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GENERALIZATIONS OF NAKAYAMA RING IV

(LEFT SERIAL RINGS WITH (*, I))

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Let R be an algebra over an algebraically closed field K with finite dimension. Under an assumption $J^4=0$, we have studied a left serial algebra with $(*, 1)$: the radical of any hollow right R-module is always a direct sum of hollow modules, in [3], where J is the Jacobson radical of R. In this case eJ/eJ^2 is square-free, i.e., a direct sum of simple modules, which are not isomorphic to one another. We shall give, in this note, a complete characterization of a left serial ring with $(*, 1)$ under the assumption: eJ/eJ^2 square-free. In the forthcoming paper [5], we shall study a left serial ring with $(*, 1)$ in general.

1. Definitions and preliminaries

In this note we only deal with a left and right artinian ring R with identity. We assume that every R-module M is a unitary right (or left) R-module and denote its Jacobson radical and socle by $J(M)$ and $\mathrm{Soc}(M)$, respectively. $|M|$ means the length of a composition series of M. If M has a unique composition series, we call M a *uniserial module*. If, for each primitive idempotent e, eR is uniserial as a right R-module, we call R a *right serial ring* (*Nakayama ring*).

We obtained a characterization of a right serial ring in terms of submodules in a direct sum of uniserial modules [1]. As a generalization of the above result, we studied the following property:

$(*, \mathrm{n})$ *Every maximal submodule of a direct sum of n hollow modules is also a direct sum of hollow modules* [2].

In this note we shall study a ring with $(*, 1)$, i.e., every factor module of eJ is a direct sum of hollow modules for each primitive idempotent e, where $J=\mathrm{J}(R)$. Concerning $(*, 1)$ we got

Lemma A ([4], Theorem 4). *Let R be a right artinian ring. Then R satisfies $(*, 1)$ for any hollow right R-module if and only if the following two conditions are fulfiled:*

1) $eJ=\sum_{i=1}^{m}\oplus A_i$, *where e is any primitive idempotent in R and the A_i are hollow.*

2) *Let* $C_i \supset D_i$ *be two submodules of* A_i *such that* C_i/D_i *is simpje. If* $f\colon C_i/D_i \approx C_j/D_j$ *for* $i \neq j$, f *or* f^{-1} *is extendible to an element in* $\mathrm{Hom}_R(A_i/D_i, A_j/D_j)$ *or* $\mathrm{Hom}_R(A_j/D_j, A_i/D_i)$.

On the other hand T. Sumioka found the following remarkable result:

Lemma B ([6], Corollary 4.2). *Let* R *be a left serial ring, then* eJ^i *is a direct sum of hollow modules as right* R*-modules for any* i.

We shall study, in this paper, only left serial rings, and so we denote the content of Lemma B by the following diagram:

$$
(1)\qquad
\begin{array}{ccccl}
 & & & & eR \\
A_1 & A_2 & \cdots & A_n & eJ \\
A_{11}\cdots A_{1n_1} & A_{21}\cdots A_{2n_2} & \cdots & A_{n1}\cdots A_{nn_n} & eJ^2
\end{array}
$$

The diagram means that $\mathrm{J}(eR)=eJ=\sum_{i=1}^{n}\oplus A_i$, $\mathrm{J}(A_k)=\sum_{i=1}^{n_k}\oplus A_{ki}$ $(k=1, 2, \cdots, n)$. (cf. [2], § 2).

Further we continue to observe (1).

$$
(2)\qquad
\begin{array}{c}
A_{11} \\
A_{111}A_{112}\cdots A_{11t_1}\cdots J(A_{11})
\end{array}
\qquad
\begin{array}{c}
A_{12} \\
A_{121}A_{122}\cdots A_{12t_2}\cdots J(A_{12})
\end{array}
$$

and repeat this process. We sometime denote $A_{i_1 i_2 \cdots i_t}$ by A_α, and define $|\alpha|=t$. Let $\alpha=i_1 i_2, \cdots, i_t$ and $\beta=j_1 j_2 \cdots j_t$. We define $\alpha>\beta$ if $t<t'$ and $i_1=j_1 \cdots, i_t=j_t$, which is nothing but $A_\alpha \supset A_\beta$. We note that $\alpha \not\gtrless \beta$ if and only if $A_\alpha \cap A_\beta = 0$.

Let x be an element in eJ. Then $x=x_1+x_2+\cdots+x_n$; $x_i \in A_i$. If $x_1 \in J(A_1)$, $x_1=x_{11}+x_{12}+\cdots+x_{1n_1}$; $x_{1i} \in A_{1i}$. Repeating this process, we obtain finally

(3) $x=z_1+z_2+\cdots+z_t$, where $z_i \in A_{\alpha_i}-J(A_{\alpha_i})$, and $\alpha_i \not\gtrless \alpha_j$ if $i \neq j$, i.e., $\sum_{i=1}^{t} A_{\alpha_i}=\sum\oplus A_{\alpha_i}$.

Finally, let e and f be primitive idempotents. By $\mathrm{T}(eJ^i f))$ (resp. $\mathrm{T}(J^i f)$) we denote the set $eJ^i f - eJ^{i+1} f$ (resp. $J^i f - J^{i+1} f$). For a hollow module A, $\bar{A}$ means $A/J(A)$.

2. Main Theorem

We shall give a characterization of a left serial ring with $(*, 1)$ as a right R-module.

Theorem. *Let* R *be a left serial ring such that* eJ/eJ^2 *is square-free for each primitive idempotent* e. *Then*

$(*, 1)$ *holds for any hollow right R-module if and only if we have the following condition: If* $A_\alpha/J(A_\alpha)\approx A_\beta/J(A_\beta)$ *for* $\alpha=i_1i_2\cdots i_k$, $\beta=j_1j_2\cdots j_s$ $(1<k<s$ *and* $i_1\neq j_1)$, $A_{\beta'}/J(A_\beta)$ *is uniterial, where* $\beta'=j_1j_2\cdots j_{s-k+1}$ *and* A_α, A_β *and* $A_{\beta'}$ *are hollow modules in* (1). (See [5], for the general case.)

Corollary. *Let R be a left serial algebra over an algebraically closed field of finite dimension. Then* $(*, 1)$ *holds for any hollow right R-module if and only if the condition in Theorem is satisfied.*

Proof. This is clear from Theorem and [3], Lemma 3.

We shall study some properties of a left serial ring. T. Sumioka has communicated us the following lemma.

Lemma 1 (T. Sumioka). *Let R be left serial. Let x (resp. y) be in* $T(eJ^if)$ *(resp.* $T(fJ^jg)$*), where e, f and g are primitive idempotents. If* $J^{i+j}g\neq 0$, $xy\in T(eJ^{i+j}g)$.

Proof. Assume that $xy\in eJ^{i+j+1}g$. Since R is left serial, $Rx=J^if$ and $Ry=J^jg$. Hence $J^{i+j}g=J^iRy=J^ify=Rxy\subset ReJ^{i+j+1}g\subset J^{i+j+1}g$. Therefore $J(J^{i+j}g)=J^{i+j}g$, which is a contradiction to the assumption $J^{i+j}g\neq 0$.

From now on for a left serial ring R A_α, $A_\beta\cdots$ are hollow modules in the diagram (1) and $A_{\alpha(i)}$ means a submodule of A_i. When we need to specify $|\alpha(i)|=t$, we denote $A_{\alpha(i)}$ by $A_{\alpha(i,t)}$.

Lemma 2. *Let R be a left serial ring and X, Y hollow right submodules in R. If* f: $X/X_1\approx Y/Y_1$ *for some* $X_1\subset X$ *and* $Y_1\subset Y$, *there exists d in R such that* $dX=Y$ *(or* $dY=X$*). If* $X_1=J(X)$ *and* $Y_1=J(Y)$, d_l, *left-sided multiplication of d, induces f. In general, if* d_l *induces f,* $dX_1\subset Y_1$. *In particular, if* $\bar{A}_{\alpha(i)}\approx \bar{A}_{\beta(j)}(|\alpha(i)|\leq|\beta(j)|)$, *we can find such d in* A_j.

Proof. Since X is hollow, we can find a generator x of X with $xe=x$ for some primitive idempotent e. Put $f(x+X_1)=y+Y_1(y\in Y)$. f being an isomorphism, y is a generator of Y and we may assume $ye=y$. Further we may assume $x\in T(J^i)$, $y\in T(J^j)$ and $i\leqslant j$ (if $j>j$, replace X by Y). Then, since Re is uniserial, there exists d in R such that $dx=y$. If $X_1=J(X)=xJ$, $dX_1=dxJ=yJ=Y_1$. Let x_1 be an element in X_1 and $x_1=xr$; $r\in R$. Then $dx_1=dxr=yr\in Y$ and $dx_1+Y_1=yr+Y_1=f(xr+X_1)=Y_1$, provided that d_l induces f. Hence $dX_1\subset Y_1$ (see the proof of [3], Theorem 3), If $\bar{A}_{\alpha(i)}\approx\bar{A}_{\beta(j)}$, $da=b$, where $A_{\alpha(i)}=aR$, $A_{\beta(j)}=bR$ and $d\in eJe$. Let $d=\Sigma d_i$: $d_i\in A_i$. Since $b=da=\Sigma d_ia\in A_j$, $b=d_ja$.

Lemma 3. *Let R be a left serial ring. Assume* $\bar{A}_{\alpha(i,t)}\approx\bar{A}_{\beta(j,t)}$ *for* $A_{\alpha(i,t)}\subset A_i$ *and* $A_{\beta(j,t)}\subset A_j$. *Then* $A_{\alpha'(i,p)}\approx A_{\beta'(j,p)}$, *provided* $A_{\alpha'(i,p)}\supset A_{\alpha(i,t)}$ *and* $A_{\beta'(j,p)}\supset A_{\beta(j,t)}$.

Proof. We show $\bar{A}_{\alpha'(i,p)} \approx \bar{A}_{\beta'(j,p)}$. From Lemma 2, there exists an element d in eRe such that $dA_{\alpha(i,t)} = A_{\beta(j,t)}$. Let $\pi_{\gamma(s)}$ be the projection of eJ^p to $A_{\gamma(s,p)}$. Then $d_l | A_{\alpha'(i,p)} = (\sum_s \pi_{\gamma(s)} d_l) | A_{\alpha'(i,p)}$, where d_l means the left-sided multiplication of d. If $\pi_{\beta'(j,p)} d_l | A_{\alpha'(i,p)}$ is an epimorphism, $\bar{A}_{\alpha'(i,p)} \approx \bar{A}_{\beta'(j,p)}$. Assume $\pi_{\beta'(j,p)}(dA_{\alpha'(i,p)}) \subset J(A_{\beta'(j,p)})$. Let $A_{\alpha(i,t)} = a_t R$ and $A_{\alpha'(i,p)} = a_p R$. Then $a_t = a_p x$; $x \in T(J^{t-p})$. $(\pi_{\beta'(i,p)}(eJ^p) \supset) A_{\beta(j,t)} = dA_{\alpha(i,t)} = da_t R = \Sigma(\pi_{\gamma(s)} da_t) R = \pi_{\beta'(j,p)}(da_p xR) \subset J(A_{\beta'(j,p)}) xR \subset eJ^{t+1}$, a contradiction. Hence $\bar{A}_{\alpha'(i,p)} \approx \bar{A}_{\beta'(j,p)}$, and so $A_{\alpha'(i,p)} \approx A_{\beta'(j,p)}$ by Lemma 2 (note $A_{\alpha'(i,p)}$, $A_{\beta'(j,p)}$ are contained in eJ^p but not in eJ^{p+1}).

Lemma 4. *Let R be a left serial ring. Assume that $eJ = \Sigma \oplus A_i$, where the A_i are hollow. Then there are no non-zero elements d_1, d_2 in eJe such that $d_1 \in A_i$, $d_2 \in A_j (i \neq j)$ provided $\bar{A}_i \approx \bar{A}_j$.*

Proof. We may assume that $d_1 \in \mathrm{T}(A_{\alpha(i)})$ and $d_2 \in \mathrm{T}(A_{\alpha(j)})$. Put $t_1 = |\alpha(i)|$ and $t_2 = |\alpha(j)|$. Then $t_1 \neq t_2$ by assumption and Lemma 3, since $\overline{d_i R} \not\approx \overline{eR}$ for $i = 1, 2$. We can choose a pair (d_1', d_2') such that $t_1' + t_2'$ is minimal $(t_1' < t_2')$. Then there exists d_0 in $A_j \cap eJ^{t_2'-t_1'}$ such that $d_0 d_1' = d_2'$ from Lemma 2, and so (d_1', d_0) gives the contradiction to the minimality.

Lemma 5. *Let A_i be as above. Assume that any pair of $\{A_1, A_2, A_3\}$ is not isomorphic to one another. Then, for any primitive idempotent g, there are no three elements $\{a_{ij_i}\}$ as follows: 1) $a_{1j_1} \in \mathrm{T}(A_1 J^{j_1})$, $a_{2j_2} \in \mathrm{T}(A_1 J^{j_2})$ and $a_{3j_3} \in \mathrm{T}(A_2 J^{j_3})$ such that $a_{ij_i} g = a_{ij_i}$ and $j_1 < \min(j_2, j_3)$ or 2) $a_{ij_i} \in A_i$ and $a_{ij_i} g = a_{ij_i}$ for $i = 1, 2, 3$.*

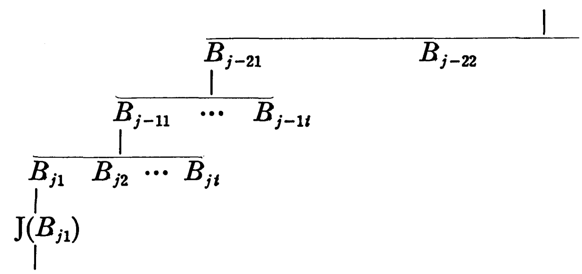
Proof. Since $Ra_{ij_i} \subset Rg$ and $ea_{ij_i} = a_{ij_i}$, this is clear from Lemmas 2 and 4.

Now we study a right artinian ring with $(*, 1)$. For the sake of simplicity, we change the expression of a direct decomosition of eJ^i.

The following lemma is the "only if" part of Theorem.

Lemma 6. *Assume that R is a right artinian ring with $(*, 1)$ as right R-modules. Put $eJ = A_1 \oplus B_1 \oplus \cdots$, $eJ^i = \sum_{k=1}^{n_i} \oplus A_{ik} \oplus \sum_{k=1}^{n_i'} \oplus B_{ik} \oplus \cdots$, where the A_1, B_1, A_{ik}, $B_{ik} \cdots$ are hollow and $A_1 \supset A_{ik}$, $B_1 \supset B_{ik}$, $\cdots$. If $\bar{A}_{i1} \approx \bar{B}_{j1}$ for $1 < i < j$, $B_{j-i+11}/J(B_{j1})$ is uniserial.*

Proof. Put $C_1 = \sum_{k=2}^{n_2} \oplus A_{2k} + \sum_{k=2}^{n_3} \oplus A_{3k} + \cdots + \sum_{k=1}^{n_i} \oplus A_{ik}$ and $D_1 = \sum_{k=2}^{n_2} \oplus A_{2k} + \cdots + \sum_{k=2}^{n_{i-1}} \oplus A_{i-1k} + \mathrm{J}(A_{i1}) + \sum_{k=2}^{n_i} \oplus A_{ik}$. Then f: $C_1/D_1 \approx A_{i1}/\mathrm{J}(A_{i1}) \approx B_{j1}/\mathrm{J}(B_{j1})$. Assume that f^{-1} is extendible to a g' in $\mathrm{Hom}_R(B_1/\mathrm{J}(B_{j1}), A_1/D_1)$. Since $B_1 J^i \supset B_1 J^{j-1} \supset B_{j1}$, $g'(B_1 J^i/\mathrm{J}(B_{j1})) \supset g'(B_{j1}/\mathrm{J}(B_{j1})) = f^{-1}(B_{j1}/\mathrm{J}(B_{j1})) \neq 0$. On the other hand, $g'(B_1J^i/\mathrm{J}(B_{j1}))\subset(A_1/D_1)J^i=0$ for $A_1J^i\subset D_1$, which is a contradiction. Hence f is extendible to a g in $\mathrm{Hom}_R(A_1/D_1, B_1/\mathrm{J}(B_{j1}))$ by Lemma A. Since A_1/D_1 is uniserial, $\mathrm{Soc}(A_1/D_1)=(A_{i1}+D_1)/D_1\approx A_{i1}/(\mathrm{J}(A_{i1})$ and $g\,|\,A_{1i}/\mathrm{J}(A_{i1})=f$ is an isomorphism, g is a monomorphism. Consider a uniserial module $\tilde{B}_{i-1}=g((A_{i-11}+D_1)/D_1)\supset B_{j1}/\mathrm{J}(B_{j1})$. Let $\mathrm{J}(B_{j-11})=B_{j1}\oplus B_{j2}\oplus\cdots\oplus B_{jt}$. Assume $B_{j2}\neq 0$.



Then $\mathrm{Soc}(B_1/B_{j1})=\sum_{k=2}^{t}\oplus\mathrm{Soc}(B_{jk})\oplus\sum_{k=2}^{t'}\oplus\mathrm{Soc}(B_{j-1k})\oplus\cdots$. Since $\mathrm{Soc}(\tilde{B}_{i-1})=B_{j1}/\mathrm{J}(B_{j1})$, $\tilde{B}_{i-1}$ is not uniserial. Hence $B_{j2}=B_{j3}=\cdots=B_{jt}=0$. Considering the same situation for $g((A_{i-21}+D_1)/D_1)$, we obtain similarly $B_{j-12}=\cdots=B_{j-1t'}=0$, where $\mathrm{J}(B_{j-21})=B_{j-11}\oplus\cdots\oplus B_{j-1t'}\cdots$. Repeating this argument, we know $B_{j-i+11}/\mathrm{J}(B_{j1})$ is uniserial.

In Lemma 6, if $i=j$, $B_1/\mathrm{J}(B_{i1})$ is not uniserial in general (see Example 1 in [5]).

Lemma 7. *Let R be left serial and $A_{\alpha(i,t)}$, $A_{\beta(i,t)}$ hollow modules in* (1). *Assume $\alpha(i, t)\neq\beta(i, t)$. Then there are no d in J such that $dA_{\alpha(i,t)}=dA_{\beta(i,t)}\neq 0$.*

Proof. Let $A_{\alpha(i,t)}=aR$, $A_{\beta(i,t)}=bR$ and assume that there exists d such that $daR=dbR$. Then there exists r in R such that $dar=db$, and so $d(ar-b)=0$. On the other hand, $ar-b\neq 0$ by assumption, and there exists a primitive idempotent g such that $db=dbg$. Further $d\in\mathrm{T}(J^s)$ for some s and $0\neq db\in\mathrm{T}(J^{t+s}g)$ by Lemma 1. Hence $d(ar-b)\in\mathrm{T}(J^{t+s}g)$, a contradiction to Lemma 1.

The following is the "if" part of Theorem.

Lemma 8. *Let R be a left serial ring with $\overline{eJ}$ square-free. If R satisfies the result in Lemma* 6, *then* $(*, 1)$ *holds for any hollow module.*

Proof. First we shall show from the assumption that

(4) there are no three distinct hollow modules $A_{\alpha(i,s)}$, $A_{\beta(i,s)}$ and $A_{\gamma(j,t)}$ such that $\bar{A}_{\alpha(i,s)}\approx\bar{A}_{\beta(i,s)}\approx\bar{A}_{\gamma(j,t)}$ and $i\neq j$, $s<t$.

Contrarily we assume that there exist such modules. Put $A_{\alpha(i,s)}=aR$, $A_{\beta(i,s)}=$

bR and $A_{\gamma(j,t)}=cR$. $A_{\gamma'(j,t-s+1)}/\mathrm{J}(A_{\gamma(j,t)})$ is unisreial for some $\gamma'(j, t-s+1)>\gamma(j, t)$ by assumption. We may assume that $ag=a$, $bg=b$ and $cg=c$ for some primitive idempotent g. Then there exists d' in $\mathrm{T}(eJ^{t-s}e)$ such that

$$(5) \qquad d'a=c$$

by Lemmas 1 and 2. Let $A_i=a_iR$ and $d'a_i=\sum_j v_j$ as in (3) ($a_ih=a_i$, $v_jh=v_j$ for some primitive idempotent h). a being in A_i, $a=a_ir$ for some r in J. Then $c=d'a=d'a_ir=\sum_j v_jr$. Hence from the observation after (2)

$$(6) \qquad c=v_kr \text{ for some } k \text{ (say 1)} \quad \text{and} \quad v_{k'}r=0 \quad \text{for } k'\neq 1\,.$$

$c=v_1r$ implies that $v_1\in A_{\delta(j)}$ and $\delta(j)>\beta(j, t)$. Further $a=a_ir$ implies $r\in \mathrm{T}(hJ^{s-1}g)$, and so $v_1\in\mathrm{T}(J^{t-s+1}h)$ by Lemma 1. Hence $v_1\in A_{\gamma'(j,t-s+1)}$ (note $A_{\gamma(j,t)}\subset A_{\gamma'(j,t-s+1)}$). Now there exists d in $\mathrm{T}(eJ^{t-s}e)\cap A_j$ with $v_1=da_i$, and so

$$(7) \qquad da=da_ir=v_1r=c \quad \text{and} \quad da_i=v_1$$

by (6). Let $b=a_ir'$ for some r' in R. Then $r'\in\mathrm{T}(J^{s-1})$ and $db=da_ir'=v_1r'\in A_{\gamma'(j,t-s+1)}\cap\mathrm{T}(J^t)=A_{\gamma(j,t)}$, since $A_{\gamma'(j,t-s+1)}/\mathrm{J}(A_{\gamma(j,t)})$ is uniserial. Hence $dA_{\alpha(i,s)}=dA_{\beta(i,s)}\neq 0$, which is a contradiction to Lemma 7. Thus we have shown (4). Now let $A_1\supset C_1\supset D_1$ and $A_2\supset C_2\supset D_2$ be submodules such that $f\colon C_1/D_1\approx C_2/D_2$ and C_1/D_1 is simple. Let $c_i=\sum_k z_{ik}$ be a generator of C_i for $i=1, 2$, where $z_{ik}\in\mathrm{T}(A_{\alpha_{ik}})$ from (3) and $z_{ik}g=z_{ik}$ for some primitive idempotent g. We choose a generator $c_i\in C_i$ with $\min_{i=1,2}\{\min_k|\alpha_{ik}|\}$. We may assume that $c_1=\sum_k z_{1k}$ and $|\alpha_{11}=11\cdots1|$ is minimal. Then $c_1=z_{11}$ $(=x)$ from (4) and Lemmas 3 and 5. We shall take the following D_1^* similarly to D_1 in the proof of Lemma 6: $D_1^*=A_{12}\oplus A_{13}\oplus\cdots\oplus A_{112}\oplus A_{113}\oplus\cdots\oplus A_{11\cdots12}\oplus\cdots(11\cdots1=\alpha_{11})$. We shall show

$$(8) \qquad D_1\subset\mathrm{J}(A_{\alpha_{11}})\oplus D_1^*.$$

Let y be an element in D_1 and $y=\sum y_i$; $y_i\in A_{\alpha_i}$ as in (3). Assume $\alpha_1\geqslant\alpha_{11}=11\cdots1$. Then there exists r in Rg such that $y_1r=x$. $x-yr=\sum_{i\geqslant 2}-y_ir$ is also a generator of C_1 and contained in D_1^*. Since $A_{\alpha_{11}}\cap D_1^*=0$, we obtain a contradiction from (4) and Lemma 5. Hence $\alpha_1<\alpha_{11}$, and so (8) is true. We choose a representative w in C_2 of $f(x)$ such that $wg=w$. Let $w=\sum_{i=1}^r w_i$ $(w_ig=w_i)$ as in (3); $w_i\in A_{\beta(2)}$. Since $w_1g=w_1$ and $|\alpha_{11}|$ is minimal, there exists d_1' in A_2 with $d_1'x=w_1$. Now we have the same situation as (5). Hence from the argument after (5), similarly to (7) there exists d_1 in $\mathrm{T}(eJ^{t-s}e)\cap A_2$ such that $d_1a_1\in\mathrm{T}(A_{\beta'(t,t-s+1)})$ and $d_1x=w_1$, where $t=|\beta_1(2)|$, $s=|\alpha_{11}|$ and $\beta'(2, t-s+1)>\beta_1(2)$. Let $\gamma(1, q)=11\cdots1$ and $\gamma'(1, q)$ be two distinct indices with $1<q\leqslant s$. Let

$p \in D_1^* \cap A_{\gamma'(1,q)}$ and a_q a generator of $A_{\gamma(1,q)}$. Then $x = a_q r'$ for some r' in R. $d_1 a_q r' = d_1 x = w_1 \neq 0$, and so $d_1 a_q$ $(\neq 0) \in \mathrm{T}(eJ^{t-s+q})$. If $d_1 p \neq 0$, $d_1 p \in \mathrm{T}(eJ^{t-s+q})$ by Lemma 1 and $d_1 p$ and $d_1 a_q$ generate a same submodule between $A_{\beta'(2,t-s+1)} = d_1 A_1$ and $A_{\beta_1(2)}$, since $A_{\beta'(2,t-s+1)}/\mathrm{J}(A_{\beta_1(2)})$ is uniserial, which is a contradiction to Lemma 7. Therefore $d_1 p = 0$ and so $d_1(D_1^*) = 0$. Similarly we can find $d_i \in A_2$ such that $d_i x = w_i$ for each w_i and $d_i(D_1^*) = 0$. Put $d^* = \sum_{i=1}^{r} d_i$. Then $d^* x = \sum d_i x = \sum w_i = w$. Let u be any element in D_1, then from (8) $u = u_1 + u_2$, where $u_1 \in \mathrm{J}(A_{\alpha_{11}})$, $u_2 \in D_1^*$. We denote $u_1 = xj$; $j \in \mathrm{J}$. $d^* u = d^* u_1 + d^* u_2 = d^* u_1 = d^* xj = wj \in C_2 J \subset D_2$. Therefore d_l^*, left-sided multiplication of d^*, is the desired extension of f.

Proposition. *Let R be a left serial ring with* $(*, 1)$. *Let* $\alpha(1, s) = 11\cdots1$, $\beta(2, t) = 211\cdots1$ $(s < t)$ *and* $|A_{\beta(2,t)}| = k$. *If* $\bar{A}_{\alpha(1,s)} \approx \bar{A}_{\beta(2,t)}$, *then* $A_{\beta'(t-s+1)} \approx A_1/D$, *where* $\beta'(2, t-s+1) = 21\cdots1$ *and after renumbering* $A_{\alpha(1\ p)}$ *for all* p, $D = A_{12} \oplus A_{13} \oplus \cdots \oplus A_{112} \oplus A_{113} \oplus \cdots \oplus \mathrm{J}(A_{\gamma(1,s+k-1)})$; $\gamma(1, s+k-1) = 11\cdots1$. *Hence* $A_{\beta'(2,t-s+1)}$ *is uniserial.*

Proof. Since $\bar{A}_{\alpha(1,s)} \approx \bar{A}_{\beta(2,t)}$, there exists d in A_2 such that $dA_{\alpha(1,s)} = A_{\beta(2,t)}$ by Lemma 2, and so $d\mathrm{J}(A_{\alpha(1,s)}) = \mathrm{J}(A_{\beta(2,t)})$. Let $\mathrm{J}(A_{\alpha(1,s)}) = A_{11\cdots1} \oplus A_{11\cdots12} \oplus \cdots$ and $\mathrm{J}(A_{\beta(2,t)}) = A_{21\cdots1} \oplus A_{21\cdots12} \oplus \cdots$. Since $d\mathrm{J}(A_{\alpha(1,s)}) = \mathrm{J}(A_{\beta(2,t)})$ and $A_{211\cdots1}$ is hollow, $\bar{A}_{21\cdots1} \approx \bar{A}_{11\cdots1k}$ for scme k. Then $A_{21\cdots1k'} = 0$ for $k' \neq 1$ by Theorem. Repeating this argument, we know that $A_{\beta(2,t)}$ is uniserial and $A_{\beta''(2,t+k-1)}$ is the socle of $A_{\beta(2,t)}$ and is isomorphic to $A_{\alpha'(1,s+k-1)}$. There exists d' in J from (7) such that $d'A_1 = A_{\beta'(2,t-s+1)}$ and $d'A_{\alpha'(1,s+k-1)} = A_{\beta''(2,t+k-1)}$. Therefore we obtain the proposition from the last part of proof of Theorem.

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