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Author(s)	Sato, Shoji
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On the Lattice Homomorphisms of Infinite Groups I

By Shoji SATO

1. Introduction. By a lattice homomorphism of a group G onto a group G' we mean a single-valued mapping ϕ of the lattice $L(G)$ of subgroups of G onto the lattice $L(G')$ of subgroups of G', which is subject to the conditions

1. $$(S_1 \cup S_2)\phi = S_1\phi \cup S_2\phi,$$
2. $$(S_1 \cap S_2)\phi = S_1\phi \cap S_2\phi$$

for every pair of subgroups S_1, S_2 of G.

We call *proper* any lattice homomorphism which is neither a lattice isomorphism nor a trivial lattice homomorphism.

D. G. Higman investigated those infinite groups that admit proper lattice homomorphisms which satisfy the stronger conditions:

1′. $$(\cup_\nu S_\nu)\phi = \cup_\nu(S_\nu\phi),$$
2′. $$(\cap_\nu S_\nu)\phi = \cap_\nu(S_\nu\phi)$$

for every (finite or infinite) set of subgroups S_ν of G.[1]

In the case of finite groups this problem was studied by M. Suzuki, G. Zappa and other authors. Our purpose is to study, under the definition 1, 2 of lattice homomorphism, the theory of lattice homomorphisms of groups mainly in the case of infinite groups.

The first difficulty we meet in our case is that there are generally neither upper kernel[2] nor lower kernel[3]. But we can prove for instance that, if the lower kernel exists, it is a normal subgroup, moreover, the ideal of $L(G)$ that is mapped to the least element of $L(G')$ has the upper bound which is normal in G. These facts will give the basis of our studies. Detailed study of this general theory will be made in the forthcoming part II.

In this part I we generalize Higman's results, that is, our problem is to investigate the conditions under which the mapping of $L(G)$ onto $L(G')$ induced by a group homomorphism of G onto G' is a lattice homomorphism which satisfies conditions 1, 2. Because every such

1) Cf. Higman [1].
2), 3) Cf. Suzuki [2].

mapping preserves unions, we need only be concerned with the conditions under which it preserves intersections. Let N be the kernel of the group homomorphism ω. Without loss in generality, we may indentify G/N with G', ω with the natural homomorphism of G onto G/N. But here we shall start with a more general mapping of $L(G)$ in itself: $S \to S \cup M$ for subgroups S of G, where M is a subgroup of G. And we shall show that M is normal and this mapping is a lattice homomorphism induced by the natural group homomorphism of G onto G/M, if the mapping is a lattice homomorphism.

Definition. A group G has property (Y) for N if there is a normal subgroup N of G such that the natural homomorphism ω of G onto G/N induces a proper lattice homomorphism of G onto G/N that satisfies the conditions 1 and 2. We shall also say that the pair G, N has property (Y). While we say that, after D. G. Higman, G has property (Z) for N if the lattice homomorphism of G onto G/N induced by ω satisfies the stronger conditions $1'$ and $2'$.

If $N=1$, ω induces a lattice isomorphism and if $N=G$, ω induces a trivial lattice homomorphism. In the definition of property (Y) these cases are naturally omitted. We assume throughout our discussion that $N \neq 1, G$.

Notation. $\{a\}$ is the cyclic group generated by a.

2. We begin with the following lemma 1 which shows that a direct generalization of our problem is impossible.

Lemma 1. *Let M be a subgroup of a (finite or infinite) group G, and ψ the mapping: $S \to S \cup M$ for subgroups S of G. If ψ is a lattice homomorphism of $L(G)$ in itself, then M is normal and hence G, M has property (Y).*

Proof. Let a be an element of G which is not contained in M. If an element b is contained in the subgroup $\{a, M\}$ which is generated by a and M, and is not contained in M, then $\{a\} \cap \{b\} \not\subset M$. For, if $\{a\} \cap \{b\} \subset M$, then $(\{a\} \cap \{b\}) \cup M = M$. But as $S \cup M$ is a lattice homomorphism, we have $(\{a\} \cap \{b\}) \cup M = (\{a\} \cup M) \cap (\{b\} \cup M) = \{b\} \cup M \neq M$, which is a contradiction. Hence, if $x \in M$ and $axa^{-1} \notin M$, then $\{a\} \cap \{axa^{-1}\} \not\subset M$ and we have $a^j = ax^i a^{-1} \notin M$ for some integers j and i. But this implies $a^j = x^i \in M$, which is a contradiction. Thus we know that, for any $a \notin M$ and $x \in M$, holds $axa^{-1} \in M$, which shows that M is normal in G, q.e.d.

The following lemma 2 corresponds to the lemma (2, 1) in Higman's paper and plays an essential rôle throughout this note. The proof is also given quite analogously.

Lemma 2. *A group G has property (Y) for N if and only if*

$$X \frown \{a\} \frown \{b\} \text{ is not empty}$$

for every coset X in G/N and every pair of elements a, b of X.

Proof. Let G, N be a pair having property (Y), and let ϕ be the lattice homomorphism of G onto G/N. If $X \frown \{a\} \frown \{b\}$ is empty for some coset X and for $a, b \in X$, we have $X \neq (\{a\} \frown \{b\})\phi$. On the other hand, $(\{a\} \frown \{b\})\phi = \{a\}\phi \frown \{b\}\phi = \{X\} \frown \{X\} = \{X\}$. This is a contradiction.

Assume conversely that a normal subgroup N of G satisfies the condition of the lemma. Let ϕ be the mapping of $L(G)$ onto $L(G/N)$ induced by the natural homomorphism $G \sim G/N$. For any pair $S, R \in L(G)$, $(S \frown R)\phi \leq S\phi \frown R\phi$ is trivial. But, for any coset $X \in S\phi \frown R\phi$ we can find naturally such elements s, r that $s \in X \frown S$, $r \in X \frown R$, then, the elements in $X \frown \{s\} \frown \{r\}$ belong to $S \frown R$. This shows $(S \frown R)\phi \ni X$, which proves $(S \frown R)\phi \geq S\phi \frown R\phi$. Hence ϕ is a lattice homomorphism, q.e.d.

Lemma 3. *If G has property (Y) for N, then every element in G/N has finite order.*

Proof. Let $X(\neq N) \in G/N$ and $a, b \in X$. We can assume $a \neq b$ because $N \neq 1$. $\{a\} \frown \{b\} \frown X$ is not empty and contains $a^i(\neq 1)$ for some integer i. If $i=1$, then $\{a\} \leq \{b\}$. This implies $b^s = a$, and hence $b^{1-s} = ba^{-1} \in N$ for some integer $s \neq 1$. If $i \neq 1$, then $a^{1-i} = aa^{-i} \in N$. In either case the order of X must be finite, q.e.d.

Theorem 1. *Let G, N be a pair having property (Y). If G has an element a of finite order that is not contained in N, then every element in G has finite order, and the pair G, N has property (Z).*

Proof. Let $a \in X(\neq N) \in G/N$ and let a have finite order. Let s be such integers that $a^s \in X$. In the set of these (finite number of) subgroups $\{a^s\}$ there is the least one, since $\bigcap_s \{a^s\} \frown X$ is not empty. Let this least one be $\{a^t\}$, then $\{a\} \frown \{b\} \geq \{a^t\}$ for every element b in X, because $\{a\} \frown \{b\} \frown X$ is not empty. This shows $a^t \in \bigcap_{b \in X} \{b\} \frown X$. Put $a^t = r$. Then we have $\{nr\} \ni r$ for any element n in N. Hence r is commutative with n, and $(nr)^i = n^i r^i = r$ for some integer i. Thus we have $n^i = r^{1-i}$, which implies that every element n in N has finite order. Then we can conclude that, according to lemma 3, every element of G has finite order.

Now we can apply the above discussion to every coset Y in G/N and see that $\bigcap_{b \in Y} \{b\} \frown Y$ is not empty, which is equivalent to the condition that the pair G, N has property (Z) as Higman proved.

Remark. Theorem 1 shows the relation between Higman's result and our case. Higman proved that, if a pair G, N has property (Z), every element in G has finite order, hence those and only those groups that satisfies the conditions in our theorem have the property (Z).

Lemma 4. *Let G, N have property (Y), and let G have elements of infinite order.*

Then $\{a\}\frown Z(n)\frown X$ is not empty for every element X in G/N and a in X, and for every element n of N, where $Z(n)$ means the centralizer of n in G. If n is of finite order, then the order of n is prime to those of elements in G/N.

Proof. Let $X=aN$ be an element of G/N and $a\notin N$. If $a^j\in\{an\}\frown\{a\}\frown X$, then a^j is commutative with an, and so with n. This implies the first part of the lemma.

Now let $n\in N$ be of finite order, and let $X=aN(\neq N)$ be an element in G/N. We can assume that a is commutative with n. Then, if $\{a\}\frown\{an\}=\{a^i\}$, we have $n^i=1$. For, if $(an)^j=a^i$, then $n^j=a^{i-j}$. But a is of infinite order (cf. th. 1), so $i=j$ and $n^i=1$. According to the condition (Y), $\{a^i\}\frown X$ is not empty, so $\{X^i\}=\{X\}$, which implies that i is prime to the order of X, q.e.d.

Lemma 5. *Let G, N have property (Y), and let $n, m\in N$ be of infinite order, then $\{m\}\frown\{n\}\neq 1$.*

Proof. Let $X=aN(\neq N)$ be an element of G/N. As can easily be seen from the proof of lemma 4, we can assume that a is commutative with both n and m. Then $\{a\}\frown\{m\}\neq 1$. For, from $\{an\}\frown\{a\}\neq 1$, we have $(an)^j=a^i$ for some integers $i, j(\neq 0)$, whence $n^j=a^{i-j}$ and $n^j\neq 1$. Similarly $\{a\}\frown\{n\}\neq 1$. Hence $\{n\}\frown\{m\}\neq 1$, because a is of infinite order, q.e.d.

Theorem 2. *Let G be a group containing elements of infinite order. A pair G, N has property (Y) if and only if*

1) *the elements of G/N are of finite order,*

2) *N contains all the elements of finite order in G, and their orders are prime to those of the elements of G/N,*

3) *$\{n\}\frown\{m\}\neq 1$ for every pair of elements m, n of N of infinite order, and*

4) *$\{a\}\frown Z(n)\frown X$ is not empty for every elements X in G/N and a in X, and for every element n in N,* (for the definition of $Z(n)$ cf. lemma 4.)

Proof. The necessity is proved in lemma 3, th. 1, lemmas 4 and 5.

To prove the sufficiency, choose an element n from N and an element $a \notin N$ from an X in G/N. According to 4), some power a^j of a is contained in $Z(n)$. From 1), 2) and 3), we have $\{an\} \frown \{a^j\} \neq 1$. Put $(an)^u = (a^j)^v$, then $(u, v) = s$ is the order of an element in N (cf. an is commutative with a^j) if u is chosen positive and as small as possible. But the order i of X is prime to u. For, $X^u = X^v$ implies $u \equiv v \bmod i$, so (u, i) divides both i and s. This means $(u, i) = 1$, according to 2).

Now $\{(an)^u\} \frown X = \{(a^j)^v\} \frown X$ is not empty. This means that $\{a\} \frown \{b\} \frown X$ is not empty for every X and its elements a, b, which is equivalent to (Y), q.e.d.

Corollary. *If a group G has property (Y) for an abelian normal subgroup N and if G contains elements of infinite order, then the rank of N is* 1.

Lemma 6. *Let N be a locally cyclic normal subgroup of a group G. If G/N has no element of infinite order and G has no element of finite order other than* 1, *then N is contained in the center of G.*

Proof. Let $a \in N, b \in G, b \notin N$ and $b^i = a$ for some integer i. If $c \in N$ and $c^s = a$ for an integer s, then $(bcb^{-1})^s = a$, this implies $bcb^{-1} = c$ because N is a torsion free abelian group. But N is generated by all these c, so is contained in the center of G, q.e.d.

Theorem 3. *Let M be the set of all elements of finite order in a group G and $G \neq M$. Then, G has property (Y) for an abelian normal subgroup N if and only if*

1) *G/N has no element of infinite order,*

2) *M is contained both in N and in the center of G, and the orders of elements in M are prime to the orders of elements in G/N,*

3) *N/M is a torsion free locally cyclic group.*

Proof. The necessity of our conditions are obvious from th. 2 and corollary to it.

To prove the sufficiency, we are only to show that our conditions imply the condition 4) in theorem 2.

G/M is torsion free and N/M is locally cyclic, and so the latter is contained in the center of G/M, according to lemma 6. Hence $N \cup X/M$ is abelian for every element X in G/N, but, as $N \cup X/N$ is a finite cyclic group, $N \cup X/M$ is necessarily an abelian group of rank 1 and so is locally cyclic. But this implies $N \cup X$ is itself an abelian group because M is contained in its center. Thus we see that N is contained in the center of G, hence condition 4) holds, q.e.d.

As a direct consequence of theorem 3, we have

Theorem 4. *If G is a (locally) free group and has property (Y) for* *some normal subgroup* $N (\neq 1, G)$, *then* G *is a (locally) cyclic group. Conversely, if* G *is a torsion free locally cyclic group, then* G *has property* (Y) *for every subgroup* N.

Remark. As can easily be seen, if G/N is of finite order in lemma 6 (or th. 3) then G is itself a locally cyclic group (or abelian group of rank 1). I could not find those non-abelian groups that satisfy the conditions in lemma 6 or in theorem 3.

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