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AN EXTENSION OF WHITNEY'S CONGRUENCE

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1. Introduction and Main results

Throughout this paper, we will work in the PL category, and all embeddings will be locally flat.

Let M be a connected and oriented 4-manifold, F a closed and connected surface of Euler characteristic $\chi(F)$. For a given embedding of F into M ($F \subset M$), let $e(M, F)$ be the normal Euler number of it, and let $[F]$ be the element in $H_2(M; \mathbb{Z}_2)$ represented by F in M . We are interested in the relation between $e(M, F)$ and $[F]$. In the case of $M = S^4$, the following theorem is well-known.

Theorem 1.1 (H. Whitney [8]: Whitney's congruence). If $M = S^4$,

$$e(M, F) + 2\chi(F) \equiv 0 \pmod{4}.$$

For some time, we assume that M is closed and $H_1(M; \mathbb{Z}) = \{0\}$. We will define a \mathbb{Z}_4 -quadratic map q from $H_2(M; \mathbb{Z}_2)$ to \mathbb{Z}_4 as follows. By the assumption $H_1(M; \mathbb{Z}) = \{0\}$, the mod 2-reduction map p_2 from $H_2(M; \mathbb{Z})$ to $H_2(M; \mathbb{Z}_2)$ is surjective. For a given element α in $H_2(M; \mathbb{Z}_2)$, we define $q(\alpha)$ by

$$q(\alpha) \equiv \tilde{\alpha} \circ \tilde{\alpha} \pmod{4},$$

where $\tilde{\alpha}$ is an element of $p_2^{-1}(\alpha)$ and \circ is the intersection form on $H_2(M; \mathbb{Z})$.

The well-definedness of q is easy to see, and q is \mathbb{Z}_4 -quadratic, i.e.,

$$q(\alpha + \beta) \equiv q(\alpha) + q(\beta) + 2(\alpha \bullet \beta) \pmod{4},$$

where \bullet is (\mathbb{Z}_2 -valued) intersection form on $H_2(M; \mathbb{Z}_2)$, and $2: \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ is the natural embedding.

Using the quadratic function q , we extend Theorem 1.1 as follows:

Theorem 1.2.

$$e(M, F) + 2\chi(F) \equiv q([F]) \pmod{4}.$$

It is well-known that if $F \subset M$ is characteristic (i.e., $[F]$ is dual to the 2nd Stiefel-Whitney class $w_2(M)$), then $\sigma(M) \equiv [F] \circ [F] \pmod{8}$, where $\sigma(M)$ is the signature of M . Thus we have

Corollary 1.3 (V.A. Rochlin [5], see also [4]: Generalized Whitney's congruence). *If $F \subset M$ is characteristic,*

$$e(M, F) + 2\chi(F) \equiv \sigma(M) \pmod{4}.$$

Theorem 1.2 can be extended to the general case in which the only assumption on M is its orientability; we need not assume that $H_1(M; \mathbb{Z}) = \{0\}$ nor that M is closed. M can even be non-compact. In fact, we can prove the following.

Theorem 1.4. *Let M be an oriented 4-manifold. A map which assigns $e(M, F) + 2\chi(F) \pmod{4}$ to an embedding $F \subset M$ induces a \mathbb{Z}_4 -quadratic map from $H_2(M; \mathbb{Z}_2)$ to \mathbb{Z}_4 . We will also call it q .*

For immersions from F into M , we have the following.

Corollary 1.5. *Let M be an oriented 4-manifold, F a closed surface immersed in M with only normal crossings. Then*

$$e(M, F) + 2\chi(F) + 2\# \text{self}(F) \equiv q([F]) \pmod{4},$$

where $\# \text{self}(F)$ is the number of self-intersection points of F .

After writing the first version of this paper, we were informed by Prof. B.-H. Li that he found a general formula which includes our theorem 1.4([3]). In fact, he works in $(2n, n)$ -dimensional case. His proof is homotopy-theoretic, on the other hand, ours is geometric.

2. A Connected sum formula

For a given embedding $F \subset M$, assume that there is a connected sum decomposition of M :

$$M = M_1 \# M_2 = \text{punc } M_1 \bigcup_{\partial} \text{punc } M_2,$$

such that each embedding $F_i \subset \text{punc } M_i$ is proper (i.e., $F_i \cap \partial(\text{punc } M_i) = \partial F_i$), where $\text{punc } M_i$ is M_i with an open 4-ball deleted, and $F_i = F \cap \text{punc } M_i$, for $i = 1, 2$. Here we assume that F intersects $\partial(\text{punc } M_i)$ transversely. The symbol \bigcup_{∂} on the right-hand side means disjoint union with boundary identified by an orientation reversing homeomorphism.

Then

$$(\partial \text{punc } M_1, \partial F_1) = (\partial \text{punc } M_2, \partial F_2) \cong (S^3, L)$$

for a certain link L in S^3 .

Let S be a (connected) Seifert Surface for L in S^3 , and regard it as being in $S^3 = \partial B^4 : S \subset S^3 = \partial B^4 \subset B^4$. Let (M_i, \hat{F}_i) denote $(\text{punc } M_i, F_i) \cup_{\partial} (B^4, S)$, for $i=1, 2$. Now, we have

Lemma 2.1 Connected Sum Formula.

Let M_i, F, \hat{F}_i be as above. Then

$$e(M_1 \# M_2, F) = e(M_1, \hat{F}_1) + e(M_2, \hat{F}_2).$$

In particular,

$$e(M_1 \# M_2, F_1 \# F_2) = e(M_1, F_1) + e(M_2, F_2).$$

Proof. Let ν be a non-zero, normal vector field over S in S^3 . We can take a transverse push-off F' of F in M such that $F' \cap (\partial \text{punc } M_1) = \nu(L)$. Then

$$\begin{aligned} e(M, F) &= \sum_{p \in F \cap F'_1} \text{sign}(p) + \sum_{p \in F \cap F'_2} \text{sign}(p) \\ &= \sum_{p \in F_1 \cap F'_1} \text{sign}(p) + \sum_{p \in F_2 \cap F'_2} \text{sign}(p), \end{aligned}$$

where F'_i is $F' \cap \text{punc } M_i$, for $i=1, 2$. On the other hand, if we regard $F'_i \cup_{\partial} \nu(S)$ as a push-off of \hat{F}_i in M_i , then

$$e(M_i, \hat{F}_i) = \sum_{p \in F'_i \cap F'_i} \text{sign}(p).$$

Thus we have the lemma. ■

3. Proof of Theorem 1.2

The proof is divided into 3 steps. We are given an embedding $F \subset M$.

(Step 1) We will show the theorem for $M = mCP^2 \# n\overline{CP^2} (m+n > 0)$. We have a standard handlebody decomposition of M :

$$M = H^0 \cup \left(\bigcup_{i=1}^{m+n} H_i^2 \right) \cup H^4,$$

where H^r is an r -handle, and fix an identification

$$h_i^2 : D^2 \times D^2 \xrightarrow{\cong} H_i^2.$$

Without loss of generality (by general position argument), we can assume the following.

- (1) $F \cap H^4 = \phi$.
- (2) $F \cap H_i^2 = h_i^2(D^2 \times \{\text{finite points in int } D^2\})$,

i.e., each component of $F \cap H_i^2$ is parallel to the core of H_i^2 .

We regard M as $S^4 \# M$ by $H^0 \cup (\bigcup_{i=1}^{m+n} H_i^2 \cup H^4) = B^4 \cup_{\partial} \text{punc } M$, and use the notation “ M, F, \hat{F}_i ” as in the last section ($M_i = S^4, M_2 = M$). Note that F_2 consists of some proper disks, and F_1 is F with some open 2-disks deleted.

We orient all the components of F_2 , and take a Seifert surface S so that the orientation of S is compatible with that of F_2 . Note that $\hat{F}_2 (= S \cup F_2)$ is an orientable closed surface.

In the situation above, we have the following equalities.

- (1) $e(M, F) = e(S^4, \hat{F}_1) + e(M, \hat{F}_2)$
- (2) $e(S^4, \hat{F}_1) + 2\chi(\hat{F}_1) \equiv 0 \pmod{4}$
- (3) $e(M, \hat{F}_2) + 2\chi(\hat{F}_2) \equiv e(M, \hat{F}_2) \equiv q([\hat{F}_2]) \pmod{4}$
- (4) $[\hat{F}_2] = [F]$ in $H_2(M; Z_2)$
- (5) $\chi(\hat{F}_1) + \chi(\hat{F}_2) \equiv \chi(F) \pmod{2}$

The first holds by the connected sum formula, the second by Theorem 1.1, the third follows from the orientability of \hat{F}_2 , and the others are easy to verify. Now the theorem in this case follows from these equalities.

(Step 2) We will prove the theorem for a simply-connected manifold. We use the following fact [7], [4: Fact (2)].

Fact. *Let M be a simply-connected, closed and oriented 4-manifold. Then there exist integers $l, m, n \geq 0$ such that*

$$M \# (l+1)\overline{CP^2} = mCP^2 \# n\overline{CP^2}.$$

For a given embedding $F \subset M$, we take a connected sum $M \# (l+1)\overline{CP^2}$ disjointly from the neighborhood of F . It is easy to see that $e(M, F), \chi(F)$ and $q([F])$ are unchanged by the connected sum. Thus the proof is reduced to the first step.

(Step 3) The general case ($H_1(M; Z) = \{0\}$). For a given embedding $F \subset M$, and an element γ of $\pi_1(M)$, we take an embedded circle c in M such that

- (1) c represents the element γ ,
- (2) $c \cap F = \phi$, and

- (3) c bounds an immersed oriented surface G in M which satisfies the following condition: for each generator x of $H_2(M; Z)$, there is a representing surface T_x such that $G \circ T_x = 0$ and $G \circ F = 0$.

This is possible because of the assumption $H_1(M; Z) = \{0\}$ and $\partial G \neq \emptyset$. We do surgery on M along c , and repeat it till $\pi_1(M)$ becomes trivial. At each surgery, $e(M, F)$, $\chi(F)$ and $q([F])$ remain unchanged. We see it as follows ([6]). Suppose that we get

$$M' = D^2 \times S^2 \cup_{\varphi|\partial} \{ M \setminus \text{int}(\varphi(S^1 \times D^3)) \}$$

from M by surgery along c , where φ is a trivialization of a tubular neighborhood of c . Then the homology of M changes into

$$H_2(M'; Z) \cong H_2(M; Z) \oplus Z\langle x \rangle \oplus Z\langle y \rangle,$$

where $x = [(D^2 \times *) \cup G]$ and $y = [* \times S^2]$. Thus the intersection form changes as

$$H_2(M'; Z) \cong H_2(M, Z) \oplus \begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix}.$$

Under the isomorphism, the correspondence of $[F]_{old}$ and $[F]_{new}$ is:

$$[F]_{new} \leftrightarrow [F]_{old} + 0 + 0.$$

Thus $q([F])$ is unchanged, and the proof is reduced to the second step. ■

4. Proof of Theorem 1.4

This proof is divided into 3 steps. We are given an embedding $F \subset M$. If M is closed and $H_1(M; Z) = \{0\}$, then Theorem 1.2 applies. We will consider other cases step by step.

(Step 1) Suppose that M is closed but $H_1(M; Z) \neq \{0\}$. We perform surgery along embedded circles c which are disjoint from F and represent non-zero elements of $H_1(M; Z)$.

Suppose that two embedded surfaces $F_1, F_2 \subset M$ satisfy $[F_1] = [F_2]$ in $H_2(M; Z_2)$ and they are in general position. In Z_2 -coefficient chain complex, we can take a 3-chain Δ^3 whose boundary is $F_1 + F_2$. We will show that we can do surgery (along c to get M' from M) so that F_1 and F_2 also satisfy $[F_1] = [F_2]$ in $H_2(M'; Z_2)$.

Since Δ has boundary $F_1 + F_2$ with Z_2 -coefficient, the small normal circle of F_i intersects Δ at an odd number of points. If necessary, by connecting c with the small normal circle along Δ , we can choose c such that the geometric intersection number $\#(c \cap \Delta)$ is even and $N(c) \cap \Delta$ consists of an even number of 3-balls B ,

where $N(c)$ is a thin tubular neighborhood of c . Then in $D^2 \times S^2$ which is to be attached to $M \setminus \text{int} N(c)$, we can take $\{\text{half as many proper arcs}\} \times S^2$ whose boundary is the same as $(\partial N(c), \partial B)$. Then it is clear that $F_1 + F_2$ bounds a new 3-chain Δ' in M' with Z_2 -coefficient.

(Step 2) Suppose that M is compact but $\partial M \neq \emptyset$.

Let DM be the double of M ($DM = M \cup_{\partial} -M$). The mapping q is already well-defined over DM by Step 1. Over M , the mapping is the composition $q \circ i_*$, where i_* is the homology homomorphism: $H_2(M; Z_2) \rightarrow H_2(DM; Z_2)$, induced by canonical inclusion i .

(Step 3) Suppose that M is non-compact. There is a sequence of countably many compact oriented 4-manifolds and inclusions:

$$M_1 \subset M_2 \subset M_3 \subset \dots \text{ such that } \bigcup_{i=1}^{\infty} M_i = M.$$

Since F is compact, there is a sufficiently large n such that $F \subset M_n$. We can apply the method in Step 2. ■

5. Proof of Corollary 1.5

We are given an immersed surface F in M with only normal crossings. For a crossing point p , we take a 4-ball neighborhood B around p . To remove the crossing at p , we cut out $\text{int} B \cap F$ from F , where $\partial B \cap F \subset \partial B$ is a Hopf link, and glue in an annulus $A \subset \partial B$. We call this new surface \tilde{F} . By the construction, $[\tilde{F}] = [F]$ in $H_2(M; Z_2)$, $\chi(\tilde{F}) \equiv \chi(F) \pmod 2$ and $\# \text{self}(\tilde{F}) = \# \text{self}(F) - 1$.

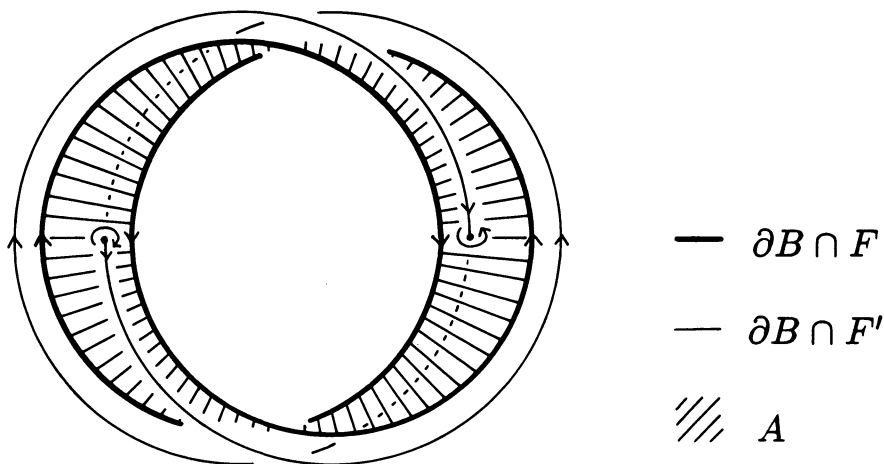


Figure 1

We show $e(M, \tilde{F}) = e(M, F) \pm 2$. Let F' be a push-off of F . We can assume that F' is parallel to F near p and in particular $\partial B \cap F'$ gives a trivial framing for each component of $\partial B \cap F$ in $\partial B \cong S^3$. Then we can take an annulus A such that $\partial A = \partial B \cap F$ and $A \cap (\partial B \cap F')$ consists of two points whose signs are the same (Figure 1). Let A' be a push-off of A which is properly embedded in B^4 such that $\partial A' = \partial B \cap F'$. If we regard $(F' \setminus \text{int } B) \cup_{\partial} A'$ as a push-off of \tilde{F} , we have the claim.

We can repeat the above process till $\# \text{self}(F)$ becomes zero without changing both sides of the congruence. Thus we can reduce the corollary to Theorem 1.2 or 1.4. ■

6. Examples

In this section, we will give two examples for Theorem 1.2.

Example 1. (I2) Let $M = m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ ($m+n > 0$), and identify its 2nd homology $H_2(M; \mathbb{Z}_2)$ with $\bigoplus_{i=1}^m \mathbb{Z}_2 \langle \xi_i \rangle \oplus \bigoplus_{j=1}^n \mathbb{Z}_2 \langle \eta_j \rangle$. For an embedding $F \subset M$, such that $[F] \equiv \sum_{i=1}^k \xi_i + \sum_{j=1}^l \eta_j$

$$e(M, F) + 2\chi(F) \equiv k - l \pmod{4}.$$

Example 2. Let $M = S^2 \times S^2$, and identify its 2nd homology $H_2(S^2 \times S^2; \mathbb{Z})$ with $\mathbb{Z} \langle x \rangle \oplus \mathbb{Z} \langle y \rangle$, where $x \circ x = y \circ y = 0$ and $x \circ y = y \circ x = 1$. Let $S^2(m) \subset S^2 \times S^2$ be an embedding of S^2 representing $1 \cdot x + m \cdot y$, which is for instance the graph of a degree m map $g_m: S^2 \rightarrow S^2$. Then

$$e(S^2 \times S^2, S^2(m)) = 2m.$$

As an element of $H_2(S^2 \times S^2; \mathbb{Z}_2) \cong \mathbb{Z}_2 \langle \underline{x} \rangle \oplus \mathbb{Z}_2 \langle \underline{y} \rangle$,

$$[S^2(m)] \equiv \begin{cases} \underline{x} & \text{if } m \text{ is even} \\ \underline{x} + \underline{y} & \text{if } m \text{ is odd} \end{cases}.$$

Thus we have

$$q([S^2(m)]) \equiv \begin{cases} 0 & \text{if } m \text{ is even} \\ 2 & \text{if } m \text{ is odd} \end{cases} \pmod{4}.$$

Example 2 shows that our main theorem is optimal in a sense.

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