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PERCOLATION ON THE PRE-SIERPINSKI GASKET

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1. Introduction and statements of results

In this paper, we regard percolation as a model of phase transitions. We are especially interested in problems near the critical point, where the phase transition occurs. We call these problems critical behaviors. Our purpose in this paper is to clarify the critical behaviors of percolation on the pre-Sierpinski gasket which has self-similarity.

Until now, studies of percolation are restricted on periodic graphs, such as $\mathbb{Z}^d$. (An exact definition of periodic graph is mentioned in Kesten [1].) There are lots of conjectures and hypotheses about critical behaviors, but many of them are still unsolved rigorously (see Grimmett [2] and references therein). In high dimension lattices $\mathbb{Z}^d$, rigorous results for critical behaviors were obtained by Hara-Slade [3]. But in low dimensions, except a work on $\mathbb{Z}^2$ by Kesten [4], few rigorous results have been proved about the existence of critical exponents and justification of the scaling, hyperscaling relations.

For critical behaviors, self-similarity of the graph plays more important role than periodicity. This is a motivation to consider percolation problems on the pre-Sierpinski gasket.

We now define the pre-Sierpinski gasket. Let $O=(0,0)$, $a_0=(1/2, \sqrt{3}/2)$, $b_0=(1,0)$. Let $F_0$ be the graph which consists of the vertices and edges of the triangle $\Delta Oa_0b_0$. Let $\{F_n\}_{n=0,1,2,\ldots}$ be the sequence of graphs given by

$$F_{n+1} = F_n \cup (F_n + 2^na_0) \cup (F_n + 2^nb_0)$$

where $A+a = \{x+a \mid x \in A\}$ and $kA = \{kx \mid x \in A\}$. Let $F = \bigcup_{n=0}^{\infty} F_n$. We call $F$ the pre-Sierpinski gasket. (Fig. 1.1) Note that $\tilde{F} = \bigcup_{n=0}^{\infty} 2^{-n}F$ become the Sierpinski gasket. Let $V$ be the set of all vertices in $F$, and $E$ the set of all edges with length 1.

We consider the Bernoulli bond percolation on the pre-Sierpinski gasket; each edges in $E$ are open with probability $p$ and closed with probability $1-p$ independently. Let $P_p$ denote its distribution. We think of open bonds as permitting to go along the bond. We write $x \leftrightarrow y$ if there is an open path from $x$ to $y$. Let $C(x) = \{y \in V : x \leftrightarrow y\}$. $C(x)$ is called the open cluster containing $x$. We denote by $C$ the open cluster containing the origin.
We define two functions in a similar way as percolations on $\mathbb{Z}^d$.

$$\theta(p) = P_p(|C| = \infty), \quad \chi(p) = E_p(|C|; |C| < \infty),$$

where $|C|$ denotes the number of vertices contained in $C$, and $E_p$ denotes the expectation with respect to $P_p$. $\theta(p)$ is called the percolation probability, and $\chi(p)$ is called the mean cluster size.

Let $p_c$ denote the critical point; that is

$$p_c = \inf\{p : \theta(p) > 0\}.$$ 

Then $p_c = 1$ for the pre-Sierpinski gasket because it is finitely ramified. We note that $\chi(p) = E_p|C|$ for $p < 1$.

The correlation length is defined by

$$\zeta(p) = \lim_{n \to \infty} \left\{ -\frac{1}{2^n} \log P_p(O \leftrightarrow a_n) \right\}^{-1}.$$ 

The existence of the limit in (1) will be proved in Section 2.

We write $f(p) \approx g(p)$ as $p \to p_0$ if $\log f(p)/\log g(p) \to 1$ as $p \to p_0$.

We now state our main theorems:

**Theorem 1.1.** $\lim_{p \to 1} \frac{\log \zeta(p)}{\log(1-p)} = \infty$, and $\lim_{p \to 1} \frac{\log \log \zeta(p)}{\log(1-p)} = -2$. 
Theorem 1.2. Let $D = \log 3 / \log 2$. Then
\[ E_p |C|^k \approx \{\xi(p)\}^D k \text{ as } p \to 1 \text{ for all } k \geq 1. \]

Remark. Our results are quite different from the results on $Z^d$ (see below). In physical literature, Theorem 1.1 was known by Gefen et al. [5] by using formal renormalization arguments. Our contribution is that we prove Theorem 1.1 rigorously.

We collect results and conjectures of the percolation on $Z^d$. It is conjectured (see [2])
\[ \xi(p) \approx |p_c - p|^{-\nu(d)} \text{ as } p \to p_c. \]

The value $\nu(d)$ is called the critical exponent. It is proved that $\nu(d) = 1/2$ for sufficiently large $d$ (Hara-Slade [3]), and conjectured $\nu(2) = 4/3$ (see [4]).

Other critical exponents considered in $Z^d$ are as follows:
\[ \chi(p) \approx |p_c - p|^{-\gamma}, \quad \frac{E_p(|C|^{k+1}; |C| < \infty)}{E_p(|C|^{k}; |C| < \infty)} \approx |p_c - p|^{-\Delta} \text{ as } p \to p_c. \]

It is conjectured for $Z^d$ that $d\nu = 2\Delta - \gamma$. This relation is one of hyperscaling relations. We note $\gamma = \Delta = \infty$ on the pre-Sierpinski gasket. So the relation $d\nu = 2\Delta - \gamma$ does not make sense on the pre-Sierpinski gasket. Accordingly we modify the hyperscaling relation as follows:
\[ \{\xi(p)\}^D \approx \frac{E_p|C|^3}{\chi(p)^2} \text{ as } p \to p_c. \]

If finite critical exponents $\nu, \gamma, \Delta$ exist, then (3) is equivalent to $d\nu = 2\Delta - \gamma$.

Remark. By Theorem 1.2, we have $E_p|C|^D \approx \{\xi(p)\}^{3D}$ and $\chi(p) \approx \{\xi(p)\}^D$. Hence the above hyperscaling relation (3) holds when we regard $D$ as the dimension of the pre-Sierpinski gasket. The value $D = \log 3 / \log 2$ coincides with the fractal dimension of the Sierpinski gasket.

In addition, we mention site percolation on the pre-Sierpinski gasket: each vertices in $V$ are determined to be open or closed independently. (Details will be given in Section 5). We define the correlation length $\hat{\xi}(p)$ in the same manner as (1). We have the result below;

Theorem 1.3. \[ \lim_{p \to 1} - \frac{\log \hat{\xi}(p)}{\log(1-p)} = \infty, \quad \text{and} \quad \lim_{p \to 1} \frac{\log(\log \hat{\xi}(p))}{\log(1-p)} = -1. \]

The critical exponent in a usual sense is also infinite in this case. But $\log \hat{\xi}(p) \approx (1 - p)^{-1}$, which is different from Theorem 1.1. We cannot see the
universality of this exponent on the pre-Sierpinski gasket.


The organization of this paper is as follows: In Section 2 we prepare for the proof of our main theorems; we construct recursion formulas of relations between events in \( F_n \) and ones in \( F_{n+1} \). In the reminder of Section 2, we prove the existence of the correlation length. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. In Section 5 we study site percolation and prove Theorem 1.3.

2. Recursion formulas and the existence of \( \xi(p) \)

We introduce two connectivity functions as follows.

\[
\Phi_n(p) = P_p(O \leftrightarrow a_n \text{ in } \Delta Oa_nb_n),
\]

\[
\Theta_n(p) = P_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } \Delta Oa_nb_n).
\]

We write \( O \leftrightarrow a_n \text{ in } \Delta Oa_nb_n \) if there is an open path from \( O \) to \( a_n \) in \( \Delta Oa_nb_n \) (contains its perimeter). We easily calculate \( \Phi_0(p) = p + p^2 - p^3 \), \( \Theta_0(p) = 3p^2 - 2p^3 \). Note that (i) \( \Phi_n(p) \geq \Theta_n(p) \) by definition, (ii) if \( O \leftrightarrow a_n \) and \( O \leftrightarrow b_n \) then we have \( a_n \leftrightarrow b_n \) automatically.

**Proposition 2.1.** For each \( n \geq 0 \) and \( 0 \leq p \leq 1 \),

\[
\Phi_{n+1}(p) = \Phi_n(p) + \Theta_n(p) + \Theta_n(p) \theta_n(p)^2 - \Theta_n(p) \theta_n(p)^3 \tag{4}
\]

\[
\Theta_{n+1}(p) = 3\Phi_n(p) \theta_n(p)^2 - 2\theta_n(p)^3 \tag{5}
\]

**Proof.** Recall \( \Delta Oa_nb_n = F_n \). Let \( F'_n = F_n + a_n \), \( F''_n = F_n + b_n \) and \( c_n = (3 \cdot 2^{n-1}, \sqrt{3} \cdot 2^{n-1}) \). Let \( A_1 \) and \( A_2 \) be events given by

\[ A_1 = \{ O \leftrightarrow a_n \text{ in } F_n \} \cap \{ a_n \leftrightarrow a_{n+1} \text{ in } F'_n \}, \]

\[ A_2 = \{ O \leftrightarrow b_n \text{ in } F_n \} \cap \{ b_n \leftrightarrow c_n \text{ in } F''_n \} \cap \{ c_n \leftrightarrow a_{n+1} \text{ in } F'_n \}. \]

Then we have

\[
\Phi_{n+1}(p) = P_p(A_1) + P_p(A_2) - P_p(A_1 \cap A_2). \tag{6}
\]

Here we used the fact that a path from \( O \) to \( a_{n+1} \) goes through \( a_n \) or \( b_n \). Since the events in \( F_n, F'_n, F''_n \) are mutually independent, \( P_p(A_1) = \Phi_n(p) \), \( P_p(A_2) = \Phi_n(p)^3 \),
$P_p(A_n^1 \cap A_n^2) = \{\Theta_n(p)\}^2 \Phi_n(p)$ (Fig. 2.1). Combining these with (6) yields (4).

We proceed to the proof of (5). Let $B_n^1, B_n^2, B_n^3$ be events given by

\[
B_n^1 = \{O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F_n\} \cap \{b_n \leftrightarrow b_{n+1} \text{ in } F_n\},
\]

\[
B_n^2 = \{O \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ and } a_n \leftrightarrow c_n \text{ in } F_n\} \cap \{c_n \leftrightarrow b_{n+1} \text{ in } F_n\},
\]

\[
B_n^3 = \{O \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow b_{n+1} \text{ and } b_n \leftrightarrow c_n \text{ in } F_n\} \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F_n\}
\]

(see Fig. 2.2).

Then we have

\[
\Theta_{n+1}(p) = P_p(B_n^1) + P_p(B_n^2) + P_p(B_n^3) - P_p(B_n^1 \cap B_n^2) - P_p(B_n^2 \cap B_n^3) - P_p(B_n^1 \cap B_n^3) + P_p(B_n^1 \cap B_n^2 \cap B_n^3).
\]

We see easily

\[
P_p(B_n^1) = P_p(B_n^2) = P_p(B_n^3) = \{\Phi_n(p)\}^2 \Theta_n(p),
\]

\[
P_p(B_n^1 \cap B_n^2) = P_p(B_n^2 \cap B_n^3) = P_p(B_n^3 \cap B_n^1) = P_p(B_n^1 \cap B_n^2 \cap B_n^3) = \{\Theta_n(p)\}^3.
\]

(5) follows from this immediately.
From now on, we assume $0 < p < 1$. We prove the existence of the limit (1), correlation length $\xi(p)$, by using these recursions.

**Proposition 2.2.** There exists $\xi(p) > 0$ such that

$$\lim_{n \to \infty} \frac{\Phi_n(p)}{\exp\left(-\frac{2^n}{\xi(p)}\right)} = 1.$$ 

**Remark.** The convergence as $n \to \infty$ in Proposition 2.2 is stronger than the convergence in (1).

**Proof.** By (4) and $\Theta_n(p) \leq \Phi_n(p)$, we have

$$\{\Phi_n(p)\}^2 \leq \Phi_{n+1}(p) \leq \{\Phi_n(p)\}^2 + \{\Phi_n(p)\}^3.$$

Hence

$$1 \leq \frac{\Phi_{n+1}(p)}{\{\Phi_n(p)\}^2} \leq 1 + \Phi_n(p).$$

Let $h_n(p) = \Phi_{n+1}(p) / \{\Phi_n(p)\}^2$. Then $1 \leq h_n(p) \leq 2$ and $\lim_{n \to \infty} h_n(p) = 1$ because $\lim_{n \to \infty} \Phi_n(p) = 0$. Now

$$\frac{1}{2^n \log \Phi_n(p)}$$
\[
\frac{1}{2^n} \log \left( \frac{\Phi_0(p)}{\Phi_1(p)} \right)^{2^n} \cdot \left( \frac{\Phi_1(p)}{\Phi_2(p)} \right)^{2^{n-1}} \cdots \left( \frac{\Phi_{n-1}(p)}{\Phi_n(p)} \right)^{2^n} \\
= \log \Phi_0(p) + \frac{1}{2} \log h_0(p) + \frac{1}{2^2} \log h_1(p) + \cdots + \frac{1}{2^n} \log h_{n-1}(p) \\
\leq \log \Phi_0(p) + \log 2.
\]

Hence \( \{ \log (\Phi_n(p)/2^n) \}_{n=0,1,2,\ldots} \) is increasing and \( \lim_{n \to \infty} \log \Phi_n(p)/2^n \) exists. Let \(-\{\xi(p)\}^{-1} = \lim_{n \to \infty} \log \Phi_n(p)/2^n \). Then

\[
-\frac{1}{\xi(p)} \geq \frac{1}{2^n} \log \Phi_n(p) = -\frac{1}{\xi(p)} \left( \frac{1}{2^n+1} \log h_n(p) + \frac{1}{2^n+2} \log h_{n+1}(p) + \cdots \right) \\
\geq -\frac{1}{\xi(p)} \frac{1}{2^n} \log H_n(p),
\]

where \( H_n(p) = \sup_{m \geq n} h_m(p) \). Therefore

\[
\exp \left( -\frac{2^n}{\xi(p)} \right) \geq \Phi_n(p) \geq \frac{1}{H_n(p)} \exp \left( -\frac{2^n}{\xi(p)} \right).
\]

Since \( \lim_{n \to \infty} H_n(p) = 1 \), we complete the proof.

**Remark.** Note that the function \( \xi(p) \) is continuous and increasing on (0, 1) from the proof above.

**Lemma 2.3.** \( \lim_{n \to \infty} P_p(O \leftrightarrow a_n) = 1 \).

**Proof.** Recall that \( \Phi_n(p) = P_p(O \leftrightarrow a_n \text{ in } F_n) \). Then

\[
P_p(O \leftrightarrow a_n) \leq P_p(O \leftrightarrow a_n \text{ in } F_n) \\
\leq P_p(O \leftrightarrow b_n \text{ in } F_n, b_n \leftrightarrow c_n \text{ in } F_n' \leftrightarrow a_n \text{ in } F_n') \\
+ P_p(O \leftrightarrow b_n \text{ in } F_n, b_n \leftrightarrow b_{n+1} \text{ in } F_n', a_n \leftrightarrow a_{n+1} \text{ in } F_n') \text{ (Fig. 2.3)} \\
= 2\{\Phi_n(p)\}^3.
\]

So

\[
1 \leq \frac{P_p(O \leftrightarrow a_n)}{\Phi_n(p)} \leq 1 + 2\{\Phi_n(p)\}^2,
\]

which implies

\[
\lim_{n \to \infty} \frac{P_p(O \leftrightarrow a_n)}{\Phi_n(p)} = 1.
\]
Combining this with Proposition 2.2 completes the proof.

3. Proof of Theorem 1.1

The next lemma is a key of the proof.

**Lemma 3.1.** There exists $\varepsilon > 0$ such that

$$2 \leq \frac{\xi(p + 3(1 - p)^3)}{\xi(p)} \leq 4 \quad \text{for} \quad 1 - \varepsilon < p < 1.$$

**Proof.** We introduce

$$\Psi_n(p) = 1 - P_p(O \leftrightarrow a_n, O \leftrightarrow b_n, a_n \leftrightarrow b_n \text{ in } F_n) = 3\Phi_n(p) - 2\Theta_n(p).$$

Here $O \leftrightarrow a_n$ in $F_n$ means that there exists no open path from $O$ to $a_n$ in $F_n$. By (4) and (5),

$$\Theta_{n+1}(p) = S(\Theta_n(p), \Psi_n(p)),$$

$$\Psi_{n+1}(p) = T(\Theta_n(p), \Psi_n(p)),$$

where $S, T : \mathbb{R}^2 \to \mathbb{R}$ are functions defined by

$$S(x,y) = -\frac{2}{3}x^3 + \frac{4}{3}x^2y + \frac{1}{3}xy^2,$$

$$T(x,y) = \frac{2}{9}x^3 + \frac{4}{3}x^2 - \frac{7}{3}x^2y + \frac{1}{3}xy + \frac{1}{9}y^3 + \frac{y^2}{3}.$$

Let $D$ be a subset of $\mathbb{R}^2$ defined by $D = \{(x,y) : 0 < x \leq y < 1\}$. We see $\partial S/\partial x$, 

...
\( \frac{\partial S}{\partial y}, \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} > 0 \) for \((x, y) \in D\). Indeed,

\[
\frac{\partial S}{\partial x} = -2x + \frac{8}{3}xy + \frac{1}{3}y^2 = 2x(y-x) + \frac{2}{3}xy + \frac{1}{3}y^2 > 0,
\]

\[
\frac{\partial S}{\partial y} = -\frac{2}{3}x + \frac{2}{3}xy > 0,
\]

\[
\frac{\partial T}{\partial x} = -2x + \frac{8}{3}xy + \frac{4}{3}y \geq \frac{2}{3}x + \frac{8}{3} - \frac{3}{3}xy + \frac{2}{3}y^2 + \frac{2}{3}y,
\]

\[
= \frac{2}{3}(y-x)^2 + \frac{8}{3}x(1-y) + \frac{2}{3}(1-x) > 0,
\]

\[
\frac{\partial T}{\partial y} = -\frac{7}{3}x + \frac{4}{3}x + \frac{1}{3}y^2 + \frac{2}{3}y
\]

\[
= \frac{4}{3}x(1-x) + \frac{1}{3}(y^2 - x^2) + \frac{2}{3}(y-x) > 0.
\]

Therefore if \((x_1, y_1), (x_2, y_2) \in D\) and \(x_1 < x_2\) and \(y_1 < y_2\), then

\( S(x_1, y_1) < S(x_2, y_2), \quad T(x_1, y_1) < T(x_2, y_2). \)

Note that \(\Psi_n(p) = \Theta_n(p) + 3(\Phi_n(p) - \Theta_n(p)) \geq \Theta_n(p)\) for all \(n\) by (8). Hence \((\Theta_n(p), \Psi_n(p)) \in D\). Calculating \(\Theta_n(p)\) and \(\Psi_n(p)\) directly from the recursions, we have

\[
\Theta_n(p) = 1 - 3(1-p)^2 - (12n-6)(1-p)^4 + 6(1-p)^5 + \cdots,
\]

\[
(10) \quad \Psi_n(p) = 1 - 3(1-p)^4 - 24n(1-p)^6 + \cdots
\]

for \(n \geq 2\). For \(1 - 1/\sqrt{3} < p < 1\), let \(\bar{p} = p + 3(1-p)^3\). Then we have

\[
\Theta_3(\bar{p}) - \Theta_2(p) = 6(1-p)^4 + 213(1-p)^6 + \cdots,
\]

\[
\Psi_3(\bar{p}) - \Psi_2(p) = 12(1-p)^6 + \cdots.
\]

Note that \(\Theta_2(p), \Psi_2(p), \Theta_3(\bar{p}), \text{ and } \Psi_3(\bar{p})\) are polynomials of finite degree. Hence we can take \(\epsilon_1 > 0\) in such a way that \(\Theta_2(p) < \Theta_3(\bar{p})\) and \(\Psi_2(p) < \Psi_3(\bar{p})\) for \(1 - \epsilon_1 < p < 1\). By (9), we have

\[
\Theta_3(p) = S(\Theta_2(p), \Psi_2(p)) < S(\Theta_3(\bar{p}), \Psi_3(\bar{p})) = \Theta_4(\bar{p}),
\]

\[
\Psi_3(p) = T(\Theta_2(p), \Psi_2(p)) < T(\Theta_3(\bar{p}), \Psi_3(\bar{p})) = \Psi_4(\bar{p}).
\]
Estimating repeatedly as above, we have \( \Theta_n(p) < \Theta_{n+1}(\bar{p}) \), \( \Psi_n(p) < \Psi_{n+1}(\bar{p}) \) for \( n \geq 2 \). Combining this with (8) yields \( \Phi_n(p) < \Phi_{n+1}(\bar{p}) \). So

\[
\frac{\log \Phi_n(p)}{2^n} < 2 \cdot \frac{\log \Phi_{n+1}(\bar{p})}{2^{n+1}}.
\]

This implies \( \zeta(p)^{-1} \geq 2 \cdot \bar{\zeta}(\bar{p})^{-1} \), that is \( \bar{\zeta}(\bar{p}) / \zeta(p) \geq 2 \) for \( 1 - \varepsilon_1 < p < 1 \).

We now proceed to the estimate from the opposite side. By using (10) and (11) again, we see

\[
\Theta_4(\bar{p}) - \Theta_2(p) = -6(1-p)^4 + 141(1-p)^6 + \cdots \\
\Psi_4(\bar{p}) - \Psi_2(p) = -12(1-p)^6 + \cdots.
\]

Hence we can take \( \varepsilon_2 > 0 \) such that \( \Theta_4(\bar{p}) < \Theta_2(p) \) and \( \Psi_4(\bar{p}) < \Psi_2(p) \) for \( 1 - \varepsilon_2 < 1 \). So we have \( \Theta_{n+2}(\bar{p}) < \Theta_n(p) \) and \( \Psi_{n+2}(\bar{p}) < \Psi_n(p) \). Therefore \( \bar{\zeta}(\bar{p}) / \zeta(p) \leq 4 \) for \( 1 - \varepsilon_2 < p < 1 \), which completes the proof.

Proof of Theorem 1.1. Let \( g(p) = \log \zeta(p) \). Since \( \zeta(p) \) is an increasing function, \( g(p) \) is also increasing. Suppose that \( p \) is sufficiently large to satisfy \( g(p) > 0 \). Let

\[
m = \lim \inf_{p \to 1} \frac{-\log g(p)}{-\log (1-p)} \geq 0, \quad M = \lim \sup_{p \to 1} \frac{-\log g(p)}{-\log (1-p)}.
\]

First, we prove \( m \geq 2 \). Suppose \( m < 2 \), and pick \( \delta > 0 \) with \( m + \delta < 2 \). Let

\[
h(x) = \frac{1}{(x-3)^{m+\delta}} - \frac{1}{x^{m+\delta}}.
\]

Applying the L'Hospital theorem, we see \( \lim_{x \to 0} h(x) = 0 \). So we take \( p_0 \) such that

\[
(12) \quad h(1-p) < \frac{1}{2} \log 2 \quad \text{for} \quad 0 < 1 - p < 1 - p_0
\]

and \( 1 - p_0 < \varepsilon \) (\( \varepsilon \) is given in Lemma 3.1.)

Let

\[
f(p) = p + 3(1-p)^3.
\]

We define \( \{p_n\}_{n=1,2,\ldots} \) by \( f(p_0) = p_1, f(p_n) = p_{n+1} \) inductively. Then \( p_0 < p_1 < \cdots < p_n < 1 \), and \( \lim_{n \to \infty} p_n = 1 \). By (13) and Lemma 3.1, we have

\[
\log 2 \leq g(p_{n+1}) - g(p_n),
\]

and hence

\[
(14) \quad g(p_0) + n \log 2 \leq g(p_n).
\]
Take \( N = N(p_0) \in N \). By assumption, there exists \( t \) such that \( p_N < t < 1 \) and

\[
\frac{-\log g(t)}{\log(1-t)} < m + \delta.
\]

For this \( t \), there exists unique \( N' = N'(t) \) such that \( p_{N'} \leq t < p_{N'} + 1 \). By (15) and \( 1 - p_{N'+1} < 1 - t \), we have

\[
(16) \quad g(t) < \frac{1}{(1-p_{N'+1})^{m+\delta}}
\]

\[
= \frac{1}{(1-p_{N'+1})^{m+\delta} - (1-p_{N'})^{m+\delta}} + \frac{1}{(1-p_{N'})^{m+\delta} - (1-p_{N'+1})^{m+\delta}} + \cdots + \frac{1}{(1-p_{N'})^{m+\delta}}
\]

\[
= h(1-p_{N'}) + h(1-p_{N'-1}) + \cdots + h(1-p_0) + \frac{1}{(1-p_0)^{m+\delta}}
\]

\[
< \frac{1}{2} (N'+1) \log 2 + \frac{1}{(1-p_0)^{m+\delta}}.
\]

The last inequality follows from (12). On the other hand, \( g(p_0) + N' \log 2 \leq g(p_{N'}) \leq g(t) \) by (14). Combining this with (16) yields

\[
(17) \quad \frac{1}{2} (N - 1) \log 2 < \frac{1}{2} (N' - 1) \log 2 < \frac{1}{(1-p_0)^{m+\delta}} - g(p_0).
\]

Here we used \( N < N' \) for the first inequality. We can pick \( N(p_0) \) so large that (17) does not hold. This yields a contradiction. Hence we have \( m \geq 2 \).

We proceed to prove \( M \leq 2 \). Suppose \( M > 2 \). Pick \( \delta > 0 \) such that \( M - \delta > 2 \). Let

\[
h(x) = \frac{1}{(x - 3x^3)^{M-\delta}} - \frac{1}{x^{M-\delta}}.
\]

Note that \( \lim_{x \to 0} h(x) = \infty \). Then by a similar argument as above, we lead a contradiction. Hence \( M \leq 2 \), which concludes \( m = M = 2 \).

4. Proof of Theorem 1.2

First, we estimate the probability \( P_p((1/9) \cdot 3^n \leq |C| \leq (9/2) \cdot 3^n) \). Let \( M = \sup \{ m : O \leftrightarrow a_m \text{ or } b_m \} \). We define two conditional probabilities

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \not\leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \not\leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \not\leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \not\leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \not\leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \not\leftrightarrow b_n \text{ in } F_n | M = n),
\]

\[
U_n(p) = P_p(O \leftrightarrow a_m, O \leftrightarrow b_n \text{ in } F_n | M = n),
\]
Clearly

\[(18) \quad 2U_d(p) + V_d(p) = 1, \]

and

\[(19) \quad V_n(p) = \frac{P_p(O \leftrightarrow a_n \leftrightarrow b_n \text{ in } F_n \cup a_{n+1}, O \leftrightarrow b_{n+1})}{P_p(M = n)} \]

Fig. 4.1

We consider the event of the numerator of (19), \( \{O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, O \leftrightarrow a_{n+1}, O \leftrightarrow b_{n+1}\} \). We divide the case into seven parts as Fig. 4.1. Since the events in \( F_n, F'_n, F''_n \) are independent, we have

\[(20) \quad V_n(p) = \frac{\Theta_n(1 - 2\Phi_n - \Phi_n^2 + 4\Phi_n\Theta_n - 2\Theta_n^2)}{P_p(M = n)} \]

Here we denoted \( \Phi_n = \Phi_n(p), \Theta_n = \Theta_n(p) \) briefly. Note that

\[(21) \quad P_p(M = n) = P_p(M \geq n) - P_p(M \geq n + 1) = 2\Phi_n - \Theta_n - (2\Phi_{n+1} - \Theta_{n+1}) = 2\Phi_n - \Theta_n - 2\Phi_n^2 - 2\Phi_n^3 + 2\Phi_n\Theta_n + 3\Phi_n^2\Theta_n - 2\Theta_n^3 \]

by (4). Hence by (18),

\[(22) \quad U_n(p) = \frac{1}{2}(1 - V_d(p)) \]
\[ P_p(M=n) = \frac{(\Phi_n - \Theta_n)(1 - \Phi_n - \Phi_n^2 + \Phi_n \Theta_n)}{P_p(M=n)}. \]

Let
\[ n_0 = n_0(p) = \sup\{ n : \Theta_n(p) \geq \frac{2}{3}\}. \]

**Lemma 4.1.** \( V_n(p) \geq \frac{2}{9} \) if \( n < n_0 \).

**Proof.** From (18), it is enough to show
\[ \frac{V_n(p)}{2U_n(p)} \geq \frac{2}{7}. \]

Let
\[ \kappa(x,y) = \frac{y(1 - 2x - x^2 + 4xy - 2y^2)}{2(x-y)(1-x-x^2+xy)}. \]

By (20) and (22), (24) follows from the following:
\[ \kappa(x,y) \geq \frac{2}{7} \text{ for } \frac{2}{3} \leq x < 1, \frac{1}{2} \frac{2}{3} \leq y < x. \]

The second condition in (25) comes from the fact that
\[ 3\Phi_n(p) - 2\Theta_n(p) = \Psi_n(p) < 1. \]

Let \( y/x = t \). Then the domain of (25) is \( 2/3 \leq x < 1/(3-2t), 2/3 \leq t < y < 1 \). And
\[ \kappa(x,t) = \frac{t}{2(1-t)} \left\{ 1 - \frac{x + (-3t + 2t^2)x^2}{1-x-(1-t)x^2} \right\}. \]

Now let
\[ \lambda(x) = \frac{x + (-3t + 2t^2)x^2}{1-x-(1-t)x^2}. \]

From a direct calculation,
\[ \lambda'(x) = \frac{(1 + 2t - 2t^2)x^2 + 2(-3t + 2t^2)x + 1}{\{1-x-(1-t)x^2\}^2}. \]
We see that if $2/3 < t < 1$, $\lambda'(x) > 0$ for $2/3 < x < 1/(3-2t)$. Therefore
\[
\kappa(x,t) > \kappa\left(\frac{1}{3-2t}, \frac{t}{3-2t}\right) = \frac{t}{5-4t} \geq \frac{2}{7}.
\]

Next, we estimate the expectation of $|C|$ on condition that $M = n$ ($n < n_0$).

**Lemma 4.2.** $E_p(|C| \mid M = n) \geq \frac{2}{9} \cdot 3^n$ if $n < n_0$.

To prove the above Lemma, we use the following inequality:

**Lemma 4.3.** For all $a \in F_n$,

\[ P_p(O \leftrightarrow a \text{ in } F_n) \geq \Phi_n(p). \]  

(27)

Proof. Besides (27), we introduce a similar inequality:

\[ P_p(a \leftrightarrow a_n \text{ or } a \leftrightarrow b_n) \geq P_p(O \leftrightarrow a_n \text{ or } O \leftrightarrow b_n) \text{ for all } a \in F_n. \]  

(28)

We prove (27) and (28) by induction at the same time. If $n = 0$, clearly both of them hold. Suppose (27) and (28) for $n = k$.

We prove (27) for $n = k + 1$ at first. By symmetry, it is sufficient to prove the cases (i) $a \in F_k$ and (ii) $a \in F'_k$.

(i) Suppose $a \in F_k$. By using (4), we see $\Phi_k(p) \geq \Phi_{k+1}(p)$. Indeed, suppose $\Phi_k(p) \geq 1/3$, then

\[ \frac{\Phi_{k+1}}{\Phi_k} = \Phi_k + \{\Phi_k\}^2 - \{\Theta_k\}^2 \]

\[ \leq \Phi_k + \{\Phi_k\}^2 - \left(\frac{3\Phi_k - 1}{2}\right)^2 \]

\[ \leq -\frac{5}{4}(1 - \Phi_k)^2 + 1 \]

\[ \leq 1. \]

Here we used (26). Combining this with assumption, we see (27) for $n = k + 1$ in this case.

(ii) Suppose $a \in F'_k$. Let $C_n^1$, $C_n^2$, $C_n^3$ be events given by

\[ C_n^1 = \{O \leftrightarrow a_n \text{ and } O \not\leftrightarrow c_n \text{ in } F_n \cup F'_n\}, \]
\[ C_n^2 = \{O \not\leftrightarrow a_n \text{ and } O \leftrightarrow c_n \text{ in } F_n \cup F'_n\}, \]
\[ C_n^3 = \{O \not\leftrightarrow a_n \text{ and } O \not\leftrightarrow c_n \text{ in } F_n \cup F'_n\}. \]
\[ C^3_n = \{ O \leftrightarrow a_n \text{ and } O \leftrightarrow c_n \text{ in } F_n \cup F'' \}. \]

We see
\[
P_p(O \leftrightarrow a \text{ in } F_{k+1})
= P_p(C^1_k)P_p(a \leftrightarrow a \text{ in } F_k) + P_p(C^2_k)P_p(c_k \leftrightarrow a \text{ in } F_k)
+ P_p(C^3_k)P_p(a_k \leftrightarrow a \text{ or } c_k \leftrightarrow a \text{ in } F_k)
\geq (\Phi_k - \Theta_k - \Theta_k)\Phi_k \cdot \Phi_k + \Phi_k \Theta_k \cdot 2(\Phi_k - \Theta_k)
= \Phi_k^2 + \Phi_k^3 - \Phi_k \Theta_k^2 = \Phi_{k+1}.\]

Here we used assumption for the inequality. We thus obtain (27) for \( n = k + 1 \). We proceed to prove (28) for \( n = k + 1 \).

(i) Suppose \( a \in F_k \). Let \( D^1_n, D^2_n, \ldots, D^5_n \) be events given by
\[
D^1_n = \{ a_n \leftrightarrow a_{n+1} \text{ or } a_n \leftrightarrow b_{n+1} \text{ in } F_k \cup F'' \},
D^2_n = \{ b_n \leftrightarrow a_{n+1} \text{ or } b_n \leftrightarrow b_{n+1} \text{ in } F_k \cup F'' \},
D^3_n = D^1_n \cap (D^2_n)',
D^4_n = (D^1_n) \cap D^2_n,
D^5_n = D^1_n \cap D^2_n'.\]
We see
\[
P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1})
= P_p(D^3_k)P_p(a \leftrightarrow a_k \text{ in } F_k) + P_p(D^4_k)P_p(a \leftrightarrow b_k \text{ in } F_k)
+ P_p(D^5_k)P_p(a \leftrightarrow a_k \text{ or } a \leftrightarrow b_k \text{ in } F_k)
\geq P_p(D^3_k)P_p(O \leftrightarrow a_k \text{ in } F_k) + P_p(D^4_k)P_p(O \leftrightarrow b_k \text{ in } F_k)
+ P_p(D^5_k)P_p(O \leftrightarrow a_k \text{ or } O \leftrightarrow b_k \text{ in } F_k)
= P_p(O \leftrightarrow a_{k+1} \text{ or } O \leftrightarrow b_{k+1})
\]
by assumption.

(ii) Suppose \( a \in F_k \). We see
\[
P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1})
\geq P_p(a \leftrightarrow a_{k+1} \text{ in } F_k)
+ P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow c_k \text{ in } F_k)P_p(c_k \leftrightarrow b_{k+1} \text{ in } F_k').\]

Here we note that
\[
P_p(a \leftrightarrow a_{k+1} \text{ and } a \leftrightarrow c_k \text{ in } F_k)
= P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow c_k \text{ in } F_k) - P_p(a \leftrightarrow a_{k+1} \text{ in } F_k)
\geq (2\Phi_k - \Theta_k) - P_p(a \leftrightarrow a_{k+1} \text{ in } F_k)
\]
by assumption. Using this and (30), we have
\[
P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1})
\]
Here we used assumption again. Now it is enough to show

\[ (31) \quad \Phi_k + \Phi_k^2 - \Phi_k \Theta_k - P_p(O \leftrightarrow a_{k+1} \text{ or } O \leftrightarrow b_{k+1}) \geq 0. \]

The left-hand side of (31) equals

\[
\Phi_k + \Phi_k^2 - \Phi_k \Theta_k - (2\Phi_{k+1} - \Theta_{k+1}) - P_p(O \leftrightarrow a_{k+1} \text{ or } O \leftrightarrow b_{k+1}) = \Phi_k(1 - \Theta_k) + 2\Phi_k - \Phi_k^2 - \Phi_k \Theta_k.
\]

By (26), we see all terms above are nonnegative. Hence the proof is completed.

Proof of Lemma 4.2.

\[ E_p(C\|M=n) = \sum_{a \in F} P_p(O \leftrightarrow a \mid M=n) \]

\[ \geq \sum_{a \in F_n} P_p(O \leftrightarrow a \mid M=n) \]

\[ \geq \sum_{a \in F_n} P_p(O \leftrightarrow a, O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, M=n). \]

Let \( D_n^6 = (D_n^6)^c \cap (D_n^6)^c \). Note that if \( M=n \) and \( O \leftrightarrow a_n, O \leftrightarrow b_n \), then \( (D_n^6)^c \) occurs. For \( a \in F_n \), we see

\[ P_p(O \leftrightarrow a, O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, M=n) \]

\[ = P_p(O \leftrightarrow a, O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, D_n^6 \text{ occurs}) \]

\[ = P_p(O \leftrightarrow a, O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n) P_p(D_n^6) \]

\[ \geq P_p(O \leftrightarrow a \text{ in } F_n) P_p(O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n) P_p(D_n^6) \]

\[ = P_p(O \leftrightarrow a \text{ in } F_n) P_p(O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, M=n). \]

Here we used FKG inequality for the forth line. Therefore

\[ E_p(C\|M=n) \geq \sum_{a \in F_n} P_p(O \leftrightarrow a \text{ in } F_n) P_p(O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, M=n) \]
\[
\sum_{a \in F_n} P_p(O \leftrightarrow a \text{ in } F_n) \geq \frac{2}{9} \sum_{a \in F_n} P_p(O \leftrightarrow a \text{ in } F_n)
\]

by Lemma 4.1. Note that \(|\{a \in V : a \in F_n\}| = (3/2)(3^n + 1)\). By virtue of Lemma 4.3, we see

\[
E_p(|C| \mid M = n) \geq \frac{2}{9} \cdot 3^n \Phi_n(p)
\]

\[
\geq \frac{2}{9} \cdot 3^n \text{ for } n < n_0.
\]

We used (23) and the fact that \(\Phi_n(p) \geq \Theta_n(p)\) for the last inequality.

We proceed to the estimate of \(P_p((1/9) \cdot 3^n \leq |C| \leq (9/2) \cdot 3^n)\).

**Lemma 4.4.** \(P_p\left(\frac{1}{9} \cdot 3^n \leq |C| \leq \frac{9}{2} \cdot 3^n\right) \geq \frac{2}{79} P_p(M = n)\) if \(n < n_0\).

**Proof.** Note that \(|C| \leq (9/2) \cdot 3^n\) if \(M = n\). Then we see the following.

\[
E_p(|C| \mid M = n)
\]

\[
= E_p(|C| \mid |C| \geq \frac{1}{9} \cdot 3^n |M = n|) + E_p(|C| \mid |C| < \frac{1}{9} \cdot 3^n |M = n|)
\]

\[
\leq \frac{9}{2} \cdot 3^n P_p(|C| \geq \frac{1}{9} \cdot 3^n |M = n|) + \frac{1}{9} \cdot 3^n P_p(|C| < \frac{1}{9} \cdot 3^n |M = n|).
\]

By Lemma 4.2, we have

\[
P_p(|C| \geq \frac{1}{9} \cdot 3^n |M = n|) \geq \frac{2}{79},
\]

thus the proof is completed.

**Lemma 4.5.** \(P_p(M = n) > \Phi_n(p)\{1 - \Phi_n(p)\}^2\) if \(n < n_0\).

**Proof.** Recall (21), that is

\[
P_p(M = n) = 2\Phi_n - \Theta_n - 2\Phi_n^2 - 2\Phi_n^3 + 2\Phi_n^2 \Theta_n + 3\Phi_n^2 \Theta_n - 2\Theta_n^3.
\]

Let \(n(y) = 2x - y - 2x^2 - 2x^3 + 2xy^2 + 3x^2y - 2y^3\). It is enough to show that \(n(y) > x(1-x)^2\) if \(2/3 \leq x < 1\), \((3x-1)/2 < y < x\). Note that
\[ \pi'(y) = -6y^2 + 4xy + 3x^2 - 1, \]

and that

\[ \pi\left( \frac{3x-1}{2} \right) = \frac{1}{2}(1-x)(9x-5) > 0, \quad \pi'(x) = x^2 - 1 < 0. \]

Hence \( \pi(y) > \min\{\pi((3x-1)/2), \pi(x), \pi((3x-1)/2) = (1-x)^2(x+3)/4 \) and \( \pi(x) = x(1-x)^2, \) so \( \pi((3x-1)/2) > \pi(x) \) for \( 2/3 < x < 1. \) This completes the proof.

Proof of Theorem 1.2. First, we estimate \( E_p[C]^k \) from below. By using Lemma 4.4 and 4.5, we see

\[
E_p[C]^k = \sum_{l=1}^{\infty} l^k P_p(|C| = l) \\
\geq \sum_{n=4,8,12,\ldots} \left( \frac{1}{9}, 3^n \right)^k P_p\left( \left| \frac{1}{9} \cdot 3^n \right| \leq |C| \leq \frac{9}{2} \cdot 3^n \right) \\
\geq \frac{1}{9^k} \sum_{m,N \in \mathbb{N}, 4m < N} 3^{4km} \Phi_{4m}(p) \{1 - \Phi_{4m}(p)\}^2.
\]

Let \( p \) be sufficiently large. Note that the function \( i(x) = x(1-x)^2 \) is decreasing in \( 2/3 < x < 1, \) and \( \Phi_{4m}(p) \leq e^{-2^{4m}/\xi(p)} \) by (7). We can see

\[
\sum_{n=0}^{n_0} 3^{4km} \Phi_{4m}(p) \{1 - \Phi_{4m}(p)\}^2 \\
\geq \sum_{4m < n_0} 3^{4km} e^{-2^{4m}/\xi(p)} (1 - e^{-2^{4m}/\xi(p)})^2 \\
\geq \int_{1}^{n_0/4} 3^{4kx} e^{-2^{4x}/\xi(p)} (1 - e^{-2^{4x}/\xi(p)})^2 dx \\
= \left( \frac{\xi(p)}{\delta_k} \right)^{2n_0 + 4/\xi(p)} \gamma [y^2 - \gamma (1 - e^{-\gamma})^2] dy.
\]

Here we set \( y = 2^x / \xi(p) \) in the last line. Note that \( \Theta_{n_0+1}(p) < 2/3, \) hence \( \Phi_{n_0+1}(p) < (1 + 2\Theta_{n_0+1}(p))/3 < 7/9 \) by (24). From (29), if \( \Phi_k(p) < 7/9, \) then \( \Phi_{k+1}(p) / \Phi_k(p) < 76/81. \) We see

\[
\Phi_{n_0+12}(p) < \left( \frac{76}{81} \right)^{11} \cdot \frac{7}{9} < \frac{1}{2}. \]
Combining this with (7), we have

$$\frac{1}{2} e^{-2^{\xi(p)+12}/\zeta(p)} \leq \Phi_{n+12}(p) \leq \frac{1}{2} e^{-7/9}.$$ 

Hence $2^{n0-4}/\zeta(p) > 2^{-16}\log(9/7)$. Since $\zeta(p) \to \infty$ as $p \to 1$, $E_p|C|^k > K_1(\zeta(p))^{D_k}$ holds if we take

$$K_1(k) = \int_{2^{-16}\log(9/7)}^{2^{-16}\log(9/7)} y^{D_k-1} e^{-y(1-e^{-y})^2} dy > 0.$$ 

Now we proceed to estimate from above. Note that $P_p(M \geq n) \leq 2\Phi_n(p)$, and we can see easily $P_p((3/2) \cdot 3^n < |C| \leq (3/2) \cdot 3^{n+1}) \leq P_p(M \geq n) \leq 2e^{-2^n/\zeta(p)}$. Hence

$$E_p|C|^k = \sum_{l=1}^{\infty} l^k P_p(|C| = l)$$

$$\leq 1 + \sum_{n=0}^{\infty} \left(\frac{3}{2} \cdot 3^{n+1}\right)^k P_p(3/2 \cdot 3^n < |C| \leq 3/2 \cdot 3^{n+1})$$

$$\leq 1 + 2 \cdot \left(\frac{9}{2}\right)^k \sum_{n=0}^{\infty} 3^{kn} e^{-2^n/\zeta(p)}.$$ 

Now

$$\int_{0}^{\infty} 3^{kn} e^{-2^n/\zeta(p)} dx = \frac{(\zeta(p))^{D_k}}{\log 2} \int_{(\zeta(p))^{-1}}^{\infty} y^{D_k-1} e^{-y} dy$$

$$\leq \frac{\Gamma(D_k)}{\log 2} \cdot (\zeta(p))^{D_k}.$$ 

So we can take $K_2(k) < \infty$ such that $E_p|C|^k < K_2(\zeta(p))^{D_k}$. \hfill \Box

5. Site percolation on the pre-Sierpinski gasket

We define the Bernoulli site percolation on the pre-Sierpinski gasket; each vertices in $V$ are open with probability $p$ and closed with $1-p$ independently. Let $\tilde{\mathcal{P}}_p$ denote its distribution. We write $x \leftrightarrow y$ if there exists a sequence of open vertices $x = x_0, x_1, \ldots, x_n = y$ such that there is a bond in $E$ which connects $x_j$ with $x_{j+1}$ for $0 \leq j \leq n-1$. We define another notations in the same manner as before. We introduce connectivity functions;

$$\tilde{\mathcal{O}}_n(p) = \tilde{\mathcal{P}}_p(O \leftrightarrow a_n \text{ in } \Delta Oa_n b_n),$$

$$\tilde{\mathcal{O}}_n(p) = \tilde{\mathcal{P}}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } \Delta Oa_n b_n).$$
We see $\Phi_0(p) = p^2$ and $\Theta_0(p) = p^3$ by definition.

**Proposition 5.1.** For each $n \geq 0$ and $0 \leq p \leq 1$,

\begin{align}
\Phi_{n+1}(p) &= p^{-1}\{\Phi_n(p)\}^2 + p^{-2}\{\Phi_n(p)\}^3 - p^{-3}\Phi_n(p)\{\Theta_n(p)\}^2, \\
\Theta_{n+1}(p) &= 3p^{-2}\{\Phi_n(p)\}^2\Theta_n(p) - 2p^{-3}\{\Theta_n(p)\}^3.
\end{align}

*Proof.* We prove (32). Let $\widetilde{A}_n^1$ and $\widetilde{A}_n^2$ be events given by

$\widetilde{A}_n^1 = \{O \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F_n'\},$

$\widetilde{A}_n^2 = \{O \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow c_n \text{ in } F_n'\} \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F_n\}.$

Then we have

\begin{equation}
\Theta_{n+1}(p) = \bar{P}_p(\widetilde{A}_n^1) + \bar{P}_p(\widetilde{A}_n^2) - \bar{P}_p(\widetilde{A}_n^1 \cap \widetilde{A}_n^2).
\end{equation}

Remark that $F_n \cap F_n' = \{a_n\}$. So we see $\bar{P}_p(\widetilde{A}_n^1) = p^{-1}\{\Phi_n(p)\}^2$. Similarly, we have $\bar{P}_p(\widetilde{A}_n^2) = p^{-2}\{\Phi_n(p)\}^3$. Thus (32) follows from (34) immediately. (33) is proved in the same way.

Let $\tilde{\Phi}_n(p) = p^{-1}\tilde{\Phi}_n(p)$ and $\tilde{\Theta}_n(p) = p^{-3}\tilde{\Theta}_n(p)$. Then we have the same recursions as (4), (5):

\begin{align}
\tilde{\Phi}_{n+1}(p) &= \{\tilde{\Phi}_n(p)\}^2 + \{\tilde{\Phi}_n(p)\}^3 - \tilde{\Phi}_n(p)\{\tilde{\Theta}_n(p)\}^2, \\
\tilde{\Theta}_{n+1}(p) &= 3\{\tilde{\Phi}_n(p)\}^2\tilde{\Theta}_n(p) - 2\{\tilde{\Theta}_n(p)\}^3.
\end{align}

Hence we see that there exists $\xi(p) > 0$ such that

\[
\lim_{n \to \infty} \frac{\tilde{\Phi}_n(p)}{\exp\{-2^n / \xi(p)\}} = 1, \quad \text{that is} \quad \lim_{n \to \infty} \frac{\bar{P}_p(O \leftrightarrow a_n)}{\exp\{-2^n / \xi(p)\}} = 1.
\]

**Lemma 5.2.** Let $\sqrt{\bar{p}} = \sqrt{p} + 6(1 - \sqrt{p})^2$. Then there exists $\varepsilon > 0$ such that

\[
2 \leq \frac{\xi(p)}{\xi(\bar{p})} \leq 4 \quad \text{for} \quad 1 - \varepsilon < p < 1.
\]

*Proof.* We use the same method as in Section 3 again. Let

\begin{equation}
\Psi_n(p) = 3\tilde{\Theta}_n(p) - 2\tilde{\Theta}_n(p).
\end{equation}

To apply (9), first we prove $(\tilde{\Theta}_n(p), \Psi_n(p)) \in D$. (Recall $D = \{(x, y) : 0 < x \leq y < 1\}$.) Since $\Psi_n(p) = \tilde{\Theta}_n(p) + 3\{\tilde{\Phi}_n(p) - \tilde{\Theta}_n(p)\}$, it is enough to prove $\tilde{\Phi}_n(p) \geq \tilde{\Theta}_n(p)$. Now

$\tilde{\Phi}_n(p) = p^{-1} \times \bar{P}_p(O \leftrightarrow a_n \text{ in } F_n)$
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\( \mathbb{P}_p(O \leftrightarrow a_n \text{ in } F_n | a_n \text{ is open}) \)
\( = \tilde{\mathbb{P}}(O \leftrightarrow a_n \text{ in } F_n | a_n, b_n \text{ are open}), \)
\( \tilde{\Theta}_n(p) = p^{-\frac{3}{2}} \times \mathbb{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n) \)
\( \leq p^{-2} \times \mathbb{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n) \)
\( = \mathbb{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n | a_n, b_n \text{ are open}). \)

Hence we have \( \tilde{\Theta}_n(p) \geq \tilde{\Theta}_n(p), \) which implies \((\tilde{\Theta}_n(p), \tilde{\Psi}_n(p)) \in D. \)
A direct calculation from (35) and (36) shows
\( \tilde{\Theta}_2(p) - \tilde{\Theta}_1(p) = 6(1 - \sqrt{p})^2 + 204(1 - \sqrt{p})^3 + \cdots, \)
\( \tilde{\Psi}_2(p) - \tilde{\Psi}_1(p) = 12(1 - \sqrt{p})^3 + \cdots, \)
\( \tilde{\Theta}_3(p) - \tilde{\Theta}_1(p) = -6(1 - \sqrt{p})^2 + 204(1 - \sqrt{p})^3 + \cdots, \)
\( \tilde{\Psi}_3(p) - \tilde{\Psi}_1(p) = -12(1 - \sqrt{p})^3 + \cdots. \)

We can take \( \varepsilon > 0 \) such that
\( \tilde{\Theta}_n(p) < \tilde{\Theta}_1(p) < \tilde{\Theta}_2(p), \tilde{\Psi}_n(p) < \tilde{\Psi}_1(p) < \tilde{\Psi}_2(p) \)
for \( 1 - \varepsilon < p < 1. \)

Now we apply (9). We have for \( n \geq 1 \) and \( 1 - \varepsilon < p < 1, \)
\( \tilde{\Theta}_n(\bar{p}) < \tilde{\Theta}_n(p) < \tilde{\Theta}_{n+1}(\bar{p}), \text{ and } \tilde{\Psi}_n(\bar{p}) < \tilde{\Psi}_n(p) < \tilde{\Psi}_{n+1}(\bar{p}). \)

We see \( \tilde{\Theta}_n(\bar{p}) < \tilde{\Theta}_n(p) < \tilde{\Theta}_{n+1}(\bar{p}) \) by (37), so we have the conclusion.

Proof of Theorem 1.3. Note that \( \bar{p} = (\sqrt{p} + 6(1 - \sqrt{p})^2)^2 = p + 3(1 - p)^2 + o((1 - p)^2) \) as \( p \to 1. \) We have Theorem 1.3 in the same way as in Section 3.

References


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