<table>
<thead>
<tr>
<th>Title</th>
<th>Existence of solutions of some nonlinear wave equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Maruo, Kenji</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 22(1) P.21-P.30</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1985</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/11409">https://doi.org/10.18910/11409</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/11409</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
</tbody>
</table>
EXISTENCE OF SOLUTIONS
OF SOME NONLINEAR WAVE EQUATIONS

KENJI MARUO

(Received June 6, 1984)

0. Introduction and Theorem

Let $H$ be a real Hilbert space and $A$ be a positive self adjoint operator in $H$. Let $\phi$ be a lower semi continuous proper convex function from $H$ to $(-\infty, \infty]$ and $\partial \phi$ be the subdifferential of $\phi$. Then we shall consider the following equation

\[
\begin{cases}
\frac{d^2}{dt^2}u + Au + \partial \phi u \geq f(\cdot, u) \\
u(0) = a, \quad \frac{d}{dt}u(0) = b \quad \text{on } [0, T]
\end{cases}
\]

where $T$ is a positive number.

The above equation was studied in Schatzman [3], [4], [5] and Maruo [2]. In this paper we prove the existence of a solution of the problem (0.1) under certain assumptions which are somewhat weaker than those of Schatzman [5] and Maruo [2].

In [5] Schatzman showed the existence and uniqueness of a solution of the following nonlinear wave equation

\[
\begin{cases}
\frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u \geq \frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u \\
u(x, t) \geq r(x), \quad u(x, 0) = u_0(x), \quad \text{for } x \in [0, 1], \\
\frac{\partial}{\partial t}u(x, t) = u_t(x) \quad \text{a.e. in } [0, 1], \\
u(0, t) = u(1, t) = 0 \quad \text{for } t \in [0, T],
\end{cases}
\]

where $r$ is a continuous given function such that $r(0) < 0$, $r(1) < 0$ and $\frac{d^2}{dx^2}r(x) \geq 0$ (in the distribution sense). Set $K = \{ f \in L_2(0, 1); f(x) \geq r(x) \}$. The equation (0.2) is rewritten as the following equation in $L_2(0, 1)$
\[
\begin{cases}
\frac{d^2 u}{dt^2} + Au + \partial I_u u \geq 0 \\
u(0) = u_0, \quad \frac{d}{dt} u(0) = u_1
\end{cases}
\]
(0.3)

where \( I(u) = \begin{cases} 0 & \text{if } u \in K \\
\infty & \text{if } u \notin K \end{cases} \) and \( A = -\frac{d^2}{dx^2} \) (Dirichlet problem).

We will show that we can apply our main theorem to this equation if only \( r \) is a continuous function satisfying \( r(0) < 0, r(1) < 0 \) to deduce the existence of a solution of (0.3). The solution of (0.3), however, does not always satisfy the locally energy conserving condition (see [5]). Hence we cannot get the uniqueness of a solution.

Let \( \Omega \) be a domain in \( \mathbb{R}^n \) with smooth boundary and consider the case \( \phi(u) = \int_0^1 |u(x)|^{p+1} dx, \quad p > 1 \). Then the equation (0.1) represents a nonlinear Klein-Gordon equation. It will be shown that if \((n+2) > p(n-2)\), the result of this paper can be applied to (0.1) in this case. Note that when \( n = 3 \) this inequality is satisfied for \( p = 3 \).

Next we shall introduce the assumptions.

Assumption 1. The following inclusion relations hold:

\[
V \subset X_1 \subset H \subset X_2 \quad \text{and} \quad X_1 \subset \{ \text{the dual space of } X_2 \}
\]

where each inclusion mapping is continuous. Moreover \( X_1 \) is separable and the inclusion mapping from \( V \) to \( X_1 \) is compact. \( H \) is dense in \( X_2 \).

Assumption 2. There exists \( z \in V \) such that

\[
(\partial \phi_x, x-z) \geq c_1 |\partial \phi_x x|_{L_\infty} - c_2
\]

for \( x \in V, \quad |x|_V \leq R \) and \( |\phi(x)| \leq R \) where \( c_1 \) and \( c_2 \) are positive constants depending only on \( R \) and \( z \).

Assumption 3. The continuous function \( f \) from \([0, T] \times H \) to \( H \) satisfies
for any $t \in [0, T]$ and $x, y \in H$

$$|f(t, x) - f(t, y)|_H \leq h(t)|x - y|_H, $$

$$\left| \frac{d}{dt} f(t, x) \right|_H \leq h(t) (1 + |x|_H)$$

where $h$ is a function belonging to $L_1(0, T)$.

**Assumption 4.** The closure of $D(\phi) \cap V$ in $H$ is equal to the closure of $D(\phi)$ in $H$.

Clearly $V$ and $X_1$ are dense in $H$. By assumption $H$ is dense in $X_2$. We use the same notation $(\cdot, \cdot)$ as the inner product of $H$ to denote the pairings between $V, X_1, X_2$ and their corresponding duals.

Now we define the solution of (0.1).

**Definition.** We say that a function $u \in C([0, T]; X_1) \cap W^1_\nu(0, T; H)$ is a solution of the equation (0.1) when it satisfies the following requirements:

1) For any $t \in [0, T]$ $u(t) \in D(\phi) \cap V$.

2) There exist weak right and left derivatives $\frac{d^\pm}{dt} u(t) \in H$ for any $t \in [0, T]$.

Moreover for any $t \in [0, T]$

$$\left| \frac{d^\pm}{dt} u(t) \right|_H + |u(t)|_H + 2 \phi(u(t)) \leq |b|_H + |a|_H + 2 \phi(a) + 2 \int_0^T (f(s, u(s)), \frac{d}{ds} u(s)) \, ds$$

(with necessary modifications at 0 and T).

3) There exists a linear functional $F$ on $C([0, T]; X_1)$ such that

$$F(v - u) \leq \int_0^T \phi(v(s)) \, ds - \int_0^T \phi(u(s)) \, ds$$

for any $v \in C([0, T]; X_1)$ and

$$\int_0^T \left( -\frac{d}{ds} u(s), \frac{d}{ds} v(s) \right) \, ds + \int_0^T (f(s, u(s)) - Au(s), v(s)) \, ds$$

$$+ (b, v(0)) - \left( \frac{d^-}{dt} u(T), v(T) \right) = F(v)$$

for any $v \in C([0, T]; X_1) \cap L_1(0, T; V) \cap W^1_\nu(0, T; H)$.

4) The initial conditions are satisfied in the following sense

$$u(0) = a, b - \frac{d}{dt} u(0) \in \partial I_{K_0} a$$

where $K_0$ is the closure of the domain of $\phi$, $I_{K_0}$ is the indicator function of
$K_0$ and $\partial I_{K_0}$ is the subdifferential of $I_{K_0}$.

We state the theorem.

**Theorem.** Let $a$ and $b$ be given elements satisfying

$$a \in V \cap D(\phi), \ b \in H.$$  

Then under the assumptions 1, 2, 3 and 4 we have at least one solution of (0.1).

To prove the above theorem we consider the following approximate equations for $\lambda > 0$

$$\begin{cases} \frac{d^2}{dt^2} u_\lambda + Au_\lambda + \partial \phi_\lambda u_\lambda = f(\cdot, u_\lambda) \\ u_\lambda(0) = a, \quad \frac{d}{dt} u_\lambda(0) = b. \end{cases}$$  

(0.4)

In the next section using a method similar to that of [2] we shall investigate the convergence of the solutions of the approximate equations (0.4). In section 2 we prove the theorem. In section 3 we show some examples.

1. Convergence of approximate solutions

In this section under the assumptions 1, 2, 3 and 4 we shall study the convergence of the solutions of (0.4). In what follows let initial values $a$ and $b$ belong to $V \cap D(\phi)$ and $H$ respectively.

First we show some properties of the approximate solutions.

**Lemma 1.** For any $\lambda > 0$ we have solutions of the problem (0.4) such that

$$u_\lambda \in C([0, T]; H) \cap L_\infty(0, T; V) \cap W^{1}_\infty(0, T; H) \cap W^{2}_\infty(0, T; V^*)$$

where $V^*$ is the dual space of $V$.

Proof. See p. 289 Barbu [1].

**Lemma 2.** We hold the following equality and inequality

1) $$| - \frac{d}{dt} u_\lambda(t) |_{H}^2 + | u_\lambda(t) |_{V}^2 + 2 \phi_\lambda(u_\lambda(t))$$

$$= | b |_{H}^2 + | a |_{V}^2 + 2 \phi_\lambda(a) + 2 \int_{0}^{t} (f(s, u_\lambda(s)), \frac{d}{ds} u_\lambda(s)) \, ds$$

2) $$| - \frac{d}{dt} u_\lambda(t) |_{H}^2 + | u_\lambda(t) |_{V}^2 + 2 \phi_\lambda(u_\lambda(t))$$

$$\leq C \left( | b |_{H}^2 + | a |_{H}^2 + | a |_{V}^2 + 1 \right)$$
where \( C_1 \) is a constant depending only on \( h \) and \( T \).

Proof. See Lemma 2.2 in [2].

**Lemma 3.** There exists a constant independent of \( \lambda \) such that

\[
\int_0^T |\partial_x \phi_\lambda u_\lambda(s)|\,dx_2\,ds \leq \text{Constant}.
\]

Proof. In the inequality of assumption 2 we put \( x = u_\lambda(t) \). From Lemma 2 the constants \( c_1 \) and \( c_2 \) are independent of \( \lambda \). Replacing \( \partial_x \phi \) by \(- (u''_\lambda + A u_\lambda - f(\cdot, u_\lambda))\), integrating over \([0, T]\), using the integration by parts and noting 2) of Lemma 2 we get the conclusion of the Lemma.

**Lemma 4.** We have a continuous function \( u \) from \([0, T]\) to \( H \) such that a subsequence \( \{u_{\lambda_j}\} \) of the sequence \( \{u_\lambda\} \) converges uniformly to \( u \) in \( H \) as \( \lambda_j \to 0 \).

Proof. In view of 2) of Lemma 2 \( |u_\lambda(t)|_V \) is uniformly bounded. Hence from the assumption 1 we know that \( \{u_\lambda(t)\} \) is a relatively compact subset of \( H \) for any \( t \in [0, T] \). From 2) of Lemma 2 \( \{u_\lambda\} \) is uniformly continuous. Thus using Ascoli-Arzela's theorem this lemma is proved.

For simplicity we denote this subsequence by \( \{u_\lambda\} \).

**Lemma 5.** There exists a subsequence \( \{\lambda_j\} \) of \( \{\lambda\} \) such that \( \{u_{\lambda_j}\} \) converges to \( u \) in \( C([0, T]; X_1) \), \( \{d/dt u_{\lambda_j}\} \) converges to \( d/dt u \) in weak*-\(L_\infty(0,T; H)\) and \( \{u_{\lambda_j}(t)\} \) weakly converges to \( u(t) \) in \( V \) for any \( t \in [0, T] \). Hence we know that

\[
u \in C([0, T]; X_1) \cap W^{1}_{\infty}(0,T; H)
\]

and

\[
u(t) \in D(\phi) \cap V \quad \text{for any} \quad t \in [0, T].
\]

Proof. From 2) of Lemma 2 it is easy to prove that some subsequence \( \{d/dt u_{\lambda_j}\} \) converges to \( d/dt u \) in weak*-\(L_\infty(0,T; H)\). Since \( \{u_{\lambda_j}(t)\} \) is bounded in \( V \) it follows from Lemma 4 that \( \{u_{\lambda_j}(t)\} \) weakly converges to \( u(t) \) in \( V \). Next we assume that there exists a sequence \( \{t_i\}_{i=1}^\infty \) such that

\[
limit_{i \to \infty} t_i = t_\infty,
\]

\[
|u(t_i) - u(t_\infty)|_X \geq \delta_0 > 0 \quad \text{and} \quad t_i \in [0, T]
\]

for \( i = 1, 2, \cdots \).

Since \( |u(t_i)|_V \) is bounded there exists a subsequence \( \{u(t_{i_j})\} \) which converges
to some element \( w \) of \( X_1 \). On the other hand \( u \) is continuous in \( H \). Hence \( w = u(t_0) \). The above results contradict (1.1). Thus \( u \) is continuous in \( X_1 \). Combining that for any \( t \in [0, T] \) \( \{u_{\lambda_j}(t)\} \) is a relatively compact set in \( X_1 \), that \( \{u_{\lambda_j}\} \) is uniformly convergent to \( u \) in \( H \) and that \( u \) is continuous in \( X_1 \) we can prove that \( \{u_{\lambda_j}\} \) is uniformly convergent to \( u \) in \( X_1 \).

For simplicity we denote the subsequence \( \{\lambda_j\} \) by \( \{\lambda\} \).

We denote \( \int_0^t \partial \phi_{\lambda} u_{\lambda}(t) \, dt \) by \( \rho_{\lambda}(t) \) for any \( t \in [0, T] \). Then \( \rho_{\lambda} \) belongs to \( W^{1}_{\infty}([0, T]; H) \).

**Lemma 6.** There exists a subsequence \( \{\lambda_{j}\} \subset \{\lambda\} \) such that for \( \alpha \in X_1 \) and \( t \in [0, T] \), \( \{(\rho_{\lambda j}(t), \alpha)\} \) converges.

Proof. From Lemma 3 for any \( \alpha \in X_1 \) we know that the total variation of the function \( \{(\rho_{\lambda}(t), \alpha)\} \) on \( [0, T] \) is uniformly bounded in \( \lambda \). Noting that \( X_1 \) is separable and using Helly’s choice theorem and the diagonal method we have a subsequence \( \{\lambda_{j}\} \) such that

\[
\lim_{\lambda_j \to \infty} (\rho_{\lambda j}(t), \alpha) \quad \text{exists for any} \quad \alpha \in X_1.
\]

For simplicity we denote \( \{\lambda_{j}\} \) by \( \{\lambda\} \).

Put

\[
\int_0^t (\partial \phi_{\lambda} u_{\lambda}(s), v(s)) \, ds = F_{\lambda, t}(v) \quad \text{for} \quad v \in C([0, T]; X_1) \quad \text{and} \quad t \in [0, T].
\]

**Lemma 7.** For each \( t \in [0, T] \) there exists a linear continuous functional \( F_t \) on \( C([0, T]; X_1) \) such that

\[
\lim_{\lambda \to \infty} F_{\lambda, t}(v) = F_t(v) \quad \text{for any} \quad v \in C([0, T]; X_1).
\]

Proof. Combining Lemmas 3 and 6 and approximating \( v \) by step functions we can prove this lemma.

**Lemma 8.** For any \( v \in C([0, T]; X_1) \) there exist

\[
\text{right lim}_{t \to t_0} F_t(v) \quad \text{and} \quad \text{left lim}_{t \to t_0} F_t(v)
\]

where \( t_0 \in [0, T] \) (with necessary modifications at 0 and \( T \)).

Proof. From Lemma 7 it follows

\[
|F_t(v) - F_t(s)| \leq \lim_{\lambda \to \infty} \int_s^t |\partial \phi_{\lambda} u_{\lambda}(\tau)| \, d\tau \cdot \sup_{0 \leq \tau \leq T} |v(\tau)|_{X_1}.
\]

Combining this inequality with Lemma 3 we see that \( F_t(v) \) is of bounded variation. Thus the lemma is proved.
We put
\[ \lim_{t \to \tau} F_t(v) = F(v). \]

**Lemma 9.** For any \( t \in [0, T] \) there exist weak right and left derivatives \( \frac{d^\pm}{dt} u(t) \) in \( H \) (with necessary modifications at 0 and \( T \)).

**Proof.** Let \( v \) be an arbitrary element of \( C^1([0, T]; X_t) \cap C([0, T]; V) \). Forming the inner product of (0.4) and \( v \) and integrating by parts we get
\[
(1.2) \quad \left( \frac{d}{dt} u_\lambda(t), v(t) \right) = (b, v(0)) + \int_0^t \left( \frac{d}{ds} u_\lambda(s), \frac{d}{ds} v(s) \right) ds + \int_0^t (f(s, u_\lambda(s)) - Au_\lambda(s), v(s)) ds - F_{\lambda, t}(v).
\]

From Lemmas 4, 5 and 7, the right side of the above equality converges for any \( t \in [0, T] \) as \( \lambda \to 0 \). Since \( \left\{ \frac{d}{dt} u_\lambda(t) \right\} \) is uniformly bounded in \( H \) in view of Lemma 2, it follows that \( \left\{ \frac{d}{dt} u_\lambda(t) \right\} \) converges weakly in \( H \) for any \( t \in [0, T] \).

Put
\[ Y_0 = \{ t \in [0, T]; \text{ weak limit } \frac{d}{dt} u_\lambda(t) = \frac{d}{dt} u(t) \}. \]

Put \( v(t) = v_0 \in V \) in (1.2). We know that the total variation on \( Y_0 \) of the right side of (1.2) is uniformly bounded for \( \lambda \). Hence the total variation on \( Y_0 \) of \( \left( \frac{d}{dt} u(t), v_0 \right) \) is bounded. Thus using that \( V \) is dense in \( H \) we have the existence of
\[
\text{weak left limit } \frac{d}{dt} u(t) \quad \text{and} \quad \text{weak right limit } \frac{d}{dt} u(t).
\]

Therefore this lemma is proved.

**Lemma 10.** Let \( v \in C([0, T]; X_t) \) and \( v(t) \in D(\phi) \) for a.e \( t \in [0, T] \). Then it follows
\[
\lim_{\lambda \to 0} \lim_{t \to T} \int_0^t (\phi_\lambda(v(s)) - \phi_\lambda(u_\lambda(s))) ds \leq \int_0^T (\phi(v(s)) - \phi(u(s))) ds.
\]

**Proof.** From Lemma 4 and 5 the sequence \( \{ J_\lambda u_\lambda(t) \} \) converges to \( u(t) \). Since \( \phi \) is lower semi continuous it follows
\[
\lim_{\lambda \to 0} \phi_\lambda(u_\lambda(t)) \geq \lim_{\lambda \to 0} \phi(J_\lambda u_\lambda(t)) \geq \phi(u(t)).
\]
From Theorem 2.2 in [1] (p. 57) we have
\[ \phi_\lambda(v(t)) \leq \phi(v(t)) \quad \text{and} \quad \lim_{\lambda \to 0} \phi_\lambda(v(t)) = \phi(v(t)). \]
Combining the above two results and Fatou's lemma we can prove our assertion.

**Lemma 11.** The function \( u \) satisfies the initial conditions in the sense stated in 4) of Definition.

**Proof.** For any \( v \in D(\phi) \cap V \) from (1.2) it follows
\[
\left( \frac{d}{dt} u(t), \, v-u(t) \right) - (b, \, v-a)
= \int_0^t \left( f(s, u(s)) - Au(s), \, v-u(s) \right) ds - \int_0^t \left( \frac{d}{ds} u(s), \frac{d}{ds} u(s) \right) ds
- F_\lambda(v-u) \equiv I_1 - I_2 + I_3.
\]
The left side of the above tends to \( \left( \frac{d^+}{dt} u(0), \, v-a \right) \) as \( t \to 0 \). From \( u \in L_\infty(0, \, T; V) \cap W_\infty^1(0, \, T; H) \) we have
\[
\lim_{i \to 0} I_1 = 0 \quad \text{and} \quad \lim_{i \to 0} I_2 = 0.
\]
On the other hand from Lemmas 5 and 7 it follows
\[
F_\lambda(v-u) = \lim_{\lambda \to 0} F_{\lambda_i}(v-u_i).
\]
Hence arguing as in the proof of Lemma 10
\[
\lim_{i \to 0} F_i(v-u) \leq \lim_{\lambda \to 0} \lim_{i \to 0} \int_0^t (\phi_\lambda(v) - \phi_{\lambda_i}(u_i)) \, ds = 0.
\]
Thus
\[
\left( \frac{d^+}{dt} u(0), \, v-a \right) \geq 0 \quad \text{for any} \quad v \in D(\phi) \cap V.
\]
Therefore using the assumption 4 we obtain
\[
b - \frac{d^+}{dt} u(0) \in \partial I_{K_0} a.
\]

2. The proof of Theorem

Combining the definition of the subdifferential and Lemma 5, 7 and 10 we have the first half of 3) in Definition of the solution. From Lemmas 4, 5, 7...
(1.2), 1) and the second half of 3) follow. Combining 2) of Lemmas 2, 5 and 9 we have 2) in Definition. From Lemma 11 we know 4). Thus the proof of the theorem is complete.

3. Examples

EXAMPLE 1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a smooth boundary. Set

\[
H = L_2(\Omega), \quad X_1 = L_{p+1}(\Omega), \quad X_2 = L_{(p+1)/p}(\Omega),
\]

\[
A = -\Delta \text{ (Dirichlet problem and)}
\]

\[
\phi(u) = \int_\Omega |u(x)|^{p+1} \, dx
\]

where \( p > 1 \).

Then we know \( \partial \phi = (p+1)|u|^{p-1} u \).

Putting \( w_\lambda = (1 + \lambda \partial \phi)^{-1} f \) we have \( \partial \phi \lambda(f) = \partial \phi(w_\lambda), \quad |w_\lambda(x)| \leq |f(x)| \) and \( w_\lambda(x) \cdot f(x) \geq 0 \). Hence it follows that

\[
(\partial \phi \lambda, f) = (p+1) \int_\Omega |w_\lambda(x)|^p |f(x)| \, dx \geq (p+1) \int_\Omega |w_\lambda(x)|^{p+1} \, dx
\]

and

\[
|\partial \phi \lambda|_{X_2} = (p+1) \left( \int_\Omega |w_\lambda(x)|^{p+1} \, dx \right)^{\frac{p}{p+1}}.
\]

Then we have

\[
(\partial \phi \lambda, f) \geq (p+1)^{\frac{1}{p+1}} |\partial \phi \lambda|_{X_2}^{(p+1)/p} > (p+1)^{\frac{1}{p+1}} (|\partial \phi \lambda|_{X_2} - 1).
\]

Thus the assumption 2 is satisfied.

If \( (n+2) > p(n-2) \) using Sobolev's lemma we know that the assumption 1 is satisfied. Since \( A = -\Delta \) (Dirichlet problem) it is easy to show the assumption 4.

EXAMPLE 2. Put \( H = L_2(\Omega), X_1 = C([0, 1]), X_2 = L_1(0, 1) \) and \( A = -\frac{d^2}{dx^2} \) (Dirichlet problem).

Let \( r \) be a continuous function on \([0, 1]\) such that \( r(0) < 0 \) and \( r(1) < 0 \). Set

\[
K = \{ f \in L_2(0, 1); f(x) \geq r(x) \text{ a.e } x \in [0, 1] \}.
\]

Let \( \phi = I_K \) which is the indicator function of \( K \). From Sobolev's lemma the assumptions 1 and 4 follow. We choose a function \( \theta \in C^1([0, 1]) \) such that \( \theta(0) = \theta(1) = 0 \) and \( \theta(x) - r(x) \geq \delta > 0 \) for any \( x \in [0, 1] \).

Since
\[ \partial \phi_\lambda f(x) = \begin{cases} 0 & \text{if } f(x) \geq r(x) \\ \lambda^{-1}(f(x) - r(x)) & \text{if } f(x) < r(x) \end{cases} \]

and \( f(x) < r(x) \) implies

\[ \theta(x) - f(x) > \theta(x) - r(x) \geq \delta_0 \]

we have

\[ (\partial \phi_\lambda f, f - \theta) \geq \delta_0 |\partial \phi_\lambda f|_{x_2}. \]

Hence the assumption 2 holds with \( z = \theta, c_1 = \delta_0 \) and \( c_2 = 0 \).

**Example 3.** Let \( K \) be a closed convex set in \( H \) with inner points and \( X_1 = H = X_2 \). Let \( A \) be a positive self adjoint operator in \( H \) and \( V \) be Domain \( (A^{1/2}) \) endowed with the graph norm of \( A^{1/2} \). If an inclusion mapping from \( V \) to \( H \) is compact it follows that the assumption 1 holds. From Lemma 2.3 in [2] we have the assumption 2.

---

**Bibliography**


Department of Mathematics
Himeji Institute of Technology
Shosha 2167, Himeji 671-22
Japan