<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Perfect categories. III. Hereditary and QF-3 categories</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Harada, Manabu</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 10(2) P.357-P.367</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1973</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/11420">https://doi.org/10.18910/11420</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/11420</td>
</tr>
</tbody>
</table>

Osaka University Knowledge Archive : OUKA
https://ir.library.osaka-u.ac.jp/repo/ouka/all/

Osaka University
PERFECT CATEGORIES III
(HEREDITARY AND QF-3 CATEGORIES)

MANABU HARADA

(Received July 24, 1972)

Recently the author has defined perfect or semi-artinian Grothendieck categories with some assumptions [8], as a generalization of categories of modules in [1].

Further he has generalized essential results in [6] to such categories [9]. This note is a continuous work to give a generalizations of results in [3], [4] and [5].

Let \( R \) be a ring with identity. R.M. Thrall defined a \( QF-3 \) algebra in [3] and many authors defined \( QF-3 \) rings and studied them (cf. [10]).

\( R \) is called right \( QF-3 \) if \( R \) has a minimal a faithful right \( R \)-module and \( R \) is called right \( QF-3^+ \) if the injective hull \( E(R_R) \) is projective, (see [2]).

We generalize those concepts to semi-perfect Grothendieck categories \( \mathcal{A} \) with generating set of finitely generated objects, (which are equivalent to group valued functor categories \((\mathbb{C}^\circ, Ab)\) by [8], Theorem 3).

We shall completely determine structures of hereditary (more weakly locally \( PP \)) and perfect \( QF-3 \) (resp. \( QF-3^+ \)) or semi-perfect and semi-artinian \( QF-3 \) (resp. \( QF-3^+ \), however this is a case of \( QF-3 \)) categories \( \mathcal{A} \). Furthermore, we shall show that \( \mathcal{A} \) is equivalent to product of \( \mathcal{A}_a \) and \( \mathcal{A}_a \) is the full subcategory \( \mathcal{M}_S^{\pm1} \), where \( S \) is the ring of upper (resp. lower) tri-angular matrices of a division ring over a well ordered set \( I \), almost all of whose entries are zero, such that if \( \mathcal{A} \) is \( QF-3 \) \( I \) has the last element (resp. if \( \mathcal{A} \) is semi-artinian \( QF-3^+ \), then \( I \) has the last element and hence, \( \mathcal{A} \) is \( QF-3 \)) and vice versa with some restrictions. Those results are generalizations of [4] and [5].

1. Preliminary results

Let \( \mathcal{A} \) be a Grothendieck category with generating set of finitely generated objects. If every object (resp. finitely generated object) has a projective cover, then \( \mathcal{A} \) is called \textit{perfect} (resp. \textit{semi-perfect}). On the other hand, if every non-zero object has the non-zero socle, \( \mathcal{A} \) is called \textit{semi-artinian}.

1) see §1.
If $\mathfrak{A}$ is semi-perfect, then $\mathfrak{A}$ has a generating set of completely indecomposable projective $\{P_\alpha\}$. Let $(\{P_\alpha\}^\circ, Ab)$ be the additive contravariant functor category of the pre-additive category $\{P_\alpha\}$ to the category $Ab$ of abelian groups. Put $R=\sum_{\alpha, \beta \in I} \oplus [P_\alpha, P_\beta]$. Then $R$ is called the induced ring from $\mathfrak{A}$ by $\{P_\alpha\}$.

By $e_\alpha$ we shall denote idempotents $1_{P_\alpha}$ in $R$. Let $\mathfrak{M}_R$ be the category of all right $R$-modules. By $\mathfrak{M}_R^+$ we denote the full subcategory of $\mathfrak{M}_R$ whose objects consist of all $M$ such that $MR=M$. Then

**Theorem A** ([8], Theorem 3). *Let $\mathfrak{A}$ be as above. Then the following are equivalent.*

1) $\mathfrak{A}$ is semi-perfect.
2) $\mathfrak{A} \cong (\{P_\alpha\}^\circ, Ab)$.
3) $\mathfrak{A} \cong \mathfrak{M}_R^+$.

In this note, we only consider a semi-perfect category $\mathfrak{A}$ and hence, $\mathfrak{A}$ will be identified with $(\{P_\alpha\}^\circ, Ab)$ or $\mathfrak{M}_R^+$ in the following. We note in this case $e_\alpha R$ corresponds to $P_\alpha$ and $e_\alpha Re_\beta \cong [P_\beta, P_\alpha]$.

We shall make use of same notations in [8] and [9] without further comments and categorical terminologies in [11]. Rings in this note do not contain identities in general.

### 2. Locally PP-categories

Let $\mathfrak{A}$ be a semi-perfect Grothendieck category with a generating set of finitely generated. If $\{P_\alpha\}$ and $\{Q_\beta\}$ are generating sets of $\mathfrak{A}$ such that $P_\alpha$ and $Q_\beta$ are completely indecomposable and projecte, then $P_\alpha$ is isomorphic to some $Q_\beta$ and vice versa by Krull-Remak-Schmidt’s theorem. Let $R$ be the induced ring from $\mathfrak{A}$ by $\{P_\alpha\}$, $R=\sum_{\alpha, \beta \in I} \oplus [P_\alpha, P_\beta]$. If $fR$ is projective in $\mathfrak{M}_R^+$ for any $\alpha$ and $\beta$, any element $f$ in $[P_\alpha, P_\beta]$, $\mathfrak{A}$ is called a locally (right) PP-category, (we called it "partially" in [3]).

This is equivalent to a fact that every functor $T_f$ in $(\{P_\alpha\}^\circ, Ab)$ defined by $T_f(P_\gamma)=fe_\gamma$ is representative for every $f \in [P_\alpha, P_\beta]$. We define similarly a left PP-category.

We can easily see from the following lemma that right PP-categories are also left PP-categories and that this definition does not depend on $\{P_\alpha\}$.

**Lemma 1.** *Let $\mathfrak{A}$ be a semi-perfect Grothendieck category with a generating set $\{P_\alpha\}$ as above. Then $\mathfrak{A}$ is locally PP if and only if any $f \in [P_\alpha, P_\beta]$ is zero or monomorphic, (cf. [9], Proposition 3).*

**Proof.** We assume that $\mathfrak{A}$ is locally PP and $0 \neq f \in [P_\alpha, P_\beta]$. Since $fe_\alpha=f$, $0 \leftarrow fR \leftarrow e_\alpha R$ is exact. Further, $e_\alpha R$ is indecomposable, and hence, $fR \cong e_\alpha R$.  


Put \( K = \text{Ker} \, f \) and \( i : K \to P \). If \( i \neq 0 \), there exists \( P, h \in [P, K] \) such that \( 0 = \text{ih} \in [P, P_a] \subseteq R \). Then \( 0 = f \text{ih} = f e_a \text{ih} \) and \( e_a \text{ih} \in e_a R \). Hence, \( \text{ih} = e_a \text{ih} = 0 \), which is a contradiction. Therefore, \( f \) is monomorphic. Conversely, if \( f \) is monomorphic, then a mapping \( \psi : fR \to e_a R (\psi(fr) = e_a r) \) is isomorphic. Hence, \( fR \) is projective in \( \mathcal{M}_R \).

As an analogy of Theorem 4 in \cite{9}, we have

**Theorem 1** ([\cite{9}]). Let \( \mathcal{A} \) be a semi-perfect Grothendieck category with generating set of finitely generated object. Then \( \mathcal{A} \) is locally PP and perfect (resp. semi-artinian) if and only if \( \mathcal{A} \) is equivalent to \([I, \mathcal{A}_i]^\ast\) (resp. \([I, \mathcal{A}_i]^\ast\)') with functors \( T_{ij} \) such that \( \psi_{kji} : T_{kj}(B) \to T_{kj}(P) \) for \( k > j > i \) (resp. \( k < j < i \)) is monomorphic, for any minimal object \( B \) in \( T_{ij}(P) \) and \( P \in \mathcal{A}_i \), where \( \mathcal{A}_i \)'s are semi-simple categories with generating sets.

**Proof.** We assume that \( \mathcal{A} \) is locally PP and \( \{P_a\} \) is a generating set of completely indecomposable projectives. Making use of Lemma 1 and the proof of Theorem 4 in \cite{9} we know that \( \mathcal{A} \) is equivalent to \([I, \mathcal{A}_i]^\ast\) (resp. \([I, \mathcal{A}_i]^\ast\)') and that \( \{P_a^{(i)} = S_i(P_i)\} \) (resp. \( \{S_i(P_i)\}\)) is a generating set in \([I, \mathcal{A}_i]^\ast\) (resp. \([I, \mathcal{A}_i]^\ast\)'), where \( \{P_i\} \) is a generating set of \( \mathcal{A}_i \) and \( P_i \) is minimal. Since \( f \in [P_a^{(i)}, P_b^{(j)}] \) is monomorphic by Lemma 1, we have the conditions in the theorem. The converse is also clear from the structure of \([I, \mathcal{A}_i]^\ast\) (resp. \([I, \mathcal{A}_i]^\ast\)') and Lemma 1.

**Remark.** If we replace a minimal objects \( B \) in the above condition by any finite coproduct of \( B_a \), it is equivalent to the condition \((*)-1)\) in Theorem 3 in \cite{9}. Hence, this fact gives us the deference between semi-hereditaty and locally PP.

We have immediately from Lemma 1. \cite{9}, Propositions 3 and 5 and their proofs

**Theorem 2.** Let \( \mathcal{A} \) be as in Theorem 1 and \( \{P_a\} \) a generating set of completely indecomposable projectives. If \( \mathcal{A} \) is locally PP, then the following are equivalent.

1) All \( P_a \) are \( J \)-nilpotent.
2) \( 1L(P_a) < \infty \) for all \( \alpha \).
3) \( \mathcal{A} \) is semi-artinian.

Furthermore, the following are equivalent.

1) \( rL(P_a) < \infty \) for all \( \alpha \).
2) \( \mathcal{A} \) is perfect, (cf. \cite{9}, Theorem 6).

**3. QF-3 categories**

Let \( \mathcal{A} \) be a Grothendieck category with generating set of projectives \( \{P_a\} \). An object \( C \) in \( \mathcal{A} \) is called faithful if for any non-zero morphism \( f : P_a \to P_b \), there exists \( g \in [P_b, C] \) such that \( gf \neq 0 \). Let \( \{Q_a\} \) be another generating set of projec-

---

2) see [8], §3.
tives and \( f' \neq 0 \in [Q, Q_\beta] \). Since \( Q \oplus Q' = \sum \oplus P_\alpha \) and \( Q_\delta \oplus Q'_\delta = \sum \oplus P_\beta \), we have a non-zero morphism \( f: \sum \oplus P_\alpha \to \sum \oplus P_\beta \) such that \( f | Q = f' \) and \( f | Q'_\delta = 0 \).

Hence, there exist \( \alpha, \beta \) such that \( p_\beta f | P_\alpha \neq 0 \), where \( p_\beta \) is the projection of \( \sum \oplus P_\beta \) to \( P_\beta \). Then we have \( g' \in [P_\beta, C] \) such that \( g'(p_\beta f | P_\alpha) = 0 \). Hence, \( g' p_\beta f = 0 \). Let \( i_{Q_\alpha} \) and \( i_{Q_\delta} \) be inclusions. Put \( g' p_\beta i_{Q_\alpha} = g \in [Q_\alpha, C] \). Then \( g' p_\beta f i_{Q_\alpha} = g' p_\beta i_{Q_\delta} f' = gf' \) and \( \text{Ker} f = Q_\delta \). Therefore, \( gf' = 0 \). Thus, we have shown that the faithfulness of \( C \) does not depend on generating sets of projectives.

Let \((\mathcal{C}, Ab)\) be the contravariant additive functor category, where \( \mathcal{C} \) is the small pre-additive category \( \{P_\alpha\} \). Then \( \mathcal{A} \) is equivalent to \( (\mathcal{A}', Ab) \). Hence \( C \) is faithful and only if the corresponding functor in the above is a faithful functor. Furthermore, \((\mathcal{C}, Ab)\) is equivalent to \( \mathfrak{M}_R^+ \), where \( R \) is the induced ring from \( \{P_\alpha\} \). Then faithful functors correspond to faithful modules in \( \mathfrak{M}_R^+ \).

An object \( M \) is called a minimal faithful if \( M \) is faithful and every faithful object is a coretract of \( M \). According to R.M. Thrall [13], we call \( \mathcal{A} \) QF-3 if \( \mathcal{A} \) contains a minimal faithful object \( M \) or equivalently, if \( \mathfrak{M}_R^+ \) has a minimal faithful module.

From now on we shall assume that \( \mathcal{A} \) is a Grothendieck category with generating set of small projectives \( P_\alpha \). Further, we shall assume that \( \mathcal{A} \) is a locally PP and semi-perfect category and hence, we may assume that all \( P_\alpha \) are completely indecomposable and \( P_\alpha \cong P_\beta \) for \( \alpha \neq \beta \).

Every object \( A \) in \( \mathcal{A} \) has an injective hull of \( A \) in \( \mathcal{A} \) (see [11], p. 89, Theorem 3.2). We denote it by \( E(A) \). If \( E(\sum \oplus P_\alpha) \) is projective, \( \mathcal{A} \) is called QF-3+ (see [2]).

Let \( Q \) be an injective envelope of \( R \) in \( \mathfrak{M}_R^+ \) and \( M \) a minimal faithful module in \( \mathfrak{M}_R^+ \). Then \( M \) is a retract of \( Q \) and hence, \( M \) is injective. Furthermore, since \( R \) is faithful, \( M \) is also a retract of \( R \). Therefore, \( M \) is projective, and injective and we may assume that \( M \) is a right ideal of \( R \).

Since \( R \) is semi-perfect, \( R = \sum \oplus e_\alpha R \) and \( e_\alpha R e_\alpha \)'s are local rings. In the proof of theorem 4 in [9], we considered indecomposable projective objects \( P \) in \( \mathfrak{M}_R^+ \) such that \( [P, e_\alpha R] = 0 \) for all \( e_\alpha R \cong P \). We call such \( P \) belonging to the first block. Contrary, if \( [e_\alpha R, P] = 0 \), \( P \) is called belonging to the last block.

**Lemma 2.** Let \( \mathcal{A} \) be a locally PP and QF-3 semi-perfect Grothendieck category and \( R \) the induced ring. Then a minimal faithful object is a coproduct of \( e_\alpha R \)'s which belong to the first block.

Proof. Since \( M \) is injective and a retract of \( \sum \oplus e_\alpha R \), \( M = \sum \oplus e_\alpha R \) by [14], Lemma 2. Further, since \( e_\alpha R \) is injective \( [e_\alpha R, eR] = 0 \) by Lemma 1 if \( e_\alpha R \cong eR \). Hence, \( e_\alpha R \) belongs to the first block.
Lemma 3. Let $\mathfrak{A}$ be as above and $\sum_j \oplus e_j R$ a minimal faithful ideal. Then for any $\delta \in I$ there exist $\varphi(\delta)$ in $J$ such that $e_{\varphi(\delta)} R_\delta \neq 0$.

Proof. Let $x$ be a non-zero element in $e_\delta R_\delta$. Since $\sum_j \oplus e_j R = \sum_j \oplus e_j R_\alpha$ is faithful, $e_{\varphi(\delta)} R_\delta x \neq 0$ for some $\varphi(\delta)$.

Let $e_\delta$ be as above. We put $R(\delta) = \{ \gamma | \in I, e_\delta R_\gamma \neq 0 \}$.

Lemma 4. Let $\mathfrak{A}$ be as above and further perfect. Then $R(\delta)$ contains the last element $\delta$ in $R(\delta)$ namely, $e_\delta R_\delta \neq 0$ and $e_\delta R$ belongs to the last block.

Proof. We assume that $R(1)$ does not contain the last element in $R(1)$. Put $N = \sum_{\gamma \in R(1)} \oplus e_\gamma R / (\sum_{\gamma} e_\gamma R_\gamma \oplus \sum_{\gamma} e_\gamma R)$ and put $N = \sum_{\gamma \in R(1)} \oplus e_\gamma R / (\sum_{\gamma} e_\gamma R)$, and $N_\gamma = \sum_{\gamma} \oplus e_\gamma R$. We shall show that $N$ is faithful in $\mathfrak{M}_R$. Let $x = \sum x_{\alpha} e_\alpha$, $x_{\alpha_\beta} e_\alpha R_\delta$ and $x_{\alpha_\beta} \neq 0$. If $\varphi(\alpha) = 1$, we take $0 = y \in e_{\varphi(\alpha)} R_\alpha e_\alpha R_\alpha R_\alpha$ and $yx \neq 0$ by Theorem 1, since $e_\delta R_\delta$ is a division ring by Lemma 1. We assume $\varphi(\alpha) = 1$. Then $\alpha \in R(1)$ and there exists $y \in e_\alpha R_\alpha$ and $0 \neq yx_{\alpha_\beta} e_\alpha R_\alpha$. Hence, $\beta \in R(1)$. Since $R(1)$ does not have the last element, we obtain $\gamma$ in $R(1)$ such that $\beta \leq \gamma$. Hence $\{ y + (\sum_{\gamma} e_\gamma R_\gamma) \} x = 0$. Therefore, $N$ is faithful and $N$ contains a submodule $N_\alpha$ which is isomorphic to $e_\alpha R$. Then $N_\alpha = n R = e_\alpha R$ and $n e_\alpha = n$. Since $e_\gamma R_\gamma = 0$ for $\gamma \neq \delta$, $n = \sum_{\gamma \in R(1)} \oplus P_\alpha$. Then $n(e_\alpha R_\alpha) = 0$ for $\gamma = \max (\gamma_\alpha)$. However, $e_\alpha R_\alpha \neq 0$. Which is a contradiction.

Theorem 3 ([4], Theorem 1). Let $\mathfrak{A}$ be a perfect or semi-perfect and semiartinian and locally PP-Grothendieck category with a generating set of small prejectives $\{ G_\gamma \}_j$. If $\mathfrak{A}$ is QF-3, there exist non-isomorphic indecomposable and projective objects $\{ P_\alpha \}_j$ (resp. $\{ Q_\beta \}_j$) such that

1) $\{ P_\alpha \}$ (resp. $\{ Q_\beta \}$) is an isomorphic representative class of the projectives in the first (resp. last) block,

2) $\sum \oplus P_\alpha$ is a minimal faithful and injective object and

3) each $P_\alpha$ contains the unique minimal subobject $S_\alpha$ which is isomorphic to $Q_\alpha$. Hence $[S_\alpha : \Delta_\alpha] = 1$ and $S_\alpha$ is projective in $\mathfrak{M}_R$ where $\Delta_\alpha = [Q_\alpha, Q_\alpha]$ is a division ring. Furthermore, any indecomposable projective is isomorphic to a subobject in some $P_\alpha$.

Proof. We shall prove the theorem on the induced ring $R = \sum \oplus e_\alpha R; e_\alpha R = e_\alpha R$ if $\alpha = \beta$. We know from Lemmas 2 and 3 that $\sum \oplus e_\gamma R$ is a minimal faithful ideal, $e_\gamma R$ belongs to the first block and $e_\gamma R$ contains a submodule $e_\gamma R_\gamma$, where $\gamma$ is the last element in $R(\gamma)$. Since $e_\gamma R_\gamma = 0$ for $\gamma = \gamma_\alpha$, $\gamma = e_\gamma R_\gamma$ is a right ideal. Put $\Delta_\gamma = e_\gamma R_\gamma$, then $\Delta_\gamma$ is a division ring by Lemma 1. $e_\gamma R$ is
indecomposable and injective. On the other hand, any $\Delta_i$-submodule of $x_i$ is a $R$-module. Hence, $[\tau_i: \Delta_i]=1$ and $x_i$ is the unique minimal subideal in $e_iR$. Since $x_i e_i R = e_i R$, $x_i$ is projective. Furthermore, $x_i e_i R$ if $i \neq j$, since $e_i R \cong e_i R_j$ and $e_j R$, $e_j R$ are injective hull of $x_i$ and $x_j$, respectively. Let $e_i R$ be in the last block. Then $e_i R \cong 0$ and $\varphi(\delta) \subseteq J$. Hence, $e_i R \cong \varphi(\delta)$. Therefore, $\{e_i R\}$ is an isomorphic representative class of projectives in the last block. Let $e \in I - J$. Then $e \varphi(\delta) = 0$ by Lemma 3. Hence, $[e_i R, e \varphi(\delta)] = 0$, which means that $e_i R$ does not belong to the first block. Furthermore, $e_i R$ is isomorphic to $e_i R$ by Lemma 1.

**Lemma 5.** Let $R$ be the induced ring from a locally $PP$-Grothendieck category with generating set $\{P_\alpha\}$ as above. We assume that $\{e_i R\}_J$ is a set of injective objects such that $E = E(R)$ in $\mathcal{W}_R$ is an essential extension of $\sum \oplus e_i R^{(K_i)}$. Then any $f \in [e_i R, E]$ is either zero or monomorphic, where $e_i R^{(K_i)} = \sum \oplus e_i R$ and $e_i$ is any primitive idempotent.

Proof. We assume $f \neq 0$. Then $x = f^{-1}(\sum \oplus e_i R) \neq 0$ for some $e_i$. Since $\sum \oplus e_i R$ is injective, $f | x$ is extended to $g \in [e_i R, \sum \oplus e_i R]$. Then $g$ is monomorphic by Lemma 1. Therefore, $f$ is monomorphic.

**Theorem 4.** Let $\mathcal{A}$ be a perfect, locally $PP$-Grothendieck category with generating set of small projectives. Then $\mathcal{A}$ is $QF-3^+$ if and only if every projective $P_j$ in the first block are injective and for any indecomposable projective $P_j$, there exists $P_j$ in $\{P_j\}$ that $[P_j, P_j] = 0$. Hence, $\{P_j\}$ is an isomorphic representative class of all projective and injective indecomposable objects.

Proof. Let $R$ be the induced ring from completely indecomposable projectives $P_j$. We assume $\mathcal{A}$ is $QF-3^+$. Then $E = E(R)$ is isomorphic to $\sum \oplus e_j R^{(K_j)}$. It is clear that $e_j R$ belongs to the first block from Lemma 1. For any projective $e_j R$, $E(e_j R) \subseteq E$. Hence, $[e_j R, e_j R] = 0$ for some $j$, which implies $\{e_j R\}$ consist of all projectives in the first block. Conversely, we assume that all projectives $\{e_i R\}_J$ in the first block are injective and have the property in the theorem. Since $[e_j R, e_i R] = 0$ for any $e_j R, E \supset \sum \oplus e_i R^{(K_i)}$ for suitable indices $K_i$. We assume $E = \sum \oplus e_j R^{(K_j)}$. Then there exists $g \in [e_j R, E]$ such that $\text{Im } g \subseteq \sum \oplus e_j R^{(K_j)}$. On the other hand, we obtain $g' \in [e_j R, E_j]$ such that $g' |\text{ Im } g = g$ from the proof of Lemma 5, where $E_j$ is a finite coproduct of $e_j R_j$'s. Then $(g - g') | E_j = 0$. Therefore, $g = g'$ by Lemma 5, which is a contradiction.

**Remark.** The fact $[e_j R, e_j R] = 0$ is equivalent to the validity of Lemma 3 for the above $\mathcal{A}$.
Theorem 4'. Let $\mathfrak{A}$ be a semi-perfect, semi-artinian and locally PP-Grothendieck category with generating set of small projectives. Then $\mathfrak{A}$ is QF-3$^+$ if and only if $\mathfrak{A}$ contains projectives $P_a$ in the first block and all of such $P_a$ are injective and for any indecomposable projective $P$, there exists $P_a$ such that $[P, P_a] \neq 0$. Hence, $\{P_a\}$ consist of all projective and injective indecomposable objects. In this case $\mathfrak{A}$ is QF-3, (cf. [2], Proposition 2 and [12], Proposition 3.1).

Proof. We assume $\mathfrak{A}$ is QF-3$. Let $S$ be the socle of $E=E(R)$ and $S=\Sigma \oplus S_i$, where $S_i$’s are minimal objects in $E$. Then $E=E(S)$ and $E_i=E(S_i)$ is indecomposable and projective by the assumption. Hence, from [8], Corollary 1 to Lemma 2, $E_i \cong e_i R$, which belongs to the first block. Let $e_a R$ be any indecomposable ideal. Then $E(e_a R) \subset E$. Hence, $[e_a R, e_a R] \neq 0$ by Lemma 1 and the proof of Lemma 5. Since each $e_j R$ has the non-zero socle, $\mathfrak{A}$ is QF-3 by the standard argument (cf. the proof of Lemma 7 below). The converse is similarly proved as in the proof of Theorem 4.

Lemma 6. Let $\mathfrak{A}$ be as in Theorem 3 (resp. Theorem 4’) and $e_j R$ in the first block. Let $\gamma$ be the last (resp. first) element in $R(1)$. Then $R(1) = C(\gamma)$. If $\mathfrak{A}$ is as Theorem 4, $R(1)^\gamma \supset C(\gamma)$ for any $\gamma \in R(1)$ and for any $\delta$ and $\delta' \in (1)$ there exists $\epsilon$ in $R(1)$ such that $e_\delta R \neq 0$ and $e_\delta' R \neq 0$, where $R(1)^\gamma = \{\alpha | \alpha \in R(1), \alpha \leq \gamma\}$ and $C(\gamma) = \{\delta | \in I, e_\delta R \neq 0\}$.

Proof. Let $\gamma$ be in $R(1)$ and $\delta$ in $(I - R(1))^\gamma$. Then $e_\delta R \neq 0$ and $\varphi(\delta) > 1$. We assume $e_\delta R = 0$. Then $e_\delta R = 0$ by Theorem 1. We take non-zero element $x, y$ in $e_\delta R$ and $e_\delta R$, respectively. Consider a mapping $\varphi: xR \rightarrow yR$ such that $\varphi(xR) = yR$. Then $\varphi$ is well defined and $R$-homomorphic by Theorem 1. Hence, $[e_\delta R, e_\delta R] \neq 0$, which is a contradiction. Therefore, $R(1) \supset C(\gamma)$. Let $x$ be a non-zero element in $e_\delta R$. Then $xR$ is a projective and indecomposable ideal in $e_\delta R$ by the assumption. Hence, $xR \cong e_\delta R$ for some $q$. Put $\varphi(x) = e_q R$. Then $\varphi(x) = \varphi(xe_\delta) = e_q R$. This implies $q \leq \gamma$ (resp. $q \geq \gamma$). Similarly, we have $q \geq \gamma$ (resp. $q \leq \gamma$). We assume $R(1)$ contains the last (resp. first) element $\eta$. Then $e_\eta R = xR = (\text{socle of } e_\eta R) \neq 0$. Hence, $R(1) = C(\gamma)$. Let $\gamma' \in R(1)$. Then $e_\gamma R$ and $e_\gamma' R$ are monomorphic to $e_\gamma R$. Since $e_\gamma R$ is injective, their images have a non-zero intersection $r$. Hence, $r e_\eta R \neq 0$ for some $\varepsilon$. Therefore, $e_\gamma R \neq 0$ and $e_\gamma' R \neq 0$.

Lemma 7 (cf. [12]). Let $\Delta$ be a division ring and $I$ a well ordered set. Let $\{e_{ij}\}_1$ be a set of matrix units. Put $R = \Sigma_{i \in I} e_{ij} \Delta$. Then $e_{ij} R$ is injective and hence, $R$ is hereditary and QF-3 in $\mathfrak{M}_R^+$. $R$ is QF-3 if and only if $I$ contains the last element.

Proof. We first note that each $e_{ij} R$ contains only right ideals of form $e_{ij} R$ $i \leq j$ and $[e_{ij} R, e_{ji} R] \approx \Delta$. Let
be a given exact diagram in $\mathcal{M}_R$. We shall extend $f$ to $M$ by the standard argument. We obtain a maximal extension $f_0: N_0 \to e_{ii} R$ such that $N_0 \supseteq N$ and $f_0|N = f$. If $M = N_0$, there exists $m$ in $M$ such that $m e_{ii} \in N_0$, since $\{e_{ii} R\}$ is a generating set. Put $M' = N_0 + m e_{ii} R$ and $\mathcal{R} = \{x \in e_{ii} R, m x \in N_0\}$. Then $\mathcal{R}$ is a right ideal in $e_{ii} R$. Hence, $\mathcal{R} = e_{jj} R$ for some $j > i$. We define $g: \mathcal{R} \to e_{ii} R$ by setting $g(x) = f_0(mx)$ for $x \in \mathcal{R}$. Then $e_{ii} | \mathcal{R}$ and $g$ are in $[\tau, e_{ii} R] = e_{ii} \Delta = \Delta$. Hence, $g = \delta(e_{ii} | \tau)$ for some $\delta$ in $\Delta$, namely $g(x) = \delta e_{ii} x$ for any $x$ in $\mathcal{R}$. Therefore, we have an extension $f_0': M' \to e_{ii} R$ by $f_0'(n_0 + m x) = f_0(n_0) + \delta e_{ii} x$. Hence, $N_0 = M$. We know from [8], Lemma 7 and [9], Proposition 1 that $R$ is perfect and $\text{J}(R) = \sum_{i,j} \oplus e_{ij} \Delta$. Since $\text{J}(R)$ is projective, $R$ is hereditary by [9], Lemma 3. Therefore, $R$ is QF-3 by Theorem 4. If $R$ is QF-3, $e_{ii} R$ is a minimal faithful module by Theorem 3. Hence, $I$ has the last element by Theorem 3. Conversely, $I$ has the last element, then $e_{ii} R$ contains the unique submodule $e_{ii} R$. It is clear that $e_{ii} R$ is faithful module. Let $M$ be a faithful module in $\mathcal{M}_R$. Then there exists $m$ in $M$ such that $m e_{ii} R = 0$. Hence, we have a monomorphism $f$ of $e_{ii} R$ to $M$ by $f(e_{ii} r) = m e_{ii} r$. Therefore, $R$ is QF-3.

**Lemma 8.** Let $\Delta$ be a division ring and $\{e_{ij}\}$ a set of matrix units. Put $S = \sum_{i,j} \oplus \Delta e_{ij}$ and $R = \sum_{i,j} \oplus \Delta e_{ii}$. Then

1) $R$ is semi-hereditary.

2) $R$ is semi-hereditary and QF-3 (or QF-3') if and only if $I$ has the last element.

3) $R$ is hereditary and QF-3' (or QF-3) if and only if $I$ is finite, (cf. [12]).

Proof. 1) Let $\mathcal{R}$ be a right ideal generated by $\{x_1, x_2, \ldots, x_n\}$. Since $x_i = \sum_{a} x_i e_a$ and $x_i e_a \in \mathcal{R}$, we may assume that $x_i e_{a_i} \in \mathcal{R}$, where $e_{a_i} = e_{a_i a_i}$. Let $\alpha_i = \max(\alpha_i)$. Considering $Re_{a_i}$ as a $\Delta$-vector space, we may assume $x_1, \ldots, x_i$ are linearly independent over $\Delta$. If $\sum_{i<j} x_i r_i = 0$ for $r_i \in R$ and $x_i r_i = 0$, then $r_i e_i = 0$ for $e_i \leq \alpha_i$. Considering in $S$, we have $\sum_{i} x_i e_{a_i} r_i e_{a_i} = 0$ and $e_{a_i} r_i e_{a_i} = 0$. Therefore, $\sum x_i R = \sum \oplus x_i R$. Put $\alpha_2 = \max(\{\alpha_i\} - \alpha_i)$. We consider a vector space $V_2$ generated by $\{\sum_{i} x_i R e_{a_2} \oplus y_1 \Delta \oplus \cdots \oplus y_k \Delta, \text{where } y_j = x_{a_2} e_{a_2} \text{ for some } k\}$. We may assume $V_2 = \sum \oplus x_i R e_{a_2} \oplus y_1 \Delta \oplus \cdots \oplus y_k \Delta$, where $y_j = x_{a_2} e_{a_2}$ for some $k$. We shall show that $\sum \oplus x_i R + \sum y_j R = \sum \oplus x_i R + \sum y_j R$. We have already shown that $\sum y_j R = \sum \oplus y_j R$. Let $\sum x_i r_i = \sum y_j r_j$; $r_i, r_j \in R$. If $r_i = 0$, $r_j e_i = 0$ for some $e_i$. Then multiplying $e_i e_{a_2}$ in the above, we have $\sum x_i e_{a_i} r_i e_i e_{a_2} = \sum y_i e_{a_2} e_i e_{a_2}$ and
Hence $\sum y_i \delta_i = \sum x_i e_{a_2} r_i e'_{a_2} = \sum x_i Re_{a_2}$, which is a contradiction. On the other hand, $x_i R \approx e_{a_i} R$, $y_j R \approx e_{a_2} R$. Repeating this argument, we show that $r$ is projective.

2) We assume that $I$ has the last element $\alpha$. We shall show that $e_{\alpha a} R$ is injective as an analogy of Lemma 7. Let $\tau$ be a right ideal in some $e_{\alpha a} R$. Put $R(\tau) = \{ \gamma \in I, \tau \gamma \cap \neq 0 \}$. If $R(\tau)$ contains the last element $\delta$ in $R(\tau)$, then $\tau = \sum e_{\alpha a} Re_{\tau'} \approx e_{\alpha} R$. Let $\varepsilon$ be the least element in $I - R(\tau)$. If $\varepsilon$ is not a limit element, $R(\tau)$ contains the element. We assume $\varepsilon$ is limit. Then $\tau = \bigcup \tau'$. We shall show $[\varepsilon, e_{\alpha a} R] \approx e_{\alpha a} R$. Let $f \in [\varepsilon, e_{\alpha a} R]$ and put $f' = f|\tau' \in [\tau', e_{\alpha a} R] \approx [e_{\tau'}, R, e_{\alpha a} R]$. Then $f' = e_{\delta'} e_{\alpha a}$ for some $\delta' \in \Delta$. For $\varepsilon' \varepsilon''$ we have $\delta' e_{\alpha a'} = f(\varepsilon') = f(\varepsilon'' e_{\alpha a'}) = \delta'' e_{\alpha a'}$. Hence, $\delta'' = \delta'$. If we put $\delta = \delta'$, $f = \delta e_{\alpha a}$. Thus, we have prepared necessary facts to use the proof of Lemma 7. Therefore, $e_{\alpha a} R$ is injective in $\mathfrak{M}_R$ and $R$ is $QF-3'$ and $QF-3$ by Theorem 4'. The converse is clear from 1) and Theorems 3 and 4'.

3) If $I$ is finite, $R$ is a hereditary and $QF-3$ artinian ring by [4], Theorem 3. We assume that $R$ is hereditary and $QF-3$ or $QF-3'$. Then $I$ has the last element by Theorem 4. We assume that $I$ contains a limit number $\alpha$. Consider $J(e_{\alpha} R) = \sum e_{\alpha} e_{\alpha} \Delta$. Let $x = \sum e_{\alpha} e_{\alpha} \Delta$. Then $x = \sum e_{\alpha} e_{\alpha} \Delta e_{\alpha} \in J(e_{\alpha} R) J(R) \subseteq J(e_{\alpha} R)$. Hence, $J(e_{\alpha} R) = J^+(e_{\alpha} R)$, which implies $J(e_{\alpha} R)$ is not projective by [8], Proposition 2. Therefore, $I$ does not contain the limit number, but contain the last element, Hence, $I$ is finite.

From the above proof and [9] Lemma 3 we have

**Corollary.** Let $R$ be as above. Then $R$ is hereditary if and only if $|I| \leq \aleph_0$ and does not contain the last element.

**Theorem 5.** Let $\mathfrak{A}$ be a perfect or semi-perfect and semi-artinian, and locally $PP$-Grothendieck category with generating set of small projectives. If $\mathfrak{A}$ is $QF-3'$ or $QF-3$, then $\mathfrak{A}$ is equivalent to $\bigoplus_{\mathfrak{A}_i}$, where $\mathfrak{A}_i$'s are of the same type as $\mathfrak{A}$ and $\mathfrak{A}_i$ is not expressed as a product of full subcategories.

Proof. Let $R$ be the induced ring from $\mathfrak{A}$ and $\sum e_{i} R$ the coproduct of projectives in the first block. We shall show $e_{i} R e_{i}' = 0$ for either $e \in R(i)$, $e' \in R(i)$ or $e \in R(i)$, $e' \in R(i)$. If $e \in R(i)$, $e_{i} R$ is monomorphic to a submodule of $e_{i} R$. Hence, $e_{i} R e_{i}' = 0$ if $e' \in R(i)$. Next, we assume $e' \in R(i)$. If $e_{i} R e_{i}' = 0$ for $e' \in R(i)$, $0 = e_{i} R e_{i}' e_{i} R e_{i} \subseteq e_{i} R e_{i}$. For some $\gamma_{\delta} \in R(i)$ (or the last (resp. first) element in $R(i)$) by Lemma 1, which contradicts to a fact $R^+(i) \supset C(\gamma_{\delta})$. Put $R_{i} = \sum e_{i} R e_{i}'$. Then $R = \bigoplus R_{i}$ as a ring by Theorems 3, 4 and 4'. It is clear that each $R_{i}$ is $QF-3'$ or $QF-3$ and directly indecomposable. Hence, we have the theorem.
From the above theorem, we may restrict ourselves to a case of indecomposable categories if $\mathfrak{A}$ is as in the theorem.

**Theorem 6.** Let $\mathfrak{A}$ be an indecomposable semi-perfect Grothendieck category with generating set of finitely generated objects. Then we have

1) $\mathfrak{A}$ is perfect, (semi-) hereditary and QF-$3^+$ (resp. QF-$3$) if and only if $\mathfrak{A}$ is equivalent to $[I, \mathcal{M}_\Delta]',$ where $I$ is a well ordered set (resp. with last element).

2) $\mathfrak{A}$ is semi-artinian, hereditary and QF-$3^+$ (or QF-$3$) if and only if $\mathfrak{A}$ is equivalent to $[I, \mathcal{M}_\Delta]',$ where $I$ is a finite set.

3) $\mathfrak{A}$ is semi-artinian, semi-hereditary and QF-$3^+$ (or QF-$3$) if and only if $\mathfrak{A}$ is equivalent to $[I, \mathcal{M}_\Delta]',$ where $I$ is a well ordered set with last element. Where $\Delta$ is a division ring and functors $T_{ij}$ in $[I, \mathcal{M}_\Delta]$ are equal to $1_{\mathcal{M}_\Delta},$ (cf. [2'], Theorem 3.2).

Proof. $[I, \mathcal{M}_\Delta]'$ is perfect, hereditary and QF-$3^+$ by Lemma 7 and [9], Theorem 3. We assume that $\mathfrak{A}$ contains the last element. $[I, \mathcal{M}_\Delta]'$ is QF-$3^+$ by Lemma 7. If $I$ is finite, $[I, \mathcal{M}_\Delta]'$ is semi-primary, hereditary and QF-$3^+$ (and QF-$3$) by Lemma 8. Finally, $[I, \mathcal{M}_\Delta]'$ is semi-artinian, semi-hereditary and QF-$3^+$ (QF-$3$) by Lemma 8 and [9], Proposition 1. Next, we assume that $\mathfrak{A}$ is one of the forms in the theorem. Let $R$ be the induced ring: $R=\bigoplus_I e_i R.$ Then $e_i R$ in the case 1) and $e_a R$ in cases 2) and 3) are in the first block by Theorems 4 and 4', respectively, where $\alpha$ is the last element in $I.$ Since, $\mathfrak{A}$ is indecomposable, $e_i R e_r$ (resp. $e_a R e_r$) $\neq 0$ for any $r \in I$ by Theorem 5, Lemma 3 and Remark. Let $\mathfrak{A}$ be hereditary (cases 1 and 2). If $[e_i R e_r; \Delta_i] \geq 2$ (resp. $[e_a R e_r; \Delta_i] \geq 2$) for any $r \in I,$ there exist linearly independent elements $x,$ $y$ over $\Delta_i = e_i R e_r.$ Then $xR + yR = xR \oplus yR$ by [9], Theorem 3, which contradicts to the indecomposability of $e_i R$ and $e_a R.$ Let $a,$ $b$ be non-zero elements in $e_i R e_r.$ As the proof of Lemma 6, a mapping $\psi: aR \rightarrow bR$ such that $\psi(a) = b$ gives a $R$-homomorphism. Furthermore, $\psi$ is extended in $[e_i R, e_i R] = \Delta_i.$ Hence $b = \delta a$ for some $\delta \in \Delta_i.$ Therefore, $[e_i R e_r; \Delta_i] = 1.$ Similarly, we obtain $[e_a R e_r; \Delta_i] = 1.$ Next, we assume $\mathfrak{A}$ is semi-hereditary and QF-$3^+$ (case 3). Then $e_a R$ is in the first block and injective. Let $x,$ $y$ be non-zero elements in $e_a R e_r.$ Then $xR + yR$ is a projective right ideal in $e_a R.$ Since $e_a R$ contains the unique minimal module and $R$ is semi-perfect, $xR + yR \cong e_i R$ for some $\delta \in I.$ Put $\psi^{-1}(e_i) = z,$ then $z \in e_a R e_b$ and $z = x \psi,$ $y = y \psi$ for $r,$ $r' \in R.$ Hence, $r = \delta$ and $x = ze_b e_r,$ $y = ze_r e_b.$ Therefore $[e_a R e_r; \Delta_i] = 1.$ Similarly to the above, we can show $[e_i R e_r; \Delta_i] = 1.$ Thus, in any cases $e_i R e_r$ (resp. $e_a R e_r$) is a simple $\Delta_i$-module. Hence, if $e_i R e_r \neq 0,$ $e_i R e_r \cong e_i R e_r \cong e_i R e_r$ implies $[e_i R e_r; \Delta_i] = [e_i R e_r; \Delta_i] = 1$ from Theorem 1. Let $x \neq 0 \in e_i R e_r.$ Then $\Delta_i$ is isomorphic to $\Delta,$ by $\xi: \delta \xi = x \xi \delta.$ First we choose non-zero elements $m_{ij}$ in $e_i R e_r.$ Then $e_j R$ is monomorphic to $\bigoplus_{k \geq j} m_{jh} \Delta$ by the multiplication of $m_{ij}$ from the left side. Hence, we can choose $m_{jh}$ in $e_j R e_h$ such that $m_{ij} m_{jh} = m_{ih}$ (if $e_j R e_h \neq 0$). Then
Perfect Categories III

Therefore, \( m_{ij}m_{jk} = m_{ik} \) if \( i \neq j \) and \( m_{jk} \neq 0 \). Thus, \( R \) is a subring of \( \bigoplus_{i,j} e_{ij} \Delta \) (resp. \( \bigoplus_{i,j} e_{ij} \Delta \)) such that all of elements of some \((i, j)\)-entries may be equal to zero, where \( \Delta \approx \Delta_i \). We assume \( e_i R e_j = 0 \) (in cases 1) and 2). Then \( i \neq 1 \) (resp. \( i \neq \alpha \)) and there exists \( \gamma \) from Lemma 6 such that \( e_i R e_\gamma = 0 \), \( e_j R e_\gamma = 0 \). Put \( e = e_{i1} + e_{i2} + e_{jj} + e_{n0} \) (resp. \( e = e_{i1} + e_{i2} + e_{jj} + e_{n0} \)). Then \( eR e = e_{i1} d e_{i1} d e_{jj} d e_{n0} d e_{n1} d e_{i1} d e_{n1} d e_{ij} d e_{jj} d e_{n0} \) is hereditary by [9], Corollary to Lemma 2 if \( R \) is hereditary. However, we can easily see that \( eR e \) is not hereditary (cf. [6], Theorem 1). Therefore, \( R = \bigoplus_{i,j} e_{ij} \Delta \) (resp. \( R = \bigoplus_{i,j} e_{ij} \Delta \)). Finally, we assume that \( R \) is semi-hereditary (case 3). Let \( \gamma < \delta \) be in \( I \). Then since \( m_{\sigma \tau} R + m_{\alpha \beta} R \) is projective, \( m_{\sigma \tau} R + m_{\alpha \beta} R = zR \) as before, where \( z \in e_{\sigma} R e_{\beta} \). Hence, \( zR = m_{\alpha \beta} R \supseteq m_{\sigma \tau} R \). Therefore, \( 0 \neq m_{\sigma \tau} = m_{\alpha \beta} e_{\gamma} d e_{\gamma} \) implies \( e_{\gamma} d e_{\gamma} \neq 0 \). Thus, \( \mathcal{A} \) is equivalent to \([I, \mathfrak{M}_\Delta]\)'s. The remaining parts are clear from Theorems 3, 4 and 4' and Lemma 8.

Osaka City University

References
