



Title	Percolation on the pre-Sierpiński gasket
Author(s)	Shinoda, Masato
Citation	Osaka Journal of Mathematics. 1996, 33(2), p. 533-554
Version Type	VoR
URL	<a href="https://doi.org/10.18910/11447">https://doi.org/10.18910/11447</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

# PERCOLATION ON THE PRE-SIERPINSKI GASKET

MASATO SHINODA

(Received April 6, 1995)

## 1. Introduction and statements of results

In this paper, we regard percolation as a model of phase transitions. We are especially interested in problems near the *critical point*, where the phase transition occurs. We call these problems *critical behaviors*. Our purpose in this paper is to clarify the critical behaviors of percolation on the pre-Sierpinski gasket which has self-similarity.

Until now, studies of percolation are restricted on *periodic* graphs, such as  $\mathbb{Z}^d$ . (An exact definition of periodic graph is mentioned in Kesten [1].) There are lots of conjectures and hypotheses about critical behaviors, but many of them are still unsolved rigorously (see Grimmett [2] and references therein). In high dimension lattices  $\mathbb{Z}^d$ , rigorous results for critical behaviors were obtained by Hara-Slade [3]. But in low dimensions, except a work on  $\mathbb{Z}^2$  by Kesten [4], few rigorous results have been proved about the existence of *critical exponents* and justification of the *scaling, hyperscaling relations*.

For critical behaviors, *self-similarity* of the graph plays more important role than periodicity. This is a motivation to consider percolation problems on the pre-Sierpinski gasket.

We now define the pre-Sierpinski gasket. Let  $O=(0,0)$ ,  $a_0=(1/2, \sqrt{3}/2)$ ,  $b_0=(1,0)$ . Let  $F_0$  be the graph which consists of the vertices and edges of the triangle  $\Delta Oa_0b_0$ . Let  $\{F_n\}_{n=0,1,2,\dots}$  be the sequence of graphs given by

$$F_{n+1} = F_n \cup (F_n + 2^n a_0) \cup (F_n + 2^n b_0)$$

where  $A+a=\{x+a \mid x \in A\}$  and  $kA=\{kx \mid x \in A\}$ . Let  $F=\bigcup_{n=0}^{\infty} F_n$ . We call  $F$  the *pre-Sierpinski gasket*. (Fig. 1.1) Note that  $\tilde{F}=\overline{\bigcup_{n=0}^{\infty} 2^{-n} F}$  become the Sierpinski gasket. Let  $V$  be the set of all vertices in  $F$ , and  $E$  the set of all edges with length 1.

We consider the Bernoulli bond percolation on the pre-Sierpinski gasket; each edges in  $E$  are *open* with probability  $p$  and *closed* with probability  $1-p$  independently. Let  $P_p$  denote its distribution. We think of open bonds as permitting to go along the bond. We write  $x \leftrightarrow y$  if there is an open path from  $x$  to  $y$ . Let  $C(x)=\{y \in V : x \leftrightarrow y\}$ .  $C(x)$  is called the *open cluster* containing  $x$ . We denote by  $C$  the open cluster containing the origin.

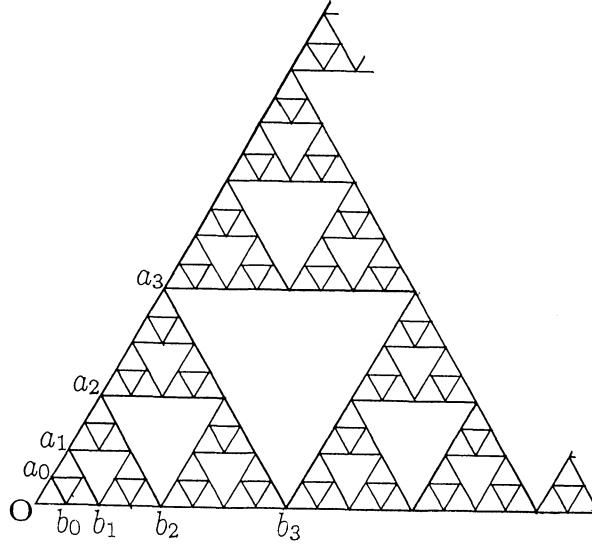


Fig. 1.1

We define two functions in a similar way as percolations on  $\mathbb{Z}^d$ .

$$\theta(p) = P_p(|C| = \infty), \quad \chi(p) = E_p(|C|; |C| < \infty),$$

where  $|C|$  denotes the number of vertices contained in  $C$ , and  $E_p$  denotes the expectation with respect to  $P_p$ .  $\theta(p)$  is called the *percolation probability*, and  $\chi(p)$  is called the *mean cluster size*.

Let  $p_c$  denote the *critical point*; that is

$$p_c = \inf\{p : \theta(p) > 0\}.$$

Then  $p_c = 1$  for the pre-Sierpinski gasket because it is finitely ramified. We note that  $\chi(p) = E_p|C|$  for  $p < 1$ .

The *correlation length* is defined by

$$(1) \quad \xi(p) = \lim_{n \rightarrow \infty} \left\{ -\frac{1}{2^n} \log P_p(O \leftrightarrow a_n) \right\}^{-1}.$$

The existence of the limit in (1) will be proved in Section 2.

We write  $f(p) \approx g(p)$  as  $p \rightarrow p_0$  if  $\log f(p)/\log g(p) \rightarrow 1$  as  $p \rightarrow p_0$ .

We now state our main theorems:

**Theorem 1.1.**  $\lim_{p \rightarrow 1} -\frac{\log \xi(p)}{\log(1-p)} = \infty$ , and  $\lim_{p \rightarrow 1} \frac{\log(\log \xi(p))}{\log(1-p)} = -2$ .

**Theorem 1.2.** *Let  $D = \log 3 / \log 2$ . Then*

$$E_p|C|^k \approx \{\xi(p)\}^{Dk} \quad \text{as } p \rightarrow 1 \quad \text{for all } k \geq 1.$$

**REMARK.** Our results are quite different from the results on  $\mathbb{Z}^d$  (see below). In physical literature, Theorem 1.1 was known by Gefen et al. [5] by using formal renormalization arguments. Our contribution is that we prove Theorem 1.1 rigorously.

We collect results and conjectures of the percolation on  $\mathbb{Z}^d$ . It is conjectured (see [2])

$$(2) \quad \xi(p) \approx |p_c - p|^{-v(d)} \quad \text{as } p \rightarrow p_c.$$

The value  $v(d)$  is called the *critical exponent*. It is proved that  $v(d) = 1/2$  for sufficiently large  $d$  (Hara-Slade [3]), and conjectured  $v(2) = 4/3$  (see [4]).

Other critical exponents considered in  $\mathbb{Z}^d$  are as follows:

$$\chi(p) \approx |p_c - p|^{-\gamma}, \quad \frac{E_p(|C|^{k+1}; |C| < \infty)}{E_p(|C|^k; |C| < \infty)} \approx |p_c - p|^{-\Delta} \quad \text{as } p \rightarrow p_c.$$

It is conjectured for  $\mathbb{Z}^d$  that  $dv = 2\Delta - \gamma$ . This relation is one of hyperscaling relations. We note  $\gamma = \Delta = \infty$  on the pre-Sierpinski gasket. So the relation  $dv = 2\Delta - \gamma$  does not make sense on the pre-Sierpinski gasket. Accordingly we modify the hyperscaling relation as follows:

$$(3) \quad \{\xi(p)\}^d \approx \frac{E_p|C|^3}{\{\chi(p)\}^2} \quad \text{as } p \rightarrow p_c.$$

If finite critical exponents  $v, \gamma, \Delta$  exist, then (3) is equivalent to  $dv = 2\Delta - \gamma$ .

**REMARK.** By Theorem 1.2, we have  $E_p|C|^3 \approx \{\xi(p)\}^{3D}$  and  $\chi(p) \approx \{\xi(p)\}^D$ . Hence the above hyperscaling relation (3) holds when we regard  $D$  as the dimension of the pre-Sierpinski gasket. The value  $D = \log 3 / \log 2$  coincides with the fractal dimension of the Sierpinski gasket.

In addition, we mention site percolation on the pre-Sierpinski gasket: each vertices in  $V$  are determined to be open or closed independently. (Details will be given in Section 5). We define the correlation length  $\hat{\xi}(p)$  in the same manner as (1). We have the result below;

$$\text{Theorem 1.3.} \quad \lim_{p \rightarrow 1} -\frac{\log \hat{\xi}(p)}{\log(1-p)} = \infty, \quad \text{and} \quad \lim_{p \rightarrow 1} \frac{\log(\log \hat{\xi}(p))}{\log(1-p)} = -1.$$

The critical exponent in a usual sense is also infinite in this case. But  $\log \hat{\xi}(p) \approx (1-p)^{-1}$ , which is different from Theorem 1.1. We cannot see the

universality of this exponent on the pre-Sierpinski gasket.

We refer to the self-avoiding walks on the Sierpinski gasket, as related works of phase transitions; Hattori-Hattori [6] and Hattori-Hattori-Kusuoka [7] construct the self-avoiding paths on two- and three-dimensional Sierpinski gasket. Before [6], Hattori-Hattori-Kusuoka [8] constructed them on the pre-Sierpinski gasket. These works also gave us a motivation to study percolation on the Sierpinski gasket.

The organization of this paper is as follows: In Section 2 we prepare for the proof of our main theorems; we construct recursion formulas of relations between events in  $F_n$  and ones in  $F_{n+1}$ . In the reminder of Section 2, we prove the existence of the correlation length. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. In Section 5 we study site percolation and prove Theorem 1.3.

## 2. Recursion formulas and the existence of $\xi(p)$

We introduce two connectivity functions as follows.

$$\begin{aligned}\Phi_n(p) &= P_p(\mathbf{O} \leftrightarrow a_n \text{ in } \Delta \mathbf{O} a_n b_n), \\ \Theta_n(p) &= P_p(\mathbf{O} \leftrightarrow a_n \text{ and } \mathbf{O} \leftrightarrow b_n \text{ in } \Delta \mathbf{O} a_n b_n).\end{aligned}$$

We write  $\mathbf{O} \leftrightarrow a_n$  in  $\Delta \mathbf{O} a_n b_n$  if there is an open path from  $\mathbf{O}$  to  $a_n$  in  $\Delta \mathbf{O} a_n b_n$  (contains its perimeter). We easily calculate  $\Phi_0(p) = p + p^2 - p^3$ ,  $\Theta_0(p) = 3p^2 - 2p^3$ . Note that (i)  $\Phi_n(p) \geq \Theta_n(p)$  by definition, (ii) if  $\mathbf{O} \leftrightarrow a_n$  and  $\mathbf{O} \leftrightarrow b_n$  then we have  $a_n \leftrightarrow b_n$  automatically.

**Proposition 2.1.** *For each  $n \geq 0$  and  $0 \leq p \leq 1$ ,*

$$(4) \quad \Phi_{n+1}(p) = \{\Phi_n(p)\}^2 + \{\Phi_n(p)\}^3 - \Phi_n(p)\{\Theta_n(p)\}^2,$$

$$(5) \quad \Theta_{n+1}(p) = 3\{\Phi_n(p)\}^2 \Theta_n(p) - 2\{\Theta_n(p)\}^3.$$

**Proof.** Recall  $\Delta \mathbf{O} a_n b_n = F_n$ . Let  $F'_n = F_n + a_n$ ,  $F''_n = F_n + b_n$ , and  $c_n = (3 \cdot 2^{n-1}, \sqrt{3} \cdot 2^{n-1})$ . Let  $A_n^1$  and  $A_n^2$  be events given by

$$\begin{aligned}A_n^1 &= \{\mathbf{O} \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F'_n\}, \\ A_n^2 &= \{\mathbf{O} \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow c_n \text{ in } F''_n\} \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F'_n\}.\end{aligned}$$

Then we have

$$(6) \quad \Phi_{n+1}(p) = P_p(A_n^1) + P_p(A_n^2) - P_p(A_n^1 \cap A_n^2).$$

Here we used the fact that a path from  $\mathbf{O}$  to  $a_{n+1}$  goes through  $a_n$  or  $b_n$ . Since the events in  $F_n, F'_n, F''_n$  are mutually independent,  $P_p(A_n^1) = \{\Phi_n(p)\}^2$ ,  $P_p(A_n^2) = \{\Phi_n(p)\}^3$ ,

$P_p(A_n^1 \cap A_n^2) = \{\Theta_n(p)\}^2 \Phi_n(p)$  (Fig. 2.1). Combining these with (6) yields (4).

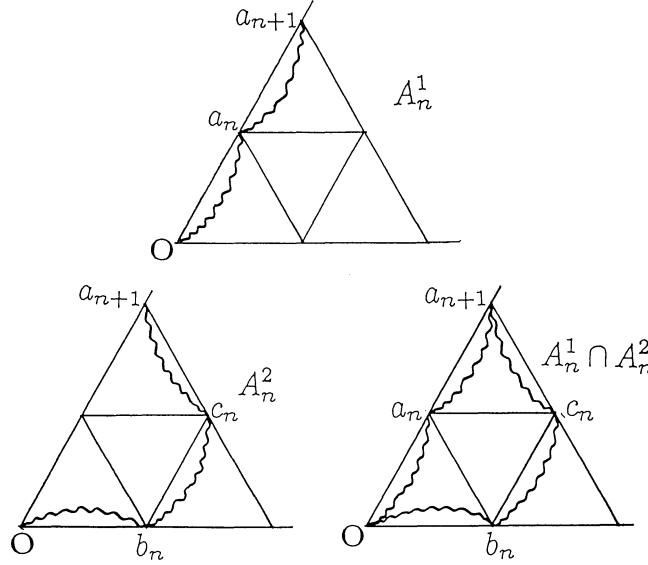


Fig. 2.1

We proceed to the proof of (5). Let  $B_n^1, B_n^2, B_n^3$  be events given by

$$B_n^1 = \{O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F'_n\} \cap \{b_n \leftrightarrow b_{n+1} \text{ in } F''_n\},$$

$$B_n^2 = \{O \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ and } a_n \leftrightarrow c_n \text{ in } F'_n\} \cap \{c_n \leftrightarrow b_{n+1} \text{ in } F''_n\},$$

$$B_n^3 = \{O \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow b_{n+1} \text{ and } b_n \leftrightarrow c_n \text{ in } F''_n\} \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F'_n\}$$

(see Fig. 2.2).

Then we have

$$\begin{aligned} \Theta_{n+1}(p) = & P_p(B_n^1) + P_p(B_n^2) + P_p(B_n^3) - P_p(B_n^1 \cap B_n^2) - P_p(B_n^2 \cap B_n^3) \\ & - P_p(B_n^3 \cap B_n^1) + P_p(B_n^1 \cap B_n^2 \cap B_n^3). \end{aligned}$$

We see easily

$$P_p(B_n^1) = P_p(B_n^2) = P_p(B_n^3) = \{\Phi_n(p)\}^2 \Theta_n(p),$$

$$P_p(B_n^1 \cap B_n^2) = P_p(B_n^2 \cap B_n^3) = P_p(B_n^3 \cap B_n^1) = P_p(B_n^1 \cap B_n^2 \cap B_n^3) = \{\Theta_n(p)\}^3.$$

(5) follows from this immediately.  $\square$

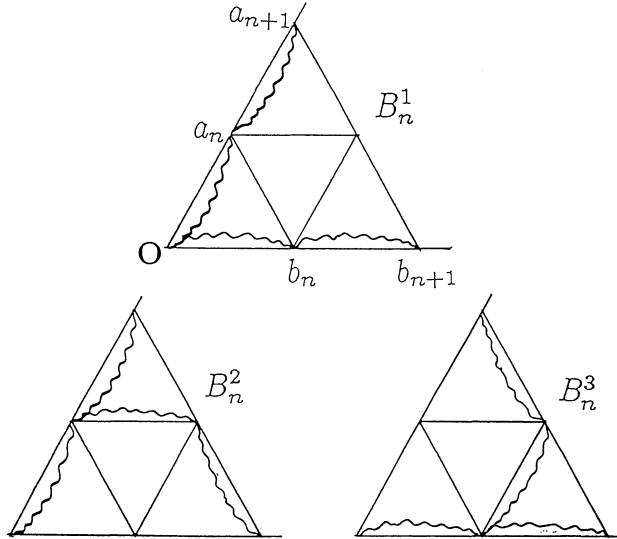


Fig. 2.2

From now on, we assume  $0 < p < 1$ . We prove the existence of the limit (1), correlation length  $\xi(p)$ , by using these recursions.

**Proposition 2.2.** *There exists  $\xi(p) > 0$  such that*

$$\lim_{n \rightarrow \infty} \frac{\Phi_n(p)}{\exp\{-2^n/\xi(p)\}} = 1.$$

**REMARK.** The convergence as  $n \rightarrow \infty$  in Proposition 2.2 is stronger than the convergence in (1).

**Proof.** By (4) and  $\Theta_n(p) \leq \Phi_n(p)$ , we have

$$\{\Phi_n(p)\}^2 \leq \Phi_{n+1}(p) \leq \{\Phi_n(p)\}^2 + \{\Phi_n(p)\}^3.$$

Hence

$$1 \leq \frac{\Phi_{n+1}(p)}{\{\Phi_n(p)\}^2} \leq 1 + \Phi_n(p).$$

Let  $h_n(p) = \Phi_{n+1}(p) / \{\Phi_n(p)\}^2$ . Then  $1 \leq h_n(p) \leq 2$  and  $\lim_{n \rightarrow \infty} h_n(p) = 1$  because  $\lim_{n \rightarrow \infty} \Phi_n(p) = 0$ . Now

$$\frac{1}{2^n} \log \Phi_n(p)$$

$$\begin{aligned}
&= \frac{1}{2^n} \log \left( \{\Phi_0(p)\}^{2^n} \cdot \frac{\{\Phi_1(p)\}^{2^{n-1}}}{\{\Phi_0(p)\}^{2^n}} \cdot \frac{\{\Phi_2(p)\}^{2^{n-2}}}{\{\Phi_1(p)\}^{2^{n-1}}} \cdots \frac{\{\Phi_n(p)\}}{\{\Phi_{n-1}(p)\}^2} \right) \\
&= \log \Phi_0(p) + \frac{1}{2} \log h_0(p) + \frac{1}{2^2} \log h_1(p) + \cdots + \frac{1}{2^n} \log h_{n-1}(p) \\
&\leq \log \Phi_0(p) + \log 2.
\end{aligned}$$

Hence  $\{\log \Phi_n(p) / 2^n\}_{n=0,1,2,\dots}$  is increasing and  $\lim_{n \rightarrow \infty} \log \Phi_n(p) / 2^n$  exists. Let  $-\{\xi(p)\}^{-1} = \lim_{n \rightarrow \infty} \log \Phi_n(p) / 2^n$ . Then

$$\begin{aligned}
-\frac{1}{\xi(p)} &\geq \frac{1}{2^n} \log \Phi_n(p) = -\frac{1}{\xi(p)} - \left( \frac{1}{2^{n+1}} \log h_n(p) + \frac{1}{2^{n+2}} \log h_{n+1}(p) + \cdots \right) \\
&\geq -\frac{1}{\xi(p)} - \frac{1}{2^n} \log H_n(p),
\end{aligned}$$

where  $H_n(p) = \sup_{m \geq n} h_m(p)$ . Therefore

$$(7) \quad \exp \left\{ -\frac{2^n}{\xi(p)} \right\} \geq \Phi_n(p) \geq \frac{1}{H_n(p)} \exp \left\{ -\frac{2^n}{\xi(p)} \right\}.$$

Since  $\lim_{n \rightarrow \infty} H_n(p) = 1$ , we complete the proof.  $\square$

**REMARK.** Note that the function  $\xi(p)$  is continuous and increasing on  $(0, 1)$  from the proof above.

**Lemma 2.3.**  $\lim_{n \rightarrow \infty} \frac{P_p(O \leftrightarrow a_n)}{\exp\{-2^n/\xi(p)\}} = 1$ .

Proof. Recall that  $\Phi_n(p) = P_p(O \leftrightarrow a_n \text{ in } F_n)$ . Then

$$\begin{aligned}
&P_p(O \leftrightarrow a_n) - P_p(O \leftrightarrow a_n \text{ in } F_n) \\
&\leq P_p(O \leftrightarrow b_n \text{ in } F_n, b_n \leftrightarrow c_n \text{ in } F_n', c_n \leftrightarrow a_n \text{ in } F_n') \\
&\quad + P_p(O \leftrightarrow b_n \text{ in } F_n, b_n \leftrightarrow b_{n+1} \text{ in } F_n', a_n \leftrightarrow a_{n+1} \text{ in } F_n') \text{ (Fig. 2.3)} \\
&= 2\{\Phi_n(p)\}^3.
\end{aligned}$$

So

$$1 \leq \frac{P_p(O \leftrightarrow a_n)}{\Phi_n(p)} \leq 1 + 2\{\Phi_n(p)\}^2,$$

which implies

$$\lim_{n \rightarrow \infty} \frac{P_p(O \leftrightarrow a_n)}{\Phi_n(p)} = 1.$$

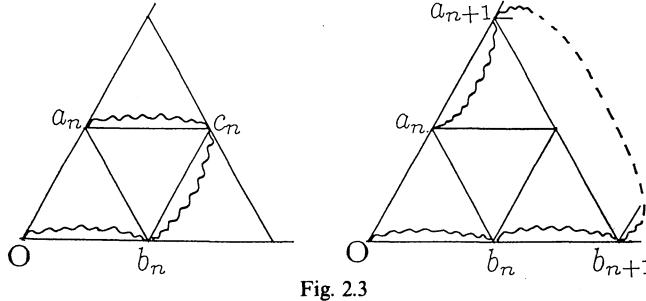


Fig. 2.3

Combining this with Proposition 2.2 completes the proof.  $\square$

### 3. Proof of Theorem 1.1

The next lemma is a key of the proof.

**Lemma 3.1.** *There exists  $\varepsilon > 0$  such that*

$$2 \leq \frac{\xi(p + 3(1-p)^3)}{\xi(p)} \leq 4 \quad \text{for} \quad 1 - \varepsilon < p < 1.$$

**Proof.** We introduce

$$\begin{aligned} (8) \quad \Psi_n(p) &= 1 - P_p(O \nleftrightarrow a_n, O \nleftrightarrow b_n, a_n \nleftrightarrow b_n \text{ in } F_n) \\ &= 3\Phi_n(p) - 2\Theta_n(p). \end{aligned}$$

Here  $O \nleftrightarrow a_n$  in  $F_n$  means that there exists no open path from  $O$  to  $a_n$  in  $F_n$ . By (4) and (5),

$$\Theta_{n+1}(p) = S(\Theta_n(p), \Psi_n(p)),$$

$$\Psi_{n+1}(p) = T(\Theta_n(p), \Psi_n(p)),$$

where  $S, T: \mathbf{R}^2 \rightarrow \mathbf{R}$  are functions defined by

$$S(x, y) = -\frac{2}{3}x^3 + \frac{4}{3}x^2y + \frac{1}{3}xy^2,$$

$$T(x, y) = \frac{2}{9}x^3 + \frac{4}{3}x^2 - \frac{7}{3}x^2y + \frac{4}{3}xy + \frac{1}{9}y^3 + \frac{1}{3}y^2.$$

Let  $D$  be a subset of  $\mathbf{R}^2$  defined by  $D = \{(x, y) : 0 < x \leq y < 1\}$ . We see  $\partial S / \partial x$ ,

$\partial S/\partial y, \partial T/\partial x, \partial T/\partial y > 0$  for  $(x, y) \in D$ . Indeed,

$$\begin{aligned} \frac{\partial S}{\partial x} &= -2x^2 + \frac{8}{3}xy + \frac{1}{3}y^2 = 2x(y-x) + \frac{2}{3}xy + \frac{1}{3}y^2 > 0, \\ \frac{\partial S}{\partial y} &= \frac{4}{3}x^2 + \frac{2}{3}xy > 0, \\ \frac{\partial T}{\partial x} &= \frac{2}{3}x^2 + \frac{8}{3}x - \frac{14}{8}xy + \frac{4}{3}y \geq \frac{2}{3}x^2 + \frac{8}{3}x - \frac{14}{3}xy + \frac{2}{3}y^2 + \frac{2}{3}y \\ &= \frac{2}{3}(y-x)^2 + \frac{8}{3}x(1-y) + \frac{2}{3}y(1-x) > 0, \\ \frac{\partial T}{\partial y} &= -\frac{7}{3}x^2 + \frac{4}{3}x + \frac{1}{3}y^2 + \frac{2}{3}y \\ &= \frac{4}{3}x(1-x) + \frac{1}{3}(y^2 - x^2) + \frac{2}{3}(y - x^2) > 0. \end{aligned}$$

Therefore if  $(x_1, y_1), (x_2, y_2) \in D$  and  $x_1 < x_2$  and  $y_1 < y_2$ , then

$$(9) \quad S(x_1, y_1) < S(x_2, y_2), \quad T(x_1, y_1) < T(x_2, y_2).$$

Note that  $\Psi_n(p) = \Theta_n(p) + 3\{\Phi_n(p) - \Theta_n(p)\} \geq \Theta_n(p)$  for all  $n$  by (8). Hence  $(\Theta_n(p), \Psi_n(p)) \in D$ . Calculating  $\Theta_n(p)$  and  $\Psi_n(p)$  directly from the recursions, we have

$$(10) \quad \begin{aligned} \Theta_n(p) &= 1 - 3(1-p)^2 - (12n-6)(1-p)^4 + 6(1-p)^5 \\ &\quad + (-48n^2 + 120n - 15)(1-p)^6 + \dots, \end{aligned}$$

$$(11) \quad \Psi_n(p) = 1 - 3(1-p)^4 - 24n(1-p)^6 + \dots$$

for  $n \geq 2$ . For  $1 - 1/\sqrt{3} < p < 1$ , let  $\tilde{p} = p + 3(1-p)^3$ . Then we have

$$\Theta_3(\tilde{p}) - \Theta_2(p) = 6(1-p)^4 + 213(1-p)^6 + \dots,$$

$$\Psi_3(\tilde{p}) - \Psi_2(p) = 12(1-p)^6 + \dots.$$

Note that  $\Theta_2(p), \Psi_2(p), \Theta_3(\tilde{p}),$  and  $\Psi_3(\tilde{p})$  are polynomials of finite degree. Hence we can take  $\varepsilon_1 > 0$  in such a way that  $\Theta_2(p) < \Theta_3(\tilde{p})$  and  $\Psi_2(p) < \Psi_3(\tilde{p})$  for  $1 - \varepsilon_1 < p < 1$ . By (9), We have

$$\Theta_3(p) = S(\Theta_2(p), \Psi_2(p)) < S(\Theta_3(\tilde{p}), \Psi_3(\tilde{p})) = \Theta_4(\tilde{p}),$$

$$\Psi_3(p) = T(\Theta_2(p), \Psi_2(p)) < T(\Theta_3(\tilde{p}), \Psi_3(\tilde{p})) = \Psi_4(\tilde{p}).$$

Estimating repeatedly as above, we have  $\Theta_n(p) < \Theta_{n+1}(\tilde{p})$ ,  $\Psi_n(p) < \Psi_{n+1}(\tilde{p})$  for  $n \geq 2$ . Combining this with (8) yields  $\Phi_n(p) < \Phi_{n+1}(\tilde{p})$ . So

$$\frac{\log \Phi_n(p)}{2^n} < 2 \cdot \frac{\log \Phi_{n+1}(\tilde{p})}{2^{n+1}}.$$

This implies  $\xi(p)^{-1} \geq 2 \cdot \xi(\tilde{p})^{-1}$ , that is  $\xi(\tilde{p}) / \xi(p) \geq 2$  for  $1 - \varepsilon_1 < p < 1$ .

We now proceed to the estimate from the opposite side. By using (10) and (11) again, we see

$$\Theta_4(\tilde{p}) - \Theta_2(p) = -6(1-p)^4 + 141(1-p)^6 + \dots$$

$$\Psi_4(\tilde{p}) - \Psi_2(p) = -12(1-p)^6 + \dots.$$

Hence we can take  $\varepsilon_2 > 0$  such that  $\Theta_4(\tilde{p}) < \Theta_2(p)$  and  $\Psi_4(\tilde{p}) < \Psi_2(p)$  for  $1 - \varepsilon_2 < 1$ . So we have  $\Theta_{n+2}(\tilde{p}) < \Theta_n(p)$  and  $\Psi_{n+2}(\tilde{p}) < \Psi_n(p)$ . Therefore  $\xi(\tilde{p}) / \xi(p) \leq 4$  for  $1 - \varepsilon_2 < p < 1$ , which completes the proof.  $\square$

**Proof of Theorem 1.1.** Let  $g(p) = \log \xi(p)$ . Since  $\xi(p)$  is an increasing function,  $g(p)$  is also increasing. Suppose that  $p$  is sufficiently large to satisfy  $g(p) > 0$ . Let

$$m = \liminf_{p \rightarrow 1} -\frac{\log g(p)}{\log(1-p)} \geq 0, \quad M = \limsup_{p \rightarrow 1} -\frac{\log g(p)}{\log(1-p)}.$$

First, we prove  $m \geq 2$ . Suppose  $m < 2$ , and pick  $\delta > 0$  with  $m + \delta < 2$ . Let

$$h(x) = \frac{1}{(x - 3x^3)^{m+\delta}} - \frac{1}{x^{m+\delta}}.$$

Applying the L'Hospital theorem, we see  $\lim_{x \rightarrow 0} h(x) = 0$ . So we take  $p_0$  such that

$$(12) \quad h(1-p) < \frac{1}{2} \log 2 \quad \text{for} \quad 0 < 1-p < 1-p_0$$

and  $1-p_0 < \varepsilon$ . ( $\varepsilon$  is given in Lemma 3.1.)

Let

$$(13) \quad f(p) = p + 3(1-p)^3.$$

We define  $\{p_n\}_{n=1,2,\dots}$  by  $f(p_0) = p_1$ ,  $f(p_n) = p_{n+1}$  inductively. Then  $p_0 < p_1 < \dots < p_n < 1$ , and  $\lim_{n \rightarrow \infty} p_n = 1$ . By (13) and Lemma 3.1, we have

$$\log 2 \leq g(p_{n+1}) - g(p_n),$$

and hence

$$(14) \quad g(p_0) + n \log 2 \leq g(p_n).$$

Take  $N=N(p_0) \in \mathbb{N}$ . By assumption, there exists  $t$  such that  $p_N < t < 1$  and

$$(15) \quad -\frac{\log g(t)}{\log(1-t)} < m + \delta.$$

For this  $t$ , there exists unique  $N'=N'(t)$  such that  $p_{N'} \leq t < p_{N'+1}$ . By (15) and  $1-p_{N'+1} < 1-t$ , we have

$$(16) \quad \begin{aligned} g(t) &< \frac{1}{(1-p_{N'+1})^{m+\delta}} \\ &= \left\{ \frac{1}{(1-p_{N'+1})^{m+\delta}} - \frac{1}{(1-p_N)^{m+\delta}} \right\} \\ &\quad + \left\{ \frac{1}{(1-p_N)^{m+\delta}} - \frac{1}{(1-p_{N'+1})^{m+\delta}} \right\} + \cdots + \frac{1}{(1-p_0)^{m+\delta}} \\ &= h(1-p_{N'}) + h(1-p_{N'-1}) + \cdots + h(1-p_0) + \frac{1}{(1-p_0)^{m+\delta}} \\ &< \frac{1}{2}(N'+1)\log 2 + \frac{1}{(1-p_0)^{m+\delta}}. \end{aligned}$$

The last inequality follows from (12). On the other hand,  $g(p_0) + N' \log 2 \leq g(p_{N'}) \leq g(t)$  by (14). Combining this with (16) yields

$$(17) \quad \frac{1}{2}(N-1)\log 2 < \frac{1}{2}(N'-1)\log 2 < \frac{1}{(1-p_0)^{m+\delta}} - g(p_0).$$

Here we used  $N < N'$  for the first inequality. We can pick  $N(p_0)$  so large that (17) does not hold. This yields a contradiction. Hence we have  $m \geq 2$ .

We proceed to prove  $M \leq 2$ . Suppose  $M > 2$ . Pick  $\delta > 0$  such that  $M - \delta > 2$ . Let

$$h(x) = \frac{1}{(x-3x^3)^{M-\delta}} - \frac{1}{x^{M-\delta}}.$$

Note that  $\lim_{x \rightarrow 0} h(x) = \infty$ . Then by a similar argument as above, we lead a contradiction. Hence  $M \leq 2$ , which concludes  $m = M = 2$ .  $\square$

#### 4. Proof of Theorem 1.2

First, we estimate the probability  $P_p((1/9) \cdot 3^n \leq |C| \leq (9/2) \cdot 3^n)$ . Let  $M = \sup\{m : O \leftrightarrow a_m \text{ or } b_m\}$ . We define two conditional probabilities

$$U_n(p) = P_p(O \leftrightarrow a_n, O \not\leftrightarrow b_n \text{ in } F_n \mid M = n),$$

$$V_n(p) = P_p(\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n \mid M=n).$$

Clearly

$$(18) \quad 2U_n(p) + V_n(p) = 1,$$

and

$$(19) \quad V_n(p) = \frac{P_p(\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n, \mathbf{O} \leftrightarrow a_{n+1}, \mathbf{O} \leftrightarrow b_{n+1})}{P_p(M=n)}.$$

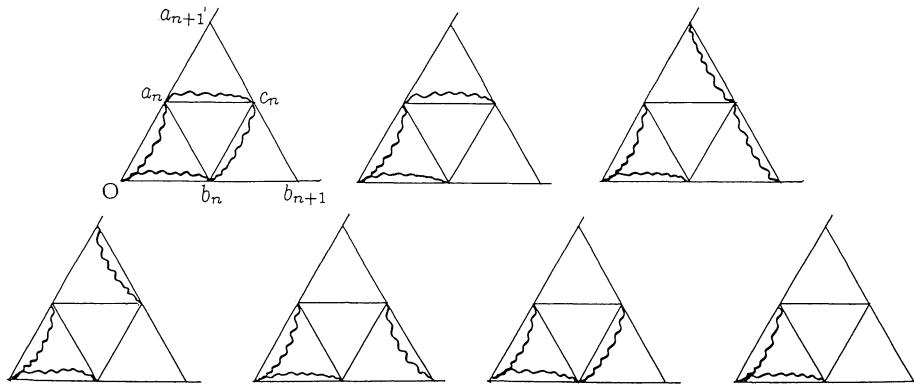


Fig. 4.1

We consider the event of the numerator of (19),  $\{\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n, \mathbf{O} \leftrightarrow a_{n+1}, \mathbf{O} \leftrightarrow b_{n+1}\}$ . We divide the case into seven parts as Fig. 4.1. Since the events in  $F_n, F'_n, F''_n$  are independent, we have

$$(20) \quad V_n(p) = \frac{\Theta_n(1 - 2\Phi_n - \Phi_n^2 + 4\Phi_n\Theta_n - 2\Theta_n^2)}{P_p(M=n)}.$$

Here we denoted  $\Phi_n = \Phi_n(p)$ ,  $\Theta_n = \Theta_n(p)$  briefly. Note that

$$\begin{aligned}
 (21) \quad & P_p(M=n) \\
 &= P_p(M \geq n) - P_p(M \geq n+1) \\
 &= 2\Phi_n - \Theta_n - (2\Phi_{n+1} - \Theta_{n+1}) \\
 &= 2\Phi_n - \Theta_n - 2\Phi_n^2 - 2\Phi_n^3 + 2\Phi_n\Theta_n^2 + 3\Phi_n^2\Theta_n - 2\Theta_n^3
 \end{aligned}$$

by (4). Hence by (18),

$$(22) \quad U_n(p) = \frac{1}{2}\{1 - V_n(p)\}$$

$$= \frac{(\Phi_n - \Theta_n)(1 - \Phi_n - \Phi_n^2 + \Phi_n \Theta_n)}{P_p(M=n)}.$$

Let

$$(23) \quad n_0 = n_0(p) = \sup \{n : \Theta_n(p) \geq \frac{2}{3}\}.$$

**Lemma 4.1.**  $V_n(p) \geq \frac{2}{9}$  if  $n < n_0$ .

Proof. From (18), it is enough to show

$$(24) \quad \frac{V_n(p)}{2U_n(p)} \geq \frac{2}{7}.$$

Let

$$\kappa(x, y) = \frac{y(1 - 2x - x^2 + 4xy - 2y^2)}{2(x - y)(1 - x - x^2 + xy)}.$$

By (20) and (22), (24) follows from the following:

$$(25) \quad \kappa(x, y) \geq \frac{2}{7} \quad \text{for} \quad \frac{2}{3} \leq x < 1, \quad \frac{1}{2}(3x - 1) < y < x.$$

The second condition in (25) comes from the fact that

$$(26) \quad 3\Phi_n(p) - 2\Theta_n(p) = \Psi_n(p) < 1.$$

Let  $y/x = t$ . Then the domain of (25) is  $2/3 \leq x < 1/(3-2t)$ ,  $2/3 \leq t < y < 1$ . And

$$\kappa(x, tx) = \frac{t}{2(1-t)} \left\{ 1 - \frac{x + (-3t + 2t^2)x^2}{1 - x - (1-t)x^2} \right\}.$$

Now let

$$\lambda(x) = \frac{x + (-3t + 2t^2)x^2}{1 - x - (1-t)x^2}.$$

From a direct calculation,

$$\lambda'(x) = \frac{(1 + 2t - 2t^2)x^2 + 2(-3t + 2t^2)x + 1}{\{1 - x - (1-t)x^2\}^2}.$$

We see that if  $2/3 \leq t < 1$ ,  $\lambda'(x) > 0$  for  $2/3 \leq x < 1/(3-2t)$ . Therefore

$$\kappa(x, tx) > \kappa\left(\frac{1}{3-2t}, \frac{t}{3-2t}\right) = \frac{t}{5-4t} \geq \frac{2}{7}. \quad \square$$

Next, we estimate the expectation of  $|C|$  on condition that  $M=n$  ( $n < n_0$ ).

**Lemma 4.2.**  $E_p(|C| \mid M=n) \geq \frac{2}{9} \cdot 3^n \quad \text{if } n < n_0.$

To prove the above Lemma, we use the following inequality:

**Lemma 4.3.** *For all  $a \in F_n$ ,*

$$(27) \quad P_p(O \leftrightarrow a \text{ in } F_n) \geq \Phi_n(p).$$

Proof. Besides (27), we introduce a similar inequality:

$$(28) \quad P_p(a \leftrightarrow a_n \text{ or } a \leftrightarrow b_n) \geq P_p(O \leftrightarrow a_n \text{ or } O \leftrightarrow b_n) \text{ for all } a \in F_n.$$

We prove (27) and (28) by induction at the same time. If  $n=0$ , clearly both of them hold. Suppose (27) and (28) for  $n=k$ .

We prove (27) for  $n=k+1$  at first. By symmetry, it is sufficient to prove the cases (i)  $a \in F_k$  and (ii)  $a \in F'_k$ .

(i) Suppose  $a \in F_k$ . By using (4), we see  $\Phi_k(p) \geq \Phi_{k+1}(p)$ . Indeed, suppose  $\Phi_k(p) \geq 1/3$ , then

$$(29) \quad \begin{aligned} \frac{\Phi_{k+1}}{\Phi_k} &= \Phi_k + \{\Phi_k\}^2 - \{\Theta_k\}^2 \\ &\leq \Phi_k + \{\Phi_k\}^2 - \left(\frac{3\Phi_k - 1}{2}\right)^2 \\ &\leq -\frac{5}{4}(1 - \Phi_k)^2 + 1 \\ &\leq 1. \end{aligned}$$

Here we used (26). Combining this with assumption, we see (27) for  $n=k+1$  in this case.

(ii) Suppose  $a \in F'_k$ . Let  $C_n^1, C_n^2, C_n^3$  be events given by

$$\begin{aligned} C_n^1 &= \{O \leftrightarrow a_n \text{ and } O \not\leftrightarrow c_n \text{ in } F_n \cup F_n''\}, \\ C_n^2 &= \{O \not\leftrightarrow a_n \text{ and } O \leftrightarrow c_n \text{ in } F_n \cup F_n''\}, \end{aligned}$$

$$C_n^3 = \{O \leftrightarrow a_n \text{ and } O \leftrightarrow c_n \text{ in } F_n \cup F_n''\}.$$

We see

$$\begin{aligned} & P_p(O \leftrightarrow a \text{ in } F_{k+1}) \\ &= P_p(C_k^1)P_p(a_k \leftrightarrow a \text{ in } F_k') + P_p(C_k^2)P_p(c_k \leftrightarrow a \text{ in } F_k') \\ & \quad + P_p(C_k^3)P_p(a_k \leftrightarrow a \text{ or } c_k \leftrightarrow a \text{ in } F_k') \\ &\geq (\Phi_k - \Theta_k) \cdot \Phi_k + (\Phi_k - \Theta_k) \Phi_k \cdot \Phi_k + \Phi_k \Theta_k \cdot 2(\Phi_k - \Theta_k) \\ &= \Phi_k^2 + \Phi_k^3 - \Phi_k \Theta_k^2 = \Phi_{k+1}. \end{aligned}$$

Here we used assumption for the inequality. We thus obtain (27) for  $n=k+1$ . We proceed to prove (28) for  $n=k+1$ .

(i) Suppose  $a \in F_k$ . Let  $D_n^1, D_n^2, \dots, D_n^5$  be events given by

$$D_n^1 = \{a_n \leftrightarrow a_{n+1} \text{ or } a_n \leftrightarrow b_{n+1} \text{ in } F_k' \cup F_k''\},$$

$$D_n^2 = \{b_n \leftrightarrow a_{n+1} \text{ or } b_n \leftrightarrow b_{n+1} \text{ in } F_k' \cup F_k''\},$$

$$D_n^3 = D_n^1 \cap (D_n^2)^c, \quad D_n^4 = (D_n^1)^c \cap D_n^2, \quad D_n^5 = D_n^1 \cap D_n^2. \quad \text{We see}$$

$$\begin{aligned} & P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1}) \\ &= P_p(D_k^3)P_p(a \leftrightarrow a_k \text{ in } F_k) + P_p(D_k^4)P_p(a \leftrightarrow b_k \text{ in } F_k) \\ & \quad + P_p(D_k^5)P_p(a \leftrightarrow a_k \text{ or } a \leftrightarrow b_k \text{ in } F_k) \\ &\geq P_p(D_k^3)P_p(O \leftrightarrow a_k \text{ in } F_k) + P_p(D_k^4)P_p(O \leftrightarrow b_k \text{ in } F_k) \\ & \quad + P_p(D_k^5)P_p(O \leftrightarrow a_k \text{ or } O \leftrightarrow b_k \text{ in } F_k) \\ &= P_p(O \leftrightarrow a_{k+1} \text{ or } O \leftrightarrow b_{k+1}) \end{aligned}$$

by assumption.

(ii) Suppose  $a \in F_k'$ . We see

$$\begin{aligned} (30) \quad & P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1}) \\ & \geq P_p(a \leftrightarrow a_{k+1} \text{ in } F_k') \\ & \quad + P_p(a \not\leftrightarrow a_{k+1} \text{ or } a \leftrightarrow c_k \text{ in } F_k')P_p(c_k \leftrightarrow b_{k+1} \text{ in } F_k''). \end{aligned}$$

Here we note that

$$\begin{aligned} & P_p(a \not\leftrightarrow a_{k+1} \text{ and } a \leftrightarrow c_k \text{ in } F_k') \\ &= P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow c_k \text{ in } F_k') - P_p(a \leftrightarrow a_{k+1} \text{ in } F_k') \\ &\geq (2\Phi_k - \Theta_k) - P_p(a \leftrightarrow a_{k+1} \text{ in } F_k') \end{aligned}$$

by assumption. Using this and (30), we have

$$P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1})$$

$$\begin{aligned}
&\geq P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k) \\
&\quad + \{(2\Phi_k - \Theta_k) - P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k)\} P_p(c_k \leftrightarrow b_{k+1} \text{ in } F''_k) \\
&= P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k)(1 - \Phi_k) + (2\Phi_k - \Theta_k)\Phi_k \\
&\geq \Phi_k(1 - \Phi_k) + 2\Phi_k^2 - \Phi_k\Theta_k = \Phi_k + \Phi_k^2 - \Phi_k\Theta_k.
\end{aligned}$$

Here we used assumption again. Now it is enough to show

$$(31) \quad \Phi_k + \Phi_k^2 - \Phi_k\Theta_k - P_p(O \leftrightarrow a_{k+1} \text{ or } O \leftrightarrow b_{k+1}) \geq 0.$$

The left-hand side of (31) equals

$$\begin{aligned}
&\Phi_k + \Phi_k^2 - \Phi_k\Theta_k - (2\Phi_{k+1} - \Theta_{k+1}) \\
&= (\Phi_k + \Phi_k^2 - \Phi_k\Theta_k) - 2(\Phi_k^2 + \Phi_k^3 - \Phi_k\Theta_k^2) + (3\Phi_k^2\Theta_k - 2\Theta_k^3) \\
&= \Phi_k(1 - \Theta_k)(1 - 3\Phi_k + 2\Theta_k) + 2(\Phi_k - \Theta_k)^2(1 - \Phi_k) \\
&\quad + 2\Theta_k(\Phi_k - \Theta_k)(1 - 2\Phi_k + \Theta_k).
\end{aligned}$$

By (26), we see all terms above are nonnegative. Hence the proof is completed.  $\square$

Proof of Lemma 4.2.

$$\begin{aligned}
E_p(|C|M=n) &= \sum_{a \in V} P_p(O \leftrightarrow a | M=n) \\
&\geq \sum_{a \in F_n} P_p(O \leftrightarrow a \text{ in } F_n | M=n) \\
&\geq \sum_{a \in F_n} \frac{P_p(O \leftrightarrow a, O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, M=n)}{P_p(M=n)}.
\end{aligned}$$

Let  $D_n^6 = (D_n^1)^c \cap (D_n^2)^c$ . Note that if  $M=n$  and  $O \leftrightarrow a_n, O \leftrightarrow b_n$ , then  $(D_n^6)^c$  occurs. For  $a \in F_n$ , we see

$$\begin{aligned}
&P_p(O \leftrightarrow a, O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, M=n) \\
&= P_p(O \leftrightarrow a, O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, D_n^6 \text{ occurs}) \\
&= P_p(O \leftrightarrow a, O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n) P_p(D_n^6) \\
&\geq P_p(O \leftrightarrow a \text{ in } F_n) P_p(O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n) P_p(D_n^6) \\
&= P_p(O \leftrightarrow a \text{ in } F_n) P_p(O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n, M=n).
\end{aligned}$$

Here we used FKG inequality for the forth line. Therefore

$$E_p(|C|M=n) \geq \sum_{a \in F_n} P_p(O \leftrightarrow a \text{ in } F_n) P_p(O \leftrightarrow a_n, O \leftrightarrow b_n \text{ in } F_n | M=n)$$

$$\geq \frac{2}{9} \sum_{a \in F_n} P_p(O \leftrightarrow a \text{ in } F_n)$$

by Lemma 4.1. Note that  $|\{a \in V : a \in F_n\}| = (3/2)(3^n + 1)$ . By virtue of Lemma 4.3, we see

$$\begin{aligned} E_p(|C| \mid M=n) &\geq \frac{2}{9} \cdot \frac{3}{2} \cdot 3^n \Phi_n(p) \\ &\geq \frac{2}{9} \cdot 3^n \quad \text{for } n < n_0. \end{aligned}$$

We used (23) and the fact that  $\Phi_n(p) \geq \Theta_n(p)$  for the last inequality.  $\square$

We proceed to the estimate of  $P_p((1/9) \cdot 3^n \leq |C| \leq (9/2) \cdot 3^n)$ .

**Lemma 4.4.**  $P_p\left(\frac{1}{9} \cdot 3^n \leq |C| \leq \frac{9}{2} \cdot 3^n\right) \geq \frac{2}{79} P_p(M=n) \quad \text{if } n < n_0.$

Proof. Note that  $|C| \leq (9/2) \cdot 3^n$  if  $M=n$ . Then we see the following.

$$\begin{aligned} &E_p(|C| \mid M=n) \\ &= E_p(|C| ; |C| \geq \frac{1}{9} \cdot 3^n \mid M=n) + E_p(|C| ; |C| < \frac{1}{9} \cdot 3^n \mid M=n) \\ &\leq \frac{9}{2} \cdot 3^n P_p(|C| \geq \frac{1}{9} \cdot 3^n \mid M=n) + \frac{1}{9} \cdot 3^n P_p(|C| < \frac{1}{9} \cdot 3^n \mid M=n). \end{aligned}$$

By Lemma 4.2, we have

$$P_p(|C| \geq \frac{1}{9} \cdot 3^n \mid M=n) \geq \frac{2}{79},$$

thus the proof is completed.  $\square$

**Lemma 4.5.**  $P_p(M=n) > \Phi_n(p)\{1 - \Phi_n(p)\}^2 \text{ if } n < n_0.$

Proof. Recall (21), that is

$$P_p(M=n) = 2\Phi_n - \Theta_n - 2\Phi_n^2 - 2\Phi_n^3 + 2\Phi_n\Theta_n^2 + 3\Phi_n^2\Theta_n - 2\Theta_n^3.$$

Let  $\pi(y) = 2x - y - 2x^2 - 2x^3 + 2xy^2 + 3x^2y - 2y^3$ . It is enough to show that  $\pi(y) > x(1-x)^2$  if  $2/3 \leq x < 1$ ,  $(3x-1)/2 < y < x$ . Note that

$$\pi'(y) = -6y^2 + 4xy + 3x^2 - 1,$$

and that

$$\pi'\left(\frac{3x-1}{2}\right) = \frac{1}{2}(1-x)(9x-5) > 0, \quad \pi'(x) = x^2 - 1 < 0.$$

Hence  $\pi(y) > \min\{\pi((3x-1)/2), \pi(x)\}$ ,  $\pi((3x-1)/2) = (1-x)^2(x+3)/4$  and  $\pi(x) = x(1-x)^2$ , so  $\pi((3x-1)/2) > \pi(x)$  for  $2/3 \leq x < 1$ . This completes the proof.  $\square$

**Proof of Theorem 1.2.** First, we estimate  $E_p|C|^k$  from below. By using Lemma 4.4 and 4.5, we see

$$\begin{aligned} E_p|C|^k &= \sum_{l=1}^{\infty} l^k P_p(|C|=l) \\ &\geq \sum_{n=4,8,12,\dots} \left(\frac{1}{9} \cdot 3^n\right)^k P_p\left(\frac{1}{9} \cdot 3^n \leq |C| \leq \frac{9}{2} \cdot 3^n\right) \\ &\geq \frac{1}{9^k} \cdot \frac{2}{79} \sum_{\substack{m \in \mathbb{N} \\ 4m < n_0}} 3^{4km} \Phi_{4m}(p) \{1 - \Phi_{4m}(p)\}^2. \end{aligned}$$

Let  $p$  be sufficiently large. Note that the function  $\iota(x) = x(1-x)^2$  is decreasing in  $2/3 \leq x < 1$ , and  $\Phi_{4m}(p) \leq e^{-2^{4m}/\xi(p)}$  by (7). We can see

$$\begin{aligned} &\sum_{\substack{m \in \mathbb{N} \\ 4m < n_0}} 3^{4km} \Phi_{4m}(p) \{1 - \Phi_{4m}(p)\}^2 \\ &\geq \sum_{\substack{m \in \mathbb{N} \\ 4m < n_0}} 3^{4km} e^{-2^{4m}/\xi(p)} (1 - e^{-2^{4m}/\xi(p)})^2 \\ &\geq \int_1^{\frac{n_0}{4}-1} 3^{4kx} e^{-2^{4x}/\xi(p)} (1 - e^{-2^{4x}/\xi(p)})^2 dx \\ &= \frac{\{\xi(p)\}^{Dk}}{4 \log 2} \int_{2^{4/\xi(p)}}^{2^{n_0-4}/\xi(p)} y^{Dk-1} e^{-y} (1 - e^{-y})^2 dy. \end{aligned}$$

Here we set  $y = 2^x/\xi(p)$  in the last line. Note that  $\Theta_{n_0+1}(p) < 2/3$ , hence  $\Phi_{n_0+1}(p) < (1 + 2\Theta_{n_0+1}(p))/3 < 7/9$  by (24). From (29), if  $\Phi_k(p) < 7/9$ , then  $\Phi_{k+1}(p)/\Phi_k(p) < 76/81$ . We see

$$\Phi_{n_0+1}(p) < \left(\frac{76}{81}\right)^{11} \cdot \frac{7}{9} < \frac{1}{2} \cdot \frac{7}{9}.$$

Combining this with (7), we have

$$\frac{1}{2}e^{-2^{n_0+12}/\xi(p)} \leq \Phi_{n_0+12}(p) < \frac{1}{2} \cdot \frac{7}{9}.$$

Hence  $2^{n_0-4}/\xi(p) > 2^{-16}\log(9/7)$ . Since  $\xi(p) \rightarrow \infty$  as  $p \rightarrow 1$ ,  $E_p|C|^k > K_1\{\xi(p)\}^{Dk}$  holds if we take

$$K_1(k) = \int_{2^{-17}\log(9/7)}^{2^{-16}\log(9/7)} y^{Dk-1} e^{-y} (1-e^{-y})^2 dy > 0.$$

Now we proceed to estimate from above. Note that  $P_p(M \geq n) \leq 2\Phi_n(p) \leq 2e^{-2^n/\xi(p)}$ , and we can see easily  $P_p((3/2) \cdot 3^n < |C| \leq (3/2) \cdot 3^{n+1}) \leq P_p(M \geq n) \leq 2e^{-2^n/\xi(p)}$ . Hence

$$\begin{aligned} E_p|C|^k &= \sum_{l=1}^{\infty} l^k P_p(|C|=l) \\ &\leq 1 + \sum_{n=0}^{\infty} \left(\frac{3}{2} \cdot 3^{n+1}\right)^k P_p\left(\frac{3}{2} \cdot 3^n < |C| \leq \frac{3}{2} \cdot 3^{n+1}\right) \\ &\leq 1 + 2 \cdot \left(\frac{9}{2}\right)^k \sum_{n=0}^{\infty} 3^{kn} e^{-2^n/\xi(p)}. \end{aligned}$$

Now

$$\begin{aligned} \int_0^{\infty} 3^{kx} e^{-2^x/\xi(p)} dx &= \frac{\{\xi(p)\}^{Dk}}{\log 2} \int_{\xi(p)^{-1}}^{\infty} y^{Dk-1} e^{-y} dy \\ &\leq \frac{\Gamma(Dk)}{\log 2} \cdot \{\xi(p)\}^{Dk}. \end{aligned}$$

So we can take  $K_2(k) < \infty$  such that  $E_p|C|^k < K_2\{\xi(p)\}^{Dk}$ .  $\square$

## 5. Site percolation on the pre-Sierpinski gasket

We define the Bernoulli site percolation on the pre-Sierpinski gasket; each vertices in  $V$  are open with probability  $p$  and closed with  $1-p$  independently. Let  $\tilde{P}_p$  denote its distribution. We write  $x \leftrightarrow y$  if there exists a sequence of open vertices  $x = x_0, x_1, \dots, x_{n-1}, x_n = y$  such that there is a bond in  $E$  which connects  $x_j$  with  $x_{j+1}$  for  $0 \leq j \leq n-1$ . We define another notations in the same manner as before. We introduce connectivity functions;

$$\begin{aligned} \tilde{\Phi}_n(p) &= \tilde{P}_p(O \leftrightarrow a_n \text{ in } \Delta O a_n b_n), \\ \tilde{\Theta}_n(p) &= \tilde{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } \Delta O a_n b_n). \end{aligned}$$

We see  $\tilde{\Phi}_0(p)=p^2$  and  $\tilde{\Theta}_0(p)=p^3$  by definition.

**Proposition 5.1.** *For each  $n \geq 0$  and  $0 \leq p \leq 1$ ,*

$$(32) \quad \tilde{\Phi}_{n+1}(p) = p^{-1}\{\tilde{\Phi}_n(p)\}^2 + p^{-2}\{\tilde{\Phi}_n(p)\}^3 - p^{-3}\tilde{\Phi}_n(p)\{\tilde{\Theta}_n(p)\}^2,$$

$$(33) \quad \tilde{\Theta}_{n+1}(p) = 3p^{-2}\{\tilde{\Phi}_n(p)\}^2\tilde{\Theta}_n(p) - 2p^{-3}\{\tilde{\Theta}_n(p)\}^3.$$

Proof. We prove (32). Let  $\tilde{A}_n^1$  and  $\tilde{A}_n^2$  be events given by

$$\tilde{A}_n^1 = \{O \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F'_n\},$$

$$\tilde{A}_n^2 = \{O \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow c_n \text{ in } F''_n\} \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F'_n\}.$$

Then we have

$$(34) \quad \tilde{\Theta}_{n+1}(p) = \tilde{P}_p(\tilde{A}_n^1) + \tilde{P}_p(\tilde{A}_n^2) - \tilde{P}_p(\tilde{A}_n^1 \cap \tilde{A}_n^2).$$

Remark that  $F_n \cap F'_n = \{a_n\}$ . So we see  $\tilde{P}_p(\tilde{A}_n^1) = p^{-1}\{\tilde{\Phi}_n(p)\}^2$ . Similarly, we have  $\tilde{P}_p(\tilde{A}_n^2) = p^{-2}\{\tilde{\Phi}_n(p)\}^3$ ,  $\tilde{P}_p(\tilde{A}_n^1 \cap \tilde{A}_n^2) = p^{-3}\{\tilde{\Theta}_n(p)\}^2\tilde{\Phi}_n(p)$ . Thus (32) follows from (34) immediately. (33) is proved in the same way.  $\square$

Let  $\hat{\Phi}_n(p) = p^{-1}\tilde{\Phi}_n(p)$  and  $\hat{\Theta}_n(p) = p^{-\frac{3}{2}}\tilde{\Theta}_n(p)$ . Then we have the same recursions as (4), (5):

$$(35) \quad \hat{\Phi}_{n+1}(p) = \{\hat{\Phi}_n(p)\}^2 + \{\hat{\Phi}_n(p)\}^3 - \hat{\Phi}_n(p)\{\hat{\Theta}_n(p)\}^2,$$

$$(36) \quad \hat{\Theta}_{n+1}(p) = 3\{\hat{\Phi}_n(p)\}^2\hat{\Theta}_n(p) - 2\{\hat{\Theta}_n(p)\}^3.$$

Hence we see that there exists  $\hat{\xi}(p) > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{\hat{\Phi}_n(p)}{\exp\{-2^n/\hat{\xi}(p)\}} = 1, \quad \text{that is} \quad \lim_{n \rightarrow \infty} \frac{\tilde{P}_p(O \leftrightarrow a_n)}{p \exp\{-2^n/\hat{\xi}(p)\}} = 1.$$

**Lemma 5.2.** *Let  $\sqrt{\tilde{p}} = \sqrt{p} + 6(1 - \sqrt{p})^2$ . Then there exists  $\varepsilon > 0$  such that*

$$2 \leq \frac{\hat{\xi}(\tilde{p})}{\hat{\xi}(p)} \leq 4 \quad \text{for} \quad 1 - \varepsilon < p < 1.$$

Proof. We use the same method as in Section 3 again. Let

$$(37) \quad \hat{\Psi}_n(p) = 3\hat{\Phi}_n(p) - 2\hat{\Theta}_n(p).$$

To apply (9), first we prove  $(\hat{\Theta}_n(p), \hat{\Psi}_n(p)) \in D$ . (Recall  $D = \{(x, y) : 0 < x \leq y < 1\}$ .) Since  $\hat{\Psi}_n(p) = \hat{\Theta}_n(p) + 3\{\hat{\Phi}_n(p) - \hat{\Theta}_n(p)\}$ , it is enough to prove  $\hat{\Phi}_n(p) \geq \hat{\Theta}_n(p)$ . Now

$$\hat{\Phi}_n(p) = p^{-1} \times \tilde{P}_p(O \leftrightarrow a_n \text{ in } F_n)$$

$$\begin{aligned}
&= \tilde{P}_p(O \leftrightarrow a_n \text{ in } F_n \mid a_n \text{ is open}) \\
&= \tilde{P}_p(O \leftrightarrow a_n \text{ in } F_n \mid a_n, b_n \text{ are open}), \\
\hat{\Theta}_n(p) &= p^{-\frac{3}{2}} \times \tilde{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n) \\
&\leq p^{-2} \times \tilde{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n) \\
&= \tilde{P}_p(O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n \mid a_n, b_n \text{ are open}).
\end{aligned}$$

Hence we have  $\hat{\Phi}_n(p) \geq \hat{\Theta}_n(p)$ , which implies  $(\hat{\Theta}_n(p), \hat{\Psi}_n(p)) \in D$ .

A direct calculation from (35) and (36) shows

$$\begin{aligned}
\hat{\Theta}_2(\tilde{p}) - \hat{\Theta}_1(p) &= 6(1 - \sqrt{p})^2 + 204(1 - \sqrt{p})^3 + \dots, \\
\hat{\Psi}_2(\tilde{p}) - \hat{\Psi}_1(p) &= 12(1 - \sqrt{p})^3 + \dots, \\
\hat{\Theta}_3(\tilde{p}) - \hat{\Theta}_1(p) &= -6(1 - \sqrt{p})^2 + 204(1 - \sqrt{p})^3 + \dots, \\
\hat{\Psi}_3(\tilde{p}) - \hat{\Psi}_1(p) &= -12(1 - \sqrt{p})^3 + \dots.
\end{aligned}$$

We can take  $\varepsilon > 0$  such that

$$\hat{\Theta}_3(\tilde{p}) < \hat{\Theta}_1(p) < \hat{\Theta}_2(\tilde{p}), \quad \hat{\Psi}_3(\tilde{p}) < \hat{\Psi}_1(p) < \hat{\Psi}_2(\tilde{p})$$

for  $1 - \varepsilon < p < 1$ .

Now we apply (9). We have for  $n \geq 1$  and  $1 - \varepsilon < p < 1$ ,

$$\hat{\Theta}_{n+2}(\tilde{p}) < \hat{\Theta}_n(p) < \hat{\Theta}_{n+1}(\tilde{p}), \quad \text{and} \quad \hat{\Psi}_{n+2}(\tilde{p}) < \hat{\Psi}_n(p) < \hat{\Psi}_{n+1}(\tilde{p}).$$

We see  $\hat{\Phi}_{n+2}(\tilde{p}) < \hat{\Phi}_n(p) < \hat{\Phi}_{n+1}(\tilde{p})$  by (37), so we have the conclusion.  $\square$

Proof of Theorem 1.3. Note that  $\tilde{p} = \{\sqrt{p} + 6(1 - \sqrt{p})^2\}^2 = p + 3(1 - p)^2 + o((1 - p)^2)$  as  $p \rightarrow 1$ . We have Theorem 1.3 in the same way as in Section 3.

#### References

- [1] H. Kesten: *Percolation Theory for Mathematicians*, Birkhauser, Boston, 1982.
- [2] G. Grimmett: *Percolation*, Springer-Verlag, New York, 1989.
- [3] T. Hara and G. Slade: *Mean field critical behavior for percolation in high dimensions*, Comm. Math. Phys. **128** (1990), 333–391.
- [4] H. Kesten: *Scaling relations for 2D-percolation*, Comm. Math. Phys. **109** (1987), 109–156.
- [5] Y. Gefen, A. Aharony, Y. Shapir and B.B. Mandelbrot: *Phase transitions on fractals II: Sierpinski gaskets*, J. Phys. A **17** (1984), 435–444.
- [6] K. Hattori and T. Hattori: *Self-avoiding process on the Sierpinski gasket*, Probab. Theory Related Fields **88** (1991), 405–428.
- [7] K. Hattori, T. Hattori and S. Kusuoka: *Self-avoiding paths on the three dimensional Sierpinski gasket*, Publ. Res. Inst. Math. Sci. **29** (1993), 455–509.

[8] K. Hattori, T. Hattori and S. Kusuoka: *Self-avoiding paths on the pre-Sierpinski gasket*, Probab. Theory Related Fields **84** (1990), 1–26.

Department of Mathematics  
Nara Women's University  
Nara 630, Japan