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HOMOGENEOUS COMPLETE INTERSECTION HODGE ALGEBRAS ON SIMPLICIAL COMPLEXES

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Introduction

Since De Concini-Eisenbud-Procesi [1] defined Hodge algebra, two special classes have been studied, one of which is an ordinal Hodge algebra and the other is a square-free Hodge algebra. An ordinal Hodge algebra (=algebra with straightening laws, ASL, for short) have been investigated in detail and we know that an ASL reflects strongly a nature of a poset.

On the other hand, let A be a square-free Hodge algebra. By [1], we can associate to A a unique simplicial complex Δ. Then A should accordingly reflect a nature of Δ. We call A a Hodge algebra on the simplicial complex Δ. The purpose of the present article is to classify the simplicial complex on which there exists a homogeneous complete intersection Hodge algebra of dimension ≤ 3. We often employ the arguments in [5].

In §1, we recall the definition of Hodge algebra and elementary definitions in topology. In §2, we give a classification of simplicial complexes Δ when there exists a homogeneous Hodge K-algebra on Δ which is a complete intersection. Its proof is given in §3.

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1. Preliminaries

Let Δ be a simplicial complex and let H be the set of vertices of Δ. We call an element of $\boldsymbol{N}^H$ a *monomial* on H, where $\boldsymbol{N}$ is the nonnegative integers and $\boldsymbol{N}^H$ is the set of $\boldsymbol{N}$-valued functions on H. Given two monomials L, M on H, we can define a product LM by assigning $LM(x)=L(x)+M(x)$ to $x \in H$. The *support* of a monomial M is the subset Supp $M=\{x \in H;\ M(x) \neq 0\}$. We define Σ_Δ by

$$\Sigma_\Delta=\{M \in \boldsymbol{N}^H;\ \text{Supp } M \text{ does not belong to } \Delta\},$$

which is an order ideal, i.e., $L \in \boldsymbol{N}^H$, $M \in \Sigma_\Delta \Rightarrow LM \in \Sigma_\Delta$. A monomial M is *standard* if M does not belong to Σ_Δ, and M is a *generator* of Σ_Δ if $M \in \Sigma_\Delta$ and M is not divisible by any other elements of Σ_Δ.

Let K be a field and let R be a K-algebra. we assume that we are given an injection i: $H \to R$. Once the injection i is fixed, we identify H with the subset $i(H)$ of R. To each monomial M on H, we can associate an element

$$i(M) = \prod_{x \in H} x^{M(x)},$$

where we understand $a^0 = 1$ for any $a \in H$. We identify $i(M)$ with M.

After [1], we introduce the following:

DEFINITION. A K-algebra R is a Hodge K-algebra on a simplicial complex Δ if the following three conditions are satisfied:
(H0) The vertex set H of Δ is a partially ordered set (a poset, for short) with respect to a partial order "$\leq$",
(H1) R admits as a K-basis the set of all standard monomials with respect to Σ_Δ,
(H2) If L is a generator of Σ_Δ, and

$$(*) \qquad L = \sum_i a_i M_i; \ a_i \in K \setminus \{0\}$$

is the unique expression for L as a linear combination of distinct standard monomials guaranteed by (H1), then for each $x \in \operatorname{Supp} L$ and each M_i, there exists $y \in \operatorname{Supp} M_i$ such that $y < x$.

The relation (*) are called the *straightening relations* for R. If the right hand-sides of all straightening relations are zero, then we call R a *Stanley-Reisner ring* of Δ and denote it by $K[\Delta]$. We say that R is *graded* if R has a graded ring structure $R = \bigoplus_{n \geq 0} R_n$ such that $R_0 = K$ and any element of H is homogeneous of positive degree. We call R *homogeneous* if R is graded and $H \subset R_1$.

A K-algebra R is a *quasi-Hodge K-algebra* on a simplicial complex Δ if R is generated by the vertex set H of Δ satisfying (H0) and if every generator of Σ_Δ is expressed as a linear comination of standard monomials which satisfies (H2).

For a simplicial complex Δ, we denote by $\Delta^r (r \geq 0)$ the r-skeleton of Δ,

$$\Delta^r = \{F \in \Delta; \ \dim F \leq r\}.$$

If $d=\dim\Delta=\max_{F\in\Delta}\dim F$, $\dot{\Delta}$ denotes Δ^{d-1}. We denote by $\Delta(n)$ the simplicial complex consisting of an n-simplex and all its faces. We say that Δ is *pure* if all maximal faces have dimension equal to $\dim\Delta$. For two simplicial complexes Δ_1 and Δ_2, their join $\Delta_1*\Delta_2$ is

$$\Delta_1*\Delta_2=\{F\cup G;\ F\in\Delta_1,\ G\in\Delta_2\}.$$

Next we recall some concept from ring theory. Let $R=\bigoplus_{n\geq 0}R_n$ be a noetherian graded ring, where $R_0=K$ and K is a field. The *Hilbert series* of R is a formal power series $H(R,t)=\sum_{n\geq 0}(\dim_K R_n)t^n$, where $\dim_K R_n$ is the dimension of R_n as a K-vector space.

2. Homogeneous Hodge algebras on simplicial complexes which are complete intersections

In this section we consider a simplicial complex on which there exists a homogeneous Hodge algebra which is a complete intersection, and classify such complexes.
We assume hereafter that K is a field, unless otherwise specified. We need the following Lemmas $1\sim 3$ in the subsequent arguements.

Lemma 1 (cf. [5, Lemma 5]). *Let R be a homogeneous Hodge K-algebra on a simplicial complex Δ. Then we have*

$$H(R,t)=\sum_{i=-1}^{d} f_i t^{i+1}(1-t)^{-i-1},$$

where $\dim\Delta=d$, *f_i denotes the number of i-face in Δ $(i=0,1,\cdots,d)$ and we put $f_{-1}=1$. In particular* $\dim R=\mathrm{d}+1$.

Lemma 2 (cf. [6. (1.4)]). *Let Δ be a simplicial complex and let R be a K-algebra. Then the following conditions are equivalent*:

(1) *R is a homogeneous Hodge K-algebra on Δ.*
(2) *R is a homogeneous quasi-Hodge K-algebra on Δ such that $H(R,t)=H(K[\Delta],t)$.*

The above lemmas are actually found in the references in the case of ASL's, but we can easily generalize the arguments to prove the above results in the present situation.

Lemma 3 [10, Cor. 3.4]. *Let $H(t)$ be a power series with integral* *coefficients. Then the following conditions are equivalent*:

(1) *$H(t)$ is the Hilbert series of a noetherian graded ring $R=\bigoplus_{n\geq 0}R_n$ such that* (*a*) $R_0=K$, (b) *R is generated by R_1 as a K-algebra*, (c) dim $R=d$, *and* (d) *R is a complete intersection.*

(2) *$H(t)$ has the form*

$$H(t)=(\textstyle\prod_{i=1}^{s}(1+t+\cdots+t^{g_i}))/(1-t)^d,$$

for some $s\geq 0$ and $g_i>0$ ($i=1,\cdots,s$).

The following is the classification of the 1-dimensional homogeneous complete intersection Hodge K-algebras on simplicial complexes.

Proposition 4. *Let R be a* 1*-dimensional homogeneous complete intersection Hodge K-algebra on a simplicial complex Δ. Then the pair of Δ and R is one of the following*:

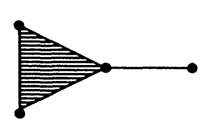
(1) $\Delta(0)$ (*one point*) *and* $R=K[\Delta(0)]=K[X]$.

(2) $\dot{\Delta}(1)$ (*two points*) *and* $R=K[\dot{\Delta}(1)]=K[X,Y]/(XY)$.

Proof. By Lemma 1 we have dim $\Delta=0$ and $H(R,t)=\{1+(f_0-1)t\}/(1-t)$, where f_0 is the number of 0-faces in Δ. Since R is complete intersection, we have $f_0=1$ or 2 by Lemma 3. If $f_0=1$, then $\Delta=\Delta(0)$. If $f_0=2$, then $\Delta=\dot{\Delta}(1)$. In both cases, we can easily show that $K[\Delta]$ is a unique Hodge K-algebra on Δ.

Next we give the classification of the 2-dimensional case.

Proposition 5. *Let Δ be a simplicial complex. If there exists a* 2*-dimensional homogeneous complete intersection Hodge K-algebra on Δ, then Δ is one of the following*:

(1) $\Delta(1)$. (2) $\dot{\Delta}(1)*\Delta(0)$. (3) $\dot{\Delta}(2)$. (4) $\dot{\Delta}(1)*\dot{\Delta}(1)$. (5) 

where a shadowed triangle is not a 2*-face of Δ.*

Conversely, if Δ is one of the above simplicial complexes, there exists a homogeneous complete intersection Hodge K-algebra on Δ.

Proof. Suppose there exists a 2-dimensional homogeneous complete intersection Hodge K-algebra on Δ. By Lemma 1 we have dim $\Delta=1$ and

$$H(R,t)=1+(f_0t)/(1\text{-}t)+(f_1t^2)/(1-t)^2,$$

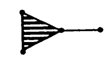
where f_i is the number of the i-faces in Δ. Since R is a complete intersection, we have

$$H(R,t)=\left(\prod_{i=1}^{r}(1+t+\cdots+t^{g_i})\right)/(1-t)^d,$$

where $r\geq 0$, $g_i>0$ $(i=1,\cdots,r)$. Equating these two expressions of $H(R,t)$, we have

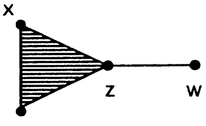
$$1+f_0t/(1-t)+f_1t^2/(1-t)^2=\left(\prod_{i=1}^{r}(1+t+\cdots+t^{g_i})\right)/(1-t)^d.$$

Hence $g_1+g_2+\cdots+g_r\leq 2$ and $(1-t)^2H(R,t)$ is classified into one of the following cases. The numbers, f_0 and f_1 are uniquely determined in each case, and we can list up the corrresponding simplicial complexes:

	$(1-t)^2H(R,t)$	f_0	f_1	The simplicial complex
(1)	1	2	1	$\Delta(1)$
(2)	$1+t$	3	2	$\dot{\Delta}(1)*\Delta(0)$
(3)	$1+t+t^2$	3	3	$\Delta(2)$
(4)	$(1+t)^2$	4	4	$\dot{\Delta}(1)*\dot{\Delta}(1)$
(5)	$(1+t)^2$	4	4	

Conversely if Δ is one of the above five, we show that there exists a homogeneous complete intersection Hodge K-algebra R on Δ. Namely if Δ is one of (1), (2), (3) or (4), $K[\Delta]$ is a such example, and in the case (5), Example 6 below gives an example.

EXAMPLE 6. Let Δ be the simplicial complex



and set $R=K[x,y,z,w]/(xw-xz,yw)$. Then R is a 2-dimensional homogeneous complete intersection Hodge K-algebra on Δ, where we define a partial order by $z<x$, $z<w$, and the straightening relations in R are $xw=xz$, $yw=0$, and $xyz=0$.

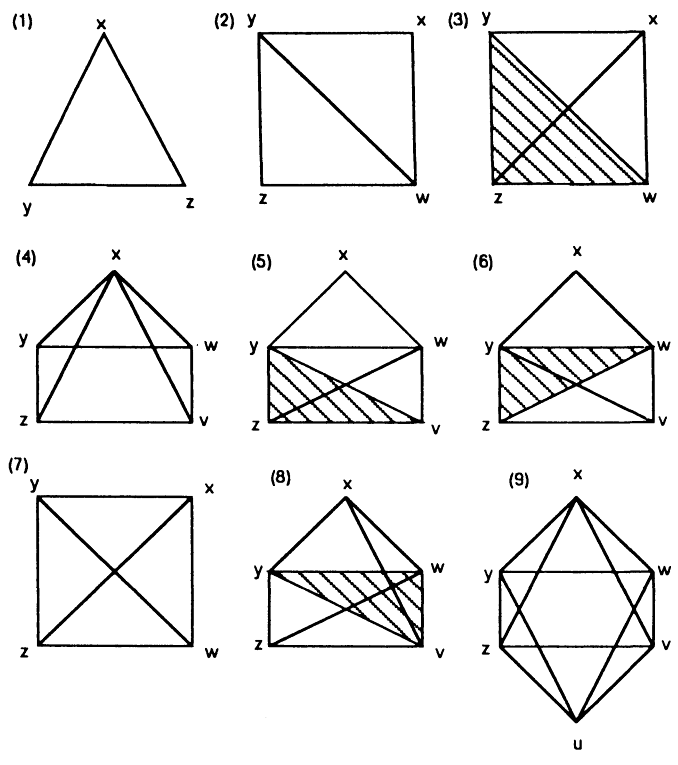
Proof. Let $v=w-z$. Then we have $R=K[x,y,v,w]/(xv,yw)$, which is a complete intersection. The generators of Σ_Δ are xw, yw and xyz. Since we have $xyz=-y(xw-xz)+x(yw)$ in $K[x,y,z,w]$, we have $xyz=0$, $xw=xz$, $yw=0$ in R. They satisfy (H2) and R is a quasi-Hodge K-algebra. Since

$$H(R,t)=(1-t^2)^2/(1-t)^4=(1+t)^2/(1-t)^2=\mathrm{H}(K[\Delta],t),$$

R is a Hodge K-algebra.

Now we state the classification of the 3-dimensional case, which is the main result in this paper.

Theorem 7. *Let Δ be a simplicial complex. If there exists a 3-dimensional homogeneous complete intersection Hodge K-algebra on Δ, then Δ is exhausted by one of the following 25 cases, where shadowed triangles are not faces of Δ.*



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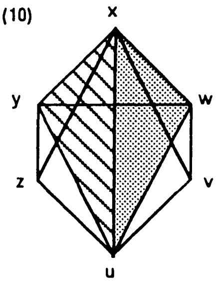

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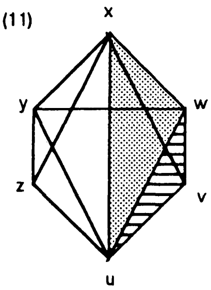

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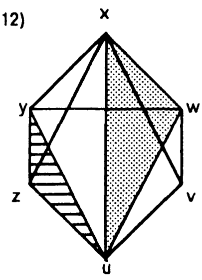

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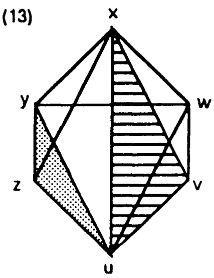

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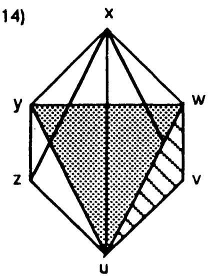

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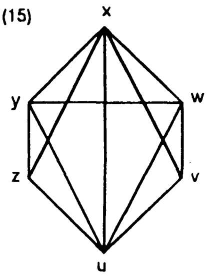
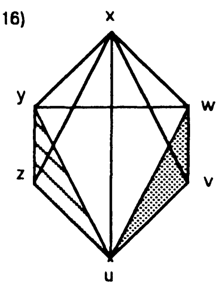


where uwy and vwx are not 2-faces.

(16)


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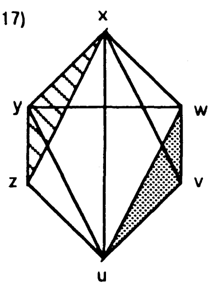

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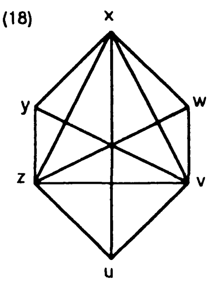

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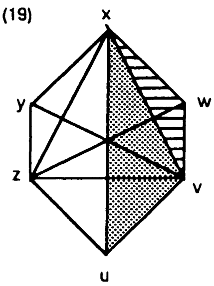

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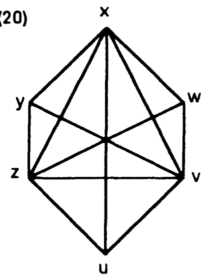

where vwz and uvx are not 2-faces

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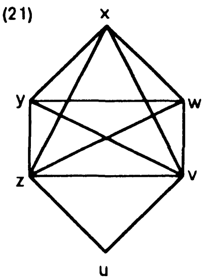
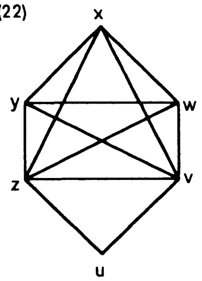


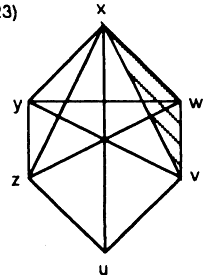
where vxy, vwx, and vwy are not 2-faces and $\#(K) \neq 2$

(22)



where xyz,vwx, and vwy are not 2-faces

(23)



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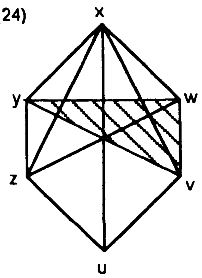

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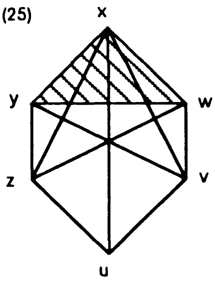


Conversely, if Δ is one of the above simplicial complexes, there exists a homogeneous complete intersection Hodge K-algebra on Δ.

EXAMPLE 8. For the above simplicial complexes, for example, we have homogeneous complete intersection Hodge K-algebras $R=A/I$ on Δ, where A are the polynomial rings $K[x;\ x$ is a vertex of $\Delta]$ and I are the following ideals:
(1) (0), (2) (xz), (3) (wyz), (4) $(vy,\ wz)$, (5) $(vx-vy,\ xz)$, (6) $(vx-wy,\ xz)$ (7) $(wxyz)$, (8) $(vwy,\ xz)$, (9) $(ux,\ vy,\ wz)$, (10) $(vy,\ vz-ux,\ wz)$, (11) $(vy+wx,\ vz-xz,\ wz+uw+yz)$, (12) $(vy-uy,\ vz-ux,\ wz)$, (13) $(vy-wy,\ vz,\ wz-ux-uz)$, (14) $(vy-wy,\ vz,\ wz-uw-ux)$, (15) $(vy,\ vz-uw,\ wz-wx-xz)$, (16) $(vy,\ vz-uv-uz,\ wz-wx)$, (17) $(vy-xy,\ vz,\ wz-uw)$, (18) $(uw-wz,\ uy,\ wy-vx)$, (19) $(uw-uv-vw,\ uy-ux-xz,\ wy)$, (20) $(uw-ux-wz+xz,\ uy-uv-yz,\ wy)$,(21) $(uw-avw,\ ux-vx,\ uy-yz)$, where $a\neq 0,1$ in K, (22) $(uw-vw,\ ux,\ uy-yz)$, (23) $(uw,\ uy-vx,\ vz-yz)$, (24) $(uw-vw,\ uy-xy,\ uz)$, (25) $(uw-wx,\ uy,\ vz)$.

We obtain the following corollary by checking one by one.

Corollary 9. *Let Δ be a simplicial complex on which there exists a 3-dimensional homogeneous complete intersection Hodge K-algebra. Then Δ is pure and the homotopy type of the geometric realization $|\Delta|$ of Δ is equal* *to that of the 2-dimensional sphere or 2-dimensional disk.*

We need the following lemma to obtain Corollary 11.

Lemma 10 [7, p180]. *Let Δ be a 2-dimensional simplical complex. Then the Stanley-Reisner ring $K[\Delta]$ of Δ is Cohen-Macaulay if and only if the following three conditions are satisfied:*

(1) *Δ is pure.*
(2) *$\tilde{H}_i(\Delta,K)=0$, $i=0,1$, where $\tilde{H}_i(\Delta,K)$ is the i-th reduced homology group of Δ.*
(3) *Every point of $|\Delta|$ has an arbitrarily small neighborhood which is connected even if any finite subset is removed.*

By Corollary 9 and Lemma 10, we obtain the following:

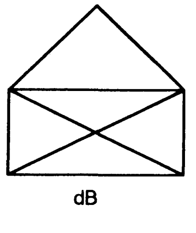
Corollary 11. *Let Δ be a simplicial complex on which there exists a 3-dimensional homogeneous complete intersection Hodge K-algebra. Then the Stanley-Reisner ring $K[\Delta]$ of Δ is Cohen-Macaulay.*

3. Proof of Theorem 7

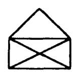
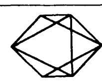
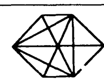
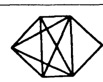
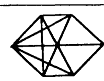
As in the proof of Proposition 5, we obtain the following table, where for the cases (dB) and (gB)$\sim$(gE) we give only 1-skeletons of the simplicial complexes.

We shall investigate the above cases about whether or not there exists a homogeneous complete intersection Hodge K-algebra on Δ. We can easily show that if Δ is the one of the case (a), (b), (c), (dA), (e), (f), and (gA), then $R=K[\Delta]$ is a complete intersection. So we have only to observe the remaining cases, i.e., (dB) and (gB)$\sim$(gE). Since the arguments are almost the same for these cases, we consider only the case (dB).

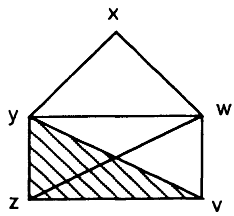
The simplicial complex Δ which we treat now has the following 1-skeleton and has $f_2=4$;



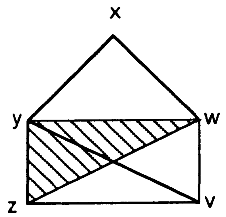
dB

	$(1-t)^3H(R,t)$	f_0	f_1	f_2	The simplicial complex
(a)	1	3	3	1	$\Delta(2)$
(b)	$1+t$	4	5	2	$\Delta(1)*\Delta(0)$
(c)	$1+t+t^2$	4	6	3	$\dot{\Delta}(2)*\Delta(0)$
(dA)	$(1+t)^2$	5	8	4	$\dot{\Delta}(1)*\dot{\Delta}(1)*\Delta(0)$
(dB)	$(1+t)^2$	5	8	4	
(e)	$1+t+t^2+t^3$	4	6	4	$\dot{\Delta}(3)$
(f)	$(1+t+t^2)(1+t)$	5	9	6	$\dot{\Delta}(2)*\dot{\Delta}(1)$
(gA)	$(1+t)^3$	6	12	8	$\dot{\Delta}(1)*\dot{\Delta}(1)*\dot{\Delta}(1)$
(gB)	$(1+t)^3$	6	12	8	
(gC)	$(1+t)^3$	6	12	8	
(gD)	$(1+t)^3$	6	12	8	
(gE)	$(1+t)^3$	6	12	8	

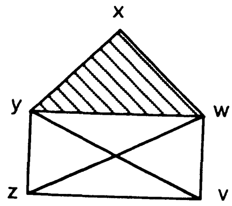
So, Δ is one of the following three upto isomorphism:



dB1



dB2



dB3

Lemma 12. *There exists a homogeneous complete intersection Hodge K-algebra on Δ if Δ is ($dB1$) or ($dB2$), but no such K-algebra structures exist if Δ is ($dB3$).*

Proof. First we consider the case that Δ is (dB3). Suppose there exists a homogeneous complete intersection Hodge K-algebra R on Δ. Then R is of the form $R=K[v,w,x,y,z]/I$, where $I=(vx-l_1,\ vz-l_2,\ wxy-l_3,\ vwyz-l_4)$. Since R is a complete intersection and dim $R=3$, the ideal I has height 2 and is generated by two elements. So we must have $I=(vx-l_1,\ vz-l_2)$. Since we have only to consider total orderings on v, w, x, y, z and since the arguments are similar, we assume y is minimal. Then $l_3=0$ by (H2). So we can write

$$wxy=(a_1v+b_1w+c_1x+d_1y+e_1z)(vx-l_1)+(a_2v+b_2w+c_2x+d_2y+e_2z)(xz-l_2)$$

where

$$\begin{aligned} l_i= & -(f_iy^2+g_iyw+h_iyz+j_iyv+k_iyx+m_iw^2 \\ & +n_iwz+p_iwv+q_iwx+r_ivz+s_iz^2+t_iv^2) \\ & (i=1,2,\ t_1=s_2=0). \end{aligned}$$

The comparison of the coefficients of the monomials on both-hand sides yields the relations as shown below:

$$\begin{aligned} vx^2 &: 0=c_1, \\ x^2z &: 0=c_2, \\ v^2x &: 0=a_1, \\ xz^2 &: 0=e_2, \\ xyz &: 0=d_2+e_1k_1, \\ vxy &: 0=d_1+a_2k_2, \\ wxz &: 0=b_2+e_1q_1, \end{aligned}$$

$$vwx: \ 0=b_1+a_2q_2,$$
$$vxz \ : \ 0=a_2+e_1.$$

Then we have

$$\begin{aligned} wxy \ : \ 1&=b_1k_1+d_1q_1+b_2k_2+d_2q_2 \\ &=-(a_2+e_1)(k_1q_2+k_2q_1) \\ &=0, \end{aligned}$$

which is a contradiction.

The rest of assertions will be ascertained by the following two examples. □

EXAMPLE 13 (The case (5) of Theorem 7). Suppose Δ is (dB1). Set

$$R=K[v,w,x,y,z]/(vx-vy, \ xz).$$

Then R is a homogeneous complete intersection Hodge K-algebra on Δ, where the straightening relations of R are given by $vx=vy$, $xz=0$ and $wyz=0$, and $y<v$, x as partial order on v, w, x, y, z.

EXAMPLE 14 (The case (6) of Theorem 7). Suppose Δ is (dB2). Set

$$R=K[v,w,x,y,z]/(vx-wy, \ xz).$$

Then R is a homogeneous complete intersection Hodge K-algebra on Δ, where the straightening relations of R are given by $vx=wy$, $xz=0$, and $wyz=0$, and $w<v$, x as partial order on v, w, x, y, z.

The cases (gB)$\sim$(gE) are handled in a similar fashion.

References

[1] C. De Concini, D. Eisenbud and C. Procesi: *Hodge algebras*, Astérisque **91** (1982).
[2] D. Eisenbud: *Introduction to algebras with straightening laws, in* "Ring Theory and Algebra III, Lect. Notes in Pure and Appl. Math. No. 55 (B.R.McDonald ed.)," Marcel Dekker, New York, 1980, 243–268.
[3] F. Harrary: "Graph Theorey," Addison-Wesley, Massachusetts, 1969.
[4] T. Hibi: *On ASL domains with* $\sharp Ind(A)\leq 2$, in "Report on the 4-th symposium on Commutative algebra" 1982, 33–42.
[5] T. Hibi and K.-I. Watanabe: *Study of three dimensional algebras with straightening laws which are Gorenstein domains I*, Hiroshima Math. J. **15** (1985), 27–54.
[6] ———: *Study of three dimensional algebras with straightening laws which are*

Gorenstein domains II, Hiroshima Math. J. **15** (1985), 321–340.
[7] M. Hochster: *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, in "Ring Theory II, Lect. Notes in Pure and Appl. Math. No.26 (B.R. McDonald and R. Morris eds.)," Marcel dekker, New York, 1977, 171–223.
[8] H. Matsumura: "Commutative ring theory," Cambridge University Press, Combridge, 1986.
[9] G. Reisner: *Cohen-Macaulay quotient of polynomial rings*, Adv. in Math. **21** (1976), 30–49.
[10] R. Stanley: *Hilbert functions of graded algebras*, Adv. in Math. **28** (1978), 57–83.
[11] ————: "Combinatorics and commutative algebra," Birkhäuser, Boston, 1983.
[12] K.-I. Watanabe: *Study of algebras with straightening laws of dimension* 2, in "Algebraic and Topological Theories—to the memory of Dr. Takehiko Miyata (M. Nagata et al. eds.)," Kinokuniya, Tokyo, 1985, 622–639.

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