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AN ESTIMATE OF THE SPECTRAL GAP FOR ZERO-RANGE-EXCLUSION DYNAMICS

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1. Introduction

This paper concerns the spectral gap for a Markovian particle system, which we call a zero-range-exclusion process. The process is a kind of lattice gas on $\mathbf{Z}^d$, which consists of particles carrying energy and whose transition mechanism is made up with a combination of dynamics for an exclusion process (for particles) and that for a zero-range process (for energy). It has two conserved quantities, the number of particles and the total energy, so that its hydrodynamic behavior must be of interest. Our process is reversible relative to certain product probability measures (serving as the grand-canonical Gibbs measures), but of non-gradient type. It will be proved that for the local process confined to a cube in $\mathbf{Z}^d$ of width n, the spectral gap is bounded below by Cn^{-2}, where C is independent of n but depends on the two order parameters, namely the number of particles per site and the energy per particle.

For the models whose grand-canonical Gibbs measures are product measures as in the present case the estimation of the spectral gap may be naturally reduced to establishing two things: one is a suitable estimate of the spectral gaps for the corresponding mean-field dynamics and the other is a certain inequality (sometimes called a moving-particle lemma) that compares a Dirichlet form for two-site dynamics of a distant pair (i.e., a pair of two sites that are far apart from each other) with a sum of those of nearest neighbor pairs (cf. [7], [2]). The former one can be obtained by adapting the arguments developed by Landim, Sethuraman and Varadhan in the paper [3] which establishes the uniform bound of the gap for zero-range processes; it has also been proved in a recent paper by Caputo [1] based on a somewhat different idea. The major ingredient in this paper therefore is a verification of the latter one, namely that of the moving-particle lemma for the present model, which is not so simple a matter as for zero-range or exclusion processes and causes the dependence on the order parameters of the constant C in the bound of the gap mentioned above. We shall also provide an indication of how to adapt the proof of [3] as well as a brief description of the approach in [1]. The uniform bound of the spectral gap for a model similar to the present one is obtained in [4], but the energy values and transition rates are uniformly bounded therein whereas they are unbounded in our model.

Our estimate of the gap, though not uniform with respect to the order parameters, is sufficient to prove a theorem (the fluctuation-dissipation equation) that is fundamental in the study of the hydrodynamic behavior of the process [6]. This theorem, originally discovered by Varadhan [7] for a stochastic Ginzburg-Landau model (cf. [8], [4] or [2] for other models), describes a structure of the quadratic form of central-limit-theorem variances and owing to this structure one can identify the bulk diffusion coefficients and prove the convergence of the equilibrium fluctuation fields to an (infinite dimensional) Ornstein-Uhlenbeck process (cf. [7], [2]).

2. The model and the result

Let Λ_n (also written $\Lambda(n)$ in sub- and superscripts) be a d-dimensional cube with width $2n+1$, centered at the origin. The lattice gases that we are to study are Markov processes on the state space $\mathbf{Z}_+^{\Lambda(n)}$, where $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$. Denote by $\eta = (\eta_x, x \in \Lambda_n)$ a generic element of $\mathbf{Z}_+^{\Lambda(n)}$, and define

$$\xi_x = I(\eta_x \geq 1),$$

where $I(A)$ is 1 or 0 according as a statement A is true or false. For an ordered pair (x, y) of two distinct sites $x,\ y \in \Lambda_n$ we define the exclusion operator $\pi_{x,y}$ and the zero-range operator $\nabla_{x,y}$ which act on a function f on $\mathbf{Z}_+^{\Lambda(n)}$ by

$$\pi_{x,y} f(\eta) = f(S_{\mathrm{ex}}^{x,y}\eta) - f(\eta) \quad \text{and} \quad \nabla_{x,y} f(\eta) = f(S_{\mathrm{zr}}^{x,y}\eta) - f(\eta)$$

where the transformation $S_{\mathrm{ex}}^{x,y}$ of configurations is defined by

$$(S_{\mathrm{ex}}^{x,y}\eta)_z = \begin{cases} \eta_y, & \text{if } z = x, \\ \eta_x, & \text{if } z = y, \\ \eta_z, & \text{otherwise,} \end{cases}$$

if $\xi_x = 1$ and $\xi_y = 0$; and $S_{\mathrm{zr}}^{x,y}\eta$ by

$$(S_{\mathrm{zr}}^{x,y}\eta)_z = \begin{cases} \eta_x - 1, & \text{if } z = x, \\ \eta_y + 1, & \text{if } z = y, \\ \eta_z, & \text{otherwise,} \end{cases}$$

if $\eta_x \geq 2$ and $\xi_y = 1$; and in the remaining case of η, both $S_{\mathrm{ex}}^{x,y}\eta$ and $S_{\mathrm{zr}}^{x,y}\eta$ are set to be η, namely

$$\begin{aligned} S_{\mathrm{ex}}^{x,y}\eta &= \eta \quad \text{if} \quad \xi_x(1 - \xi_y) = 0, \\ S_{\mathrm{zr}}^{x,y}\eta &= \eta \quad \text{if} \quad I(\eta_x \geq 2)\xi_y = 0. \end{aligned}$$

We shall interpret ξ_x as the indicator of occupation of the site x by a particle and η_x as the energy possessed by the particle.

Given two non-negative functions $c_{\rm ex}$ and $c_{\rm zr}$ on $\mathbf{Z}_+$, we define

$$L_{x,y} = c_{\rm ex}(\eta_x)(1 - \xi_y)\pi_{x,y} + c_{\rm zr}(\eta_x)\xi_y\nabla_{x,y}.$$

It may be worth noticing that the factors $1 - \xi_y$ and ξ_y on the right-hand side are superfluous because from our definitions of $S_{\rm ex}^{x,y}$ and $S_{\rm zr}^{x,y}$ it follows that $\pi_{x,y} = \xi_x(1 - \xi_y)\pi_{x,y}$ and $\nabla_{x,y} = I(\eta_x \geq 2)\xi_y\nabla_{x,y}$. (Here we put them to stress the condition for possible transitions of a configuration.)

Let

$$\Lambda_n^* = \{b = (x, y) : x, y \in \Lambda_n, \ \ |x - y| = 1\},$$

namely Λ_n^* denotes the set of all directed bonds connecting two neighboring sites in Λ_n. Here $|x| := \sum_{i=1}^d |x_i|$ for $x = (x^1, \dots, x^d) \in \mathbf{Z}^d$. For $b = (x, y) \in \Lambda_n^*$ we write π_b, $S_{\rm ex}^b$, L_b, etc. for $\pi_{x,y}$, $S_{\rm ex}^{x,y}$, $L_{x,y}$ etc. Then the infinitesimal generator $L_{\Lambda(n)}$ of our lattice gas on Λ_n is given by

$$L_{\Lambda(n)} = \sum_{b \in \Lambda^*(n)} L_b.$$

The process is regarded as a gas of particles having energy as alluded to above. The site x is occupied by a particle if $\xi_x = 1$ and vacant otherwise. Each particle has energy which takes discrete values $1, 2, \dots$ and for which η_x stands. A particle at site x jumps to a nearest neighbor site at rate $c_{\rm ex}(\eta_x)$ if it is vacant. Between two neighboring particles the energies are transferred unit by unit according to the same stochastic rule as that of the zero range processes. It is assumed that for some positive constant a_0, $c_{\rm ex}(k) \geq a_0$ for $k \geq 1$ and $c_{\rm zr}(k) \geq a_0$ for $k \geq 2$. This especially implies that the lattice gas on Λ_n with both the number of particles and the total energy being specified is ergodic. We call the Markov process generated by $L_{\Lambda(n)}$ the zero-range-exclusion process. For sake of convenience we set

$$c_{\rm ex}(0) = 0 \quad \text{and} \quad c_{\rm zr}(0) = c_{\rm zr}(1) = 0.$$

We need some technical conditions on the functions $c_{\rm ex}$ and $c_{\rm zr}$:

$$|c_{\rm zr}(k) - c_{\rm zr}(k+1)| \leq a_1 \quad \text{for all} \quad k \geq 1; \tag{1}$$

$$c_{\rm zr}(k) - c_{\rm zr}(l) \geq a_2 \quad \text{whenever} \quad k \geq l + k_0; \tag{2}$$

$$c_{\rm ex} \geq a_3 c_{\rm zr}, \tag{3}$$

where a_1, a_2 and a_3 are positive constants and k_0 is a positive integer.

Take a pair of constants $0 < p < 1$ and $\alpha > 0$ and let $\nu_{p,\alpha} = \nu_{p,\alpha}^{\Lambda(n)}$ denote the product probability measure on $\mathbf{Z}_+^{\Lambda(n)}$ whose marginal laws are given by

$$\nu_{p,\alpha}(\{\eta : \eta_x = l\}) := \begin{cases} 1-p & \text{if } l = 0, \\ p\dfrac{1}{Z_\alpha} & \text{if } l = 1, \\ p\dfrac{1}{Z_\alpha}\dfrac{\alpha^{l-1}}{c_{\mathrm{zr}}(2)c_{\mathrm{zr}}(3)\cdots c_{\mathrm{zr}}(l)} & \text{if } l \geq 2, \end{cases}$$

for all x. Here Z_α is the normalizing constant:

$$Z_\alpha := 1 + \sum_{l=2}^{\infty} \frac{\alpha^{l-1}}{c_{\mathrm{zr}}(2)c_{\mathrm{zr}}(3)\cdots c_{\mathrm{zr}}(l)}$$

and α is supposed to be less than the radius of convergence of the power series on the right-hand side above. Our lattice gases are reversible relative to the measures $\nu_{p,\alpha}$, (namely $L_{\Lambda(n)}$ is symmetric relative to each of them), as is easily shown (see (4) below). For each pair of positive integers $m \leq n$ and $E \geq m$ the lattice gas which consists of m particles whose total energy is E is ergodic. The invariant measure is the conditional law:

$$P_{n,m,E}[\ \cdot\ ] := \frac{\nu_{p,\alpha}(\ \cdot\ \cap \{\eta : |\xi| = m,\ \ |\eta| = E\})}{\nu_{p,\alpha}(\{\eta : |\xi| = m,\ \ |\eta| = E\})}.$$

Here

$$|\xi| = \sum_{x \in \Lambda(n)} \xi_x \quad \text{and} \quad |\eta| = \sum_{x \in \Lambda(n)} \eta_x.$$

This definition does not depend on a choice of the pair $p,\ \alpha$. We denote by $E_{n,m,E}$ the corresponding expectation.

The reversibility is equivalent to the detailed balance condition, namely the following set of conditions:

$$c_{\mathrm{zr}}(\eta_x)\xi_y P_{n,m,E}\{\eta\} = c_{\mathrm{zr}}(\eta_y + 1)I(\eta_x \geq 2)P_{n,m,E}\{S_{\mathrm{zr}}^{x,y}\eta\} \tag{4}$$

and

$$P_{n,m,E}\{\eta\} = P_{n,m,E}\{S_{\mathrm{ex}}^{x,y}\eta\}, \tag{5}$$

both of which are valid for any $n,\ m,\ E \in \mathbf{Z}_+$ $(n,\ \ E \geq m)$, for any two distinct sites $x,\ y \in \Lambda_n$ and for any configuration η on Λ_n. Here $P\{\eta\}$ denotes the P-measure of the one point set $\{\eta\}$. From (4) it follows that for any functions f and g of η,

$$E_{n,m,E}[c_{\mathrm{zr}}(\eta_x)\xi_y f(S_{\mathrm{zr}}^{x,y}\eta)g(\eta)] = E_{n,m,E}[c_{\mathrm{zr}}(\eta_y)\xi_x f(\eta)g(S_{\mathrm{zr}}^{y,x}\eta)]$$

(here ξ_x or ξ_y cannot be removed); an analogous relation involving $S_{\mathrm{ex}}^{x,y}$ also holds. The Dirichlet form

$$D_{n,m,E}\{f\} := -E_{n,m,E}[f L_{\Lambda(n)} f]$$

is accordingly written as

$$\frac{1}{2}\sum_{x\in\Lambda_n}\sum_{|y-x|=1} E_{n,m,E}\left[c_{\mathrm{ex}}(\eta_x)(\pi_{x,y}f)^2 + c_{\mathrm{zr}}(\eta_x)(\nabla_{x,y}f)^2\right].$$

The objective of this paper is to find a suitable bound of the variance

$$\mathcal{V}_{n,m,E}(f) = E_{n,m,E}\left[(f - E_{n,m,E}[f])^2\right]$$

by means of the Dirichlet form $D_{n,m,E}\{f\}$ as stated in the following theorem.

Theorem 1. *Suppose that the conditions* (1) *through* (3) *are satisfied. Then there exists a constant C such that for all positive integers n, m and E, satisfying $m \le |\Lambda_n|$ and $E \ge m$, and for all real functions f on $\mathbf{Z}_+^{\Lambda(n)}$,*

$$\mathcal{V}_{n,m,E}(f) \le C \cdot \left(\frac{|\Lambda_n|}{m}\right)^2 \frac{E}{m} \cdot n^2 D_{n,m,E}\{f\}. \tag{6}$$

Here $|\Lambda|$ denotes the cardinality of a set Λ.

REMARK. i) We shall actually prove that

$$\mathcal{V}_{n,m,E}(f) \le Cn^2\left(\frac{|\Lambda_n|}{m}\right)^2\left\{D_{n,m,E}\{f\} + \frac{E}{m}\sum_{b\in\Lambda^*(n)} E_{n,m,E}[(\pi_b f)^2]\right\}.$$

(If $d = 1$, the factor $(|\Lambda_n|/m)^2$ on the right-hand side may be deleted.)
ii) It is natural to have the factor n^2 (the square of the length of the underlying physical space) in the bound (6), while the dependence on $m/|\Lambda_n|$ and E/m of the right-hand side of it might not be intrinsic and seems to be caused by shortcoming of the method. The non-uniformity of the bound, however, would not be serious obstruction in its application to the hydrodynamic limit of the model. In fact, by applying Theorem 1, we can determine the structure of the quadratic form of central-limit-theorem variances and thereby identify the limit of the equilibrium fluctuation field under the hydrodynamic scaling [6]. We have difficulty for proving the hydrodynamic limit itself because of the lack of sufficient moment bounds.

An outline of the proof of Theorem 1. We fix n, m and E, and simply write $\mathbf{E}[\,\cdot\,]$ and $D[\,\cdot\,]$ for $E_{n,m,E}[\,\cdot\,]$ and $D_{n,m,E}[\,\cdot\,]$, respectively. We take the conditional expectation given the occupation variable

$$\xi := \{\xi_x : x \in \Lambda_n\},$$

which we denote by $\mathbf{E}[\,\cdot\,|\xi]$. Then

$$\mathcal{V}_{n,m,E}(f) \le 2\mathbf{E}\Big[\big(f - \mathbf{E}[f|\xi]\big)^2\Big] + 2E\Big[\big(\mathbf{E}[f|\xi] - \mathbf{E}[f]\big)^2\Big]. \tag{7}$$

The second term on the right-hand side is easy to dispose of. Since $\mathbf{E}[f|\xi]$ is a function of ξ, we can use a spectral gap estimate for the simple exclusion process to see that

$$\mathbf{E}\Big[(\mathbf{E}[f|\xi] - \mathbf{E}[f])^2\Big] \le C_0 n^2 \mathbf{E}\Big[\sum_{b\in\Lambda^*(n)} \big(\pi_b \mathbf{E}[f|\xi]\big)^2\Big]$$

(cf. [5]). For $x,\ y \in \Lambda_n$ the operator $\pi_{x,y}$ and the conditional expectation $\mathbf{E}[\cdot|\xi]$ commute since ξ is distributed uniformly on its configuration space under $P_{n,m,E}$. On using Jensen's inequality, the last term therefore is at most $C_0 n^2 \sum_b \mathbf{E}[(\pi_b f)^2]$. Thus

$$\mathbf{E}\Big[\big(\mathbf{E}[f|\xi] - \mathbf{E}[f]\big)^2\Big] \le C_0 n^2 \sum_{b\in\Lambda^*(n)} \mathbf{E}[(\pi_b f)^2]. \tag{8}$$

Estimation of the first term of (7) is made by applying the following two lemmas.

Lemma 2. *Suppose that the conditions* (1) *and* (2) *hold true. Then there exists a constant C such that*

$$\mathbf{E}\Big[\big(f - \mathbf{E}[f|\xi]\big)^2 \Big| \xi\Big] \le C\frac{1}{m}\mathbf{E}\Big[\sum_{x,y(\ne)} c_{\mathrm{zr}}(\eta_x)\xi_y \big(\nabla_{x,y} f\big)^2 \Big| \xi\Big].$$

Here the summation on the right-hand side extends over all ordered pairs $(x,y) \in \Lambda_n \times \Lambda_n$ such that $x \ne y$.

Lemma 3. *Suppose that the conditions* (1) *and* (3) *hold true. Then there exists a constant C such that*

$$\frac{1}{|\Lambda_n|}\mathbf{E}\Big[\sum_{x,y(\ne)} c_{\mathrm{zr}}(\eta_x)\xi_y \big(\nabla_{x,y} f\big)^2\Big] \le Cn^2\frac{|\Lambda_n|}{m}\Big\{D\{f\} + \frac{E}{m}\sum_{b\in\Lambda^*(n)} \mathbf{E}[(\pi_b f)^2]\Big\}.$$

We shall prove Lemmas 2 and 3 in Sections 4 and 3 respectively. By applying Lemmas 2 and 3 in turn, the first expectation on the right-hand side of (7) admits the bound

$$\mathbf{E}\Big[\big(f - \mathbf{E}[f|\xi]\big)^2\Big] \le C'' n^2 \frac{|\Lambda_n|^2}{m^2}\Big\{D\{f\} + \frac{E}{m}\sum_{b\in\Lambda_n^*} E[(\pi_b f)^2]\Big\}.$$

The inequality of Theorem 1 is now obtained by combining this with (8).

REMARK. Lemma 3 is an averaged version of the moving-particle lemma for energy exchange (by the zero-range interaction) in our mixed dynamics. Without such averaging our model would not admit any relevant bound written by means of Dirichlet forms only, while the zero-range or exclusion processes do (cf. Lemma 4 in Section 3). Owing to the inequality (8) we do not need the corresponding one for particle exchange, which seems not easy to prove if $d = 1$.

3. Proof of Lemma 3

As in the outline of the proof of Theorem 1 given in the preceding section we fix n, m, E, and simply write $\mathbf{P}[\,\cdot\,]$, $\mathbf{E}[\,\cdot\,]$ and $D\{\,\cdot\,\}$ for $P_{n,m,E}[\,\cdot\,]$, $E_{n,m,E}[\,\cdot\,]$ and $D_{n,m,E}\{\,\cdot\,\}$, respectively.

Let $\gamma(x, y)$ denote the canonical path from x to y. By this we mean that

$$\gamma(x, y) = \{z_i : 0 \le i \le |x - y|\}, \tag{9}$$

with $z_i = (z_i^1, \dots, z_i^d)$ which are defined in the following way: for $k = 1, \dots, d$, $r(k) = \sum_{l=1}^{k} |x^l - y^l|$ and

$$z_i^k = x^k \quad \text{for} \quad i = 1, \dots, r(k-1); \quad z_i^k = y^k \quad \text{for} \quad i = r(k), \dots, d;$$
$$z_i^k = x^k + (i - r(k-1))\frac{y^k - x^k}{|y^k - x^k|} \quad \text{for} \quad r(k-1) < i \le r(k),$$

namely $\gamma(x, y)$ denotes the (shortest) path of successive nearest neighbor sites that goes from x to y, moving firstly along the first coordinate axis up to the $r(1)$-th step, secondly along the second coordinate axis up to the $r(2)$-th step, and so on. For the following lemma we do not need any of the conditions (1) to (3) imposed on c_{ex} and c_{zr}.

Lemma 4. *There exists a constant C such that for any $x, y \in \Lambda_n$ $(x \ne y)$,*

$$\begin{aligned}
&\mathbf{E}\Big[c_{\mathrm{zr}}(\eta_x)\xi_y[\nabla_{xy} f]^2\Big] \\
&\quad \le C|x - y| \sum_{z,v \in \gamma(x,y):|z-v|=1} \mathbf{E}\Big[c_{\mathrm{zr}}(\eta_z)\Big([\nabla_{z,v} f]^2 + [\pi_{z,v} f]^2\Big)\xi_y\Big] \\
&\qquad + C|x - y| \sum_{z,v \in \gamma(x,y):|z-v|=1} \mathbf{E}\Big[[\pi_{z,v} f]^2 c_{\mathrm{zr}}(\eta_y)\Big].
\end{aligned}$$

Proof. Let us define the transformation $S^{x,y}$ of η by

$$S^{x,y}\eta := \begin{cases} S_{\mathrm{ex}}^{x,y}\eta & \text{if } \xi_x = 1,\ \xi_y = 0, \\ S_{\mathrm{zr}}^{x,y}\eta & \text{if } \eta_x \geq 2,\ \xi_y = 1, \\ \eta & \text{otherwise,} \end{cases}$$

Let $z(i) = z_i$ $(0 \leq i \leq |x-y|)$ be the i-th site from x on the canonical path $\gamma(x,y)$ as defined just after (9). Put

$$r = |x-y|$$

and suppose that $\eta_x \geq 2$ and $\xi_y = 1$. Then the transformation $\eta \mapsto S_{\mathrm{zr}}^{x,y}\eta$ is achieved first by applying the transformations $S^{z(i-1),z(i)}$, $1 \leq i \leq r$, successively along the canonical path $\gamma(x,y)$ until arriving at the site y and then, in the return trip starting at $z(r-1)$, by applying the transformations $S_{\mathrm{ex}}^{z(2r-i),z(2r-i-1)}$, $r+1 \leq i \leq 2r-1$ to recover the original configuration between x and y; formally

$$\begin{aligned} S_{\mathrm{zr}}^{x,y}\eta = {} & S_{\mathrm{ex}}^{z(1),z(0)} \circ S_{\mathrm{ex}}^{z(2),z(1)} \circ \cdots \circ S_{\mathrm{ex}}^{z(r-1),z(r-2)} \\ & \circ S^{z(r-1),z(r)} \circ S^{z(r-2),z(r-1)} \circ \cdots \circ S^{z(0),z(1)}\eta. \end{aligned}$$

Let us define $T_i\eta$ by $T_0\eta := \eta$,

$$T_i\eta = S^{z(i-1),z(i)}T_{i-1}\eta \quad \text{for} \quad 1 \leq i \leq r,$$

and

$$T_i\eta = S_{\mathrm{ex}}^{z(2r-i),z(2r-i-1)}T_{i-1}\eta, \quad \text{for} \quad r+1 \leq i \leq 2r-1.$$

Then the relation above may be written as

$$S_{\mathrm{zr}}^{x,y}\eta = T_{2r-1}\eta.$$

It follows from the reversibility relations (4) and (5) that for any two distinct sites z, u

$$\begin{aligned} c_{\mathrm{zr}}(\eta_z)\mathbf{P}\{\eta\} &= c_{\mathrm{zr}}(\eta_z)\mathbf{P}\{\eta\}\xi_u + c_{\mathrm{zr}}(\eta_z)\mathbf{P}\{\eta\}(1-\xi_u) \\ &= c_{\mathrm{zr}}(\eta_u+1)\mathbf{P}\{S_{\mathrm{zr}}^{z,u}\eta\} + c_{\mathrm{zr}}(\eta_z)(1-\xi_u)\mathbf{P}\{S_{\mathrm{ex}}^{z,u}\eta\} \end{aligned}$$

whenever $\eta_z \geq 2$; hence

$$c_{\mathrm{zr}}(\eta_z)\mathbf{P}\{\eta\} = c_{\mathrm{zr}}((S^{z,u}\eta)_u)\mathbf{P}\{S^{z,u}\eta\}I(\eta_z \geq 2). \tag{10}$$

One notices that the argument of c_{zr} is unaltered (namely $(S^{z,u}\eta)_u = \eta_z$) if the jump is done by exclusion rule.

By repeated applications of (10), we see that

$$c_{\mathrm{zr}}(\eta_x)\mathbf{P}\{\eta\} = c_{\mathrm{zr}}((T_i\eta)_{z(i)})\mathbf{P}\{T_i\eta\}I(\eta_x \geq 2), \quad (T_i\eta)_y = \eta_y$$

for $0 \leq i \leq r-1$, and that if $\eta_x \geq 2$ and $\xi_y = 1$, then

$$c_{\mathrm{zr}}(\eta_x)\mathbf{P}\{\eta\} = c_{\mathrm{zr}}((T_i\eta)_y)\mathbf{P}\{T_i\eta\}, \quad (T_i\eta)_y = \eta_y + 1$$

for $r \leq i \leq 2r-1$. (In the last stage of the onward trip one unit of energy is handed over by a particle at $z(r-1)$ to that at $y = z(r)$ when an application of the rule (10) changes the argument of c_{zr} to $\eta_y + 1$, which since then remains to be $\eta_y + 1$ all the way back.) Using these equalities, we have

$$\begin{aligned}
&\mathbf{E}\left[c_{\mathrm{zr}}(\eta_x)\xi_y[\nabla_{x,y}f]^2\right]\\
&\leq \mathbf{E}\left[c_{\mathrm{zr}}(\eta_x)\left(\textstyle\sum_{i=1}^{2r-1}(f(T_i\eta) - f(T_{i-1}\eta))\right)^2\xi_y\right]\\
&\leq 2r\sum_{i=1}^{2r-1}\mathbf{E}\left[c_{\mathrm{zr}}(\eta_x)[f(T_i\eta) - f(T_{i-1}\eta)]^2\xi_y\right]\\
&\leq 2r\sum_{i=1}^{r}\sum_{\eta}\mathbf{P}\{T_i\eta\}c_{\mathrm{zr}}((T_i\eta)_{z(i)})[f(T_i\eta) - f(T_{i-1}\eta)]^2 I((T_i\eta)_y > 0)\\
&\quad + 2r\sum_{i=r+1}^{2r-1}\sum_{\eta}\mathbf{P}\{T_i\eta\}c_{\mathrm{zr}}((T_i\eta)_y)[f(T_i\eta) - f(T_{i-1}\eta)]^2.
\end{aligned}$$

Since T_i is one to one on the set $\{\eta : \eta_x \geq 2,\ \xi_y = 1\}$, the right-most member equals

$$\begin{aligned}
&2r\sum_{i=1}^{r}\mathbf{E}\left[c_{\mathrm{zr}}(\eta_{z(i)})\left([\nabla_{z(i),z(i-1)}f]^2 + [\pi_{z(i),z(i-1)}f]^2\right)\xi_y\right]\\
&\quad + 2r\sum_{i=r+1}^{2r-1}\mathbf{E}\left[[\pi_{z(i),z(i-1)}f]^2 c_{\mathrm{zr}}(\eta_y)\right].
\end{aligned}$$

Thus we obtain the required inequality. □

Proof of Lemma 3. First we prove Lemma 3 in the case $d = 1$. We sum up both sides of the inequality of Lemma 4 over x and y, and dominate $|x - y|$ by $2n$ to see that

$$\begin{aligned}
&\sum_{x,y(\neq)}\mathbf{E}\left[c_{\mathrm{zr}}(\eta_x)\xi_y[\nabla_{x,y}f]^2\right]\\
&\leq Cn\sum_{x,y(\neq)}\sum_{u,v\in\gamma(x,y):|u-v|=1}\mathbf{E}\left[c_{\mathrm{zr}}(\eta_u)([\nabla_{u,v}f]^2 + [\pi_{u,v}f]^2)\xi_y\right]
\end{aligned} \tag{11}$$

$$+ Cn \sum_{x,y(\neq)} \sum_{u,v\in\gamma(x,y):|u-v|=1} \mathbf{E}\Big[[\pi_{u,v} f]^2 c_{\mathrm{zr}}(\eta_y)\Big].$$

Taking summation on x, y first and applying the inequalities $\sum \xi_y \le m$, $\sum \eta_y \le E$ and $c_{\mathrm{zr}} \le c_{\mathrm{ex}}/a_3$ we derive the inequality

$$\sum_{x,y(\neq)} \mathbf{E}\Big[c_{\mathrm{zr}}(\eta_x)\xi_y[\nabla_{x,y} f]^2\Big] \le C'n^2\Big(mD\{f\} + E\sum_{x=1}^{n-1} \mathbf{E}[(\pi_{x,x+1} f)^2]\Big),$$

which in turn implies the inequality of Lemma 3.

In the case $d \ge 2$ the argument made above is inadequate. This is because in the summation over the bonds (u, v) there occurs concentration on particular ones which depend on y (so that if (u, v) is fixed first, the multiplicity of y significantly varies with (u, v)) even if we choose any number of paths γ from x to y any different ways and make averaging over them. However, for the first term on the right-hand side of (11) we obtain, by dominating ξ_y by 1, the bound

$$C'n^2|\Lambda_n|D\{f\}, \tag{12}$$

as is easily observed by applying the inequality

$$\sum_{x,y(\neq)} \sum_{u,v\in\gamma(x,y):|u-v|=1} A(u, v) \le 2n|\Lambda_n| \sum_{u,v\in\Lambda_n|u-v|=1} A(u, v)$$

valid for every non-negative function $A(u, v)$. For the second term we cannot follow suit: we have anyhow to dispose of $c_{\mathrm{zr}}(\eta_y)$. To this end we use the following variant of Lemma 4.

Lemma 5. *There exists a constant C such that for any x, $y \in \Lambda_n$ $(x \neq y)$,*

$$\begin{aligned} &\mathbf{E}\Big[c_{\mathrm{zr}}(\eta_x)\xi_y[\nabla_{x,y} f]^2 I\Big(1 \le \eta_y \le \frac{2E}{m}\Big)\Big] \\ &\quad \le C|x-y| \sum_{z,v\in\gamma(x,y):|z-v|=1} \mathbf{E}\Big[c_{\mathrm{zr}}(\eta_z)([\nabla_{z,v} f]^2 + [\pi_{z,v} f]^2)\xi_y\Big] \\ &\qquad + \frac{E}{m} C|x-y| \sum_{z,v\in\gamma(x,y):|z-v|=1} \mathbf{E}\Big[[\pi_{z,v} f]^2 \xi_y\Big], \end{aligned}$$

where $\gamma(x, y)$ is the canonical path from x to y.

Proof. We may proceed along the same lines as in the proof of Lemma 4 but with the function $g(\eta_y) := I(1 \le \eta_y \le 2E/m)$ being inserted in all the expectations appearing there except for the last formula consisting of two lines in which we insert $g(\eta_y - 1)$ instead of $g(\eta_y)$. $\square$

We resume the proof of Lemma 3. Using the inequality

$$\frac{m}{2} \leq \sum_{x \in \Lambda(n)} I\left(1 \leq \eta_x \leq \frac{2E}{m}\right)$$

and making decomposition $\nabla_{x,y} f = f \circ S^{x,y} - f \circ S^{x,w} + \nabla_{x,w} f$ in turn, we dominate $\sum_{x,y(\neq)} \mathbf{E}\left[c_{\mathrm{zr}}(\eta_x)\xi_y[\nabla_{x,y} f]^2\right]$ by

$$\begin{aligned}
&\frac{2}{m} \sum_{x,y(\neq)} \sum_{w} \mathbf{E}\left[c_{\mathrm{zr}}(\eta_x)\xi_y I\left(1 \leq \eta_w \leq \frac{2E}{m}\right)[\nabla_{x,y} f]^2\right] \\
&\leq \frac{4}{m} \sum_{x,y,w \in \Lambda_n} \left\{\mathbf{E}\left[c_{\mathrm{zr}}(\eta_x) I\left(1 \leq \eta_w \leq \frac{2E}{m}\right)[\nabla_{x,w} f]^2\right]\right. \\
&\qquad \left. + \mathbf{E}\left[c_{\mathrm{zr}}(\eta_y) I\left(1 \leq \eta_w \leq \frac{2E}{m}\right)[\nabla_{y,w} f]^2\right]\right\},
\end{aligned}$$

where Schwarz inequality and the reversibility (with a special care in the case $w = x$) are applied for the last inequality and $\nabla_{u,v} f$ is understood to identically vanish if $u = v$. Noticing the symmetric role of x and y and then using Lemma 5, we can dominate the last member by $C' n^2 (|\Lambda_n|^2/m) D\{f\}$ plus

$$C' |\Lambda_n|^2 \frac{1}{m} \cdot \frac{E}{m} n^2 \sum_{u,v \in \Lambda_n : |u-v|=1} \mathbf{E}\left[(\pi_{u,v} f)^2\right]$$

by the same computation as is done for obtaining the bound (12). The proof of Lemma 3 is complete. □

4. Mean field interaction

Let $J_m := \{1, 2, \ldots, m\}$. In this section we consider a meanfield type zero-range process on $\{1, 2, \ldots\}^{J_m}$ with jump rate c_{zr}. Its generator is defined by

$$\bar{L}_m^{\mathrm{zr}} f(\eta) := \frac{1}{m} \sum_{x,y \in J_m(\neq)} c_{\mathrm{zr}}(\eta_x) \nabla_{x,y} f(\eta).$$

(Note that here $\eta_x \geq 1$ for all η.) It is reversible relative to the product measure (the grand canonical measure), denoted by ν_α, whose marginal distribution at site x equals the conditional law of that under $\nu_{p,\alpha}$ given $\xi_x = 1$ for every x. We denote by $P_{m,E}[\ \cdot\ ]$ the conditional measures given by

$$E_{m,E}[\ \cdot\ ] = E^{\nu_\alpha}[\ \cdot\ |\eta_1 + \cdots + \eta_m = E]$$

and the corresponding Dirichlet forms by $\bar{D}_{m,E}^{\mathrm{zr}}\{f\}$, namely

$$\bar{D}_{m,E}^{\mathrm{zr}}\{f\} := -E_{m,E}[f \bar{L}_m^{\mathrm{zr}} f].$$

Lemma 2 clearly follows from the following proposition.

Proposition 6. *Suppose that the rate function c_{zr} satisfies the conditions* (1) *and* (2). *Then there exists a constant C such that*

$$E_{m,E}[(f - E_{m,E}[f])^2] \leq C\bar{D}^{\mathrm{zr}}_{m,E}\{f\}.$$

Proposition 6 is the mean field version of the result for the zero-range processes that is established by Landim, Sethuraman and Varadhan [3]. Let $D^{\mathrm{zr}}_{m,E}$ denote the Dirichlet form for the zero-range process under the canonical measure $P_{m,E}$. Then they have shown that

$$\mathcal{V}_{m,E}(f) := E_{m,E}[(f - E_{m,E}[f])^2] \leq C_1 m^2 D^{\mathrm{zr}}_{m,E}\{f\}. \tag{13}$$

It is easy to see that $\bar{D}^{\mathrm{zr}}_{m,E}\{f\} \leq C_2 m^2 D^{\mathrm{zr}}_{m,E}\{f\}$. Hence (13) follows from Proposition 6, but actually the latter is a corollary of the proof of the former given in [3]: indeed, the proof of Proposition 6 can be carried out by adapting the proof of (13) for replacement of $m^2 D^{\mathrm{zr}}_{m,E}\{f\}$ with $\bar{D}^{\mathrm{zr}}_{m,E}\{f\}$ in various steps of it. In below we indicate some main points for the adaptation of the proof. In a recent paper [1] Proposition 6 is proved in another approach, which we shall describe briefly at the end of this section. We shall omit "zr" from the notations $\bar{L}^{\mathrm{zr}}_m$, $\bar{D}^{\mathrm{zr}}_{m,E}$ and c_{zr} and let $\nabla_{x,y} f = 0$ if $x = y$.

Proof of Proposition 6. As in [3] we proceed by induction on m to prove that for each m there exists a constant W_{m-1} such that if $2 \leq k \leq m-1$, then

$$\mathcal{V}_{k,E}(f) \leq W_{m-1}\bar{D}_{k,E}\{f\} \tag{14}$$

for all $E \geq k$ and $f = f(\eta_1, \ldots, \eta_k)$. In the case $m = 3$ (implying $k = 2$) the assertion (14) follows from Lemmas 2.1 and 2.2 of [3]) (a bound for one-site spectral gaps).

Suppose that $m > 3$ and (14) holds for $k = 2, \ldots, m-1$. Let $E_{m,E}[f] = 0$. We then observe that $\mathcal{V}_{m,E}(f) = E_{m,E}[f^2] = I + II$ where

$$I = \frac{1}{m}\sum_{x\in J_m} E_{m,E}[(f - E_{m,E}[f|\eta_x])^2],$$

$$II = \frac{1}{m}\sum_{x\in J_m} E_{m,E}[(E_{m,E}[f|\eta_x])^2].$$

Regarding f as the function of variables $(\eta_y, y \neq x)$ with η_x fixed, we denote it by $f|_{\eta_x}$. Then by integrating over the variables $(\eta_y, y \neq x)$ first and applying (14),

$$I = \frac{1}{m}\sum_{x=1}^{m} E_{m,E}[\mathcal{V}_{m-1,E-\eta_x}(f|_{\eta_x})]$$

$$\le \frac{1}{m}\sum_{x\in J_m} W_{m-1}E_{m,E}\left[\bar{D}_{m-1,E-\eta_x}\{f|_{\eta_x}\}\right]$$

$$\le W_{m-1}\frac{1}{m}\sum_{x=1}^{m}\frac{1}{m-1}\sum_{u,v\in J_m\setminus\{x\}} E_{m,E}\left[c(\eta_u)[\nabla_{u,v}f]^2\right]$$

$$= \left(1-\frac{1}{m-1}\right)W_{m-1}\bar{D}_{m,E}\{f\}. \tag{15}$$

Proceeding as in [3] with suitable modifications (with the help of Lemmas 2.1, 2.2 and 2.3 and Eq (3.1) of [3]) one will deduce that

$$II \le (C''W_{m-1}+1)\bar{D}_{m,E}\{f\}. \tag{16}$$

It is immediate from this and (15) that there exists a constant W_m such that $\mathcal{V}_{k,E}(f) \le W_m\bar{D}_{k,E}\{f\}$ for $k = 2,\ldots,m$, which completes our induction argument.

In what follows we denote by the same symbol W_m the best possible one (namely the minimum) among such constants. Notice that (15) remains valid with this choice of constants. It remains to prove that the sequence W_m is bounded. For this end we apply another fundamental result of [3] (Proposition 3.1) in the following version of it:

Proposition 7. *For every $\varepsilon > 0$ there exists an positive integer m_o and a constant $C = C(\varepsilon)$ such that*

$$\frac{1}{E_{m,E}[c(\eta_1)]}\left(\mathcal{V}_{m,E}\left(f;\frac{1}{m}\sum_{y=1}^{m}c(\eta_y)\right)\right)^2 \le \frac{C}{m}\bar{D}_{m,E}\{f\}+\frac{\varepsilon}{m}\mathcal{V}_{m,E}(f)$$

for all $m \ge m_o$ and $E \ge m$ and for all real functions f of η.

Proof. The proof is essentially the same as that of Proposition 3.1 of [3]. In the latter the set J_m is divided into intervals $B(i)$ of size l or $l+1$ for each l. For the present purpose of mean field estimation J_m is partitioned into subsets $B(i)$ of cardinality l or $l+1$. We consider all such partitions and take the average over them. Since the distribution of $\{\eta_y : y \in B(i)\}$ under the law $P_{E,m}$ does not depend on the shape of $B(i)$, the all the relevant computations are carried through with these partitions in the same way as with the partitions into intervals. □

Employing (15) and Proposition 7 together with the arguments made for deriving (16) one will deduce

$$\mathcal{V}_{m,E}(f) \le \left(1-\frac{1}{m-1}\right)W_{m-1}\bar{D}_{m,E}\{f\}+\frac{C'''}{m}\bar{D}_{m,E}\{f\}+\frac{C'\varepsilon}{m-1}\mathcal{V}_{m,E}(f).$$

Taking ε small enough we infer that W_m must satisfy

$$W_m \leq \left\{1 - \frac{1}{2m}\right\} W_{m-1} + \frac{C_1}{m} \quad \text{for all} \quad m \geq m_o$$

with a constant C_1 independent of m. This is possible only if $W_m \leq 2C_1$ for all m since W_m cannot decrease. Proposition 6 is thus obtained. □

On Caputo's approach. Caputo [1] found out a remarkable inequality that holds in product spaces obeying certain conditions (its proof relies upon some key results in [3] in the case of zero-range processes). The Caputo inequality corresponding to the zero-range process may read that if $E_{m,E}[f] = 0$, then

$$\sum_{x \in J_m} E_{m,E}[(E_{m,E}[f|\eta_x])^2] \leq \left(1 + O\left(\frac{1}{m^\delta}\right)\right) E_{m,E}[f^2]$$

or, what amounts to the same thing,

$$I = E_{m,E}[f^2] - II \geq \left(1 - \frac{1}{m-1}\right)\left(1 - \frac{\beta}{m^{1+\delta}}\right) E_{m,E}[f^2]$$

for m sufficiently large, where β and δ are some positive constants less than one. This combined with (15) shows that if m is large enough, then W_m may be taken so that $W_m \leq W_{m-1}/(1 - \beta m^{-1-\delta})$, which immediately yields Proposition 6.

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