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PERCOLATION ON THE PRE-SIERPINSKI GASKET

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1. Introduction and statements of results

In this paper, we regard percolation as a model of phase transitions. We are especially interested in problems near the *critical point*, where the phase transition occurs. We call these problems *critical behaviors*. Our purpose in this paper is to clarify the critical behaviors of percolation on the pre-Sierpinski gasket which has self-similarity.

Until now, studies of percolation are restricted on *periodic* graphs, such as \mathbb{Z}^d . (An exact definition of periodic graph is mentioned in Kesten [1].) There are lots of conjectures and hypotheses about critical behaviors, but many of them are still unsolved rigorously (see Grimmett [2] and references therein). In high dimension lattices \mathbb{Z}^d , rigorous results for critical behaviors were obtained by Hara-Slade [3]. But in low dimensions, except a work on \mathbb{Z}^2 by Kesten [4], few rigorous results have been proved about the existence of *critical exponents* and justification of the *scaling, hyperscaling relations*.

For critical behaviors, *self-similarity* of the graph plays more important role than periodicity. This is a motivation to consider percolation problems on the pre-Sierpinski gasket.

We now define the pre-Sierpinski gasket. Let $O=(0,0)$, $a_0=(1/2, \sqrt{3}/2)$, $b_0=(1,0)$. Let F_0 be the graph which consists of the vertices and edges of the triangle ΔOa_0b_0 . Let $\{F_n\}_{n=0,1,2,\dots}$ be the sequence of graphs given by

$$F_{n+1} = F_n \cup (F_n + 2^n a_0) \cup (F_n + 2^n b_0)$$

where $A+a=\{x+a \mid x \in A\}$ and $kA=\{kx \mid x \in A\}$. Let $F=\bigcup_{n=0}^{\infty} F_n$. We call F the *pre-Sierpinski gasket*. (Fig. 1.1) Note that $\tilde{F}=\bigcup_{n=0}^{\infty} 2^{-n}F$ become the Sierpinski gasket. Let V be the set of all vertices in F , and E the set of all edges with length 1.

We consider the Bernoulli bond percolation on the pre-Sierpinski gasket; each edges in E are *open* with probability p and *closed* with probability $1-p$ independently. Let P_p denote its distribution. We think of open bonds as permitting to go along the bond. We write $x \leftrightarrow y$ if there is an open path from x to y . Let $C(x)=\{y \in V: x \leftrightarrow y\}$. $C(x)$ is called the *open cluster* containing x . We denote by C the open cluster containing the origin.

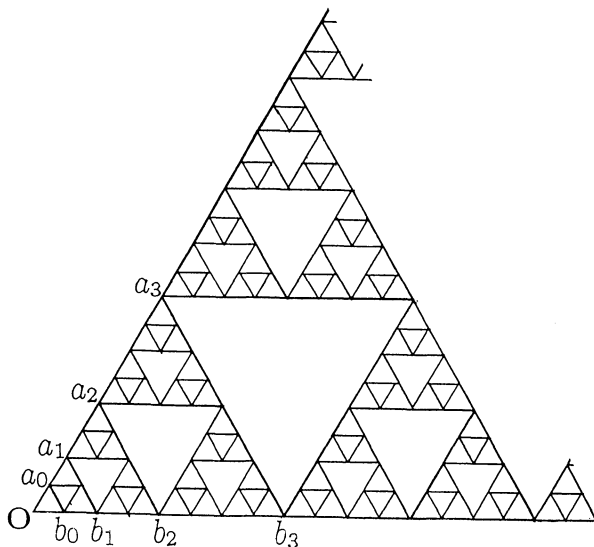


Fig. 1.1

We define two functions in a similar way as percolations on \mathbf{Z}^d .

$$\theta(p) = P_p(|C| = \infty), \quad \chi(p) = E_p(|C|; |C| < \infty),$$

where $|C|$ denotes the number of vertices contained in C , and E_p denotes the expectation with respect to P_p . $\theta(p)$ is called the *percolation probability*, and $\chi(p)$ is called the *mean cluster size*.

Let p_c denote the *critical point*; that is

$$p_c = \inf\{p : \theta(p) > 0\}.$$

Then $p_c = 1$ for the pre-Sierpinski gasket because it is finitely ramified. We note that $\chi(p) = E_p|C|$ for $p < 1$.

The *correlation length* is defined by

$$(1) \quad \xi(p) = \lim_{n \rightarrow \infty} \left\{ -\frac{1}{2^n} \log P_p(O \leftrightarrow a_n) \right\}^{-1}.$$

The existence of the limit in (1) will be proved in Section 2.

We write $f(p) \approx g(p)$ as $p \rightarrow p_0$ if $\log f(p) / \log g(p) \rightarrow 1$ as $p \rightarrow p_0$.

We now state our main theorems:

Theorem 1.1. $\lim_{p \rightarrow 1} -\frac{\log \xi(p)}{\log(1-p)} = \infty$, and $\lim_{p \rightarrow 1} \frac{\log(\log \xi(p))}{\log(1-p)} = -2$.

Theorem 1.2. *Let $D = \log 3 / \log 2$. Then*

$$E_p |C|^k \approx \{\xi(p)\}^{Dk} \quad \text{as } p \rightarrow 1 \quad \text{for all } k \geq 1.$$

REMARK. Our results are quite different from the results on Z^d (see below). In physical literature, Theorem 1.1 was known by Gefen et al. [5] by using formal renormalization arguments. Our contribution is that we prove Theorem 1.1 rigorously.

We collect results and conjectures of the percolation on Z^d . It is conjectured (see [2])

$$(2) \quad \xi(p) \approx |p_c - p|^{-\nu(d)} \quad \text{as } p \rightarrow p_c.$$

The value $\nu(d)$ is called the *critical exponent*. It is proved that $\nu(d) = 1/2$ for sufficiently large d (Hara-Slade [3]), and conjectured $\nu(2) = 4/3$ (see [4]).

Other critical exponents considered in Z^d are as follows:

$$\chi(p) \approx |p_c - p|^{-\gamma}, \quad \frac{E_p(|C|^{k+1}; |C| < \infty)}{E_p(|C|^k; |C| < \infty)} \approx |p_c - p|^{-\Delta} \quad \text{as } p \rightarrow p_c.$$

It is conjectured for Z^d that $d\nu = 2\Delta - \gamma$. This relation is one of hyperscaling relations. We note $\gamma = \Delta = \infty$ on the pre-Sierpinski gasket. So the relation $d\nu = 2\Delta - \gamma$ does not make sense on the pre-Sierpinski gasket. Accordingly we modify the hyperscaling relation as follows:

$$(3) \quad \{\xi(p)\}^d \approx \frac{E_p |C|^3}{\{\chi(p)\}^2} \quad \text{as } p \rightarrow p_c.$$

If finite critical exponents ν, γ, Δ exist, then (3) is equivalent to $d\nu = 2\Delta - \gamma$.

REMARK. By Theorem 1.2, we have $E_p |C|^3 \approx \{\xi(p)\}^{3D}$ and $\chi(p) \approx \{\xi(p)\}^D$. Hence the above hyperscaling relation (3) holds when we regard D as the dimension of the pre-Sierpinski gasket. The value $D = \log 3 / \log 2$ coincides with the fractal dimension of the Sierpinski gasket.

In addition, we mention site percolation on the pre-Sierpinski gasket: each vertices in V are determined to be open or closed independently. (Details will be given in Section 5). We define the correlation length $\hat{\xi}(p)$ in the same manner as (1). We have the result below;

$$\textbf{Theorem 1.3.} \quad \lim_{p \rightarrow 1} -\frac{\log \hat{\xi}(p)}{\log(1-p)} = \infty, \quad \text{and} \quad \lim_{p \rightarrow 1} \frac{\log(\log \hat{\xi}(p))}{\log(1-p)} = -1.$$

The critical exponent in a usual sense is also infinite in this case. But $\log \hat{\xi}(p) \approx (1-p)^{-1}$, which is different from Theorem 1.1. We cannot see the

universality of this exponent on the pre-Sierpinski gasket.

We refer to the self-avoiding walks on the Sierpinski gasket, as related works of phase transitions; Hattori-Hattori [6] and Hattori-Hattori-Kusuoka [7] construct the self-avoiding paths on two- and three-dimensional Sierpinski gasket. Before [6], Hattori-Hattori-Kusuoka [8] constructed them on the pre-Sierpinski gasket. These works also gave us a motivation to study percolation on the Sierpinski gasket.

The organization of this paper is as follows: In Section 2 we prepare for the proof of our main theorems; we construct recursion formulas of relations between events in F_n and ones in F_{n+1} . In the reminder of Section 2, we prove the existence of the correlation length. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. In Section 5 we study site percolation and prove Theorem 1.3.

2. Recursion formulas and the existence of $\xi(p)$

We introduce two connectivity functions as follows.

$$\Phi_n(p) = P_p(O \leftrightarrow a_n \text{ in } \Delta O a_n b_n),$$

$$\Theta_n(p) = P_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } \Delta O a_n b_n).$$

We write $O \leftrightarrow a_n$ in $\Delta O a_n b_n$ if there is an open path from O to a_n in $\Delta O a_n b_n$ (contains its perimeter). We easily calculate $\Phi_0(p) = p + p^2 - p^3$, $\Theta_0(p) = 3p^2 - 2p^3$. Note that (i) $\Phi_n(p) \geq \Theta_n(p)$ by definition, (ii) if $O \leftrightarrow a_n$ and $O \leftrightarrow b_n$ then we have $a_n \leftrightarrow b_n$ automatically.

Proposition 2.1. *For each $n \geq 0$ and $0 \leq p \leq 1$,*

$$(4) \quad \Phi_{n+1}(p) = \{\Phi_n(p)\}^2 + \{\Phi_n(p)\}^3 - \Phi_n(p)\{\Theta_n(p)\}^2,$$

$$(5) \quad \Theta_{n+1}(p) = 3\{\Phi_n(p)\}^2\Theta_n(p) - 2\{\Theta_n(p)\}^3.$$

Proof. Recall $\Delta O a_n b_n = F_n$. Let $F'_n = F_n + a_n$, $F''_n = F_n + b_n$, and $c_n = (3 \cdot 2^{n-1}, \sqrt{3} \cdot 2^{n-1})$. Let A_n^1 and A_n^2 be events given by

$$A_n^1 = \{O \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F'_n\},$$

$$A_n^2 = \{O \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow c_n \text{ in } F''_n\} \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F'_n\}.$$

Then we have

$$(6) \quad \Phi_{n+1}(p) = P_p(A_n^1) + P_p(A_n^2) - P_p(A_n^1 \cap A_n^2).$$

Here we used the fact that a path from O to a_{n+1} goes through a_n or b_n . Since the events in F_n, F'_n, F''_n are mutually independent, $P_p(A_n^1) = \{\Phi_n(p)\}^2$, $P_p(A_n^2) = \{\Phi_n(p)\}^3$,

$P_p(A_n^1 \cap A_n^2) = \{\Theta_n(p)\}^2 \Phi_n(p)$ (Fig. 2.1). Combining these with (6) yields (4).

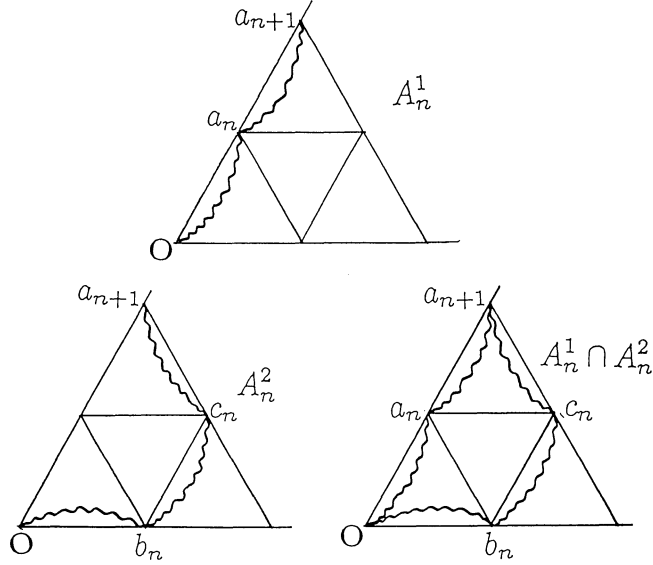


Fig. 2.1

We proceed to the proof of (5). Let B_n^1, B_n^2, B_n^3 be events given by

$$\begin{aligned} B_n^1 &= \{O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F'_n\} \\ &\quad \cap \{b_n \leftrightarrow b_{n+1} \text{ in } F''_n\}, \\ B_n^2 &= \{O \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ and } a_n \leftrightarrow c_n \text{ in } F'_n\} \\ &\quad \cap \{c_n \leftrightarrow b_{n+1} \text{ in } F''_n\}, \\ B_n^3 &= \{O \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow b_{n+1} \text{ and } b_n \leftrightarrow c_n \text{ in } F'_n\} \\ &\quad \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F''_n\} \end{aligned}$$

(see Fig. 2.2).

Then we have

$$\begin{aligned} \Theta_{n+1}(p) &= P_p(B_n^1) + P_p(B_n^2) + P_p(B_n^3) - P_p(B_n^1 \cap B_n^2) - P_p(B_n^2 \cap B_n^3) \\ &\quad - P_p(B_n^3 \cap B_n^1) + P_p(B_n^1 \cap B_n^2 \cap B_n^3). \end{aligned}$$

We see easily

$$\begin{aligned} P_p(B_n^1) &= P_p(B_n^2) = P_p(B_n^3) = \{\Phi_n(p)\}^2 \Theta_n(p), \\ P_p(B_n^1 \cap B_n^2) &= P_p(B_n^2 \cap B_n^3) = P_p(B_n^3 \cap B_n^1) = P_p(B_n^1 \cap B_n^2 \cap B_n^3) = \{\Theta_n(p)\}^3. \end{aligned}$$

(5) follows from this immediately. □

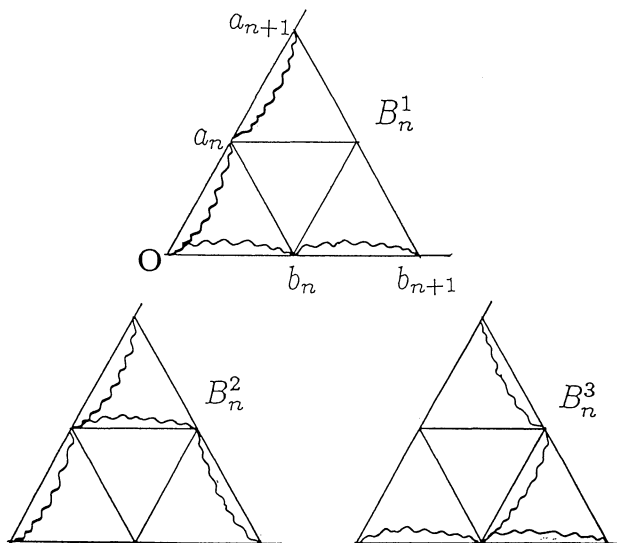


Fig. 2.2

From now on, we assume $0 < p < 1$. We prove the existence of the limit (1), correlation length $\xi(p)$, by using these recursions.

Proposition 2.2. *There exists $\xi(p) > 0$ such that*

$$\lim_{n \rightarrow \infty} \frac{\Phi_n(p)}{\exp\{-2^n / \xi(p)\}} = 1.$$

REMARK. The convergence as $n \rightarrow \infty$ in Proposition 2.2 is stronger than the convergence in (1).

Proof. By (4) and $\Theta_n(p) \leq \Phi_n(p)$, we have

$$\{\Phi_n(p)\}^2 \leq \Phi_{n+1}(p) \leq \{\Phi_n(p)\}^2 + \{\Phi_n(p)\}^3.$$

Hence

$$1 \leq \frac{\Phi_{n+1}(p)}{\{\Phi_n(p)\}^2} \leq 1 + \Phi_n(p).$$

Let $h_n(p) = \Phi_{n+1}(p) / \{\Phi_n(p)\}^2$. Then $1 \leq h_n(p) \leq 2$ and $\lim_{n \rightarrow \infty} h_n(p) = 1$ because $\lim_{n \rightarrow \infty} \Phi_n(p) = 0$. Now

$$\frac{1}{2^n} \log \Phi_n(p)$$

$$\begin{aligned}
&= \frac{1}{2^n} \log \left(\{\Phi_0(p)\}^{2^n} \cdot \frac{\{\Phi_1(p)\}^{2^{n-1}}}{\{\Phi_0(p)\}^{2^n}} \cdot \frac{\{\Phi_2(p)\}^{2^{n-2}}}{\{\Phi_1(p)\}^{2^{n-1}}} \cdots \frac{\Phi_n(p)}{\{\Phi_{n-1}(p)\}^2} \right) \\
&= \log \Phi_0(p) + \frac{1}{2} \log h_0(p) + \frac{1}{2^2} \log h_1(p) + \cdots + \frac{1}{2^n} \log h_{n-1}(p) \\
&\leq \log \Phi_0(p) + \log 2.
\end{aligned}$$

Hence $\{\log \Phi_n(p)/2^n\}_{n=0,1,2,\dots}$ is increasing and $\lim_{n \rightarrow \infty} \log \Phi_n(p)/2^n$ exists. Let $-\{\xi(p)\}^{-1} = \lim_{n \rightarrow \infty} \log \Phi_n(p)/2^n$. Then

$$\begin{aligned}
-\frac{1}{\xi(p)} &\geq \frac{1}{2^n} \log \Phi_n(p) = -\frac{1}{\xi(p)} - \left(\frac{1}{2^{n+1}} \log h_n(p) + \frac{1}{2^{n+2}} \log h_{n+1}(p) + \cdots \right) \\
&\geq -\frac{1}{\xi(p)} - \frac{1}{2^n} \log H_n(p),
\end{aligned}$$

where $H_n(p) = \sup_{m \geq n} h_m(p)$. Therefore

$$(7) \quad \exp \left\{ -\frac{2^n}{\xi(p)} \right\} \geq \Phi_n(p) \geq \frac{1}{H_n(p)} \exp \left\{ -\frac{2^n}{\xi(p)} \right\}.$$

Since $\lim_{n \rightarrow \infty} H_n(p) = 1$, we complete the proof. \square

REMARK. Note that the function $\xi(p)$ is continuous and increasing on $(0, 1)$ from the proof above.

Lemma 2.3. $\lim_{n \rightarrow \infty} \frac{P_p(\mathcal{O} \leftrightarrow a_n)}{\exp\{-2^n/\xi(p)\}} = 1.$

Proof. Recall that $\Phi_n(p) = P_p(\mathcal{O} \leftrightarrow a_n \text{ in } F_n)$. Then

$$\begin{aligned}
&P_p(\mathcal{O} \leftrightarrow a_n) - P_p(\mathcal{O} \leftrightarrow a_n \text{ in } F_n) \\
&\leq P_p(\mathcal{O} \leftrightarrow b_n \text{ in } F_n, b_n \leftrightarrow c_n \text{ in } F_n'', c_n \leftrightarrow a_n \text{ in } F_n') \\
&\quad + P_p(\mathcal{O} \leftrightarrow b_n \text{ in } F_n, b_n \leftrightarrow b_{n+1} \text{ in } F_n'', a_n \leftrightarrow a_{n+1} \text{ in } F_n') \text{ (Fig. 2.3)} \\
&= 2\{\Phi_n(p)\}^3.
\end{aligned}$$

So

$$1 \leq \frac{P_p(\mathcal{O} \leftrightarrow a_n)}{\Phi_n(p)} \leq 1 + 2\{\Phi_n(p)\}^2,$$

which implies

$$\lim_{n \rightarrow \infty} \frac{P_p(\mathcal{O} \leftrightarrow a_n)}{\Phi_n(p)} = 1.$$

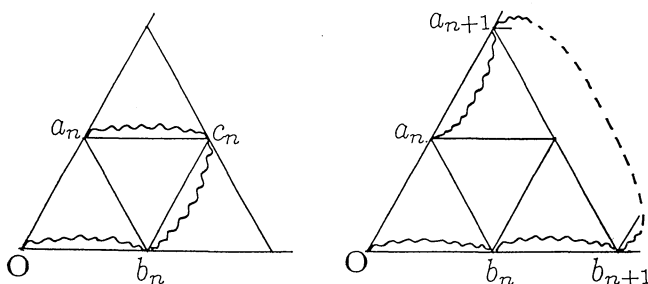


Fig. 2.3

Combining this with Proposition 2.2 completes the proof. \square

3. Proof of Theorem 1.1

The next lemma is a key of the proof.

Lemma 3.1. *There exists $\varepsilon > 0$ such that*

$$2 \leq \frac{\xi(p + 3(1-p)^3)}{\xi(p)} \leq 4 \quad \text{for} \quad 1 - \varepsilon < p < 1.$$

Proof. We introduce

$$(8) \quad \begin{aligned} \Psi_n(p) &= 1 - P_p(O \nleftrightarrow a_n, O \nleftrightarrow b_n, a_n \nleftrightarrow b_n \text{ in } F_n) \\ &= 3\Phi_n(p) - 2\Theta_n(p). \end{aligned}$$

Here $O \nleftrightarrow a_n$ in F_n means that there exists no open path from O to a_n in F_n . By (4) and (5),

$$\Theta_{n+1}(p) = S(\Theta_n(p), \Psi_n(p)),$$

$$\Psi_{n+1}(p) = T(\Theta_n(p), \Psi_n(p)),$$

where $S, T: \mathbf{R}^2 \rightarrow \mathbf{R}$ are functions defined by

$$S(x, y) = -\frac{2}{3}x^3 + \frac{4}{3}x^2y + \frac{1}{3}xy^2,$$

$$T(x, y) = \frac{2}{9}x^3 + \frac{4}{3}x^2 - \frac{7}{3}x^2y + \frac{4}{3}xy + \frac{1}{9}y^3 + \frac{1}{3}y^2.$$

Let D be a subset of \mathbf{R}^2 defined by $D = \{(x, y): 0 < x \leq y < 1\}$. We see $\partial S / \partial x$,

$\partial S/\partial y, \partial T/\partial x, \partial T/\partial y > 0$ for $(x, y) \in D$. Indeed,

$$\frac{\partial S}{\partial x} = -2x^2 + \frac{8}{3}xy + \frac{1}{3}y^2 = 2x(y-x) + \frac{2}{3}xy + \frac{1}{3}y^2 > 0,$$

$$\frac{\partial S}{\partial y} = \frac{4}{3}x^2 + \frac{2}{3}xy > 0,$$

$$\begin{aligned} \frac{\partial T}{\partial x} &= \frac{2}{3}x^2 + \frac{8}{3}x - \frac{14}{8}xy + \frac{4}{3}y \geq \frac{2}{3}x^2 + \frac{8}{3}x - \frac{14}{3}xy + \frac{2}{3}y^2 + \frac{2}{3}y \\ &= \frac{2}{3}(y-x)^2 + \frac{8}{3}x(1-y) + \frac{2}{3}y(1-x) > 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial T}{\partial y} &= -\frac{7}{3}x^2 + \frac{4}{3}x + \frac{1}{3}y^2 + \frac{2}{3}y \\ &= \frac{4}{3}x(1-x) + \frac{1}{3}(y^2 - x^2) + \frac{2}{3}(y - x^2) > 0. \end{aligned}$$

Therefore if $(x_1, y_1), (x_2, y_2) \in D$ and $x_1 < x_2$ and $y_1 < y_2$, then

$$(9) \quad S(x_1, y_1) < S(x_2, y_2), \quad T(x_1, y_1) < T(x_2, y_2).$$

Note that $\Psi_n(p) = \Theta_n(p) + 3\{\Phi_n(p) - \Theta_n(p)\} \geq \Theta_n(p)$ for all n by (8). Hence $(\Theta_n(p), \Psi_n(p)) \in D$. Calculating $\Theta_n(p)$ and $\Psi_n(p)$ directly from the recursions, we have

$$(10) \quad \begin{aligned} \Theta_n(p) &= 1 - 3(1-p)^2 - (12n-6)(1-p)^4 + 6(1-p)^5 \\ &\quad + (-48n^2 + 120n - 15)(1-p)^6 + \dots, \end{aligned}$$

$$(11) \quad \Psi_n(p) = 1 - 3(1-p)^4 - 24n(1-p)^6 + \dots$$

for $n \geq 2$. For $1 - 1/\sqrt{3} < p < 1$, let $\tilde{p} = p + 3(1-p)^3$. Then we have

$$\begin{aligned} \Theta_3(\tilde{p}) - \Theta_2(p) &= 6(1-p)^4 + 213(1-p)^6 + \dots, \\ \Psi_3(\tilde{p}) - \Psi_2(p) &= 12(1-p)^6 + \dots. \end{aligned}$$

Note that $\Theta_2(p)$, $\Psi_2(p)$, $\Theta_3(\tilde{p})$, and $\Psi_3(\tilde{p})$ are polynomials of finite degree. Hence we can take $\varepsilon_1 > 0$ in such a way that $\Theta_2(p) < \Theta_3(\tilde{p})$ and $\Psi_2(p) < \Psi_3(\tilde{p})$ for $1 - \varepsilon_1 < p < 1$. By (9), We have

$$\begin{aligned} \Theta_3(p) &= S(\Theta_2(p), \Psi_2(p)) < S(\Theta_3(\tilde{p}), \Psi_3(\tilde{p})) = \Theta_4(\tilde{p}), \\ \Psi_3(p) &= T(\Theta_2(p), \Psi_2(p)) < T(\Theta_3(\tilde{p}), \Psi_3(\tilde{p})) = \Psi_4(\tilde{p}). \end{aligned}$$

Estimating repeatedly as above, we have $\Theta_n(p) < \Theta_{n+1}(\tilde{p})$, $\Psi_n(p) < \Psi_{n+1}(\tilde{p})$ for $n \geq 2$. Combining this with (8) yields $\Phi_n(p) < \Phi_{n+1}(\tilde{p})$. So

$$\frac{\log \Phi_n(p)}{2^n} < 2 \cdot \frac{\log \Phi_{n+1}(\tilde{p})}{2^{n+1}}.$$

This implies $\xi(p)^{-1} \geq 2 \cdot \xi(\tilde{p})^{-1}$, that is $\xi(\tilde{p}) / \xi(p) \geq 2$ for $1 - \varepsilon_1 < p < 1$.

We now proceed to the estimate from the opposite side. By using (10) and (11) again, we see

$$\Theta_4(\tilde{p}) - \Theta_2(p) = -6(1-p)^4 + 141(1-p)^6 + \dots$$

$$\Psi_4(\tilde{p}) - \Psi_2(p) = -12(1-p)^6 + \dots$$

Hence we can take $\varepsilon_2 > 0$ such that $\Theta_4(\tilde{p}) < \Theta_2(p)$ and $\Psi_4(\tilde{p}) < \Psi_2(p)$ for $1 - \varepsilon_2 < 1$. So we have $\Theta_{n+2}(\tilde{p}) < \Theta_n(p)$ and $\Psi_{n+2}(\tilde{p}) < \Psi_n(p)$. Therefore $\xi(\tilde{p}) / \xi(p) \leq 4$ for $1 - \varepsilon_2 < p < 1$, which completes the proof. \square

Proof of Theorem 1.1. Let $g(p) = \log \xi(p)$. Since $\xi(p)$ is an increasing function, $g(p)$ is also increasing. Suppose that p is sufficiently large to satisfy $g(p) > 0$. Let

$$m = \liminf_{p \rightarrow 1} -\frac{\log g(p)}{\log(1-p)} \geq 0, \quad M = \limsup_{p \rightarrow 1} -\frac{\log g(p)}{\log(1-p)}.$$

First, we prove $m \geq 2$. Suppose $m < 2$, and pick $\delta > 0$ with $m + \delta < 2$. Let

$$h(x) = \frac{1}{(x - 3x^3)^{m+\delta}} - \frac{1}{x^{m+\delta}}.$$

Applying the L'Hospital theorem, we see $\lim_{x \rightarrow 0} h(x) = 0$. So we take p_0 such that

$$(12) \quad h(1-p) < \frac{1}{2} \log 2 \quad \text{for} \quad 0 < 1-p < 1-p_0$$

and $1-p_0 < \varepsilon$. (ε is given in Lemma 3.1.)

Let

$$(13) \quad f(p) = p + 3(1-p)^3.$$

We define $\{p_n\}_{n=1,2,\dots}$ by $f(p_0) = p_1$, $f(p_n) = p_{n+1}$ inductively. Then $p_0 < p_1 < \dots < p_n < 1$, and $\lim_{n \rightarrow \infty} p_n = 1$. By (13) and Lemma 3.1, we have

$$\log 2 \leq g(p_{n+1}) - g(p_n),$$

and hence

$$(14) \quad g(p_0) + n \log 2 \leq g(p_n).$$

Take $N = N(p_0) \in \mathbb{N}$. By assumption, there exists t such that $p_N < t < 1$ and

$$(15) \quad -\frac{\log g(t)}{\log(1-t)} < m + \delta.$$

For this t , there exists unique $N' = N'(t)$ such that $p_{N'} \leq t < p_{N'+1}$. By (15) and $1 - p_{N'+1} < 1 - t$, we have

$$(16) \quad \begin{aligned} g(t) &< \frac{1}{(1-p_{N'+1})^{m+\delta}} \\ &= \left\{ \frac{1}{(1-p_{N'+1})^{m+\delta}} - \frac{1}{(1-p_{N'})^{m+\delta}} \right\} \\ &\quad + \left\{ \frac{1}{(1-p_{N'})^{m+\delta}} - \frac{1}{(1-p_{N'-1})^{m+\delta}} \right\} + \cdots + \frac{1}{(1-p_0)^{m+\delta}} \\ &= h(1-p_{N'}) + h(1-p_{N'-1}) + \cdots + h(1-p_0) + \frac{1}{(1-p_0)^{m+\delta}} \\ &< \frac{1}{2}(N'+1)\log 2 + \frac{1}{(1-p_0)^{m+\delta}}. \end{aligned}$$

The last inequality follows from (12). On the other hand, $g(p_0) + N' \log 2 \leq g(p_{N'}) \leq g(t)$ by (14). Combining this with (16) yields

$$(17) \quad \frac{1}{2}(N-1)\log 2 < \frac{1}{2}(N'-1)\log 2 < \frac{1}{(1-p_0)^{m+\delta}} - g(p_0).$$

Here we used $N < N'$ for the first inequality. We can pick $N(p_0)$ so large that (17) does not hold. This yields a contradiction. Hence we have $m \geq 2$.

We proceed to prove $M \leq 2$. Suppose $M > 2$. Pick $\delta > 0$ such that $M - \delta > 2$. Let

$$h(x) = \frac{1}{(x-3x^3)^{M-\delta}} - \frac{1}{x^{M-\delta}}.$$

Note that $\lim_{x \rightarrow 0} h(x) = \infty$. Then by a similar argument as above, we lead a contradiction. Hence $M \leq 2$, which concludes $m = M = 2$. \square

4. Proof of Theorem 1.2

First, we estimate the probability $P_p((1/9) \cdot 3^n \leq |C| \leq (9/2) \cdot 3^n)$. Let $M = \sup\{m: \mathcal{O} \leftrightarrow a_m \text{ or } b_m\}$. We define two conditional probabilities

$$U_n(p) = P_p(\mathcal{O} \leftrightarrow a_n, \mathcal{O} \not\leftrightarrow b_n \text{ in } F_n | M = n),$$

$$V_n(p) = P_p(O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n | M=n).$$

Clearly

$$(18) \quad 2U_n(p) + V_n(p) = 1,$$

and

$$(19) \quad V_n(p) = \frac{P_p(O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, O \nleftrightarrow a_{n+1}, O \nleftrightarrow b_{n+1})}{P_p(M=n)}.$$

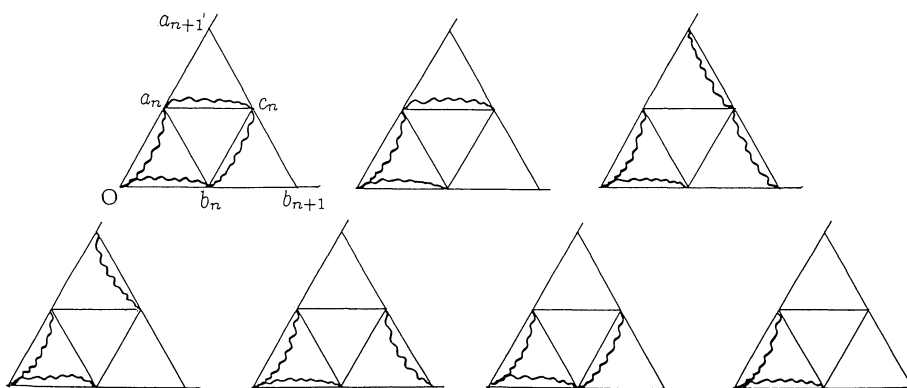


Fig. 4.1

We consider the event of the numerator of (19), $\{O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, O \nleftrightarrow a_{n+1}, O \nleftrightarrow b_{n+1}\}$. We divide the case into seven parts as Fig. 4.1. Since the events in F_n, F'_n, F''_n are independent, we have

$$(20) \quad V_n(p) = \frac{\Theta_n(1 - 2\Phi_n - \Phi_n^2 + 4\Phi_n\Theta_n - 2\Theta_n^2)}{P_p(M=n)}.$$

Here we denoted $\Phi_n = \Phi_n(p)$, $\Theta_n = \Theta_n(p)$ briefly. Note that

$$(21) \quad \begin{aligned} P_p(M=n) &= P_p(M \geq n) - P_p(M \geq n+1) \\ &= 2\Phi_n - \Theta_n - (2\Phi_{n+1} - \Theta_{n+1}) \\ &= 2\Phi_n - \Theta_n - 2\Phi_n^2 - 2\Phi_n^3 + 2\Phi_n\Theta_n^2 + 3\Phi_n^2\Theta_n - 2\Theta_n^3 \end{aligned}$$

by (4). Hence by (18),

$$(22) \quad U_n(p) = \frac{1}{2}\{1 - V_n(p)\}$$

$$= \frac{(\Phi_n - \Theta_n)(1 - \Phi_n - \Phi_n^2 + \Phi_n \Theta_n)}{P_p(M=n)}.$$

Let

$$(23) \quad n_0 = n_0(p) = \sup\{n : \Theta_n(p) \geq \frac{2}{3}\}.$$

Lemma 4.1. $V_n(p) \geq \frac{2}{9}$ if $n < n_0$.

Proof. From (18), it is enough to show

$$(24) \quad \frac{V_n(p)}{2U_n(p)} \geq \frac{2}{7}.$$

Let

$$\kappa(x, y) = \frac{y(1 - 2x - x^2 + 4xy - 2y^2)}{2(x - y)(1 - x - x^2 + xy)}.$$

By (20) and (22), (24) follows from the following:

$$(25) \quad \kappa(x, y) \geq \frac{2}{7} \quad \text{for} \quad \frac{2}{3} \leq x < 1, \quad \frac{1}{2}(3x - 1) < y < x.$$

The second condition in (25) comes from the fact that

$$(26) \quad 3\Phi_n(p) - 2\Theta_n(p) = \Psi_n(p) < 1.$$

Let $y/x = t$. Then the domain of (25) is $2/3 \leq x < 1/(3 - 2t)$, $2/3 \leq t < y < 1$.
And

$$\kappa(x, tx) = \frac{t}{2(1 - t)} \left\{ 1 - \frac{x + (-3t + 2t^2)x^2}{1 - x - (1 - t)x^2} \right\}.$$

Now let

$$\lambda(x) = \frac{x + (-3t + 2t^2)x^2}{1 - x - (1 - t)x^2}.$$

From a direct calculation,

$$\lambda'(x) = \frac{(1 + 2t - 2t^2)x^2 + 2(-3t + 2t^2)x + 1}{\{1 - x - (1 - t)x^2\}^2}.$$

We see that if $2/3 \leq t < 1$, $\lambda'(x) > 0$ for $2/3 \leq x < 1/(3-2t)$. Therefore

$$\kappa(x, tx) > \kappa\left(\frac{1}{3-2t}, \frac{t}{3-2t}\right) = \frac{t}{5-4t} \geq \frac{2}{7}. \quad \square$$

Next, we estimate the expectation of $|C|$ on condition that $M=n$ ($n < n_0$).

Lemma 4.2. $E_p(|C| | M=n) \geq \frac{2}{9} \cdot 3^n$ if $n < n_0$.

To prove the above Lemma, we use the following inequality:

Lemma 4.3. For all $a \in F_n$,

$$(27) \quad P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n) \geq \Phi_n(p).$$

Proof. Besides (27), we introduce a similar inequality:

$$(28) \quad P_p(a \leftrightarrow a_n \text{ or } a \leftrightarrow b_n) \geq P_p(\mathbf{O} \leftrightarrow a_n \text{ or } \mathbf{O} \leftrightarrow b_n) \text{ for all } a \in F_n.$$

We prove (27) and (28) by induction at the same time. If $n=0$, clearly both of them hold. Suppose (27) and (28) for $n=k$.

We prove (27) for $n=k+1$ at first. By symmetry, it is sufficient to prove the cases (i) $a \in F_k$ and (ii) $a \in F'_k$.

(i) Suppose $a \in F_k$. By using (4), we see $\Phi_k(p) \geq \Phi_{k+1}(p)$. Indeed, suppose $\Phi_k(p) \geq 1/3$, then

$$\begin{aligned} (29) \quad \frac{\Phi_{k+1}}{\Phi_k} &= \Phi_k + \{\Phi_k\}^2 - \{\Theta_k\}^2 \\ &\leq \Phi_k + \{\Phi_k\}^2 - \left(\frac{3\Phi_k - 1}{2}\right)^2 \\ &\leq -\frac{5}{4}(1 - \Phi_k)^2 + 1 \\ &\leq 1. \end{aligned}$$

Here we used (26). Combining this with assumption, we see (27) for $n=k+1$ in this case.

(ii) Suppose $a \in F'_k$. Let C_n^1, C_n^2, C_n^3 be events given by

$$\begin{aligned} C_n^1 &= \{\mathbf{O} \leftrightarrow a_n \text{ and } \mathbf{O} \not\leftrightarrow c_n \text{ in } F_n \cup F_n''\}, \\ C_n^2 &= \{\mathbf{O} \not\leftrightarrow a_n \text{ and } \mathbf{O} \leftrightarrow c_n \text{ in } F_n \cup F_n''\}, \end{aligned}$$

$$C_n^3 = \{O \leftrightarrow a_n \text{ and } O \leftrightarrow c_n \text{ in } F_n \cup F_n''\}.$$

We see

$$\begin{aligned} & P_p(O \leftrightarrow a \text{ in } F_{k+1}) \\ &= P_p(C_k^1)P_p(a_k \leftrightarrow a \text{ in } F_k') + P_p(C_k^2)P_p(c_k \leftrightarrow a \text{ in } F_k') \\ &\quad + P_p(C_k^3)P_p(a_k \leftrightarrow a \text{ or } c_k \leftrightarrow a \text{ in } F_k') \\ &\geq (\Phi_k - \Phi_k \Theta_k) \cdot \Phi_k + (\Phi_k - \Theta_k) \Phi_k \cdot \Phi_k + \Phi_k \Theta_k \cdot 2(\Phi_k - \Theta_k) \\ &= \Phi_k^2 + \Phi_k^3 - \Phi_k \Theta_k^2 = \Phi_{k+1}. \end{aligned}$$

Here we used assumption for the inequality. We thus obtain (27) for $n=k+1$. We proceed to prove (28) for $n=k+1$.

(i) Suppose $a \in F_k$. Let $D_n^1, D_n^2, \dots, D_n^5$ be events given by

$$D_n^1 = \{a_n \leftrightarrow a_{n+1} \text{ or } a_n \leftrightarrow b_{n+1} \text{ in } F_k' \cup F_k''\},$$

$$D_n^2 = \{b_n \leftrightarrow a_{n+1} \text{ or } b_n \leftrightarrow b_{n+1} \text{ in } F_k' \cup F_k''\},$$

$$D_n^3 = D_n^1 \cap (D_n^2)^c, \quad D_n^4 = (D_n^1)^c \cap D_n^2, \quad D_n^5 = D_n^1 \cap D_n^2. \quad \text{We see}$$

$$\begin{aligned} & P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1}) \\ &= P_p(D_k^3)P_p(a \leftrightarrow a_k \text{ in } F_k) + P_p(D_k^4)P_p(a \leftrightarrow b_k \text{ in } F_k) \\ &\quad + P_p(D_k^5)P_p(a \leftrightarrow a_k \text{ or } a \leftrightarrow b_k \text{ in } F_k) \\ &\geq P_p(D_k^3)P_p(O \leftrightarrow a_k \text{ in } F_k) + P_p(D_k^4)P_p(O \leftrightarrow b_k \text{ in } F_k) \\ &\quad + P_p(D_k^5)P_p(O \leftrightarrow a_k \text{ or } O \leftrightarrow b_k \text{ in } F_k) \\ &= P_p(O \leftrightarrow a_{k+1} \text{ or } O \leftrightarrow b_{k+1}) \end{aligned}$$

by assumption.

(ii) Suppose $a \in F_k'$. We see

$$\begin{aligned} (30) \quad & P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1}) \\ &\geq P_p(a \leftrightarrow a_{k+1} \text{ in } F_k') \\ &\quad + P_p(a \not\leftrightarrow a_{k+1} \text{ or } a \leftrightarrow c_k \text{ in } F_k')P_p(c_k \leftrightarrow b_{k+1} \text{ in } F_k''). \end{aligned}$$

Here we note that

$$\begin{aligned} & P_p(a \not\leftrightarrow a_{k+1} \text{ and } a \leftrightarrow c_k \text{ in } F_k') \\ &= P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow c_k \text{ in } F_k') - P_p(a \leftrightarrow a_{k+1} \text{ in } F_k') \\ &\geq (2\Phi_k - \Theta_k) - P_p(a \leftrightarrow a_{k+1} \text{ in } F_k') \end{aligned}$$

by assumption. Using this and (30), we have

$$P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1})$$

$$\begin{aligned}
&\geq P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k) \\
&\quad + \{(2\Phi_k - \Theta_k) - P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k)\} P_p(c_k \leftrightarrow b_{k+1} \text{ in } F''_k) \\
&= P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k)(1 - \Phi_k) + (2\Phi_k - \Theta_k)\Phi_k \\
&\geq \Phi_k(1 - \Phi_k) + 2\Phi_k^2 - \Phi_k\Theta_k = \Phi_k + \Phi_k^2 - \Phi_k\Theta_k.
\end{aligned}$$

Here we used assumption again. Now it is enough to show

$$(31) \quad \Phi_k + \Phi_k^2 - \Phi_k\Theta_k - P_p(\mathbf{O} \leftrightarrow a_{k+1} \text{ or } \mathbf{O} \leftrightarrow b_{k+1}) \geq 0.$$

The left-hand side of (31) equals

$$\begin{aligned}
&\Phi_k + \Phi_k^2 - \Phi_k\Theta_k - (2\Phi_{k+1} - \Theta_{k+1}) \\
&= (\Phi_k + \Phi_k^2 - \Phi_k\Theta_k) - 2(\Phi_k^2 + \Phi_k^3 - \Phi_k\Theta_k^2) + (3\Phi_k^2\Theta_k - 2\Theta_k^3) \\
&= \Phi_k(1 - \Theta_k)(1 - 3\Phi_k + 2\Theta_k) + 2(\Phi_k - \Theta_k)^2(1 - \Phi_k) \\
&\quad + 2\Theta_k(\Phi_k - \Theta_k)(1 - 2\Phi_k + \Theta_k).
\end{aligned}$$

By (26), we see all terms above are nonnegative. Hence the proof is completed. \square

Proof of Lemma 4.2.

$$\begin{aligned}
E_p(|C||M=n) &= \sum_{a \in V} P_p(\mathbf{O} \leftrightarrow a \mid M=n) \\
&\geq \sum_{a \in F_n} P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n \mid M=n) \\
&\geq \sum_{a \in F_n} \frac{P_p(\mathbf{O} \leftrightarrow a, \mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n, M=n)}{P_p(M=n)}.
\end{aligned}$$

Let $D_n^6 = (D_n^1)^c \cap (D_n^2)^c$. Note that if $M=n$ and $\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n$, then $(D_n^6)^c$ occurs. For $a \in F_n$, we see

$$\begin{aligned}
&P_p(\mathbf{O} \leftrightarrow a, \mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n, M=n) \\
&= P_p(\mathbf{O} \leftrightarrow a, \mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n, D_n^6 \text{ occurs}) \\
&= P_p(\mathbf{O} \leftrightarrow a, \mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n) P_p(D_n^6) \\
&\geq P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n) P_p(\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n) P_p(D_n^6) \\
&= P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n) P_p(\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n, M=n).
\end{aligned}$$

Here we used FKG inequality for the forth line. Therefore

$$E_p(|C||M=n) \geq \sum_{a \in F_n} P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n) P_p(\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n \mid M=n)$$

$$\geq \frac{2}{9} \sum_{a \in F_n} P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n)$$

by Lemma 4.1. Note that $|\{a \in V : a \in F_n\}| = (3/2)(3^n + 1)$. By virtue of Lemma 4.3, we see

$$\begin{aligned} E_p(|C| | M=n) &\geq \frac{2}{9} \cdot \frac{3}{2} \cdot 3^n \Phi_n(p) \\ &\geq \frac{2}{9} \cdot 3^n \quad \text{for } n < n_0. \end{aligned}$$

We used (23) and the fact that $\Phi_n(p) \geq \Theta_n(p)$ for the last inequality. \square

We proceed to the estimate of $P_p((1/9) \cdot 3^n \leq |C| \leq (9/2) \cdot 3^n)$.

Lemma 4.4. $P_p\left(\frac{1}{9} \cdot 3^n \leq |C| \leq \frac{9}{2} \cdot 3^n\right) \geq \frac{2}{79} P_p(M=n) \quad \text{if } n < n_0.$

Proof. Note that $|C| \leq (9/2) \cdot 3^n$ if $M=n$. Then we see the following.

$$\begin{aligned} &E_p(|C| | M=n) \\ &= E_p(|C|; |C| \geq \frac{1}{9} \cdot 3^n | M=n) + E_p(|C|; |C| < \frac{1}{9} \cdot 3^n | M=n) \\ &\leq \frac{9}{2} \cdot 3^n P_p(|C| \geq \frac{1}{9} \cdot 3^n | M=n) + \frac{1}{9} \cdot 3^n P_p(|C| < \frac{1}{9} \cdot 3^n | M=n). \end{aligned}$$

By Lemma 4.2, we have

$$P_p(|C| \geq \frac{1}{9} \cdot 3^n | M=n) \geq \frac{2}{79},$$

thus the proof is completed. \square

Lemma 4.5. $P_p(M=n) > \Phi_n(p) \{1 - \Phi_n(p)\}^2 \quad \text{if } n < n_0.$

Proof. Recall (21), that is

$$P_p(M=n) = 2\Phi_n - \Theta_n - 2\Phi_n^2 - 2\Phi_n^3 + 2\Phi_n\Theta_n^2 + 3\Phi_n^2\Theta_n - 2\Theta_n^3.$$

Let $\pi(y) = 2x - y - 2x^2 - 2x^3 + 2xy^2 + 3x^2y - 2y^3$. It is enough to show that $\pi(y) > x(1-x)^2$ if $2/3 \leq x < 1$, $(3x-1)/2 < y < x$. Note that

$$\pi'(y) = -6y^2 + 4xy + 3x^2 - 1,$$

and that

$$\pi'\left(\frac{3x-1}{2}\right) = \frac{1}{2}(1-x)(9x-5) > 0, \quad \pi'(x) = x^2 - 1 < 0.$$

Hence $\pi(y) > \min\{\pi((3x-1)/2), \pi(x)\}$, $\pi((3x-1)/2) = (1-x)^2(x+3)/4$ and $\pi(x) = x(1-x)^2$, so $\pi((3x-1)/2) > \pi(x)$ for $2/3 \leq x < 1$. This completes the proof. \square

Proof of Theorem 1.2. First, we estimate $E_p|C|^k$ from below. By using Lemma 4.4 and 4.5, we see

$$\begin{aligned} E_p|C|^k &= \sum_{l=1}^{\infty} l^k P_p(|C|=l) \\ &\geq \sum_{n=4,8,12,\dots} \left(\frac{1}{9} \cdot 3^n\right)^k P_p\left(\frac{1}{9} \cdot 3^n \leq |C| \leq \frac{9}{2} \cdot 3^n\right) \\ &\geq \frac{1}{9^k} \cdot \frac{2}{79} \sum_{\substack{m \in N \\ 4m < n_0}} 3^{4km} \Phi_{4m}(p) \{1 - \Phi_{4m}(p)\}^2. \end{aligned}$$

Let p be sufficiently large. Note that the function $\iota(x) = x(1-x)^2$ is decreasing in $2/3 \leq x < 1$, and $\Phi_{4m}(p) \leq e^{-2^{4m}/\xi(p)}$ by (7). We can see

$$\begin{aligned} &\sum_{\substack{m \in N \\ 4m < n_0}} 3^{4km} \Phi_{4m}(p) \{1 - \Phi_{4m}(p)\}^2 \\ &\geq \sum_{\substack{m \in N \\ 4m < n_0}} 3^{4km} e^{-2^{4m}/\xi(p)} (1 - e^{-2^{4m}/\xi(p)})^2 \\ &\geq \int_1^{n_0-1} 3^{4kx} e^{-2^{4x}/\xi(p)} (1 - e^{-2^{4x}/\xi(p)})^2 dx \\ &= \frac{\{\xi(p)\}^{Dk}}{4 \log 2} \int_{2^{4/\xi(p)}}^{2^{n_0-4/\xi(p)}} y^{Dk-1} e^{-y} (1 - e^{-y})^2 dy. \end{aligned}$$

Here we set $y = 2^x / \xi(p)$ in the last line. Note that $\Theta_{n_0+1}(p) < 2/3$, hence $\Phi_{n_0+1}(p) < (1 + 2\Theta_{n_0+1}(p))/3 < 7/9$ by (24). From (29), if $\Phi_k(p) < 7/9$, then $\Phi_{k+1}(p) / \Phi_k(p) < 76/81$. We see

$$\Phi_{n_0+12}(p) < \left(\frac{76}{81}\right)^{11} \cdot \frac{7}{9} < \frac{1}{2} \cdot \frac{7}{9}.$$

Combining this with (7), we have

$$\frac{1}{2}e^{-2^{n_0+12}/\xi(p)} \leq \Phi_{n_0+12}(p) < \frac{1}{2} \cdot \frac{7}{9}.$$

Hence $2^{n_0-4}/\xi(p) > 2^{-16}\log(9/7)$. Since $\xi(p) \rightarrow \infty$ as $p \rightarrow 1$, $E_p|C|^k > K_1\{\xi(p)\}^{Dk}$ holds if we take

$$K_1(k) = \int_{2^{-17}\log(9/7)}^{2^{-16}\log(9/7)} y^{Dk-1} e^{-y} (1-e^{-y})^2 dy > 0.$$

Now we proceed to estimate from above. Note that $P_p(M \geq n) \leq 2\Phi_n(p) \leq 2e^{-2^n/\xi(p)}$, and we can see easily $P_p((3/2) \cdot 3^n < |C| \leq (3/2) \cdot 3^{n+1}) \leq P_p(M \geq n) \leq 2e^{-2^n/\xi(p)}$. Hence

$$\begin{aligned} E_p|C|^k &= \sum_{l=1}^{\infty} l^k P_p(|C|=l) \\ &\leq 1 + \sum_{n=0}^{\infty} \left(\frac{3}{2} \cdot 3^{n+1}\right)^k P_p\left(\frac{3}{2} \cdot 3^n < |C| \leq \frac{3}{2} \cdot 3^{n+1}\right) \\ &\leq 1 + 2 \cdot \left(\frac{9}{2}\right)^k \sum_{n=0}^{\infty} 3^{kn} e^{-2^n/\xi(p)}. \end{aligned}$$

Now

$$\begin{aligned} \int_0^{\infty} 3^{kx} e^{-2^{x/\xi(p)}} dx &= \frac{\{\xi(p)\}^{Dk}}{\log 2} \int_{\xi(p)^{-1}}^{\infty} y^{Dk-1} e^{-y} dy \\ &\leq \frac{\Gamma(Dk)}{\log 2} \cdot \{\xi(p)\}^{Dk}. \end{aligned}$$

So we can take $K_2(k) < \infty$ such that $E_p|C|^k < K_2\{\xi(p)\}^{Dk}$. \square

5. Site percolation on the pre-Sierpinski gasket

We define the Bernoulli site percolation on the pre-Sierpinski gasket; each vertices in V are open with probability p and closed with $1-p$ independently. Let \tilde{P}_p denote its distribution. We write $x \leftrightarrow y$ if there exists a sequence of open vertices $x = x_0, x_1, \dots, x_{n-1}, x_n = y$ such that there is a bond in E which connects x_j with x_{j+1} for $0 \leq j \leq n-1$. We define another notations in the same manner as before. We introduce connectivity functions;

$$\tilde{\Phi}_n(p) = \tilde{P}_p(O \leftrightarrow a_n \text{ in } \Delta O a_n b_n),$$

$$\tilde{\Theta}_n(p) = \tilde{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } \Delta O a_n b_n).$$

We see $\tilde{\Theta}_0(p)=p^2$ and $\tilde{\Theta}_0(p)=p^3$ by definition.

Proposition 5.1. *For each $n \geq 0$ and $0 \leq p \leq 1$,*

$$(32) \quad \tilde{\Theta}_{n+1}(p) = p^{-1}\{\tilde{\Theta}_n(p)\}^2 + p^{-2}\{\tilde{\Theta}_n(p)\}^3 - p^{-3}\tilde{\Theta}_n(p)\{\tilde{\Theta}_n(p)\}^2,$$

$$(33) \quad \tilde{\Theta}_{n+1}(p) = 3p^{-2}\{\tilde{\Theta}_n(p)\}^2\tilde{\Theta}_n(p) - 2p^{-3}\{\tilde{\Theta}_n(p)\}^3.$$

Proof. We prove (32). Let \tilde{A}_n^1 and \tilde{A}_n^2 be events given by

$$\tilde{A}_n^1 = \{O \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F_n'\},$$

$$\tilde{A}_n^2 = \{O \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow c_n \text{ in } F_n''\} \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F_n'\}.$$

Then we have

$$(34) \quad \tilde{\Theta}_{n+1}(p) = \tilde{P}_p(\tilde{A}_n^1) + \tilde{P}_p(\tilde{A}_n^2) - \tilde{P}_p(\tilde{A}_n^1 \cap \tilde{A}_n^2).$$

Remark that $F_n \cap F_n' = \{a_n\}$. So we see $\tilde{P}_p(\tilde{A}_n^1) = p^{-1}\{\tilde{\Theta}_n(p)\}^2$. Similarly, we have $\tilde{P}_p(\tilde{A}_n^2) = p^{-2}\{\tilde{\Theta}_n(p)\}^3$, $\tilde{P}_p(\tilde{A}_n^1 \cap \tilde{A}_n^2) = p^{-3}\{\tilde{\Theta}_n(p)\}^2\tilde{\Theta}_n(p)$. Thus (32) follows from (34) immediately. (33) is proved in the same way. \square

Let $\hat{\Phi}_n(p) = p^{-1}\tilde{\Phi}_n(p)$ and $\hat{\Theta}_n(p) = p^{-\frac{3}{2}}\tilde{\Theta}_n(p)$. Then we have the same recursions as (4), (5):

$$(35) \quad \hat{\Phi}_{n+1}(p) = \{\hat{\Phi}_n(p)\}^2 + \{\hat{\Phi}_n(p)\}^3 - \hat{\Phi}_n(p)\{\hat{\Theta}_n(p)\}^2,$$

$$(36) \quad \hat{\Theta}_{n+1}(p) = 3\{\hat{\Phi}_n(p)\}^2\hat{\Theta}_n(p) - 2\{\hat{\Theta}_n(p)\}^3.$$

Hence we see that there exists $\xi(p) > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{\hat{\Phi}_n(p)}{\exp\{-2^n / \xi(p)\}} = 1, \quad \text{that is} \quad \lim_{n \rightarrow \infty} \frac{\tilde{P}_p(O \leftrightarrow a_n)}{p \exp\{-2^n / \xi(p)\}} = 1.$$

Lemma 5.2. *Let $\sqrt{\tilde{p}} = \sqrt{p} + 6(1 - \sqrt{p})^2$. Then there exists $\varepsilon > 0$ such that*

$$2 \leq \frac{\xi(\tilde{p})}{\xi(p)} \leq 4 \quad \text{for} \quad 1 - \varepsilon < p < 1.$$

Proof. We use the same method as in Section 3 again. Let

$$(37) \quad \hat{\Psi}_n(p) = 3\hat{\Phi}_n(p) - 2\hat{\Theta}_n(p).$$

To apply (9), first we prove $(\hat{\Theta}_n(p), \hat{\Psi}_n(p)) \in D$. (Recall $D = \{(x, y) : 0 < x \leq y < 1\}$.) Since $\hat{\Psi}_n(p) = \hat{\Theta}_n(p) + 3\{\hat{\Phi}_n(p) - \hat{\Theta}_n(p)\}$, it is enough to prove $\hat{\Phi}_n(p) \geq \hat{\Theta}_n(p)$. Now

$$\hat{\Phi}_n(p) = p^{-1} \times \tilde{P}_p(O \leftrightarrow a_n \text{ in } F_n)$$

$$\begin{aligned}
&= \tilde{P}_p(O \leftrightarrow a_n \text{ in } F_n | a_n \text{ is open}) \\
&= \tilde{P}_p(O \leftrightarrow a_n \text{ in } F_n | a_n, b_n \text{ are open}), \\
\hat{\Theta}_n(p) &= p^{-\frac{3}{2}} \times \tilde{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n) \\
&\leq p^{-2} \times \tilde{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n) \\
&= \tilde{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n | a_n, b_n \text{ are open}).
\end{aligned}$$

Hence we have $\hat{\Phi}_n(p) \geq \hat{\Theta}_n(p)$, which implies $(\hat{\Theta}_n(p), \hat{\Psi}_n(p)) \in D$.

A direct calculation from (35) and (36) shows

$$\begin{aligned}
\hat{\Theta}_2(\tilde{p}) - \hat{\Theta}_1(p) &= 6(1 - \sqrt{p})^2 + 204(1 - \sqrt{p})^3 + \dots, \\
\hat{\Psi}_2(\tilde{p}) - \hat{\Psi}_1(p) &= 12(1 - \sqrt{p})^3 + \dots, \\
\hat{\Theta}_3(\tilde{p}) - \hat{\Theta}_1(p) &= -6(1 - \sqrt{p})^2 + 204(1 - \sqrt{p})^3 + \dots, \\
\hat{\Psi}_3(\tilde{p}) - \hat{\Psi}_1(p) &= -12(1 - \sqrt{p})^3 + \dots.
\end{aligned}$$

We can take $\varepsilon > 0$ such that

$$\hat{\Theta}_3(\tilde{p}) < \hat{\Theta}_1(p) < \hat{\Theta}_2(\tilde{p}), \quad \hat{\Psi}_3(\tilde{p}) < \hat{\Psi}_1(p) < \hat{\Psi}_2(\tilde{p})$$

for $1 - \varepsilon < p < 1$.

Now we apply (9). We have for $n \geq 1$ and $1 - \varepsilon < p < 1$,

$$\hat{\Theta}_{n+2}(\tilde{p}) < \hat{\Theta}_n(p) < \hat{\Theta}_{n+1}(\tilde{p}), \quad \text{and} \quad \hat{\Psi}_{n+2}(\tilde{p}) < \hat{\Psi}_n(p) < \hat{\Psi}_{n+1}(\tilde{p}).$$

We see $\hat{\Phi}_{n+2}(\tilde{p}) < \hat{\Phi}_n(p) < \hat{\Phi}_{n+1}(\tilde{p})$ by (37), so we have the conclusion. \square

Proof of Theorem 1.3. Note that $\tilde{p} = \{\sqrt{p} + 6(1 - \sqrt{p})^2\}^2 = p + 3(1 - p)^2 + o((1 - p)^2)$ as $p \rightarrow 1$. We have Theorem 1.3 in the same way as in Section 3.

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