# GENERALIZATIONS OF NAKAYAMA RING IV

## (LEFT SERIAL RINGS WITH (*, I))

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Let $R$ be an algebra over an algebraically closed field $K$ with finite dimension. Under an assumption $J^4=0$, we have studied a left serial algebra with (*, 1): the radical of any hollow right $R$-module is always a direct sum of hollow modules, in [3], where $J$ is the Jacobson radical of $R$. In this case $eJ/eJ^2$ is square-free, i.e., a direct sum of simple modules, which are not isomorphic to one another. We shall give, in this note, a complete characterization of a left serial ring with (*, 1) under the assumption: $eJ/eJ^2$ square-free. In the forthcoming paper [5], we shall study a left serial ring with (*, 1) in general.

## 1. Definitions and preliminaries

In this note we only deal with a left and right artinian ring $R$ with identity. We assume that every $R$-module $M$ is a unitary right (or left) $R$-module and denote its Jacobson radical and socle by $J(M)$ and $\mathrm{Soc}(M)$, respectively. $|M|$ means the length of a composition series of $M$. If $M$ has a unique composition series, we call $M$ a *uniserial module*. If, for each primitive idempotent $e$, $eR$ is uniserial as a right $R$-module, we call $R$ a *right serial ring (Nakayama ring)*.

We obtained a characterization of a right serial ring in terms of submodules in a direct sum of uniserial modules [1]. As a generalization of the above result, we studied the following property:

(*, n) *Every maximal submodule of a direct sum of n hollow modules is also a direct sum of hollow modules* [2].

In this note we shall study a ring with (*, 1), i.e., every factor module of $eJ$ is a direct sum of hollow modules for each primitive idempotent $e$, where $J=J(R)$. Concerning (*, 1) we got

**Lemma A** ([4], Theorem 4). *Let $R$ be a right artinian ring. Then $R$ satisfies (*, 1) for any hollow right $R$-module if and only if the following two conditions are fulfiled:*

1) $eJ=\sum_{i=1}^{m}\oplus A_i$, *where $e$ is any primitive idempotent in $R$ and the $A_i$ are hollow.*

2) *Let $C_i \supset D_i$ be two submodules of $A_i$ such that $C_i/D_i$ is simpje. If $f: C_i/D_i \approx C_j/D_j$ for $i \neq j$, $f$ or $f^{-1}$ is extendible to an element in $\mathrm{Hom}_R(A_i/D_i, A_j/D_j)$ or $\mathrm{Hom}_R(A_j/D_j, A_i/D_i)$.*

On the other hand T. Sumioka found the following remarkable result:

**Lemma B** ([6], Corollary 4.2). *Let $R$ be a left serial ring, then $eJ^i$ is a direct sum of hollow modules as right $R$-modules for any $i$.*

We shall study, in this paper, only left serial rings, and so we denote the content of Lemma B by the following diagram:

$$
\begin{array}{cccc}
& & & eR \\
& \overline{\rule{3cm}{0pt}\;|\;\rule{3cm}{0pt}} & & \\
(1)\quad A_1 & A_2 & \cdots & A_n & eJ \\
| & | & & | & \\
\overline{A_{11}\cdots A_{1n_1}}\; \overline{A_{21}\cdots A_{2n_2}}\; \cdots\; \overline{A_{n1}\cdots A_{nn_n}} & & & eJ^2 \\
| \quad\quad | \quad\quad | \quad\quad | \quad\quad | \quad\quad | & & &
\end{array}
$$

The diagram means that $J(eR)=eJ=\sum_{i=1}^{n}\oplus A_i$, $J(A_k)=\sum_{i=1}^{n_k}\oplus A_{ki}$ $(k=1, 2, \cdots, n)$. (cf. [2], § 2).

Further we continue to observe (1).

$$
\begin{array}{cc}
A_{11} & A_{12} \\
| & | \\
(2)\quad \overline{A_{111}A_{112}\cdots A_{11t_1}\cdots J(A_{11})} & \overline{A_{121}A_{122}\cdots A_{12t_2}\cdots J(A_{12})} \\
| \quad | \quad\quad | & | \quad | \quad\quad |
\end{array}
$$

and repeat this process. We sometime denote $A_{i_1 i_2 \cdots i_t}$ by $A_\alpha$, and define $|\alpha|=t$. Let $\alpha=i_1 i_2, \cdots, i_t$ and $\beta=j_1 j_2 \cdots j_{t'}$. We define $\alpha > \beta$ if $t < t'$ and $i_1=j_1 \cdots, i_t=j_t$, which is nothing but $A_\alpha \supset A_\beta$. We note that $\alpha \ngtr \beta$ if and only if $A_\alpha \cap A_\beta = 0$.

Let $x$ be an element in $eJ$. Then $x=x_1+x_2+\cdots+x_n$; $x_i \in A_i$. If $x_1 \in J(A_1)$, $x_1=x_{11}+x_{12}+\cdots+x_{1n_1}$; $x_{1i} \in A_{1i}$. Repeating this process, we obtain finally

(3) $\quad x=z_1+z_2+\cdots+z_t$, where $z_i \in A_{\alpha_i}-J(A_{\alpha_i})$, and $\alpha_i \ngtr \alpha_j$ if $i \neq j$, i.e.;
$$\sum_{i=1}^{t} A_{\alpha_i}=\sum\oplus A_{\alpha_i}.$$

Finally, let $e$ and $f$ be primitive idempotents. By $T(eJ^if))$ (resp. $T(J^if)$) we denote the set $eJ^if-eJ^{i+1}f$ (resp. $J^if-J^{i+1}f$). For a hollow module $A$, $\bar{A}$ means $A/J(A)$.

## 2. Main Theorem

We shall give a characterization of a left serial ring with $(*, 1)$ as a right $R$-module.

**Theorem.** *Let $R$ be a left serial ring such that $eJ/eJ^2$ is square-free for each primitive idempotent $e$. Then*

($*$, 1) *holds for any hollow right $R$-module if and only if we have the following condition: If $A_\alpha/J(A_\alpha) \approx A_\beta/J(A_\beta)$ for $\alpha = i_1 i_2 \cdots i_k$, $\beta = j_1 j_2 \cdots j_s$ ($1 < k < s$ and $i_1 \neq j_1$), $A_{\beta'}/J(A_{\beta'})$ is uniterial, where $\beta' = j_1 j_2 \cdots j_{s-k+1}$ and $A_\alpha$, $A_\beta$ and $A_{\beta'}$ are hollow modules in* (1). (See [5], for the general case.)

**Corollary.** *Let $R$ be a left serial algebra over an algebraically closed field of finite dimension. Then* ($*$, 1) *holds for any hollow right $R$-module if and only if the condition in Theorem is satisfied.*

Proof. This is clear from Theorem and [3], Lemma 3.

We shall study some properties of a left serial ring. T. Sumioka has communicated us the following lemma.

**Lemma 1** (T. Sumioka). *Let $R$ be left serial. Let $x$ (resp. $y$) be in $T(eJ^i f)$ (resp. $T(fJ^j g)$), where $e$, $f$ and $g$ are primitive idempotents. If $J^{i+j}g \neq 0$, $xy \in T(eJ^{i+j}g)$.*

Proof. Assume that $xy \in eJ^{i+j+1}g$. Since $R$ is left serial, $Rx = J^i f$ and $Ry = J^j g$. Hence $J^{i+j}g = J^i Ry = J^i fy = Rxy \subset ReJ^{i+j+1}g \subset J^{i+j+1}g$. Therefore $J(J^{i+j}g) = J^{i+j}g$, which is a contradiction to the assumption $J^{i+j}g \neq 0$.

From now on for a left serial ring $R$ $A_\alpha$, $A_\beta \cdots$ are hollow modules in the diagram (1) and $A_{\alpha(i)}$ means a submodule of $A_i$. When we need to specify $|\alpha(i)| = t$, we denote $A_{\alpha(i)}$ by $A_{\alpha(i,t)}$.

**Lemma 2.** *Let $R$ be a left serial ring and $X$, $Y$ hollow right submodules in $R$. If $f: X/X_1 \approx Y/Y_1$ for some $X_1 \subset X$ and $Y_1 \subset Y$, there exists $d$ in $R$ such that $dX = Y$ (or $dY = X$). If $X_1 = J(X)$ and $Y_1 = J(Y)$, $d_l$, left-sided multiplication of $d$, induces $f$. In general, if $d_l$ induces $f$, $dX_1 \subset Y_1$. In particular, if $\bar{A}_{\alpha(i)} \approx \bar{A}_{\beta(j)}(|\alpha(i)| \leq |\beta(j)|)$, we can find such $d$ in $A_j$.*

Proof. Since $X$ is hollow, we can find a generator $x$ of $X$ with $xe = x$ for some primitive idempotent $e$. Put $f(x + X_1) = y + Y_1(y \in Y)$. $f$ being an isomorphism, $y$ is a generator of $Y$ and we may assume $ye = y$. Further we may assume $x \in T(J^i)$, $y \in T(J^j)$ and $i \leq j$ (if $j > j$, replace $X$ by $Y$). Then, since $Re$ is uniserial, there exists $d$ in $R$ such that $dx = y$. If $X_1 = J(X) = xJ$, $dX_1 = dxJ = yJ = Y_1$. Let $x_1$ be an element in $X_1$ and $x_1 = xr$; $r \in R$. Then $dx_1 = dxr = yr \in Y$ and $dx_1 + Y_1 = yr + Y_1 = f(xr + X_1) = Y_1$, provided that $d_l$ induces $f$. Hence $dX_1 \subset Y_1$ (see the proof of [3], Theorem 3). If $\bar{A}_{\alpha(i)} \approx \bar{A}_{\beta(j)}$, $da = b$, where $A_{\alpha(i)} = aR$, $A_{\beta(j)} = bR$ and $d \in eJe$. Let $d = \Sigma d_i$: $d_i \in A_i$. Since $b = da = \Sigma d_i a \in A_j$, $b = d_j a$.

**Lemma 3.** *Let $R$ be a left serial ring. Assume $\bar{A}_{\alpha(i,t)} \approx \bar{A}_{\beta(j,t)}$ for $A_{\alpha(i,t)} \subset A_i$ and $A_{\beta(j,t)} \subset A_j$. Then $A_{\alpha'(i,p)} \approx A_{\beta'(j,p)}$, provided $A_{\alpha'(i,p)} \supset A_{\alpha(i,t)}$ and $A_{\beta'(j,p)} \supset A_{\beta(j,t)}$.*

Proof. We show $\bar{A}_{\alpha'(i,p)} \approx \bar{A}_{\beta'(j,p)}$. From Lemma 2, there exists an element $d$ in $eRe$ such that $dA_{\alpha(i,t)} = A_{\beta(j,t)}$. Let $\pi_{\gamma(s)}$ be the projection of $eJ^p$ to $A_{\gamma(s,p)}$. Then $d_l | A_{\alpha'(i,p)} = (\sum_s \pi_{\gamma(s)}d_l) | A_{\alpha'(i,p)}$, where $d_l$ means the left-sided multiplication of $d$. If $\pi_{\beta'(j,p)}d_l | A_{\alpha'(i,p)}$ is an epimorphism, $\bar{A}_{\alpha'(i,p)} \approx \bar{A}_{\beta'(j,p)}$. Assume $\pi_{\beta'(j,p)}(dA_{\alpha'(i,p)}) \subset J(A_{\beta'(j,p)})$. Let $A_{\alpha(i,t)} = a_t R$ and $A_{\alpha'(i,p)} = a_p R$. Then $a_t = a_p x$; $x \in T(J^{t-p})$. $(\pi_{\beta'(i,p)}(eJ^p) \supset)A_{\beta(j,t)} = dA_{\alpha(i,t)} = da_t R = \sum(\pi_{\gamma(s)}da_t)R = \pi_{\beta'(j,p)}(da_p xR) \subset J(A_{\beta'(j,p)})xR \subset eJ^{t+1}$, a contradiction. Hence $\bar{A}_{\alpha'(i,p)} \approx \bar{A}_{\beta'(j,p)}$, and so $A_{\alpha'(i,p)} \approx A_{\beta'(j,p)}$ by Lemma 2 (note $A_{\alpha'(i,p)}$, $A_{\beta'(j,p)}$ are contained in $eJ^p$ but not in $eJ^{p+1}$).

**Lemma 4.** *Let $R$ be a left serial ring. Assume that $eJ = \sum \oplus A_i$, where the $A_i$ are hollow. Then there are no non-zero elements $d_1$, $d_2$ in $eJe$ such that $d_1 \in A_i$, $d_2 \in A_j (i \neq j)$ provided $\bar{A}_i \approx \bar{A}_j$.*

Proof. We may assume that $d_1 \in T(A_{\alpha(i)})$ and $d_2 \in T(A_{\alpha(j)})$. Put $t_1 = |\alpha(i)|$ and $t_2 = |\alpha(j)|$. Then $t_1 \neq t_2$ by assumption and Lemma 3, since $\overline{d_i R} \not\approx \overline{eR}$ for $i = 1, 2$. We can choose a pair $(d_1', d_2')$ such that $t_1' + t_2'$ is minimal $(t_1' < t_2')$. Then there exists $d_0$ in $A_j \cap eJ^{t_2'-t_1'}$ such that $d_0 d_1' = d_2'$ from Lemma 2, and so $(d_1', d_0)$ gives the contradiction to the minimality.

**Lemma 5.** *Let $A_i$ be as above. Assume that any pair of $\{A_1, A_2, A_3\}$ is not isomorphic to one another. Then, for any primitive idempotent $g$, there are no three elements $\{a_{ij_i}\}$ as follows: 1) $a_{1j_1} \in T(A_1 J^{j_1})$, $a_{2j_2} \in T(A_1 J^{j_2})$ and $a_{3j_3} \in T(A_2 J^{j_3})$ such that $a_{ij_i}g = a_{ij_i}$ and $j_1 < \min(j_2, j_3)$ or 2) $a_{ij_i} \in A_i$ and $a_{ij_i}g = a_{ij_i}$ for $i = 1, 2, 3$.*

Proof. Since $Ra_{ij_i} \subset Rg$ and $ea_{ij_i} = a_{ij_i}$, this is clear from Lemmas 2 and 4.

Now we study a right artinian ring with $(*, 1)$. For the sake of simplicity, we change the expression of a direct decomosition of $eJ^i$.

The following lemma is the "only if" part of Theorem.

**Lemma 6.** *Assume that $R$ is a right artinian ring with $(*, 1)$ as right $R$-modules. Put $eJ = A_1 \oplus B_1 \oplus \cdots$, $eJ^i = \sum_{k=1}^{n_i} \oplus A_{ik} \oplus \sum_{k=1}^{n_i'} \oplus B_{ik} \oplus \cdots$, where the $A_1$, $B_1$, $A_{ik}$, $B_{ik} \cdots$ are hollow and $A_1 \supset A_{ik}$, $B_1 \supset B_{ik}$, $\cdots$. If $\bar{A}_{i1} \approx \bar{B}_{j1}$ for $1 < i < j$, $B_{j-i+11}/J(B_{j1})$ is uniserial.*

Proof. Put $C_1 = \sum_{k=2}^{n_2} \oplus A_{2k} + \sum_{k=2}^{n_3} \oplus A_{3k} + \cdots + \sum_{k=1}^{n_i} \oplus A_{ik}$ and $D_1 = \sum_{k=2}^{n_2} \oplus A_{2k} + \cdots + \sum_{k=2}^{n_{i-1}} \oplus A_{i-1k} + J(A_{i1}) + \sum_{k=2}^{n_i} \oplus A_{ik}$. Then $f: C_1/D_1 \approx A_{i1}/J(A_{i1}) \approx B_{j1}/J(B_{j1})$. Assume that $f^{-1}$ is extendible to a $g'$ in $\mathrm{Hom}_R(B_1/J(B_{j1}), A_1/D_1)$. Since $B_1 J^i \supset B_1 J^{i-1} \supset B_{j1}$, $g'(B_1 J^i/J(B_{j1})) \supset g'(B_{j1}/J(B_{j1})) = f^{-1}(B_{j1}/J(B_{j1})) \neq 0$. On the other

hand, $g'(B_1J^i/J(B_{j1}))\subset(A_1/D_1)J^i=0$ for $A_1J^i\subset D_1$, which is a contradiction. Hence $f$ is extendible to a $g$ in $\mathrm{Hom}_R(A_1/D_1, B_1/J(B_{j1}))$ by Lemma A. Since $A_1/D_1$ is uniserial, $\mathrm{Soc}\,(A_1/D_1)=(A_{i1}+D_1)/D_1\approx A_{i1}/(J(A_{i1})$ and $g\,|\,A_{1i}/J(A_{i1})=f$ is an isomorphism, $g$ is a monomorphism. Consider a uniserial module $\tilde{B}_{i-1}=g((A_{i-11}+D_1)/D_1)\supset B_{j1}/J(B_{j1})$. Let $J(B_{j-11})=B_{j1}\oplus B_{j2}\oplus\cdots\oplus B_{jt}$. Assume $B_{j2}\neq 0$.

$$
\begin{array}{ccc}
 & & \rule{2cm}{0.4pt}\,|\, \\
\overline{B_{j-21}} & & B_{j-22} \\
| & & \\
\overline{B_{j-11}}\quad\cdots\quad B_{j-1t} & & \\
| & & \\
\overline{B_{j1}\quad B_{j2}\quad\cdots\quad B_{jt}} & & \\
| & & \\
J(B_{j1}) & & \\
| & &
\end{array}
$$

Then $\mathrm{Soc}(B_1/B_{j1})=\sum\limits_{k=2}^{t}\oplus\mathrm{Soc}(B_{jk})\oplus\sum\limits_{k=2}^{t'}\oplus\mathrm{Soc}(B_{j-1k})\oplus\cdots$. Since $\mathrm{Soc}(\tilde{B}_{i-1})=B_{j1}/J(B_{j1})$, $\tilde{B}_{i-1}$ is not uniserial. Hence $B_{j2}=B_{j3}=\cdots=B_{jt}=0$. Considering the same situation for $g((A_{i-21}+D_1)/D_1)$, we obtain similarly $B_{j-12}=\cdots=B_{j-1t'}=0$, where $J(B_{j-21})=B_{j-11}\oplus\cdots\oplus B_{j-1t'}\cdots$. Repeating this argument, we know $B_{j-i+11}/J(B_{j1})$ is uniserial.

In Lemma 6, if $i=j$, $B_1/J(B_{i1})$ is not uniserial in general (see Example 1 in [5]).

**Lemma 7.** *Let $R$ be left serial and $A_{\alpha(i,t)}$, $A_{\beta(i,t)}$ hollow modules in* (1). *Assume $\alpha(i, t)\neq\beta(i, t)$. Then there are no $d$ in $J$ such that $dA_{\alpha(i,t)}=dA_{\beta(i,t)}\neq 0$.*

Proof. Let $A_{\alpha(i,t)}=aR$, $A_{\beta(i,t)}=bR$ and assume that there exists $d$ such that $daR=dbR$. Then there exists $r$ in $R$ such that $dar=db$, and so $d(ar-b)=0$. On the other hand, $ar-b\neq 0$ by assumption, and there exists a primitive idempotent $g$ such that $db=dbg$. Further $d\in T(J^s)$ for some $s$ and $0\neq db\in T(J^{t+s}g)$ by Lemma 1. Hence $d(ar-b)\in T(J^{t+s}g)$, a contradiction to Lemma 1.

The following is the "if" part of Theorem.

**Lemma 8.** *Let $R$ be a left serial ring with $e\overline{J}$ square-free. If $R$ satisfies the result in Lemma 6, then* $(*, 1)$ *holds for any hollow module.*

Proof. First we shall show from the assumption that

( 4 )  there are no three distinct hollow modules $A_{\alpha(i,s)}$, $A_{\beta(i,s)}$ and $A_{\gamma(j,t)}$ such that $\overline{A}_{\alpha(i,s)}\approx\overline{A}_{\beta(i,s)}\approx\overline{A}_{\gamma(j,t)}$ and $i\neq j$, $s<t$.

Contrarily we assume that there exist such modules. Put $A_{\alpha(i,s)}=aR$, $A_{\beta(i,s)}=$

$bR$ and $A_{\gamma(j,t)}=cR$. $A_{\gamma'(j,t-s+1)}/J(A_{\gamma(j,t)})$ is unisreial for some $\gamma'(j, t-s+1)>$ $\gamma(j, t)$ by assumption. We may assume that $ag=a$, $bg=b$ and $cg=c$ for some primitive idempotent $g$. Then there exists $d'$ in $T(eJ^{t-s}e)$ such that

$$(5) \qquad\qquad d'a = c$$

by Lemmas 1 and 2. Let $A_i=a_iR$ and $d'a_i=\sum_j v_j$ as in (3) ($a_ih=a_i$, $v_jh=v_j$ for some primitive idempotent $h$). $a$ being in $A_i$, $a=a_ir$ for some $r$ in $J$. Then $c=d'a=d'a_ir=\sum_j v_jr$. Hence from the observation after (2)

$$(6) \qquad c=v_kr \text{ for some } k \text{ (say 1)} \quad \text{and} \quad v_{k'}r = 0 \quad \text{for } k'\neq 1.$$

$c=v_1r$ implies that $v_1\in A_{\delta(j)}$ and $\delta(j)>\beta(j,t)$. Further $a=a_ir$ implies $r\in$ $T(hJ^{s-1}g)$, and so $v_1\in T(J^{t-s+1}h)$ by Lemma 1. Hence $v_1\in A_{\gamma'(j,t-s+1)}$ (note $A_{\gamma(j,t)}\subset A_{\gamma'(j,t-s+1)}$). Now there exists $d$ in $T(eJ^{t-s}e)\cap A_j$ with $v_1=da_i$, and so

$$(7) \qquad\qquad da = da_ir = v_1r = c \quad \text{and} \quad da_i = v_1$$

by (6). Let $b=a_ir'$ for some $r'$ in $R$. Then $r'\in T(J^{s-1})$ and $db=da_ir'=$ $v_1r'\in A_{\gamma'(j,t-s+1)}\cap T(J^t)=A_{\gamma(j,t)}$, since $A_{\gamma'(j,t-s+1)}/J(A_{\gamma(j,t)})$ is uniserial. Hence $dA_{\alpha(i,s)}=dA_{\beta(i,s)}\neq 0$, which is a contradiction to Lemma 7. Thus we have shown (4). Now let $A_1\supset C_1\supset D_1$ and $A_2\supset C_2\supset D_2$ be submodules such that $f: C_1/D_1\approx C_2/D_2$ and $C_1/D_1$ is simple. Let $c_i=\sum_k z_{ik}$ be a generator of $C_i$ for $i=1, 2$, where $z_{ik}\in T(A_{\alpha_{ik}})$ from (3) and $z_{ik}g=z_{ik}$ for some primitive idempotent $g$. We choose a generator $c_i\in C_i$ with $\min_{i=1,2}\{\min_k |\alpha_{ik}|\}$. We may assume that $c_1=\sum_k z_{1k}$ and $|\alpha_{11}=11\cdots1|$ is minimal. Then $c_1=z_{11}$ ($=x$) from (4) and Lemmas 3 and 5. We shall take the following $D_1^*$ similarly to $D_1$ in the proof of Lemma 6: $D_1^*=A_{12}\oplus A_{13}\oplus\cdots\oplus A_{112}\oplus A_{113}\oplus\cdots\oplus A_{11\cdots12}\oplus\cdots(11\cdots1=\alpha_{11})$. We shall show

$$(8) \qquad\qquad D_1\subset J(A_{\alpha_{11}})\oplus D_1^*.$$

Let $y$ be an element in $D_1$ and $y=\sum y_i$; $y_i\in A_{\alpha_i}$ as in (3). Assume $\alpha_1\geqslant\alpha_{11}=$ $11\cdots1$. Then there exists $r$ in $Rg$ such that $y_1r=x$. $x-yr=\sum_{i\geqslant2}-y_ir$ is also a generator of $C_1$ and contained in $D_1^*$. Since $A_{\alpha_{11}}\cap D_1^*=0$, we obtain a contradiction from (4) and Lemma 5. Hence $\alpha_1<\alpha_{11}$, and so (8) is true. We choose a representative $w$ in $C_2$ of $f(x)$ such that $wg=w$. Let $w=\sum_{i=1}^r w_i(w_ig=w_i)$ as in (3); $w_i\in A_{\beta(2)}$. Since $w_1g=w_1$ and $|\alpha_{11}|$ is minimal, there exists $d_1'$ in $A_2$ with $d_1'x=w_1$. Now we have the same situation as (5). Hence from the argument after (5), similarly to (7) there exists $d_1$ in $T(eJ^{t-s}e)\cap A_2$ such that $d_1a_1\in$ $T(A_{\beta'(t,t-s+1)})$ and $d_1x=w_1$, where $t=|\beta_1(2)|$, $s=|\alpha_{11}|$ and $\beta'(2,t-s+1)>\beta_1(2)$. Let $\gamma(1,q)=11\cdots1$ and $\gamma'(1,q)$ be two distinct indices with $1<q\leqslant s$. Let

$p \in D_1^* \cap A_{\gamma'(1,q)}$ and $a_q$ a generator of $A_{\gamma(1,q)}$. Then $x = a_q r'$ for some $r'$ in $R$. $d_1 a_q r' = d_1 x = w_1 \neq 0$, and so $d_1 a_q (\neq 0) \in \mathrm{T}(eJ^{t-s+q})$. If $d_1 p \neq 0$, $d_1 p \in \mathrm{T}(eJ^{t-s+q})$ by Lemma 1 and $d_1 p$ and $d_1 a_q$ generate a same submodule between $A_{\beta'(2,t-s+1)} = d_1 A_1$ and $A_{\beta_1(2)}$, since $A_{\beta'(2,t-s+1)} / \mathrm{J}(A_{\beta_1(2)})$ is uniserial, which is a contradiction to Lemma 7. Therefore $d_1 p = 0$ and so $d_1(D_1^*) = 0$. Similarly we can find $d_i \in A_2$ such that $d_i x = w_i$ for each $w_i$ and $d_i(D_1^*) = 0$. Put $d^* = \sum_{i=1}^{r} d_i$. Then $d^* x = \sum d_i x = \sum w_i = w$. Let $u$ be any element in $D_1$, then from (8) $u = u_1 + u_2$, where $u_1 \in \mathrm{J}(A_{\alpha_{11}})$, $u_2 \in D_1^*$. We denote $u_1 = xj$; $j \in J$. $d^* u = d^* u_1 + d^* u_2 = d^* u_1 = d^* xj = wj \in C_2 J \subset D_2$. Therefore $d_1^*$, left-sided multiplication of $d^*$, is the desired extension of $f$.

**Proposition.** *Let $R$ be a left serial ring with $(*, 1)$. Let $\alpha(1, s) = 11 \cdots 1$, $\beta(2, t) = 211 \cdots 1$ $(s < t)$ and $|A_{\beta(2,t)}| = k$. If $\bar{A}_{\alpha(1,s)} \approx \bar{A}_{\beta(2,t)}$, then $A_{\beta'(t-s+1)} \approx A_1/D$, where $\beta'(2, t-s+1) = 21 \cdots 1$ and after renumbering $A_{\alpha(1,p)}$ for all $p$, $D = A_{12} \oplus A_{13} \oplus \cdots \oplus A_{112} \oplus A_{113} \oplus \cdots \oplus \mathrm{J}(A_{\gamma(1,s+k-1)})$; $\gamma(1, s+k-1) = 11 \cdots 1$. Hence $A_{\beta'(2,t-s+1)}$ is uniserial.*

Proof. Since $\bar{A}_{\alpha(1,s)} \approx \bar{A}_{\beta(2,t)}$, there exists $d$ in $A_2$ such that $dA_{\alpha(1,s)} = A_{\beta(2,t)}$ by Lemma 2, and so $d\mathrm{J}(A_{\alpha(1,s)}) = \mathrm{J}(A_{\beta(2,t)})$. Let $\mathrm{J}(A_{\alpha(1,s)}) = A_{11\cdots1} \oplus A_{11\cdots12} \oplus \cdots$ and $\mathrm{J}(A_{\beta(2,t)}) = A_{21\cdots1} \oplus A_{21\cdots12} \oplus \cdots$. Since $d\mathrm{J}(A_{\alpha(1,s)}) = \mathrm{J}(A_{\beta(2,t)})$ and $A_{21\cdots1}$ is hollow, $\bar{A}_{21\cdots1} \approx \bar{A}_{11\cdots1k}$ for some $k$. Then $A_{21\cdots1k'} = 0$ for $k' \neq 1$ by Theorem. Repeating this argument, we know that $A_{\beta(2,t)}$ is uniserial and $A_{\beta''(2,t+k-1)}$ is the socle of $A_{\beta(2,t)}$ and is isomorphic to $A_{\alpha'(1,s+k-1)}$. There exists $d'$ in $J$ from (7) such that $d'A_1 = A_{\beta'(2,t-s+1)}$ and $d'A_{\alpha'(1,s+k-1)} = A_{\beta''(2,t+k-1)}$. Therefore we obtain the proposition from the last part of proof of Theorem.

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