambda(s), v(s)) ds = F_{\lambda,t}(v) \text{ for } v \in C([0, T]; X_1) \text{ and } t \in [0, T].$$

Lemma 7. *For each $t \in [0, T]$ there exists a linear continuous functional F_t on $C([0, T]; X_1)$ such that*

$$\lim_{\lambda \rightarrow 0} F_{\lambda,t}(v) = F_t(v) \text{ for any } v \in C([0, T]; X_1).$$

Proof. Combining Lemmas 3 and 6 and approximating v by step functions we can prove this lemma.

Lemma 8. *For any $v \in C([0, T]; X_1)$ there exist*

$$\text{right } \lim_{t \rightarrow t_0} F_t(v) \text{ and } \text{left } \lim_{t \rightarrow t_0} F_t(v)$$

where $t_0 \in [0, T]$ (with necessary modifications at 0 and T).

Proof. From Lemma 7 it follows

$$|F_t(v) - F_s(v)| \leq \lim_{\lambda \rightarrow 0} \int_s^t |\partial \phi_\lambda u_\lambda(\tau)|_{X_2} d\tau \cdot \sup_{0 \leq \tau \leq T} |v(\tau)|_{X_1}.$$

Combining this inequality with Lemma 3 we see that $F_t(v)$ is of bounded variation. Thus the lemma is proved.

We put

$$\lim_{t \rightarrow T} F_t(v) = F(v).$$

Lemma 9. *For any $t \in [0, T]$ there exist weak right and left derivatives $\frac{d^\pm}{dt} u(t)$ in H (with necessary modifications at 0 and T).*

Proof. Let v be an arbitrary element of $C^1([0, T]; X_1) \cap C([0, T]; V)$. Forming the inner product of (0.4) and v and integrating by parts we get

$$(1.2) \quad \left(\frac{d}{dt} u_\lambda(t), v(t) \right) = (b, v(0)) + \int_0^t \left(\frac{d}{ds} u_\lambda(s), \frac{d}{ds} v(s) \right) ds \\ + \int_0^t (f(s, u_\lambda(s)) - Au_\lambda(s), v(s)) ds - F_{\lambda,t}(v).$$

From Lemmas 4, 5 and 7, the right side of the above equality converges for any $t \in [0, T]$ as $\lambda \rightarrow 0$. Since $\left\{ \frac{d}{dt} u_\lambda(t) \right\}$ is uniformly bounded in H in view of Lemma 2, it follows that $\left\{ \frac{d}{dt} u_\lambda(t) \right\}$ converges weakly in H for any $t \in [0, T]$.

Put

$$Y_0 = \{t \in [0, T]; \text{weak } \lim_{\lambda \rightarrow 0} \frac{d}{dt} u_\lambda(t) = \frac{d}{dt} u(t)\}.$$

Put $v(t) = v_0 \in V$ in (1.2). We know that the total variation on Y_0 of the right side of (1.2) is uniformly bounded for λ . Hence the total variation on Y_0 of $\left(\frac{d}{dt} u(t), v_0 \right)$ is bounded. Thus using that V is dense in H we have the existence of

$$\text{weak left } \lim_{t \in Y_0, t \rightarrow t_0} \frac{d}{dt} u(t) \quad \text{and} \quad \text{weak right } \lim_{t \in Y_0, t \rightarrow t_0} \frac{d}{dt} u(t).$$

Therefore this lemma is proved.

Lemma 10. *Let $v \in C([0, T]; X_1)$ and $v(t) \in D(\phi)$ for a.e $t \in [0, T]$. Then it follows*

$$\overline{\lim}_{t \rightarrow T} \overline{\lim}_{\lambda \rightarrow 0} \int_0^t (\phi_\lambda(v(s)) - \phi_\lambda(u_\lambda(s))) ds \leq \int_0^T (\phi(v(s)) - \phi(u(s))) ds.$$

Proof. From Lemma 4 and 5 the sequence $\{J_\lambda u_\lambda(t)\}$ converges to $u(t)$. Since ϕ is lower semi continuous it follows

$$\lim_{\lambda \rightarrow 0} \phi_\lambda(u_\lambda(t)) \geq \lim_{\lambda \rightarrow 0} \phi(J_\lambda u_\lambda(t)) \geq \phi(u(t)).$$

From Theorem 2.2 in [1] (p. 57) we have

$$\phi_\lambda(v(t)) \leq \phi(v(t)) \text{ and } \lim_{\lambda \rightarrow 0} \phi_\lambda(v(t)) = \phi(v(t)).$$

Combining the above two results and Fatou's lemma we can prove our assertion.

Lemma 11. *The function u satisfies the initial conditions in the sense stated in 4) of Definition.*

Proof. For any $v \in D(\phi) \cap V$ from (1.2) it follows

$$\begin{aligned} & \left(\frac{d}{dt} u(t), v - u(t) \right) - (b, v - a) \\ &= \int_0^t (f(s, u(s)) - Au(s), v - u(s)) \, ds - \int_0^t \left(\frac{d}{ds} u(s), \frac{d}{ds} u(s) \right) \, ds \\ & - F_t(v - u) \equiv I_1 - I_2 + I_3. \end{aligned}$$

The left side of the above tends to $\left(\frac{d^+}{dt} u(0) - b, v - a \right)$ as $t \rightarrow 0$. From $u \in L_\infty(0, T; V) \cap W_\infty^1(0, T; H)$ we have

$$\lim_{t \rightarrow 0} I_1 = 0 \text{ and } \lim_{t \rightarrow 0} I_2 = 0.$$

On the other hand from Lemmas 5 and 7 it follows

$$F_t(v - u) = \lim_{\lambda \rightarrow 0} F_{\lambda, t}(v - u_\lambda).$$

Hence arguing as in the proof of Lemma 10

$$\lim_{t \rightarrow 0} F_t(v - u) \leq \overline{\lim}_{t \rightarrow 0} \lim_{\lambda \rightarrow 0} \int_0^t (\phi_\lambda(v) - \phi_\lambda(u_\lambda(s))) \, ds = 0.$$

Thus

$$\left(\frac{d^+}{dt} u(0) - b, v - a \right) \geq 0 \text{ for any } v \in D(\phi) \cap V.$$

Therefore using the assumption 4 we obtain

$$b - \frac{d^+}{dt} u(0) \in \partial I_{K_0} a.$$

2. The proof of Theorem

Combining the definition of the subdifferential and Lemma 5, 7 and 10 we have the first half of 3) in Definition of the solution. From Lemmas 4, 5, 7

(1.2), 1) and the second half of 3) follow. Combining 2) of Lemmas 2, 5 and 9 we have 2) in Definition. From Lemma 11 we know 4). Thus the proof of the theorem is complete.

3. Examples

EXAMPLE 1. Let Ω be a bounded domain in R^n with a smooth boundary. Set

$$\begin{aligned} H &= L_2(\Omega), X_1 = L_{p+1}(\Omega), X_2 = L_{(p+1)/p}(\Omega), \\ A &= -\Delta \text{ (Dirichlet problem) and} \\ \phi(u) &= \int_{\Omega} |u(x)|^{p+1} dx \end{aligned}$$

where $p > 1$.

Then we know $\partial\phi u = (p+1)|u|^{p-1}u$.

Putting $w_{\lambda} = (1 + \lambda \partial\phi)^{-1}f$ we have $\partial\phi_{\lambda}(f) = \partial\phi(w_{\lambda})$, $|w_{\lambda}(x)| \leq |f(x)|$ and $w_{\lambda}(x) \cdot f(x) \geq 0$. Hence it follows that

$$(\partial\phi_{\lambda}f, f) = (p+1) \int_{\Omega} |w_{\lambda}(x)|^p |f(x)| dx \geq (p+1) \int_{\Omega} |w_{\lambda}(x)|^{p+1} dx$$

and

$$|\partial\phi_{\lambda}f|_{x_2} = (p+1) \left(\int_{\Omega} |w_{\lambda}(x)|^{p+1} dx \right)^{p/(p+1)}.$$

Then we have

$$(\partial\phi_{\lambda}f, f) \geq (p+1)^{-1/p} |\partial\phi_{\lambda}f|_{x_2}^{(p+1)/p} > (p+1)^{-1/p} (|\partial\phi_{\lambda}f|_{x_2} - 1).$$

Thus the assumption 2 is satisfied.

If $(n+2) > p(n-2)$ using Sobolev's lemma we know that the assumption 1 is satisfied. Since $A = -\Delta$ (Dirichlet problem) it is easy to show the assumption 4.

EXAMPLE 2. Put $H = L_2(0, 1)$, $X_1 = C([0, 1])$, $X_2 = L_1(0, 1)$ and $A = -\frac{d^2}{dx^2}$

(Dirichlet problem).

Let r be a continuous function on $[0, 1]$ such that $r(0) < 0$ and $r(1) < 0$. Set

$$K = \{f \in L_2(0, 1); f(x) \geq r(x) \text{ a.e } x \in [0, 1]\}.$$

Let $\phi = I_K$ which is the indicator function of K . From Sobolev's lemma the assumptions 1 and 4 follow. We choose a function $\theta \in C^1([0, 1])$ such that $\theta(0) = \theta(1) = 0$ and $\theta(x) - r(x) \geq \delta_0 > 0$ for any $x \in [0, 1]$.

Since

$$\partial\phi_\lambda f(x) = \begin{cases} 0 & \text{if } f(x) \geq r(x) \\ \lambda^{-1}(f(x) - r(x)) & \text{if } f(x) < r(x), \end{cases}$$

and $f(x) < r(x)$ implies

$$\theta(x) - f(x) > \theta(x) - r(x) \geq \delta_0$$

we have

$$(\partial\phi_\lambda f, f - \theta) \geq \delta_0 |\partial\phi_\lambda f|_{x_2}.$$

Hence the assumption 2 holds with $z = \theta$, $c_1 = \delta_0$ and $c_2 = 0$.

EXAMPLE 3. Let K be a closed convex set in H with inner points and $X_1 = H = X_2$. Let A be a positive self adjoint operator in H and V be Domain $(A^{1/2})$ endowed with the graph norm of $A^{1/2}$. If an inclusion mapping from V to H is compact it follows that the assumption 1 holds. From Lemma 2.3 in [2] we have the assumption 2.

Bibliography

- [1] V. Barbu: Nonlinear semigroup and differential equations in Banach space, Noordhoff International, 1976
- [2] K. Maruo: *On certain nonlinear differential equations of second order in time*, to appear
- [3] M. Schatzman: *Sur une classe de problemes hyperboliques nonlineaires*, C.R. Acad. Sci Paris **277** (1973), A 671-A 674.
- [4] M. Schatzman: *A class of nonlinear differential equations of second order in time*, Nonlinear Anal., Theory, Meth. Appl. **2** (1978), 335-373.
- [5] M. Schatzman: *A hyperbolic problem of second order with unilateral constraints; the vibrating string with a concave obstacle*, J. Math. Anal. Appl. **73** (1980), 138-191.

Department of Mathematics
Himeji Institute of Technology
Shosha 2167, Himeji 671-22
Japan