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<th>Actions of special unitary groups on a product of complex projective spaces</th>
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<tr>
<td>Author(s)</td>
<td>Uchida, Fuichi</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 20(3) P.513-P.520</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1983</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/11436">https://doi.org/10.18910/11436</a></td>
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<td>DOI</td>
<td>10.18910/11436</td>
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Osaka University
0. Introduction

Let $X$ be a connected closed orientable $C^\infty$ manifold which admits a non-trivial smooth $SU(n)$ action. Suppose

$H^*(X; \mathbb{Q}) = \mathbb{Q}[u, v]/(u^{s+1}, v^{t+1})$, $\deg u = \deg v = 2$,

that is, the cohomology ring of $X$ is isomorphic to that of a product $P_a(C) \times P_b(C)$ of complex projective spaces, where $\mathbb{Q}$ is the field of rational numbers.

We shall show the following result.

Theorem. On the above situation, suppose

$1 \leq b < a < a+b \leq 2n-3$.

Then, $a = n-1$ and $X$ is equivariantly diffeomorphic to $P_{n-1}(C) \times Y$, where $Y$ is a connected closed orientable manifold whose rational cohomology ring is isomorphic to that of $P_b(C)$, and $SU(n)$ acts naturally on $P_{n-1}(C)$ and trivially on $Y$.

1. Preliminary lemmas

We prepare the following lemmas.

Lemma 1.1. Let $G$ be a closed connected proper subgroup of $SU(n)$ such that $g = \dim SU(n)/G \leq 4n-6$. Then it is one of the following up to an inner automorphism of $SU(n)$.

(i) $SU(n-k) \subset G \subset S(U(k) \times U(n-k))$, $n \geq 2k$; $k=1, 2$ or 3.

(ii) \[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
n & G & g & 4n-6 & n & G & g & 4n-6 \\
\hline
6 & Sp(3) & 14 & 18 & 4 & SO(4) & 9 & 10 \\
5 & Sp(2) & 14 & 14 & 4 & Sp(2) & 5 & 10 \\
5 & NSp(2) & 13 & 14 & 3 & SO(3) & 5 & 6 \\
5 & SO(5) & 14 & 14 & 3 & T^2 & 6 & 6 \\
\hline
\end{array}
\]

*) Supported in part by Grant-in-Aid for Scientific Research 56540005.
Here $\text{NSp}(2)$ denotes the normalizer of $\text{Sp}(2)$ in $\text{SU}(5)$.

The proof is a routine work by a standard method [2, 3], so we omit it.

**Lemma 1.2.** Suppose $n \geq 3$ and $k \leq 4n - 6$. Then a non-trivial real representation of $\text{SU}(n)$ of degree $k$ is equivalent to $(\mu_n)_R \oplus \theta^{0-2n}$ or $\pi \oplus \theta^{0-n}$ (for $n = 4$). Here $(\mu_n)_R : \text{SU}(n) \to \text{O}(2n)$ is a standard inclusion, $\pi : \text{SU}(4) \to \text{SO}(6)$ is a double covering, and $\theta^i$ is a trivial representation of degree $i$.

Proof. The proof is also a routine work by a standard method [3], but we give a proof for completeness. Denote by $L_1, L_2, \ldots, L_{n-3}$ the standard fundamental weights of $\text{SU}(n)$. Then there is a one-to-one correspondence between complex irreducible representations of $\text{SU}(n)$ and sequences $(a_1, \ldots, a_{n-1})$ of non-negative integers such that $a_1 L_1 + \cdots + a_{n-1} L_{n-1}$ is the highest weight of a corresponding representation. Denote by $d(a_1 L_1 + \cdots + a_{n-1} L_{n-1})$ the degree of the complex irreducible representation of $\text{SU}(n)$ with the highest weight $a_1 L_1 + \cdots + a_{n-1} L_{n-1}$. Notice that if $a_i \geq a'_i$ for $i = 1, \ldots, n - 1$, then $d(a_1 L_1 + \cdots + a_{n-1} L_{n-1}) \geq d(a'_1 L_1 + \cdots + a'_{n-1} L_{n-1})$ and the equality holds only if $a_i = a'_i$ for $i = 1, \ldots, n - 1$. The degree can be computed by Weyl's dimension formula. We obtain

$$
\begin{align*}
\text{for } 1 \leq i \leq n-1, & \quad d(L_i) = n_{C_i} \quad \text{where } C_i \text{ is the } i\text{-th lowest weight of } \text{SU}(n), \\
& \quad d(2L_i) = d(2L_{n-2}) = n^2(n^2-1)/12, \quad d(L_i + L_{n-1}) = n^2 - 1, \\
& \quad d(L_{n-1} + L_{n-2}) = n(n+1)(n-2)/2, \\
& \quad d(L_2 + L_{n-2}) = n^2(n-1)(n-3)/4, \\
& \quad d(L_{n-2} + L_{n-3}) = n^2(n^2-1)/3, \\
& \quad d(3L_1) = d(3L_{n-1}) = n(n+1)(n+2)/6.
\end{align*}
$$

(i) Suppose $n \geq 5$. Then a non-trivial complex irreducible representation of degree $4n - 6$ is equivalent to one of the following: $\mu_n$, $\mu_n^*$, $\Lambda^2(\mu_n)$, $\Lambda^2(\mu_n^*)$, where $\mu_n^*$ is the conjugate representation and $\Lambda^2(\ )$ is the second exterior product. Therefore a non-trivial self-conjugate complex representation of degree $4n - 6$ is equivalent to $\mu_n + \mu_n^* \oplus \text{trivial}$, which has a real form $(\mu_n)_R \oplus \text{trivial}$.

(ii) Suppose $n = 4$. Then a non-trivial complex irreducible representation of degree $4n - 6 = 10$ is equivalent to one of the following: $\mu_4$, $\mu_4^*$, $\Lambda^2(\mu_4)$, $\Lambda^2(\mu_4^*)$, $S^2(\mu_4)$, $S^2(\mu_4^*)$, where $S^2(\ )$ is the second symmetric product. Therefore a non-trivial self-conjugate complex representation of degree $10$ is equivalent to $\mu_4 \oplus \mu_4^* \oplus \text{trivial}$ or $\Lambda^2(\mu_4) \oplus \text{trivial}$. They have a real form $(\mu_4)_R \oplus \text{trivial}$ and $\pi \oplus \text{trivial}$, respectively.

(iii) Suppose $n = 3$. Then a non-trivial complex irreducible representation of degree $4n - 6 = 6$ is equivalent to one of the following: $\mu_3$, $\mu_3^*$, $S^2(\mu_3)$, $S^2(\mu_3^*)$. 


Therefore a non-trivial self-conjugate complex representation of degree \( \leq 6 \) is equivalent to \( \mu_3 \oplus \mu_3^* \), which has a real form \( (\mu_3)_R \). q.e.d.

**Notations.** In the following sections, let \( K^0 \) denote the identity component of a closed subgroup \( K \) of \( SU(n) \), and \( N(K) \) denote the normalizer of \( K \) in \( SU(n) \). Let \( \chi(X) \) denote the Euler characteristic of a manifold \( X \).

2. **Smooth \( SU(n) \) actions**

Throughout this section, suppose that \( X \) is a connected closed orientable manifold with a non-trivial smooth \( SU(n) \) action such that \( \dim X \leq 4n - 6 \). Denote by \( (H) \) the principal isotropy type.

**Proposition 2.1.** Suppose \( n=5 \) and \( H^0=NSp(2) \). Then \( \chi(X)=0 \). In fact, \( X \) has only one orbit type \( SU(5)/NSp(2) \).

Proof. Since \( N(NSp(2))=NSp(2) \), it follows that \( H=NSp(2) \) and \( X \) has no exceptional orbits. Now we shall show that \( X \) has no singular orbits. It is clear for \( \dim X=13 \). Suppose that \( \dim X=14 \) and \( X \) has a singular orbit. Then the orbit type must be \( SU(5)/S(U(1) \times U(4)) \) by Lemmas 1.1 and 1.2. Considering the slice representation, we obtain a covering projection of \( SU(4)/center \) onto \( SO(6) \). But, there is no injection of \( \pi_1(SU(4)/center)=\mathbb{Z}_2 \times \mathbb{Z}_2 \) into \( \pi_1(SO(6))=\mathbb{Z}_2 \), and hence there is no covering projection of \( SU(4)/center \) onto \( SO(6) \). Therefore, \( X \) has no singular orbits. q.e.d.

The next three propositions can be easily proved.

**Proposition 2.2.** Suppose that \( H^0 \) is one of the following: \( Sp(3), n=6; Sp(2), n=5; Sp(2), n=4; SO(5), n=5; SO(4), n=4; SO(3), n=3. \) Then, \( X \) has no singular orbits and \( \chi(X)=0 \).

**Proposition 2.3.** Suppose \( n=3 \) and \( H^0=\mathbb{T} \). Then \( SU(3) \) acts transitively on \( X \).

**Proposition 2.4.** Suppose \( n \geq 6 \) and \( SU(n-3) \subset H^0 \subset SU(U(3) \times U(n-3)). \) Then \( n=6 \) and \( X=SU(6)/S(U(3) \times U(3)) \).

The remaining possibilities are the followings:

\[ SU(n-k) \subset H^0 \subset S(U(k) \times U(n-k)); k=1, 2. \]

In these cases, considering the slice representation, we can prove that \( SU(n-j) \subset K^0 \subset S(U(j) \times U(n-j)); j=0, 1 \text{ or } 2, \) for any singular isotropy type \( (K) \). Denote

\[ F_{(k)} = \{ x \in X | SU(n-k) \subset SU(n) \subset S(U(k) \times U(n-k)) \}, \]

\[ X_{(k)} = SU(n) \cdot F_{(k)}. \]
Then \( X = X_{(\omega)} \cup X_{(1)} \cup X_{(2)} \) for the remaining cases.

**Proposition 2.5.** If \( X_{(2)} \) is non-empty, then \( X_{(\omega)} \) and \( X_{(1)} \) are empty.

**Proof.** Since \( X_{(2)} \) is non-empty, we have \( n \geq 4 \) and

\[
SU(n-2) \subset H^0 \subset S(U(2) \times U(n-2)).
\]

Suppose that \( X_{(1)} \) is non-empty. Let \( \sigma_y \) be the slice representation at \( y \in F_{(1)} \). Then

\[
\deg \sigma_y = \dim X - \dim SU(n) \cdot y \leq 2n - 4 < 4(n-1) - 6.
\]

Hence we obtain \( \sigma_y | SU(n-1) = (\mu_{n-1}) \oplus \text{trivial} \), by (*) and Lemma 1.2. Let \( \rho_y \) be the isotropy representation at \( y \) in the orbit \( SU(n) \cdot y \). Then \( \rho_y | SU(n-1) = (\mu_{n-1}) \oplus \text{trivial} \), and hence

\[
\text{codim } F_{(1)} \text{ at } y = 4n - 4 > 4n - 6.
\]

This is a contradiction, and hence \( X_{(1)} \) is empty. Similarly we can prove that \( X_{(\omega)} \) is empty. q.e.d.

**Proposition 2.6.** Suppose \( X = X_{(2)} \) and \( \chi(X) \neq 0 \). Then \( X = SU(n)/S(U(2) \times U(n-2)) \) or \( X = SU(n)/SU(n-2) \times_S S^2 \), where \( W = S(U(2) \times U(n-2))/SU(n-2) \equiv U(2) \).

**Proof.** Since \( X = X_{(2)} \), we obtain an equivariant decomposition \( X = SU(n)/SU(n-2) \times_{w} F_{(2)} \), where \( F_{(2)} \) is a connected closed orientable manifold on which \( W \) acts smoothly. The conditions \( \dim X \leq 4n - 6 \) and \( \chi(X) \neq 0 \) imply that \( \dim F_{(2)} \leq 2 \) and \( \chi(F_{(2)}) = 0 \). Hence we have a desired result. q.e.d.

Put \( G_{n,2} = SU(n)/S(U(2) \times U(n-2)) \). For the case \( X = SU(n)/SU(n-2) \times w S^2 \), there is a fibration: \( S^2 \to X \xrightarrow{\pi} G_{n,2} \). Suppose that the \( W \) action on \( S^2 \) is non-transitive. Then the \( W \) action on \( S^2 \) has a fixed point, and hence the above fibration has an equivariant cross-section \( s \).

**Proposition 2.7.** On the above situation, there is an element of \( H^4(X; \mathbb{Q}) \) which is not a linear combination of \( x_j^2 \); \( x_j \in H^2(X; \mathbb{Q}) \).

**Proof.** Let \( c_1 \) and \( c_2 \) be the first and the second Chern classes of the canonical 2-plane bundle over \( G_{n,2} \), respectively. Suppose that \( \pi^*(c_2) \) is represented as

\[
\pi^*(c_2) = \sum_j a_j x_j^2; \ a_j \in \mathbb{Q}, \ x_j \in H^2(X; \mathbb{Q}).
\]

Then \( c_2 = \pi^*(c_2) = \sum_j a_j (s^* x_j)^2 = a c_1^2 \) for some \( a \in \mathbb{Q} \), and hence \( c_2 \) and \( c_1^2 \) are linearly dependent in \( H^4(G_{n,2}; \mathbb{Q}) \). This is a contradiction. Hence \( \pi^*(c_2) \) is
ACTIONS OF SPECIAL UNITARY GROUPS

REMARK. Suppose \( H^*(X; \mathbb{Q}) = \mathbb{Q}[u, v]/(u^{a+1}, v^{b+1}), \) \( \deg u = \deg v = 2. \) Then any element of \( H^i(X; \mathbb{Q}) \) is represented as

\[
p u^2 + q uv + r v^2 = p u^2 + q(u+v)^2 - q'(u-v)^2 + r v^2,
\]

where \( p, q, r \in \mathbb{Q} \) and \( q' = q/4. \)

Suppose next that the \( W \) action on \( S^2 \) is transitive. Then \( X = SU(n)/SU(n-2) \times SU(n)/SU(1) \times SU(1) \times U(n-2). \) Define

\[
X_1 = \{(x_1, \ldots, x_n) \times (y_1, \ldots, y_n) \in P_{n-1} \times P_{n-1} | \sum_{i=1}^{n} x_i y_i = 0\}.
\]

Then \( X_1 \) is invariant under the natural diagonal \( SU(n) \) action on \( P_{n-1}(C) \times P_{n-1}(C), \) and we have \( X_1 = SU(n)/SU(1) \times SU(1) \times U(n-2). \) Considering \( X_1 \) as a projective space bundle over \( P_{n-1}(C), \) we have a ring structure: \( H^*(X_1; \mathbb{Q}) = \mathbb{Q}[c, t]/(c^n, \sum_i c^i t^{n-i-1}), \) \( \deg c = \deg t = 2. \)

**Proposition 2.8.** Let \( X_1 = SU(n)/SU(1) \times SU(1) \times U(n-2) \) and \( u \in H^2(X_1; \mathbb{Q}). \) If \( u^{n-1} = 0, \) then \( u = 0. \)

Proof. Any element of \( H^2(X_1; \mathbb{Q}) \) is represented as \( u = p c + q t; p, q \in \mathbb{Q}. \) Suppose \( u^{n-1} = 0. \) Then we have

\[
q^{n-1} = n-1 c_k p^{n-1} q^k, k = 0, 1, \ldots, n-2,
\]

Hence we obtain \( p = q = 0. \) q.e.d.

**3. Proof of the theorem**

Throughout this section, suppose that \( X \) is a connected closed orientable manifold with a non-trivial smooth \( SU(n) \) action, and \( H^*(X; \mathbb{Q}) = \mathbb{Q}[u, v]/(u^{a+1}, v^{b+1}); \) \( \deg u = \deg v = 2. \) Moreover, suppose

\[
1 \leq b \leq a < n \leq a+b \leq 2n-3.
\]

By arguments and notations in Section 2, the possibility remains only when \( X = X(\omega) \cup X(\eta). \)

**Proposition 3.1.** \( X(\omega) \) is empty.

Proof. Suppose that \( X(\omega) \) is non-empty. Let \( U \) be an invariant closed tubular neighborhood of \( X(\omega) \) in \( X, \) and let \( E = X - \text{int } U. \) Put \( Y = E \cap F(\omega). \) Then \( Y \) is a connected compact orientable manifold with non-empty boundary \( \partial Y, \) and \( U(1) \) acts naturally on \( Y. \) Since there is a natural diffeomorphism: \( E = SU(n)/SU(n-1) \times U(1) Y = S^{2n-1} \times U(1) Y, \) we obtain
Let $i : E \to X$ be the inclusion. Then, $i^* : H^*(X; \mathbb{Q}) \to H^*(E; \mathbb{Q})$ is an isomorphism for each $t \leq 2n - 2$, because the codimension of each connected component of $X_{(1)}$ is $2n$ by Lemma 1.2. By the Gysin sequence of the principal $U(1)$ bundle $p : S^{2n-1} \times Y \to E$, we obtain an exact sequence:

$$0 \to H^{2k-1}(S^{2k-1} \times Y) \to H^{2k-2}(E) \to H^{2k}(E) \to H^{2k}(S^{2k-1} \times Y) \to 0.$$ 

Hence we obtain $\text{rank } H^0(_Y) \leq \text{rank } H^0(Y) + \text{rank } H^{2k-1}(Y) - \text{rank } H^{2k}(Y) = 1$. Therefore $\partial Y$ is empty; this is a contradiction. q.e.d.

Consequently we obtain $X = X_{(1)} = S^{2n-1} \times U(1)F_{(1)}$.

**Proposition 3.2.** $a = n - 1$ and $H^*(F_{(1)}; \mathbb{Q}) = H^*(P_n(C); \mathbb{Q})$.

Proof. Since $n \cdot \chi(F_{(1)}) = \chi(X) = (a+1)(b+1) 
eq 0$, the $U(1)$ action on $F_{(1)}$ has a fixed point $y_0$. Consider the following commutative diagram:

$$\begin{array}{ccc}
S^{2n-1} & \xrightarrow{i} & S^{2n-1} \times F_{(1)} \\
\downarrow{\pi} & & \downarrow{\pi} \\
P_{n-1}(C) & \xrightarrow{i} & X \xrightarrow{\bar{q}} P_{n-1}(C).
\end{array}$$

Here $\pi$, $p$ are projections of the principal $U(1)$ bundles, $q$ is the projection to the first factor, $i$ is an inclusion defined by $i(x) = (x, y_0)$, and $\bar{q}$ are induced mappings. Denote by $e(\cdot)$ the Euler class of a principal $U(1)$ bundle. We can represent as $e(p) = ku + jv$; $k, j \in \mathbb{Q}$. Then

$$0 = \bar{q}^*(e(\pi))^n = e(p)^n = (ku + jv)^n = \sum_i C_i k^{n-i} j^i u^{n-i} v^i,$$

and hence $C_i k^{n-i} j^i = 0$ for $n - a \leq i \leq b$. Hence we obtain $kj = 0$. Suppose $k = 0$. Then

$$0 = e(\pi)^{n-1} = i^*(e(p))^{n-1} = i^*(j^{n-1} v^{n-1}) = 0,$$

because $v^{k+1} = 0$ and $b \leq n - 2$. This is a contradiction. Therefore $e(p) = ku$ ($k \neq 0$). Since $i^*(\langle ku \rangle^{n-1}) = e(\pi)^{n-1} \neq 0$, we obtain $u^{n-1} \neq 0$ and hence $a = n - 1$. Next, considering the Gysin sequence of the principal $U(1)$ bundle $p : S^{2n-1} \times Y \to X$ and the ring structure of $H^*(X; \mathbb{Q})$, we obtain $H^*(F_{(1)}; \mathbb{Q}) = H^*(P_n(C); \mathbb{Q})$. 
Proposition 3.3. The $U(1)$ action on $F_{(1)}$ is trivial.

Proof. Suppose that the $U(1)$ action on $F_{(1)}$ is non-trivial, and let $Y$ be the fixed point set. Consider the following commutative diagram:

$$
\begin{array}{cccc}
H^i(S^{2n-1} \times V_{(1)} F_{(1)}) & \xrightarrow{j^*} & H^i(S^n \times V_{(1)} F_{(1)}) & \xrightarrow{L^*} & S^{-1}H^*(S^n \times V_{(1)} F_{(1)}) \\
\downarrow i^* & & \downarrow i^* & & \downarrow S^{-1}i^* \\
H^i(S^{2n-1} \times V_{(1)} Y) & \xrightarrow{j_Y^*} & H^i(S^n \times V_{(1)} Y) & \xrightarrow{L_Y^*} & S^{-1}H^*(S^n \times V_{(1)} Y).
\end{array}
$$

Here $i$, $i_\omega$, $j$, $j_Y$ are natural inclusions; $L$, $L_Y$ are localization homomorphisms; $S^{-1}$ is a localization by the Euler class of the universal principal $U(1)$ bundle. It is known that $S^{-1}i^*$ is an isomorphism [1]. Since $H^{odd}(F_{(1)}; \mathbb{Q})=0$, we have that $j^*$ is surjective and $L$ is injective, in particular, $i^*$ is injective. On the other hand, $j_Y^*$ is isomorphic for each $t \leq 2n-2$.

Now we shall show that $\omega^{b+1}=0$ implies $\omega^b=0$ for $\omega \in H^i(S^{2n-1} \times V_{(1)} F_{(1)}; \mathbb{Q})$. We can represent as $i^*(\omega) = p_1^*(\alpha) + p_2^*(\beta)$ for some $\alpha \in H^i(P_{n-1}(C))$, $\beta \in H^i(Y)$, where $p_1$, $p_2$ are projections from $S^{2n-1} \times V_{(1)} Y = P_{n-1}(C) \times Y$ to each factor. Then

$$0 = k^*i^*(\omega^{b+1}) = (k^*(p_1^*(\alpha) + p_2^*(\beta)))^{b+1} = \alpha^{b+1},$$

where $k: P_{n-1}(C) \to P_{n-1}(C) \times Y$ is an inclusion defined by $k(x) = (x, \ast)$. Since $b \leq n-2$, we obtain $\alpha = 0$, and hence $i^*(\omega) = p_2^*(\beta)$. Therefore $i^*(\omega^b) = p_2^*(\beta^b) = 0$, because dim $Y < 2b = \dim F_{(1)}$. Since $j^*$ is surjective, there is an element $\bar{w} \in H^i(S^n \times V_{(1)} F_{(1)}; \mathbb{Q})$ such that $j^*\bar{w} = \omega$. Then

$$j_Y^*i_Y^*(\bar{w}^b) = i_Y^*j_Y^*(\bar{w}^b) = i^*(\bar{w}^b) = 0,$$

and hence $\bar{w}^b = 0$, because $j_Y^*i_Y^*$ is injective for the degree $2b$ ($\leq 2n-2$). Then $\omega^b = j^*(\bar{w}^b) = 0$.

On the other hand, $X = S^{2n-1} \times V_{(1)} F_{(1)}$ and $H^*(X; \mathbb{Q}) = \mathbb{Q}[u, v]/(u^{b+1}, v^{b+1})$, where $a = n-1$. There is an element $v \in H^2(X; \mathbb{Q})$ such that $v^{b+1} = 0$ but $v^b \neq 0$. This is a contradiction. q.e.d.

Summarizing the above propositions, we obtain $X = P_{n-1}(C) \times Y$ as $SU(n)$ manifolds, where $Y$ is a connected closed orientable manifold with trivial $SU(n)$ action, and $H^*(Y; \mathbb{Q}) = H^*(P_{(1)} C; \mathbb{Q})$. This completes the proof of the theorem stated in Introduction.

4. Concluding remark

We give examples [2] of a manifold whose rational cohomology ring is isomorphic to that of $P_{(1)} C$. 

\[ \]
EXAMPLE 1. Let $p$ be a positive integer. There is a connected closed orientable $C^\infty$ manifold $Y_1$ such that

$$H^*(Y_1; \mathbb{Q}) = H^*(\mathbb{C}P_k; \mathbb{Q}) \quad \text{and} \quad \pi_1(Y_1) = \mathbb{Z}/p\mathbb{Z}$$

for each $k \geq 3$.

EXAMPLE 2. Let $G$ be a finitely presentable group such that $H^1(G; \mathbb{Z}) = 0$, where $\mathbb{Z}$ is the ring of integers. There is a connected closed orientable $C^\infty$ manifold $Y_2$ such that

$$H^*(Y_2; \mathbb{Z}) = H^*(\mathbb{C}P_k; \mathbb{Z}) \quad \text{and} \quad \pi_1(Y_2) = G$$

for each $k \geq 5$.

References

