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**SPECTRAL AND SCATTERING THEORY FOR SECOND-
ORDER DIFFERENTIAL OPERATORS WITH
OPERATOR-VALUED COEFFICIENTS**

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Introduction

Let us consider a differential operator

$$(0.1) \quad L = -\frac{d^2}{dr^2} + B(r) + C(r), \quad r \in (0, \infty) = I,$$

where for each $r \in I$ $B(r)$ is a non-negative self-adjoint operator in a Hilbert space X with domain $\mathcal{D}(B(r)) = D$ constant in r , and $C(r)$ is a symmetric operator with domain $\mathcal{D}(C(r)) = D$ and $\text{s-lim}_{r \rightarrow \infty} C(r)x = 0$ for any $x \in D$. L acts on X -valued functions $f(r)$ on I . By defining the domain of L appropriately L can be regarded as a self-adjoint operator in $L^2(I, X, dr)$. In the present paper we shall develop a spectral and scattering theory for the self-adjoint operator L .

Jäger [3] constructs an eigenfunction expansion for L with $C(r)x = 0(r^{-3/2-\varepsilon})$ ($\varepsilon > 0$) for any $x \in D$ as $r \rightarrow \infty$. He shows the existence of the “eigenoperator” $\Gamma(r, k)$ associated with L . $\Gamma(r, k)$ is a bounded, linear operator on X for each pair $(r, k) \in I \times (R - \{0\})$ such that

$$(0.2) \quad -\frac{d^2}{dr^2}\Gamma(r, k)x + \Gamma(r, k)(B(r) + C(r))x = k^2\Gamma(r, k)x$$

holds for any $x \in D$. The “generalized Fourier transforms” \mathcal{F}^\pm are defined by

$$(0.3) \quad \mathcal{F}^\pm f(k) = \int_I \Gamma^\pm(r, k)f(r)dr, \quad \Gamma^\pm(r, k) = \pm ik\sqrt{\frac{2}{\pi}}\Gamma(r, \pm k).$$

\mathcal{F}^\pm transform $L^2(I, X, dr)$ into $L^2((0, \infty), X, dk)$ and satisfy for any Borel set Δ in I

$$(0.4) \quad E(\Delta) = (\mathcal{F}^\pm)^* \chi_{\sqrt{\Delta}} \mathcal{F}^\pm,$$

where $E(\lambda)$ is the resolution of the identity associated with L , $\chi_{\sqrt{\Delta}}$ is the characteristic function of $\sqrt{\Delta} = \{k > 0 / k^2 \in \Delta\}$, and $(\mathcal{F}^\pm)^*$ are the adjoints of \mathcal{F}^\pm

in $L^2((0, \infty), X, dk)$. These results imply that the spectrum of L is absolutely continuous on $(0, \infty)$. He has also shown that the spectrum of L is discrete on $(-\infty, 0)$. Jäger's methods are based on the fact that the principle of limiting absorption holds for the operator L . For the notion of the principle of limiting absorption see, for example, Eidus [2].

We shall treat in this paper the operator L with $C(r)x=0(r^{-1-\varepsilon})$ ($\varepsilon > 0$) for any $x \in D$ as $r \rightarrow \infty$. The other conditions imposed on $B(r)$ and $C(r)$ are the same as those in Jäger [3].

In [7] the author has justified the principle of limiting absorption for L with $C(r)x=0(r^{-1-\varepsilon})$. The results in [7] will be summarized in §1. These will be used as main tools throughout in this paper. In §2 and §3 we shall give a spectral decomposition of L using the generalized Fourier transforms \mathcal{F}^\pm . In our case the existence of the eigenoperator is not shown. But with the aid of the results of Jäger [3] we can construct \mathcal{F}^\pm such that (0.4) holds good.

§4 and §5 are devoted to investigating the properties of L using the techniques of stationary methods in perturbation theory. We shall make use of some results given in Kato and Kuroda [6]. L will be considered to be the perturbed operator of

$$(0.5) \quad L_0 = -\frac{d^2}{dr^2} + B(r).$$

In §4 we shall define and study two families of operators $\{G^\pm(\lambda)\}_{\lambda > 0}$ and $\{H^\pm(\lambda)\}_{\lambda > 0}$. Roughly speaking, $G^\pm(\lambda)$ and $H^\pm(\lambda)$ are defined by

$$(0.6) \quad \begin{cases} G^\pm(\lambda) = \lim_{z \rightarrow \lambda \pm i0} (L - z)(L_0 - z)^{-1}, \\ H^\pm(\lambda) = \lim_{z \rightarrow \lambda \pm i0} (L_0 - z)(L - z)^{-1}. \end{cases}$$

The results obtained in §4 will be used in the following sections. We shall show in §5 that the generalized Fourier transforms \mathcal{F}^\pm corresponding to L are *orthogonal*, i.e., \mathcal{F}^\pm transform $L^2(I, X, dr)$ onto $L^2((0, \infty), X, dk)$, under the assumption that the generalized Fourier transforms \mathcal{F}_0^\pm associated with L_0 are orthogonal. We shall define the wave operators $W^\pm = W^\pm(L, L_0)$ and study their properties in §6. It will be shown that W^\pm are *complete* and that we have

$$(0.7) \quad W^\pm = (\mathcal{F}^\pm)^* \mathcal{F}^\pm.$$

As an application we shall consider the Schrödinger operator $-\Delta + q(x)$ in the whole \mathbf{R}^n ($n \geq 3$) in §7. In this case we have

$$(0.8) \quad \begin{cases} X = L^2(S^{n-1}) \\ B(r) = \frac{1}{r^2} \left\{ -\Delta_n + \frac{(n-1)(n-3)}{4} \right\}, \\ C(r) = q(r\omega), \quad (\omega \in S^{n-1}), \end{cases}$$

where S^{n-1} is the $(n-1)$ -sphere and Λ_n is the Laplace-Beltrami operator on S^{n-1} . $q(x)$ is assumed to behave like $0(|x|^{-1-\varepsilon})$ ($\varepsilon > 0$). All the results obtained in the preceding sections are valid for the Schrödinger operator $-\Delta + q(x)$. In particular the spectrum of $-\Delta + q(x)$ on $(0, \infty)$ is absolutely continuous.

Throughout in this paper we shall follow Saitō [7] as to the notations. Here we recall some of them.

X — a Hilbert space with the norm $\| \cdot \|$ and inner product (\cdot, \cdot) .

$H^\beta(J, X)$ — the Hilbert space $L^2(J, X, (1 + |r|)^\beta dr)$ with the inner product

$$(0.9) \quad ((f, g))_{\beta, J} = \int_J (f(r), g(r))(1 + |r|)^\beta dr$$

and norm

$$(0.10) \quad \|f\|_\beta = [((f, f))_\beta]^{1/2}.$$

Here β is a real number and J is an open interval.

$C^m(J, Y)$ — the set of all Y -valued functions on J having m strong continuous derivatives. Here m is a non-negative integer, J is as above, and Y is a subset of a topological vector space. In particular we set $C^m(J, C) = C^m(J)$.

$C_0^m(J, Y)$ — the set of all $f(r) \in C^m(J, Y)$ with a compact carrier in J .

$C^{2, \beta}(J, X)(C_0^{2, \beta}(J, X))$ — the linear space spanned by the set of all $\varphi \in C^2(J, X)$ having the form $\varphi(r) = \psi(r)x$, where $x \in D$, $\psi \in C^2(J)$ ($\psi \in C_0^2(J)$), and

$$(0.11) \quad \int_J \{ |\psi'(r)|^2 + |\psi(r)|^2 (1 + |B(r)x|^2) \} dr < \infty.$$

Here for each $r \in J$ $B(r)$ is a non-negative operator in X with domain $\mathcal{D}(B(r)) = D$ constant in r .

$H^{1, B}(J, X)(H_0^{1, B}(J, X))$ — the Hilbert space obtained by the completion of $C^{2, B}(J, X)(C_0^{2, B}(J, X))$ using the norm

$$(0.12) \quad \|\varphi\|_{B, J} = [\int_J \{ |\varphi'(r)|^2 + |B^{1/2}(r)\varphi(r)|^2 + |\varphi(r)|^2 \} dr]^{1/2}.$$

The inner product is given by

$$(0.13) \quad ((f, g))_{B, J} = \int_J \{ (f'(r), g'(r)) + (B^{1/2}(r)f(r), B^{1/2}(r)g(r)) + (f(r), g(r)) \} dr$$

$\mathcal{U}(J)$ — the set of all linear, continuous functionals ℓ on $H_0^{1, B}(J, X)$. $\mathcal{U}(J)$ is a Banach space with the norm

$$(0.14) \quad \|\ell\|_J = \sup \{ |\langle \ell, \varphi \rangle| ; \varphi \in C_0^{2, B}(J, X), \|\varphi\|_{B, J} = 1 \}.$$

$\mathcal{U}_\alpha(J)$ —the subspace of $\mathcal{U}(J)$ satisfying

$$(0.15) \quad \|\ell\|_{\alpha, J} = \sup \{ |\langle \ell, (1+|r|)^{\alpha/2} \varphi \rangle| ; \varphi \in C_0^{2, B}(J, X), \|\varphi\|_{B, J} = 1 \} < \infty.$$

$\mathcal{U}_\alpha(J)$ is a Banach space with the norm $\|\cdot\|_{\alpha, J}$. Here α is a positive number.

$\text{loc } H^B(\bar{I}, X)$ —the set of all X -valued functions $f(r)$ on $I = (0, \infty)$ such that $f \in H^B((0, n), X)$ for every $n = 1, 2, \dots$. Here \bar{I} means the closure of I , i.e., $\bar{I} = [0, \infty)$.

$\text{loc } H^{1, B}(\bar{I}, X)(\text{loc } H_0^{1, B}(\bar{I}, X))$ —the set of all X -valued function f on I such that $\psi_n f \in H^{1, B}(I, X)$ ($\psi_n f \in H_0^{1, B}(I, X)$) for every $n = 1, 2, \dots$, where $\psi_n \in C^1(I)$ satisfying $0 \leq n \leq 1$ and

$$(0.16) \quad \psi_n(r) = \begin{cases} 1 & \text{for } 0 < r \leq n, \\ 0 & \text{for } r \geq n+1. \end{cases}$$

1. Preliminaries

Let $I = (0, \infty)$. For each $r \in I$ $B(r)$ and $C(r)$ are operators in a Hilbert space X with its norm $|\cdot|$ and inner product (\cdot, \cdot) . The following conditions are imposed on $B(r)$ and $C(r)$:

Assumption 1.1.¹⁾ (B-1) For each $r \in I$ $B(r)$ is a non-negative, self-adjoint operator in X whose domain $\mathcal{D}(B(r)) = D$ does not depend on r . We have $B(r)x \in C^0(I, X)$ for any $x \in D$.

(B-2) Let $x, y \in D$. Then $(B(r)x, y) \in C^2(I)$. For any compact interval I_0 in I there is a constant $c_1(I_0)$ such that

$$(1.1) \quad \left| \frac{d^j}{dr^j} (B(s)x, y) \right| \leq c_1(I_0) (|x| + |B^{1/2}(r)x|) (|y| + |B^{1/2}(r)y|)$$

for any $r, s \in I_0$ and $j = 1, 2$.

(B-3) There exist constants $\rho > 0$ and $\beta > 1$ such that

$$(1.2) \quad -\frac{d}{dr} (B(r)x, x) \geq \frac{\beta}{r} (B(r)x, x)$$

holds for any $x \in D$ and any $r \geq \rho$.

(B-4) The natural imbedding

$$(1.3) \quad H_0^{1, B}((0, b), X) \rightarrow H^0((0, b), X)$$

is compact for each $b \in I$.

1) Assumption 1.1 is the same as Assumptions 1.1 and 1.2 in Saito [7].

(C-1) For each $r \in I$ $C(r)$ is a symmetric operator with $\mathcal{D}(C(r)) = D$. We have $C(r)x \in C^1(I, X)$ for any $x \in D$.

(C-2). For any compact interval I_0 in I there exists a constant $c_2(I_0) > 0$ such that

$$(1.4) \quad \left| \frac{d}{dr} C(r)x \right| \leq c_2(I_0) (|x| + |B^{1/2}(r)x|)$$

for any $x \in D$.

(C-3) There exist constants $c_0 > 0$ and $0 < \varepsilon < 1$ such that

$$(1.5) \quad |C(r)x| \leq c_0(1+r)^{-1-\varepsilon} (|x| + |B^{1/2}(r)x|)$$

holds for any $x \in D$ and any $r \in I$.

Denote by \mathfrak{A}_0 the set of all X -valued functions $\varphi(r)$ on \bar{I} satisfying the following (i)~(iii):

$$(1.6) \quad \begin{cases} \text{(i)} & \varphi(r) \in D \quad (r \in I). \\ \text{(ii)} & \varphi \in C^2(I, X) \cap C^1(\bar{I}, X) \text{ and the carrier of } \varphi \text{ is compact in } \bar{I}. \\ \text{(iii)} & \varphi \in H_0^{1,B}(I, X) \text{ and } L_0\varphi \in H^0(I, X). \end{cases}$$

It is easy to see that we have

$$(1.7) \quad C_0^{2,B}(I, X) \subset \mathfrak{A}_0 \subset H_0^{1,B}(I, X).$$

We define differential operators M and M_0 in $H^0(I, X)$ by

$$(1.8) \quad \begin{cases} \mathcal{D}(M_0) = \mathfrak{A}_0 \\ M_0\varphi = L_0\varphi = -\frac{d^2}{dr^2}\varphi + B(r)\varphi \end{cases}$$

and

$$(1.9) \quad \begin{cases} \mathcal{D}(M) = \mathfrak{A}_0 \\ M\varphi = L\varphi = L_0\varphi + C(r)\varphi = -\frac{d^2}{dr^2}\varphi + B(r)\varphi + C(r)\varphi. \end{cases}$$

obviously M_0 is symmetric and non-negative definite. It follows from (1.5) that

$$(1.10) \quad \begin{aligned} \|C(\cdot)\varphi\|_0^2 &= \int_I |C(r)\varphi(r)|^2 dr \leq \int_I c_0^2 (|\varphi(r)| + |B^{1/2}(r)\varphi(r)|)^2 dr \\ &\leq 2c_0^2 \int_I (|\varphi(r)|^2 + |B^{1/2}(r)\varphi(r)|^2) dr \\ &\leq 2c_0^2 \|\varphi\|_B^2 = 2c_0^2 ((M_0\varphi, \varphi))_0 + 2c_0^2 \|\varphi\|_0^2 \\ &\leq c_0^2 \left(\alpha \|M_0\varphi\|_0^2 + \frac{1}{\alpha} \|\varphi\|_0^2 \right) + 2c_0^2 \|\varphi\|_0^2 \quad (\varphi \in \mathfrak{A}_0), \end{aligned}$$

2) Here and in the sequel we put $\|\cdot\|_{B,I} = \|\cdot\|_B$ and $\|\cdot\|_{B,I} = \|\cdot\|_B$ for simplicity.

where $\alpha > 0$ is arbitrary and we have used integration by parts. Hence M is symmetric and bounded below. We denote by A_0 and A the Friedrichs extensions of M_0 and M , respectively. Then we have

$$(1.11) \quad \begin{cases} C_0^{2,B}(I, X) \subset \mathcal{D}(A_0) = H_0^{1,B}(I, X) \cap \mathcal{D}(M_0^*) \subset H_0^{1,B}(I, X) \\ C_0^{2,B}(I, X) \subset \mathcal{D}(A) = H_0^{1,B}(I, X) \cap \mathcal{D}(M^*) \subset H_0^{1,B}(I, X). \end{cases}$$

where M_0^* and M^* are the adjoints of M_0 and M in $H^0(I, X)$, respectively.

In the remainder of this section we state some results on the differential equation $(L - k^2)v = f$ with a sort of radiation condition. These are proved in [7] and will be used in the following sections. First we give the next

DEFINITION 1.2. Let $\ell \in \mathcal{U}(I)$, $u \in H^{1,B}(I, X)$ ³⁾ and $k \in \mathbf{C}^+ = \{k/k \in \mathbf{C}, \operatorname{Im} k \geq 0, \operatorname{Re} k \neq 0\}$ be given. Then $v \in \operatorname{loc} H^{1,B}(\bar{I}, X)$ is called the *radiative function for $\{L, k, \ell, u\}$* , if the following three conditions hold:

- (a) $v - u \in \operatorname{loc} H_0^{1,B}(\bar{I}, X)$.
- (b) $v' - ikv \in H^{-1+\varepsilon}(I, X)$ (the “radiation condition”).
- (c) For all $\varphi \in C_0^{2,B}(I, X)$, we have

$$(1.12) \quad ((v, (L - \bar{k}^2)\varphi))_0 = \langle \ell, \varphi \rangle.$$

We can prove under the assumptions (B-1)~(B-3) and (C-1)~(C-3) that the radiative function for given $\{k, \ell, u\} \in \mathbf{C}^+ \times \mathcal{U}(I) \times H^{1,B}(I, X)$ is unique ([7], Therem 2.2). We can also prove under Assumption 1.1 that for given $\{k, \ell, u\} \in \mathbf{C}^+ \times \mathcal{U}_{1+\varepsilon}(I) \times H^{1,B}(I, X)$ there exists a unique radiative function $v = v(\cdot, k, \ell, u)$ for $\{L, k, \ell, u\}$ which belongs to $H^{-1-\varepsilon}(I, X) \cap \operatorname{loc} H^{1,B}(\bar{I}, X)$, and that the mapping

$$(1.13) \quad \begin{aligned} \Sigma : \mathbf{C}^+ \times \mathcal{U}_{1+\varepsilon}(I) \times H^{1,B}(I, X) &\ni \{k, \ell, u\} \\ &\rightarrow v(\cdot, k, \ell, u) \in H^{-1-\varepsilon}(I, X) \cap \operatorname{loc} H^{1,B}(\bar{I}, X) \end{aligned}$$

is continuous as a mapping from $\mathbf{C}^+ \times \mathcal{U}_{1+\varepsilon}(I) \times H^{1,B}(I, X)$ into $H^{-1-\varepsilon}(I, X)$ and is also continuous as a mapping from $\mathbf{C}^+ \times \mathcal{U}_{1+\varepsilon}(I) \times H^{1,B}(I, X)$ into $\operatorname{loc} H^{1,B}(\bar{I}, X)$ ([7], Theorems 3.7 and 3.8). Under Assumption 1.1 an a priori estimate for radiative functions is obtained as follows ([7], Lemma 3.4): Let $v = v(\cdot, k, \ell)$ be the radiative function for $\{L, k, \ell, 0\}$, where k belongs to a compact set K in \mathbf{C}^+ and $\ell \in \mathcal{U}_{1+\varepsilon}(I)$. Then there exists a constant δ_0 , depending only on K and L , such that

$$(1.14) \quad \|v\|_{-1-\varepsilon} + \|v' - ikv\|_{-1+\varepsilon} + \|B^{1/2}v\|_{-1+\varepsilon} \leq \delta_0 \|\ell\|_{1+\varepsilon}.$$

3) For the definition of $\mathcal{U}(I)$ and $H^{1,B}(I, X)$ see the list of the notations in the Introduction.

2. The resolvent kernel $G(r, s, k)$ for A

We are ready to define the resolvent kernel $G(r, s, k)$ for A , which satisfies

$$(2.1) \quad (L - k^2)G(r, s, k) = \delta(r - s),$$

where $\delta(r)$ is Dirac's distribution. To this end for $s \in \bar{I}$ and $x \in X$ we shall introduce a linear functional $\ell[s, x]$ on $H_0^{1, B}(I, X)$ by

$$(2.2) \quad \langle \ell[s, x], u \rangle = (x, u(s)) \quad (u \in H_0^{1, B}(I, X)).$$

We can show that the inequalities

$$(2.3) \quad \begin{cases} |\varphi(s) - \varphi(t)|^2 \leq |s - t| \|\varphi\|_B^2, \\ |\varphi(s)|^2 \leq 2\|\varphi\|_B^2 \end{cases}$$

hold for any $\varphi \in C_0^{2, B}(I, X)$ and any $s, t \in \bar{I}$.⁴⁾ In fact we have

$$(2.4) \quad \begin{aligned} |\varphi(s) - \varphi(t)|^2 &= \left| \int_s^t \varphi(r) dr \right|^2 \leq |s - t| \left| \int_s^t |\varphi'(r)|^2 dr \right| \\ &\leq |s - t| \|\varphi\|_B^2 \end{aligned}$$

Next, putting $|\varphi(t)| = \min_{s \leq r \leq s+1} |\varphi(r)|$, we have

$$(2.5) \quad \begin{aligned} |\varphi(s)|^2 &\leq 2\{|\varphi(s) - \varphi(t)|^2 + |\varphi(t)|^2\} \\ &\leq 2\{ |s - t| \int_s^t |\varphi'(r)|^2 dr + \int_s^{s+1} |\varphi(t)|^2 dr \} \\ &\leq 2 \int_s^{s+1} \{ |\varphi'(r)|^2 + |\varphi(r)|^2 \} dr \\ &\leq 2\|\varphi\|_B^2. \end{aligned}$$

Since $C_0^{2, B}(I, X)$ is dense in $H_0^{1, B}(I, X)$, it follows from (2.3) that $H_0^{1, B}(I, X)$ is continuously imbedded in $C^0(\bar{I}, X)$. Hence $\ell[s, x]$ can be regarded as a linear functional on $H_0^{1, B}(I, X)$. Further, we can see from (2.3) that

$$(2.6) \quad \begin{cases} \|\ell[s, x]\| = \sup_{\|\varphi\|_B=1} |(x, \varphi(s))| \leq |x| \cdot \sup_{\|\varphi\|_B=1} |\varphi(s)| \leq \sqrt{2} |x|, \\ \|\ell[s, x]\|_{1+\varepsilon} \leq \sqrt{2} (1+s)^{(1+\varepsilon)/2} |x|, \end{cases}$$

which implies that $\ell[s, x] \in \mathcal{U}_{1+\varepsilon}(I)$.

Lemma 2.1. *Let us assume Assumption 1.1. Let $v = v(\cdot, s, k, x)$ be the radiative function for $\{L, k, \ell[s, x], 0\}$, where $k \in \mathbf{C}^+$, $s \in \bar{I}$ and $x \in X$. Then for any $R > 0$ there exists a constant $\delta_2 = \delta_2(R, k) > 0$ such that we have*

$$(2.7) \quad \|v\|_{B, (0, R)} \leq \delta_2 (1+s)^{(1+\varepsilon)/2} |x|$$

4) See Jäger [3], p. 69.

and

$$(2.8) \quad \max_{0 \leq r \leq R} |v(r)| \leq \sqrt{2} \delta_2 (1+s)^{(1+\varepsilon)/2} |x|.$$

$\delta_2(R, k)$ is bounded when R moves in a bounded set in I and k moves in a compact set in \mathbf{C}^+ .

Proof. Put $I_1 = (0, R)$ and $I_2 = (0, R+1)$. It follows from Lemma 1.5 in [7] that we have

$$(2.9) \quad \|v\|_{B, I_1} \leq c \{ \|v\|_{0, I_2} + \|l[s, x]\| \}$$

with a constant $c = c(R, k) > 0$ which is bounded when the pair (R, k) moves in a bounded set in $I \times \mathbf{C}$. Using (1.14) we have

$$(2.10) \quad \|v\|_{0, I_2} \leq (1+R)^{(1+\varepsilon)/2} \|v\|_{-1-\varepsilon} \leq \delta_0 (1+R)^{(1+\varepsilon)/2} \|l[s, x]\|_{1+\varepsilon}$$

with a constant $\delta_0 = \delta_0(k)$ which is bounded when k moves in a compact set in \mathbf{C}^+ . Hence we obtain from (2.9), (2.10) and (2.6)

$$\begin{aligned} (2.11) \quad \|v\|_{B, I} &\leq c \{ \delta_0 (1+R)^{(1+\varepsilon)/2} \|l[s, x]\|_{1+\varepsilon} + \|l[s, x]\| \} \\ &\leq c \{ \delta_0 (1+R)^{(1+\varepsilon)/2} \sqrt{2} (1+s)^{(1+\varepsilon)/2} |x| + \sqrt{2} |x| \} \\ &= \sqrt{2} c \{ \delta_0 (1+R)^{(1+\varepsilon)/2} + (1+s)^{(-1-\varepsilon)/2} \} (1+s)^{(1+\varepsilon)/2} |x| \\ &\leq \sqrt{2} c \{ \delta_0 (1+R)^{(1+\varepsilon)/2} + 1 \} (1+s)^{(1+\varepsilon)/2} |x|, \end{aligned}$$

which means (2.7) with $\delta_2 = \sqrt{2} c \{ \delta_0 (1+R)^{(1+\varepsilon)/2} + 1 \}$. (2.8) follows from (2.7) and (2.3). Q. E. D.

In view of the above lemma we can give the following definition.

DEFINITION 2.2. For each triple $(r, s, k) \in \bar{I} \times \bar{I} \times \mathbf{C}^+$ we define a bounded linear operator $G(r, s, k)$ on X by $G(r, s, k)x = v(r, s, k, x)$, where $v(r, s, k, x)$ is the radiative function for $\{L, k, l[s, x], 0\}$. $G(r, s, k)$ is said to be the *resolvent kernel* for A .

For the properties of the resolvent kernel $G(r, s, k)$ we can show almost the same results as given in §6 of Jäger [3].

Theorem 2.3. Let us assume Assumption 1.1. Let $G(r, s, k)$ be the resolvent kernel for A . Then we have the following (i)~(v):

(i) $G(\cdot, s, k)x \in \text{loc } H_0^{1, B}(\bar{I}, X) \cap H^{-1-\varepsilon}(I, X)$ for $s \in \bar{I}$ and $x \in X$. $G(\cdot, s, k)x$ is continuous on $\bar{I} \times \mathbf{C}^+ \times X$ both in $\text{loc } H_0^{1, B}(\bar{I}, X)$ and in $H^{-1-\varepsilon}(I, X)$. $G(r, s, k)x$ is a continuous X -valued function on $\bar{I} \times \bar{I} \times \mathbf{C}^+ \times X$, too.

(ii) $G(0, r, k) = G(r, 0, k) = 0$ for any $r \in \bar{I}$.

(iii) For any $(s, k, x) \in \bar{I} \times \mathbf{C} \times X$ $G(\cdot, s, k)x \in C^2(I - \{s\}, D)$ and $G(\cdot, s, k)x$ satisfies the equation

$$(2.12) \quad (L - k^2)G(\cdot, s, k)x = 0 \quad \text{in } I - \{s\}.$$

(iv) Denote by $R(z) = (A - z)^{-1}$ the resolvent of A . Then we have

$$(2.13) \quad R(k^2)f(r) = \int_I G(r, s, k)f(s)ds,$$

where $k \in \mathbf{C}^+$, $\text{Im } k > 0$ and $f \in H^0(I, X)$ with a compact carrier in \bar{I} .

(v) We have

$$(2.14) \quad G(r, s, k)^* = G(s, r, -\bar{k})$$

for any triple $(r, s, k) \in \bar{I} \times \bar{I} \times \mathbf{C}^+$, where $G(r, s, k)^*$ is the adjoint of $G(r, s, k)$ in X .

Proof. Since we have from (2.2) and (2.3)

$$(2.15) \quad \begin{aligned} & |\langle \ell[s, x], (1+r)^{(1+\varepsilon)/2}\varphi \rangle - \langle \ell[t, x], (1+r)^{(1+\varepsilon)/2}\varphi \rangle| \\ &= |(1+s)^{(1+\varepsilon)/2}(x, \varphi(s)) - (1+t)^{(1+\varepsilon)/2}(y, \varphi(t))| \\ &\leq \sqrt{2} |(1+s)^{(1+\varepsilon)/2} - (1+t)^{(1+\varepsilon)/2}| |x| \|\varphi\|_B + \sqrt{2} (1+t)^{(1+\varepsilon)/2} |x - y| \|\varphi\|_B \\ &\quad + (1+t)^{(1+\varepsilon)/2} |s - t| |y| \|\varphi\|_B \quad (\varphi \in C_0^{2,B}(I, X)), \end{aligned}$$

we see that $\ell[s, x]$ is a $U_{1+\varepsilon}(I)$ -valued continuous function on $\bar{I} \times X$. Hence (i) follows from Theorem 3.7 in [7]. We obtain $G(0, r, k) = 0$ from the fact that $G(\cdot, r, k)x \in \text{loc } H_0^{1,B}(\bar{I}, X)$ for any $x \in X$. It follows from the uniqueness of the radiative solution that $G(r, 0, k) = 0$. (iii) follows from the regularity theorem in Jäger [3] (Satz 3.1).

Next let us show (iv). It follows from (1.11) that we have

$$(2.16) \quad \begin{cases} R(k^2)f \in H_0^{1,B}(I, X) \\ ((R(k^2)f, (L - \bar{k}^2)\varphi))_0 = ((f, \varphi))_0 \quad (\varphi \in C_0^{2,B}(I, X)), \end{cases}$$

where $k \in \mathbf{C}^+$, $\text{Im } k > 0$ and $f \in H^0(I, X)$. Hence $R(k^2)f$ is the radiative function for $\{L, k, \ell[f], 0\}$. Further, assume that the carrier of f is compact in \bar{I} and is contained in $[0, R]$. Then we shall show that

$$(2.17) \quad V(r) = \int_0^R G(r, s, k)f(s)ds$$

is the radiative function for $\{L, k, \ell[f], 0\}$, too. In fact we can easily see for any $\varphi \in C_0^{2,B}(I, X)$

$$(2.18) \quad \begin{aligned} ((V, (L - \bar{k}^2)\varphi))_0 &= \int_I dr \left(\int_0^R G(r, s, k)f(s)ds, (L - \bar{k}^2)\varphi(r) \right) \\ &= \int_0^R ds \left((G(\cdot, s, k)f(s), (L - \bar{k}^2)\varphi(\cdot)) \right)_0 \\ &= \int_0^R (f(s), \varphi(s))ds \end{aligned}$$

$$= ((f, \varphi))_0.$$

The following estimate is sufficient to see that $V \in H^{-1-\varepsilon}(I, X)$:

$$\begin{aligned}
 (2.19) \quad & \int_I (1+r)^{-1-\varepsilon} |V(r)|^2 dr \\
 &= \int_I (1+r)^{-1-\varepsilon} dr \int_0^R ds \int_0^R dt (G(r, s, k)f(s), G(r, t, k)f(t)) \\
 &\leq 2 \int_0^R ds \int_0^R dt \int_I (1+r)^{-1-\varepsilon} \{ |G(r, s, k)f(s)|^2 \\
 &\quad + |G(r, t, k)f(t)|^2 \} dr \\
 &= 4R \int_0^R \|G(\cdot, s, k)f(s)\|_{-1-\varepsilon}^2 ds \\
 &\leq 4R\delta_0^2 \int_0^R \|\ell[s, f(s)]\|_{1+\varepsilon}^2 ds \\
 &\leq 4R\delta_0^2 \int_0^R 2(1+s)^{1+\varepsilon} |f(s)|^2 ds < \infty,
 \end{aligned}$$

where we have made use of (1.14) and (2.6). Similarly we can show that $V' - ikV \in H^{-1+\varepsilon}(I, X)$. Therefore we have $V(r) = R(k^2)f(r)$ by the uniqueness theorem of the radiative function (Theorem 2.2 in [7]), which completes the proof of (iv).

Finally we shall prove (v). If $k \in \mathbf{C}^+$ and $\operatorname{Im} k > 0$, then (2.14) follows from (2.13) and the fact that $R(k^2)^* = R(\bar{k}^2)$. Since $G(r, s, k)x$ is continuous on \mathbf{C}^+ , (2.14) is also true for $k \in \mathbf{C}^+$, $\operatorname{Im} k = 0$. Q.E.D.

Let $C_j(r)$, $j = 1, 2$, be operator-valued functions on I satisfying (C-1)~(C-3) in Assumption 1.1. We set

$$(2.20) \quad \begin{cases} M_j = -\frac{d^2}{dr^2} + B(r) + C_j(r) \\ \mathcal{D}(M_j) = \mathfrak{A}_0 \end{cases}$$

and we denote by A_j the Friedrichs extensions of M_j , $j = 1, 2$. $G_j(r, s, k)$, $j = 1, 2$, are the resolvent kernels for A_j . We shall show a formula involving $G_1(r, s, k)$ and $G_2(r, s, k)$ which will be used in §3.

Lemma 2.4⁵⁾. *Let $C_1(r)$ and $C_2(r)$ be as above. Let $B(r)$ satisfy (B-1)~(B-4) in Assumption 1.1. Let $k \in \mathbf{R} - \{0\}$. Then we have*

$$\begin{aligned}
 (2.21) \quad & (\{G_1(r, s, k) - G_2(r, s, -k)\}x, y) \\
 &= \left(\frac{d}{dt} G_1(t, s, k)x, G_2(t, r, k)y \right) - \left(G_1(t, s, k)x, \frac{d}{dt} G_2(t, r, k)y \right) \\
 &\quad + \int_0^t (\{C_2(\tau) - C_1(\tau)\}G_1(\tau, s, k)x, G_2(\tau, r, k)y) d\tau,
 \end{aligned}$$

5) Cf. Lemma 6.1 in Jäger [3].

where $x, y \in X$, $r, s \in I$, and $t > \max(r, s)$.

Proof. Let $\psi \in C^1(\mathbf{R})$ satisfying $0 \leq \psi \leq 1$ and

$$(2.22) \quad \psi(r) = \begin{cases} 1 & (r \leq 0) \\ 0 & (r \geq 1). \end{cases}$$

Set $\psi_n(\tau) = \psi_{n,t}(\tau) = \psi(n(\tau - t))$ and $I_0 = (0, t+1)$. Then we can see that

$$(2.23) \quad \psi_n(\cdot)G_j(\cdot, s, k)x \in H_0^{1,B}(I_0, X),$$

where $j = 1, 2$, $s \in I$, $k \in \mathbf{C}^+$ and $x \in X$. Using integration by parts we rewrite the relation

$$(2.24) \quad ((G_1(\cdot, s, k)x, (L_1 - \bar{k}^2)\varphi))_0 = (x, \varphi(s))$$

to obtain

$$(2.25) \quad ((G_1(\cdot, s, k)x, \varphi(\cdot)))_{B, I_0} + (((C_1(\cdot) - 1)G_1(\cdot, s, k)x, \varphi(\cdot)))_{0, I_0} = (x, \varphi(s))$$

for any $\varphi \in C_0^{2,B}(I_0, X)$. Since both sides of (2.25) can be extended to bounded, anti-linear functionals on $H_0^{1,B}(I_0, X)$ for fixed $s \in I$, $k \in \mathbf{C}^+$, $x \in X$, (2.25) holds good for any $\varphi \in H_0^{1,B}(I_0, X)$. Hence we can put $\varphi = \psi_n G_2(\cdot, r, k)y$ in (2.25) to see

$$(2.26) \quad \begin{aligned} & ((G_1(\cdot, s, k)x, \psi_n(\cdot)G_2(\cdot, r, k)y))_{B, I_0} \\ & + (((C_1(\cdot) - 1)G_1(\cdot, s, k)x, \psi_n(\cdot)G_2(\cdot, r, k)y))_{0, I_0} \\ & = (x, \psi_n(s)G_2(s, r, k)y) \\ & = (G_2(r, s, -k)x, y) \end{aligned}$$

for $r, s \in (0, t)$, $x, y \in X$ and $k \in \mathbf{R} - \{0\}$, where we used Therem 2.3, (v). Similarly we obtain

$$(2.27) \quad \begin{aligned} & ((\psi_n(\cdot)G_1(\cdot, s, k)x, G_2(\cdot, r, k)y))_{B, I_0} \\ & + ((\psi_n G_1(\cdot, s, k)x, (C_2(\cdot) - 1)G_2(\cdot, r, k)y))_{0, I_0} \\ & = (G_1(r, s, k)x, y). \end{aligned}$$

Combining (2.26) and (2.27), we have

$$(2.28) \quad \begin{aligned} & (\{G_1(r, s, k) - G_2(r, s, -k)\}x, y) \\ & = \int_0^\infty \psi_n(\tau) (\{C_2(\tau) - C_1(\tau)\}G_1(\tau, s, k)x, G_2(\tau, r, k)y) d\tau \\ & + \int_0^\infty \psi_n'(\tau) \left\{ \left(G_1(\tau, s, k)x, \frac{d}{d\tau}G_2(\tau, r, k)y \right) \right. \\ & \left. - \left(\frac{d}{d\tau}G_1(\tau, s, k)x, G_2(\tau, r, k)y \right) \right\} d\tau. \end{aligned}$$

Noting that

$$(2.29) \quad \begin{cases} \psi_n(\tau) \rightarrow \begin{cases} 1 & \text{for } \tau \leq t \\ 0 & \text{for } \tau > t, \end{cases} \\ \psi_n'(\tau) \rightarrow -\delta(\tau - t), \end{cases}$$

as $n \rightarrow \infty$, we obtain (2.21) from (2.28).

Q.E.D.

3. The spectral representation for A

This section is devoted to constructing a spectral decomposition for A by means of a “generalized Fourier transform”. We start with the results of Jäger [3]. The following theorem on the spectrum of A has been proved in Jäger [3], §4.

Theorem 3.1. *Let us assume Assumption 1.1. (i) Then A is bounded below with the lower bound $\kappa_0 \leq 0$. (ii) On $(\kappa_0, 0)$ the continuous spectrum of A is absent. The negative eigenvalues, if they exist, are of finite multiplicity and are discrete in the sense that they form an isolated set having no limit point other than the origin 0. (iii) We have $\sigma_e(A) \subset (0, \infty)$, where $\sigma_e(A)$ means the essential spectrum of A . There exists no positive eigenvalue of A . If in addition there exists $x_0 \in D$ which satisfies*

$$(3.1) \quad \begin{cases} |x_0| = 1, \\ s - \lim_{r \rightarrow \infty} B(r)x_0 = 0, \end{cases}$$

then we have $\sigma_e(A) = [0, \infty)$.

If we assume instead of (1.5) that we have for any $x \in D$ and any $r \in I$

$$(3.2) \quad |c(r)x| \leq \tilde{c}_0(1 + |r|)^{-3/2 - \varepsilon_0}(|x| + |B^{1/2}(r)x|)$$

with constants $\tilde{c}_0 > 0$, $\varepsilon_0 > 0$, then an eigenfunction expansion for A has been obtained by Jäger [3], §6. Jäger’s results are summarized as follows:

Theorem 3.2. *Let us assume (B-1) (B-4) and (C-1), (C-2) in Assumption 1.1 and (3.2). Then for $k \in \mathbf{R} - \{0\}$, $r \in \bar{I}$ and $x \in X$ the strong limit*

$$(3.3) \quad \Gamma(r, k)x = s - \lim_{t \rightarrow \infty} e^{-itk}G(t, r, k)x$$

exists. $\Gamma(r, k)$ is a bounded linear operator on X for each $r \in \bar{I}$ and $k \in \mathbf{R} - \{0\}$. The adjoint $\Gamma^*(r, k)$ of $\Gamma(r, k)$ satisfies

$$(3.4) \quad \begin{cases} \Gamma^*(\cdot, k)x \in C^0(\bar{I}, X) \cap \text{loc } H_0^{1, B}(\bar{I}, X), \\ \Gamma^*(\cdot, k)x \in C^2(I, D), \\ (L - k^2)\Gamma^*(r, k)x = 0 \quad (r \in I), \\ \Gamma^*(0, k)x = 0, \end{cases}$$

for $x \in X$ and $k \in \mathbf{R} - \{0\}$. Set

$$(3.5) \quad \begin{cases} \Gamma^\pm(r, k) = \pm \sqrt{\frac{2}{\pi}} ik \Gamma(r, \pm k) \\ \mathcal{Q}^\pm f(k) = \int_I \Gamma^\pm(r, k) f(r) dr, \end{cases}$$

where $f \in C_0^0(\bar{I}, X)$.⁶⁾ Then $\mathcal{Q}^\pm f \in L_2((0, \infty), X, dk)$ and \mathcal{Q}^\pm have unique extenstions \mathcal{F}^\pm to $H^0(I, X)$ which are bounded linear operators on $H^0(I, X)$ into $L_2((0, \infty), X, dk)$. Denote the resolution of the identity for A by $E(\lambda)$. We have for $0 < \lambda_1 < \lambda_2 \leq \infty$ and $f, g \in H^0(I, X)$

$$(3.6) \quad ((E((\lambda_1, \lambda_2))f, g))_0 = \int_{\lambda_1 < k < \lambda_2} ((\mathcal{F}^\pm f)(k), (\mathcal{F}^\pm g)(k)) dk.$$

Hence the spectrum of A is absolutely continuous on $(0, \infty)$.

Let us assume (C-3) in place of (3.2) again. For each $n = 1, 2, \dots$ we take $\psi_n \in C^1(I)$ satisfying $0 \leq \psi_n \leq 1$ and

$$(3.7) \quad \psi_n(r) = \begin{cases} 1 & \text{for } 0 < r \leq n, \\ 0 & \text{for } r \geq n+1. \end{cases}$$

A_n , $n = 1, 2, \dots$, denote the Friedrichs extensions of the operators M_n which are defined by

$$(3.8) \quad \begin{cases} \mathcal{D}(M_n) = \mathfrak{A}_0 \\ M_n = -\frac{d^2}{dr^2} + B(r) + C_n(r), \quad C_n(r) = \psi_n(r) C(r), \end{cases}$$

respectively. We can easily see that for each $n = 1, 2, \dots$, $C_n(r)$ satisfies (3.2) and that the sequence $C_n(r)$ satisfies Assumption 4.1 in [7]. For each $n = 1, 2, \dots$, $G_n(r, s, k)$ is the resolvent kernel for A_n . $\Gamma_n(r, k)$, $\Gamma_n^\pm(r, k)$, \mathcal{Q}_n^\pm , \mathcal{F}_n^\pm and $E_n(\lambda)$ are defined as in Theorem 3.2.

Lemma 3.3. *We have*

$$(3.9) \quad \begin{aligned} (\{G_n(r, s, k) - G_m(r, s, -k)\}x, y) &= 2ik(\Gamma_n(s, k)x, \Gamma_m(r, k)y) \\ &+ \int_0^\infty (\{C_m(\tau) - C_n(\tau)\}G_n(\tau, s, k)x, G_m(\tau, r, k)y) d\tau \\ & \quad (n, m = 1, 2, \dots), \end{aligned}$$

where $k \in \mathbf{R} - \{0\}$, $r, s \in \bar{I}$ and $x, y \in X$.

6) $C_0^0(\bar{I}, X)$ denotes the set of all X -valued, continuous functions on $\bar{I} = [0, \infty)$ with a compact carrier in \bar{I} .

Proof. We obtain from Lemma 2.4

$$\begin{aligned}
 (3.10) \quad & (\{G_n(r, s, k) - G_m(r, s, -k)\}x, y) \\
 &= 2ik(G_n(t, s, k)x, G_m(t, r, k)y) \\
 &+ \int_0^t (\{C_n(\tau) - C_m(\tau)\}G_n(\tau, s, k)x, G_m(\tau, r, k)y) d\tau \\
 &+ \left\{ \left(\frac{d}{dt}G_n(t, s, k)x - ikG_n(t, s, k)x, G_m(t, r, k)y \right) \right. \\
 &\quad \left. - \left(G_n(t, s, k)x, \frac{d}{dt}G_m(t, r, k)y - ikG_m(t, r, k)y \right) \right\} \\
 &= K_1(t) + K_2(t) + K_3(t).
 \end{aligned}$$

First it follows from (1.5) that

$$\begin{aligned}
 (3.11) \quad & |(\{C_n(\tau) - C_m(\tau)\}G_n(\tau, s, k)x, G_m(\tau, r, k)y)| \\
 &\leq 2(1+\tau)^{-1-\varepsilon}c_0(|G_n(\tau, s, k)x| + |B^{1/2}(\tau)G_n(\tau, s, k)x|) |G_m(\tau, r, k)y| \\
 &\leq 4(1+\tau)^{-1-\varepsilon}c_0\{2|G_n(\tau, s, k)x|^2 + 2|B^{1/2}(\tau)G_n(\tau, r, k)x|^2 \\
 &\quad + |G_m(\tau, r, k)y|^2\} \\
 &\in L^1(I, d\tau),
 \end{aligned}$$

where we have noted that $G_n(\cdot, s, k)x, G_m(\cdot, r, k)y \in H^{-1-\varepsilon}(I, X)$ and $B^{1/2}(\cdot)G_n(\cdot, s, k)x \in H^{-1+\varepsilon}(I, X)$ by (1.14). Hence we have

$$(3.12) \quad \lim_{t \rightarrow \infty} K_2(t) = \int_0^\infty (\{C_n(\tau) - C_m(\tau)\}G_n(\tau, s, k)x, G_m(\tau, r, k)y) d\tau$$

Since $G'_n(\cdot, s, k)x - ikG_n(\cdot, s, k)x, G'_m(\cdot, r, k)y - ikG_m(\cdot, r, k)y \in H^{-1+\varepsilon}(I, X)$ and $G_n(\cdot, s, k)x, G_m(\cdot, r, k)y \in H^{-1-\varepsilon}(I, X)$, we have $\lim_{j \rightarrow \infty} K_3(t_j) = 0$ along some sequence $\{t_j\}$, $t_j \rightarrow \infty$. Finally we see from (3.3)

$$\begin{aligned}
 (3.13) \quad \lim_{t \rightarrow \infty} K_1(t) &= \lim_{t \rightarrow \infty} 2ik(e^{-itk}G_n(t, s, k)x, e^{-itk}G_m(t, r, k)y) \\
 &= 2ik(\Gamma_n(s, k)x, \Gamma_m(r, k)y).
 \end{aligned}$$

Thus we obtain (3.9). Q.E.D.

Now we are in a position to show that $\mathcal{Q}_n^\pm f$ is a Cauchy sequence in $\text{loc } L_2(\bar{I}, X, dk)$ for any $f \in C_0^0(\bar{I}, X)$. Then $E(\lambda)$, the resolution of the identity for A , will be represented by unique extensions \mathcal{F}^\pm of $\mathcal{Q}^\pm = \lim_{n \rightarrow \infty} \mathcal{Q}_n^\pm$.

Theorem 3.4. *Let us assume Assumption 1.1.*

(i) *Then for any $f \in C_0^0(\bar{I}, X)$ there exist*

$$(3.14) \quad \mathcal{Q}^\pm f = \lim_{n \rightarrow \infty} \mathcal{Q}_n^\pm f \quad \text{in } L_2((\alpha, \beta), X, dk),$$

where α and β are arbitrary numbers such that $0 < \alpha < \beta < \infty$. For $0 < \lambda_1 < \lambda_2 <$

∞ and $f, g \in C_0^0(\bar{I}, X)$ we have

$$(3.15) \quad ((E((\lambda_1, \lambda_2))f, g)) = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{Q}^\pm f(k), \mathcal{Q}^\pm g(k)) dk,$$

where $E(\lambda)$ is the resolution of the identity for A . Hence \mathcal{Q}^\pm are bounded, linear operators from $C_0^0(\bar{I}, X)$ contained in $H^0(I, X)$ into $L_2((0, \infty), X, dk)$.

(ii) Let \mathcal{F}^\pm be unique extensions of \mathcal{Q}^\pm to $H^0(I, X)$, respectively. Then for $\Delta = (\lambda_1, \lambda_2)$, $0 \leq \lambda_1 < \lambda_2 \leq \infty$ we have

$$(3.16) \quad E(\Delta) = (\mathcal{F}^\pm)^* \chi_{\sqrt{\Delta}} \mathcal{F}^\pm,$$

where $\chi_{\sqrt{\Delta}}$ is the characteristic function of $(\sqrt{\lambda_1}, \sqrt{\lambda_2})$ and $(\mathcal{F}^\pm)^*$ are the adjoints of \mathcal{F}^\pm acting from $L_2((0, \infty); X, dk)$ into $H^0(I, X)$, respectively.

The following corollary directly follows from (3.16).

Corollary 3.5. *The spectrum of A is absolutely continuous on $(0, \infty)$.*

For the proof of Theorem 3.4. we need

Lemma 3.6. *We have*

$$(3.17) \quad ((E((\lambda_1, \lambda_2))f, g))_0 = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} \frac{k dk}{\pi i} \int_I \int_I dr ds (\{G(r, s, k) - G(r, s, -k)\} f(s), g(r)),$$

where $0 < \lambda_1 < \lambda_2 < \infty$ and $f, g \in H^0(I, X)$ with compact carriers.

Proof. Let us start with the well-known relation⁷⁾

$$(3.18) \quad ((E(\lambda_1, \lambda_2))f, g))_0 = \lim_{\eta \downarrow 0} \lim_{\mu \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1+\eta}^{\lambda_2-\eta} ((\{R(\lambda + i\mu) - R(\lambda - i\mu)\} f, g))_0 d\lambda.$$

Using (2.13), $R(\lambda + i\mu)f$ can be represented by the resolvent kernel $G(r, s, k)$. Hence (3.18) becomes

$$(3.19) \quad \begin{aligned} & ((E((\lambda_1, \lambda_2))f, g))_0 \\ &= \lim_{\eta \downarrow 0} \lim_{\mu \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1+\eta}^{\lambda_2-\eta} ((\int_I \{G(\cdot, s, \sqrt{\lambda + i\mu}) - G(\cdot, s, \sqrt{\lambda - i\mu})\} f(s) ds, \\ & \quad g))_0 d\lambda, \quad 8) \\ &= \lim_{\eta \downarrow 0} \lim_{\mu \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1+\eta}^{\lambda_2-\eta} \int_0^R \int_0^R (\{G(r, s, \sqrt{\lambda + i\mu}) - G(r, s, \sqrt{\lambda - i\mu})\} f(s), \\ & \quad g(r)) ds dr d\lambda, \end{aligned}$$

7) See, for example, Dunford & Schwartz [1], p. 1202.

8) Here and in the sequel by \sqrt{z} is meant the branch of the square root of z with $\text{Im } \sqrt{z} \geq 0$.

where R is taken so large that the carriers of f and g are contained in $[0, R]$. Since $G(\cdot, s, \sqrt{\lambda \pm i\mu})f(s)$ is the radiative function for $\{L, \lambda \pm i\mu, \ell[s, f(s)], 0\}$, we can apply Theorem 3.7 in [7] to show that $G(r, s, \sqrt{\lambda \pm i\mu})f(s)$ is uniformly bounded for $(r, s, \lambda, \mu) \in [0, R] \times [0, R] \times [\lambda_1, \lambda_2] \times [0, 1]$ and

$$(3.20) \quad \lim_{\eta \downarrow 0} G(r, s, \sqrt{\lambda \pm i\mu})f(s) = G(r, s, \pm \sqrt{\lambda})f(s) \quad \text{in } X.$$

(3.17) follows from (3.19) and (3.20). Q.E.D.

Proof of Theorem 3.4. Since we have

$$(3.21) \quad \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} |\mathcal{Q}_n^\pm f - \mathcal{Q}_m^\pm f|^2 dk = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} |\mathcal{Q}_n^\pm f(k)|^2 dk + \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} |\mathcal{Q}_m^\pm f(k)|^2 dk \\ - \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{Q}_n^\pm f(k), \mathcal{Q}_m^\pm f(k)) dk - \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{Q}_m^\pm f(k), \mathcal{Q}_n^\pm f(k)) dk,$$

in order to show that $\{\mathcal{Q}_n^\pm f\}_{n=1}^\infty$ are Cauchy sequences in $L_2((\sqrt{\lambda_1}, \sqrt{\lambda_2}), X, dk)$ it suffices to show

$$(3.22) \quad \lim_{n, m \rightarrow \infty} \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{Q}_n^\pm f(k), \mathcal{Q}_m^\pm f(k)) dk = ((E(\lambda_1, \lambda_2))f, f)_0.$$

Recalling (3.9) of Lemma 3.3 and (3.5), we have

$$(3.23) \quad \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{Q}_n^\pm f(k), \mathcal{Q}_m^\pm f(k)) dk = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} dk \int_I \int_I dr ds (\Gamma_n^\pm(s, k) f(s), \\ \Gamma_m^\pm(r, k) f(r)) \\ = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} \frac{2k^2}{\pi} dk \int_0^R \int_0^R dr ds (\Gamma_n(s, \pm k) f(s), \Gamma_m(r, \pm k) f(r)) \\ = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} \pm \frac{k}{\pi i} dk \int_0^R \int_0^R dr ds [(\{G_n(r, s, \pm k) - G_m(r, s, \mp k)\} f(s), f(r)) \\ + \int_0^\infty (\{C_n(\tau) - C_m(\tau)\} G_n(\tau, s, \pm k) f(s), G_m(\tau, r, \pm k) f(r)) d\tau] \\ = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} \pm \frac{k dk}{i\pi} \int_0^R \int_0^R dr ds [K_{n,m}^{(1)}(r, s, R) + K_{n,m}^{(2)}(r, s, k)],$$

where we take $R > 0$ such that $[0, R]$ contains the carrier of f . Let us show that $\lim_{n, m \rightarrow \infty} K_{n,m}^{(2)} = 0$ and $|K_{n,m}^{(2)}(r, s, k)|$ is uniformly bounded for $(r, s, k) \in [0, R] \times [0, R] \times [\sqrt{\lambda_1}, \sqrt{\lambda_2}]$ and $n, m = 1, 2, \dots$. Using (1.5) we estimate $K_{n,m}^{(2)}(r, s, k)$ as follows:

$$(3.24) \quad |K_{n,m}^{(2)}| \leq c_0 \int_I |\psi_n(\tau) - \psi_m(\tau)| (1 + \tau)^{-1-\varepsilon} (|G_n(\tau, s, \pm k) f(s)| \\ + |B^{1/2}(\tau) G_n(\tau, s, \pm k) f(s)|) \times |G_m(\tau, r, \pm k) f(r)| d\tau \\ \leq 8c_0 \int_I |\psi_n(\tau) - \psi_m(\tau)| (1 + \tau)^{-1-\varepsilon} \{ |G_n(\tau, s, \pm k) f(s)| -$$

$$\begin{aligned}
& -|G(\tau, s, \pm k)f(s)|^2 + |G_m(\tau, r, \pm k)f(r) - G(\tau, r, \pm k)f(r)|^2 \\
& + |G(\tau, s, \pm k)f(s)|^2 + |G(\tau, r, \pm k)f(r)|^2 \\
& + |B^{1/2}(\tau)G_n(\tau, s, \pm k)f(s)|^2 \} d\tau \\
\leq & 8c_0 [\{ |G_n(\cdot, s, \pm k)f(s) - G(\cdot, s, \pm k)f(s)|^2_{-1-\varepsilon} \\
& + |G_m(\cdot, r, \pm k)f(r) - G(\cdot, r, \pm k)f(r)|^2_{-1-\varepsilon} \} \\
& + \int_{\varepsilon(m,n)}^{\infty} (1+\tau)^{-1-\varepsilon} \{ |G(\tau, s, \pm k)f(s)|^2 + |G(\tau, r, \pm k)f(r)|^2 \} d\tau \\
& + \xi(m, n)^{-2\varepsilon} |B^{1/2}(\cdot)G_n(\cdot, s, \pm k)f(s)|^2_{-1+\varepsilon}] \\
= & Q_1 + Q_2 + Q_3,
\end{aligned}$$

where we put $\xi(m, n) = \min(m, n)$. We see from (2.15) that $\ell[s, f(s)]$ is a $\mathcal{U}_{1+\varepsilon}(I)$ -valued, continuous function on $[0, R]$ such that

$$(3.25) \quad |||\ell[s, f(s)]|||_{1+\varepsilon} \leq \sqrt{2} (1+s)^{(1+\varepsilon)/2} |f(s)|.$$

Hence, applying Theorem 4.2 in [7], we see

$$(3.26) \quad Q_3 \leq c(\xi(m, n))^{-2\varepsilon} \sqrt{2} (1+s)^{(1+\varepsilon)/2} |f(s)| \rightarrow 0 \quad (m, n \rightarrow \infty)$$

uniformly on $(s, k) \in [0, R] \times [\sqrt{\lambda_1}, \sqrt{\lambda_2}]$, where c is a positive constant. Putting $v_n = G_n(\cdot, s, \pm k)f(s)$ and $\ell_n(s, k) \equiv \ell[s, f(s)]$ (or $v_m = G_m(\cdot, r, \pm k)f(r)$ and $\ell_m(r, k) \equiv \ell[r, f(r)]$) in (ii) of Theorem 4.3 in [7], we obtain

$$(3.27) \quad \begin{cases} ||G_n(\cdot, s, \pm k)f(s) - G(\cdot, s, \pm k)f(s)||_{-1-\varepsilon} \rightarrow 0, & n \rightarrow \infty, \\ ||G_m(\cdot, r, \pm k)f(r) - G(\cdot, r, \pm k)f(r)||_{-1-\varepsilon} \rightarrow 0, & m \rightarrow \infty \end{cases}$$

uniformly on $(r, s, k) \in [0, R] \times [0, R] \times [\sqrt{\lambda_1}, \sqrt{\lambda_2}]$, which implies the uniform convergence of $Q_1 \rightarrow 0$. It follows from (1.14) and (3.25) that Q_2 is uniformly bounded for $(r, s, k) \in [0, R] \times [0, R] \times [\sqrt{\lambda_1}, \sqrt{\lambda_2}]$ and $n, m = 1, 2, \dots$, and $Q_2 \rightarrow 0$ as $n, m \rightarrow \infty$. Thus we have shown that $K_{n,m}^{(2)}(r, s, k)$ is uniformly bounded and $K_{n,m}^{(2)}(r, s, k) \rightarrow 0$ as $n, m \rightarrow \infty$. For $K_{n,m}^{(1)}(r, s, k)$ we obtain from (3.27)

$$\begin{aligned}
(3.28) \quad & \int_0^R \int_0^R K_{n,m}^{(1)}(r, s, k) dr ds \\
& \rightarrow \int_0^R \int_0^R (\{G(r, s, \pm k) - G(r, s, \mp k)\} f(s), f(r)) dr ds
\end{aligned}$$

as $n, m \rightarrow \infty$ uniformly for $k \in [\sqrt{\lambda_1}, \sqrt{\lambda_2}]$. Therefore with the aid of Lemma 3.6 we have

$$\begin{aligned}
(3.29) \quad & \lim_{n, m \rightarrow \infty} \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{Q}_n^{\pm} f(k), \mathcal{Q}_m^{\pm} f(k)) dk \\
& = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} \frac{k dk}{i\pi} \int_0^R \int_0^R dr ds (\{G(r, s, k) - G(r, s, -k)\} f(s), f(r)) \\
& = ((E(\lambda_1, \lambda_2) f, f))_0,
\end{aligned}$$

which implies the convergence of $\{\mathcal{Q}_n^\pm f\}$.

We define $\mathcal{Q}_\pm f$ for $f \in C_0^0(\bar{I}, X)$ and $0 < \alpha < \beta < \infty$ by

$$(3.30) \quad \mathcal{Q}^\pm f = \lim_{n \rightarrow \infty} \mathcal{Q}_n^\pm f \quad \text{in } L_2((\alpha, \beta), X, dk)$$

Then putting $m=n$ in (3.22), we have

$$(3.31) \quad \begin{aligned} ((E(\lambda_1, \lambda_2))f, f)_0 &= \lim_{n \rightarrow \infty} \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} |\mathcal{Q}_n^\pm f(k)|^2 dk \\ &= \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} |\mathcal{Q}^\pm f(k)|^2 dk, \end{aligned}$$

which implies that $\mathcal{Q}^\pm f \in L_2((0, \infty), X, dk)$ and

$$(3.32) \quad ((E((\lambda_1, \lambda_2))f, g)_0 = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{Q}^\pm f(k), \mathcal{Q}^\pm g(k)) dk,$$

for $f, g \in C_0^0(\bar{I}, X)$ and $0 \leq \lambda_1 < \lambda_2 \leq \infty$. Thus (i) of Theorem 3.4 has been proved. (ii) follows directly from (i). Q.E.D.

4. The operators $G_n^\pm(\lambda)$ and $H_n^\pm(\lambda)$

In this and the following two sections we study the property of A from the standpoint of perturbation theory. Stationary methods⁹⁾ are useful for our purpose. In particular we shall make use of the results of Kato and Kuroda [6]. This section is a preliminary one. We shall define a family of operators $G_n^\pm(\lambda)$, $H_n^\pm(\lambda)$ ($\lambda > 0$, $n=1, 2, \dots$) and study the properties of them.

Let $C_n(r)$, L_n , A_n , \mathcal{F}_n^\pm etc. be as in §3. Let $L_0 = -\frac{d^2}{dr^2} + B(r)$ and A_0 be as in §1. Starting with L_0 , we can define the resolvent $R_0(z)$, the resolvent kernel $G_0(r, s, k)$, and the generalized Fourier transforms \mathcal{F}_0^\pm . In this section we put $C(r) = C_\infty(r)$, $L = L_\infty$, $A = A_\infty$, $R(z) = R_\infty(z)$ etc.

Let us set

$$(4.1) \quad \mathbf{C}_1 = \mathbf{C} - (-\infty, 0], \text{ i.e., } \mathbf{C}_1 = \{z \in \mathbf{C} / \sqrt{z} \in \mathbf{C}^+\}$$

and

$$(4.2) \quad \mathbf{C}_{1,+} = \{z \in \mathbf{C}_1 / \text{Im } z \geq 0\}, \quad \mathbf{C}_{1,-} = \{z \in \mathbf{C}_1 / \text{Im } z \leq 0\}.$$

Then $R_n(z, f) = R_n(\cdot, z, f)$ is defined as the radiative function for $\{L_n, \sqrt{z}, \mathcal{F}_n^\pm, 0\}$, where $z \in \mathbf{C}_1$, $f \in H^{1+\epsilon}(I, X)$ and $n=0, 1, 2, \dots, \infty$. By the regularity theorem of Jäger [3] (Satz 3.1) $C_m(r)R_n(r, z, f)$ is well-defined for any $f \in H^{1+\epsilon}(I, X) \cap H^{1+\epsilon}(I, X)$, where $n, m=0, 1, 2, \dots, \infty$.

9) For the literature of stationary methods see Kato [5] and Kato and Kuroda [6].

Lemma 4.1. *Let us assume Assumption 1.1. Let K be a compact set in \mathbf{C}_1 . Let $f \in H^{1,B}(I, X) \cap H^{1+\varepsilon}(I, X)$ and $z \in K$.*

(i) *Then there exists a constant $\delta_3 > 0$ such that*

$$(4.3) \quad \|C_m R_n(z, f)\|_{1+\varepsilon} \leq \delta_3 \|f\|_{1+\varepsilon} \quad (n, m = 0, 1, 2, \dots, \infty),$$

where δ_3 depends only on K .

(ii) *$C_m R_n(z, f)$ is an $H^{1+\varepsilon}(I, X)$ -valued continuous function on both $\mathbf{C}_{1,+}$ and $\mathbf{C}_{1,-}$.*

The proof is easy from Theorem 4.2 in [7] and we omit it.

It follows from Lemma 4.1 that the operator $C_m R_n(z, \cdot)$ from $H^{1,B}(I, X) \cap H^{1+\varepsilon}(I, X)$ into $H^{1+\varepsilon}(I, X)$ is uniquely extended to a bounded linear operator $Q_{m,n}(z)$ from $H^{1+\varepsilon}(I, X)$ into $H^{1+\varepsilon}(I, X)$. $Q_{m,n}(z)f$ is an $H^{1+\varepsilon}(I, X)$ -valued continuous function on both $\mathbf{C}_{1,+}$ and $\mathbf{C}_{1,-}$.

DEFINITION 4.2. For each $n = 1, 2, \dots, \infty$ and each $z \in \mathbf{C}_1$ bounded linear operators $G_n(z), H_n(z)$ from $H^{1+\varepsilon}(I, X)$ into $H^{1+\varepsilon}(I, X)$ are defined by

$$(4.4) \quad \begin{cases} G_n(z) = 1 + Q_{n,0}(z), \\ H_n(z) = 1 - Q_{n,n}(z). \end{cases}$$

In particular for $\lambda > 0$ we put

$$(4.5) \quad G_n^\pm(\lambda) = G_n(\lambda \pm i0) \quad \text{and} \quad H_n^\pm(\lambda) = H_n(\lambda \pm i0).$$

Lemma 4.3. *Let $G_n(z)$ and $H_n(z)$, $n = 1, 2, \dots, \infty$, be as above.*

(i) *Then we have for $z \in \mathbf{C} - \mathbf{R}$ and $f \in H^{1+\varepsilon}(I, X)$*

$$(4.6) \quad G_n(z)f = (A_n - z)R_n(z)f \quad \text{and} \quad H_n(z)f = (A_0 - z)R_n(z)f.$$

(ii) *For any $n = 1, 2, \dots, \infty$ and any $f \in H^{1+\varepsilon}(I, X)$, $G_n(z)f$ and $H_n(z)f$ are $H^{1+\varepsilon}(I, X)$ -valued, continuous functions on both $\mathbf{C}_{1,+}$ and $\mathbf{C}_{1,-}$.*

(iii) *Let $z \in \mathbf{C}_1$. Then we have*

$$(4.7) \quad G_n(z)H_n(z) = H_n(z)G_n(z) = 1 \quad \text{on } H^{1+\varepsilon}(I, X).$$

In particular we obtain for $\lambda > 0$

$$(4.8) \quad G_n^\pm(\lambda)H_n^\pm(\lambda) = H_n^\pm(\lambda)G_n^\pm(\lambda) = 1.$$

(iv) *For any $f \in H^{1+\varepsilon}(I, X)$ we have*

$$(4.9) \quad \begin{cases} \lim_{n \rightarrow \infty} G_n(z)f = G_\infty(z)f \\ \lim_{n \rightarrow \infty} H_n(z)f = H_\infty(z)f \end{cases}$$

in $H^{1+\varepsilon}(I, X)$ uniformly on any compact set in \mathbf{C}_1

Proof. Since we have

$$(4.10) \quad R_n(z, f) = R_n(z)f \quad (z \in \mathbf{C} - \mathbf{R}, f \in H^{1+\varepsilon}(I, X)),$$

it follows that

$$(4.11) \quad \begin{aligned} G_n(z)f &= \{1 + C_n R_0(z)\}f = \{1 + ((A_n - z) - (A_0 - z))R_0(z)\}f \\ &= (A_n - z)R_0(z)f \quad (f \in H^{1+\varepsilon}(I, X)). \end{aligned}$$

Similarly we have $H_n(z) = (L_0 - z)R_n(z)$ on $H^{1+\varepsilon}(I, X)$. (ii) follows from (ii) of Lemma 4.1. If $z \in \mathbf{C} - \mathbf{R}$, (4.7) follows from (4.6). Taking account of (ii), we can see that (4.7) is true for $z = \lambda > 0$. (iv) follows from (ii) of Theorem 4.3 in [7]. Q. E. D

Next we show some formulas involving $G_n(z)$, $H_n(z)$ and the radiative solutions $R_n(z, f)$. Let $\lambda > 0$ and $n = 0, 1, 2, \dots, \infty$. Then, according to Kato and Kuroda [6], we introduce a *bilinear form* $e_n(\lambda; \cdot, \cdot)$ on $H^{1+\varepsilon}(I, X) \times H^{1+\varepsilon}(I, X)$ by

$$(4.12) \quad e_n(\lambda; f, g) = \frac{1}{2\pi i} ((R_n(\lambda + i0, f) - R_n(\lambda - i0, f), g))_0.$$

Since $R_n(\lambda \pm i0, f) \in H^{-1-\varepsilon}(I, X)$, the right-hand side of (4.8) is well-defined. We can easily see that $e_n(\cdot; \cdot, \cdot)$ is a continuous function on $(0, \infty) \times H^{1+\varepsilon}(I, X) \times H^{1+\varepsilon}(I, X)$. It follows from (3.18), (4.12) and the continuity in z of the radiative function $R_n(z, f)$ that we have $e(\cdot; f, g) \in L^1((0, \infty); d\lambda)$ and

$$(4.13) \quad ((E_n(\Delta)f, g))_0 = \int_{\Delta} e_n(\lambda; f, g) d\lambda,$$

where Δ is a Borel set in $(0, \infty)$ and $f, g \in H^{1+\varepsilon}(I, X)$. The bilinear form $e_n(\lambda; f, g)$ is called the *spectral form* for $E_n(\lambda)$.

Lemma 4.4. (i) Let $z \in \mathbf{C}_1$, $f \in H^{1+\varepsilon}(I, X)$ and $n = 1, 2, \dots, \infty$. Then we have

$$(4.14) \quad R_n(z, f) = R_0(z, H_n(z)f),$$

and

$$(4.15) \quad R_0(z, f) = R_n(z, G_n(z)f)$$

in $H^{-1-\varepsilon}(I, X)$. In particular we have for $\lambda > 0$

$$(4.16) \quad \begin{cases} R_n(\lambda \pm i0, f) = R_0(\lambda \pm i0, H_n^{\pm}(\lambda)f) & \text{in } H^{-1-\varepsilon}(I, X) \\ R_0(\lambda \pm i0, f) = R_n(\lambda \pm i0, G_n^{\pm}(\lambda)f) \end{cases}$$

(ii) Let $\lambda > 0$ and let $n = 1, 2, 3, \dots, \infty$. Then the relations

$$(4.17) \quad \begin{cases} e_n(\lambda; G_n^\pm(\lambda)f, G_n^\pm(\lambda)g) = e_0(\lambda; f, g) \\ e_0(\lambda; H_n^\pm(\lambda)f, G_n^\pm(\lambda)g) = e_n(\lambda; f, g) \end{cases}$$

hold for any pair of $(f, g) \in H^{1+\epsilon}(I, X) \times H^{1+\epsilon}(I, X)$.

Proof. Let us prove (i). The both sides of (4.14) are $H^{-1-\epsilon}(I, X)$ -valued continuous functions on $H^{1+\epsilon}(I, X)$, and hence it suffices to show (4.14) for $f \in H^{1,B}(I, X) \cap H^{1+\epsilon}(I, X)$. For $\varphi \in C_0^{2,B}(I, X)$ we have

$$(4.18) \quad \begin{aligned} & ((R_0(z, H_n(z)f), (L_n - \bar{z})\varphi))_0 \\ &= ((R_0(z, H_n(z)f), (L_0 - \bar{z} + C_n)\varphi))_0 \\ &= ((H_n(z)f, \varphi))_0 + ((C_n R_0(z, H_n(z)f), \varphi))_0 \\ &= ((G_n(z)H_n(z)f, \varphi))_0 \\ &= ((f, \varphi))_0, \end{aligned}$$

i.e., $R_0(z, H_n(z)f)$ is the radiative function for $\{L_n, \sqrt{z}, \ell[f], 0\}$. It follows from the uniqueness of the radiative function (Theorem 2.2 in [7]) that we have $R_0(z, H_n(z)f) = R_n(z, f)$. In a similar way we can prove (4.15).

Next let us prove (ii). For $z \in \mathbb{C} - \mathbf{R}$ and $f, g \in H^{1+\epsilon}(I, X) \cap H^{1,B}(I, X)$ we have

$$(4.19) \quad \begin{aligned} & ((\{R_n(z) - R_n(\bar{z})\}G_n(z)f, G_n(z)g))_0 \\ &= ((R_n(z)G_n(z)f, G_n(z)g))_0 - ((G_n(z)f, R_n(z)G_n(z)g))_0 \\ &= ((R_0(z)f, \{1 + C_n R_0(z)\}g))_0 - ((\{1 + C_n R_0(z)\}f, R_0(z)g))_0 \\ &= ((\{R_0(z) - R_0(\bar{z})\}f, g))_0. \end{aligned}$$

Here we have made use of (4.6) and the relation $Q_{n,0}(z)f = C_n R_0(z)f$. Letting $z \rightarrow \lambda + i0$ in (4.19), we obtain

$$(4.20) \quad e_n(\lambda; G_n^\pm(\lambda)f, G_n^\pm(\lambda)g) = e_0(\lambda; f, g).$$

The both sides of (4.20) are bounded forms on $H^{1+\epsilon}(I, X) \times H^{1-\epsilon}(I, X)$, and hence (4.20) holds for all $f, g \in H^{1+\epsilon}(I, X)$. The other relations in (4.17) are obtained similarly. Q.E.D.

Finally we shall show some formulas involving \mathcal{F}_n^\pm , \mathcal{F}_0^\pm , $G_n^\pm(\lambda)$ and $H_n^\pm(\lambda)$.

Lemma 4.5. (i) Let $k > 0$ and $n = 1, 2, \dots$ ($n \neq \infty$). Let $f \in C_0^0(\bar{I}, X)$. Then we have

$$(4.21) \quad \mathcal{F}_0^\pm(H_n^\pm(k^2)f)(k) = \mathcal{F}_n^\pm f(k)$$

and

$$(4.22) \quad \mathcal{F}_n^\pm(G_n^\pm(k^2)f)(k) = \mathcal{F}_0^\pm f(k).$$

(ii) Let $n=1, 2, \dots, \infty$ and let Δ be a Borel set in $(0, \infty)$. Let $f, g \in H^{1+\epsilon}(I, X)$. Then we have

$$(4.23) \quad \begin{aligned} ((\mathcal{F}_n^\pm \chi_{\sqrt{\Delta}} \mathcal{F}_0^\pm f, g))_0 &= \int_{\Delta} e_n(\lambda; G_n^\pm(\lambda)f, g)d\lambda \\ &= \int_{\Delta} e_n(\lambda; f, H_n^\pm(\lambda)g)d\lambda, \end{aligned}$$

where $\chi_{\sqrt{\Delta}}$ is the characteristic function of $\sqrt{\Delta} = \{k > 0 / k^2 \in \Delta\}$.

Proof. First let us prove (i). We prove (4.21) only, since (4.22) can be shown similarly. Note that

$$(4.24) \quad R_n(z, g) = \int_I G_n(r, s, \sqrt{z})g(s)ds \quad (n = 0, 1, 2, \dots, \infty, z \in \mathbf{C}_1)$$

hold for $g \in H^0(I, X)$ with a compact carrier in \bar{I} , and that the carrier of $H_n^\pm(\lambda)f$ is compact in \bar{I} for $n=1, 2, \dots$, and $f \in C_0^0(\bar{I}, X)$, as can be seen from the fact that $C_n(r)=0$ for $r \geq n+1$. Then from the definition of \mathcal{F}_n^\pm and (4.16) of Lemma 4.4 we see

$$(4.25) \quad \begin{aligned} \mathcal{F}_n^\pm f(k) &= \int_I \Gamma_n^\pm(r, k)f(r)dr \\ &= \pm \sqrt{\frac{2}{\pi}} ik \lim_{t \rightarrow \infty} [e^{\pm itk} \int_I G_n(t, r, \pm k)f(r)dr] \\ &= \pm \sqrt{\frac{2}{\pi}} ik \lim_{t \rightarrow \infty} [e^{\pm itk} R_n(t, k^2 \pm i0, f)] \\ &= \pm \sqrt{\frac{2}{\pi}} ik \lim_{t \rightarrow \infty} [e^{\pm itk} R_0(t, k^2 \pm i0, H_n^\pm(k^2)f)] \\ &= \pm \sqrt{\frac{2}{\pi}} ik \lim_{t \rightarrow \infty} [e^{\pm itk} \int_I G_0(t, r, \pm k)(H_n^\pm(k^2)f)(r)dr] \\ &= \pm \sqrt{\frac{2}{\pi}} ik \int_I \Gamma_0(r, \pm k)(H_n^\pm(k^2)f)(r)dr \\ &= \mathcal{F}_0^\pm(H_n^\pm(k^2)f)(r). \end{aligned}$$

Next let us prove (ii). Obviously we may assume that $\Delta = (\lambda_1, \lambda_2)$, $0 < \lambda_1 < \lambda_2 < \infty$. Since we obtain from (4.5) and Theorems 4.2 and 4.3 in [7] that for fixed $f, g \in H^{1+\epsilon}(I, X)$ $e_n(\lambda; G_n(\lambda)f, g)$ is uniformly bounded for $\lambda \in \Delta$ and $n=1, 2, \dots$, and $e_n(\lambda; G_n^\pm(\lambda)f, g) \rightarrow e_\infty(\lambda; G_\infty^\pm(\lambda)f, g)$ as $n \rightarrow \infty$, we may assume that $n \neq \infty$. Both sides of (4.23) are bounded bilinear forms on $H^{1+\epsilon}(I, X) \times H^{1+\epsilon}(I, X)$, and hence we may assume that $f, g \in C_0^0(\bar{I}, X)$. Thus it suffices to show (4.23) for $n \neq \infty$, $\Delta = (\lambda_1, \lambda_2)$, $0 < \lambda_1 < \lambda_2 < \infty$, and $f, g \in C_0^0(\bar{I}, X)$.

Using (4.12) and the continuity of $((\{R_n(\lambda + i\mu) - R_n(\lambda - i\mu)\})(G_n^\pm(\lambda)f), g)_0$ on $\{(\lambda, \mu) / \lambda_1 \leq \lambda \leq \lambda_2, 0 \leq \mu \leq 1\}$, we have

$$\begin{aligned}
(4.26) \quad & \int_{\Delta} e_n(\lambda; G_n^{\pm}(\lambda)f, g) d\lambda \\
&= \int_{\Delta} \frac{1}{2\pi i} ((R_n(\lambda + i0, G_n^{\pm}(\lambda)f) - R_n(\lambda - i0, G_n^{\pm}(\lambda)f, g))_0 d\lambda \\
&= \lim_{\mu \rightarrow 0} \frac{1}{2\pi i} \int_{\Delta} I(\lambda, \mu) d\lambda,
\end{aligned}$$

where $I(\lambda, \mu) = ((\{R_n(\lambda + i\mu) - R_n(\lambda - i\mu)\}(G_n^{\pm}(\lambda)f), g))_0$. $I(\lambda, \mu)$ is calculated as follows:

$$\begin{aligned}
(4.27) \quad I(\lambda, \mu) &= \int_0^{\infty} (\mathcal{F}_n^{\pm} \{R_n(\lambda + i\mu) - R_n(\lambda - i\mu)\}(G_n^{\pm}(\lambda)f)(k), \mathcal{F}_n^{\pm} g(k)) dk \\
&+ \sum_j ((\{R_n(\lambda + i\mu) - R_n(\lambda - i\mu)\}(G_n^{\pm}(\lambda)f), \varphi_{n,j})_0 ((\varphi_{n,j}, g))_0 \\
&= \int_0^{\infty} \frac{2i\mu}{(k^2 - \lambda)^2 + \mu^2} (\mathcal{F}_n^{\pm}(G_n^{\pm}(\lambda)f)(k), \mathcal{F}_n^{\pm} g(k)) dk \\
&+ \sum_j \frac{2i\mu}{(\lambda_{n,j} - \lambda)^2 + \mu^2} ((G_n^{\pm}(\lambda)f, \varphi_{n,j})_0 ((\varphi_{n,j}, g))_0 \\
&= I_1(\lambda, \mu) + I_2(\lambda, \mu),
\end{aligned}$$

where $-\infty < \lambda_{n,1} \leq \lambda_{n,2} \leq \dots \leq \lambda_{n,j} \leq \dots \leq 0$ are the eigenvalues of A_n and for each j $\varphi_{n,j}$ is the normalized eigenfunction of A_n associated with $\lambda_{n,j}$. Here we have made use of the relation $(\mathcal{F}_n^{\pm})^* \mathcal{F}_n^{\pm} f + \sum_j (f, \varphi_{n,j}) \varphi_{n,j} = f$ for $f \in H^0(I, X)$ (Theorems 3.1 and 3.4 in §3). Since we can easily show that $I_2(\lambda, \mu) \rightarrow 0$, $\mu \rightarrow 0$, uniformly in $\lambda \in [\lambda_1, \lambda_2]$, we have

$$\begin{aligned}
(4.28) \quad & \int_{\Delta} e_n(\lambda; G_n^{\pm}(\lambda)f, g) d\lambda \\
&= \lim_{\mu \rightarrow 0} \int_{\Delta} \frac{1}{\pi} \int_I \frac{\mu}{(k^2 - \lambda)^2 + \mu^2} (\mathcal{F}_n^{\pm}(G_n^{\pm}(\lambda)f)(k), \mathcal{F}_n^{\pm} g(k)) dk d\lambda
\end{aligned}$$

Noting that

$$\begin{aligned}
(4.29) \quad & \int_{\Delta} \int_I \frac{1}{\pi} \frac{\mu}{(k^2 - \lambda)^2 + \mu^2} |(\mathcal{F}_n^{\pm}(G_n^{\pm}(\lambda)f)(k), \mathcal{F}_n^{\pm} g(k))| dk d\lambda \\
&\leq \frac{1}{\pi \mu} \int_{\Delta} \|G_n^{\pm}(\lambda)f\|_0 d\lambda \|g\|_0 < \infty,
\end{aligned}$$

we can change the order of integration to obtain

$$\begin{aligned}
(4.30) \quad & \int_{\Delta} e_n(\lambda; G_n^{\pm}(\lambda)f, g) d\lambda \\
&= \int_0^{\infty} \lim_{\mu \rightarrow 0} \int_{\Delta} \frac{1}{\pi} \frac{\mu}{(k^2 - \lambda)^2 + \mu^2} (\mathcal{F}_n^{\pm}(G_n^{\pm}(\lambda)f)(k), \mathcal{F}_n^{\pm} g(k)) d\lambda dk \\
&= \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{F}_n^{\pm}(G_n^{\pm}(k^2)f)(k), \mathcal{F}_n^{\pm} g(k)) dk
\end{aligned}$$

$$= \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{F}_n^\pm f(k), \mathcal{F}_n^\pm g(k)) dk = (((\mathcal{F}_n^\pm)^* \chi_{\sqrt{\lambda_1} \leq |k| \leq \sqrt{\lambda_2}} f, g))_0,$$

where we have made use of (i) and the well-known relation¹⁰⁾ for a continuous function $h(\lambda)$

$$(4.31) \quad \frac{1}{\pi} \lim_{\mu \downarrow 0} \int_a^\beta \frac{\mu}{(\alpha - \lambda)^2 + \mu^2} h(\lambda) d\lambda = \begin{cases} 0, & \text{if } a \notin (\alpha, \beta) \\ h(a), & \text{if } a \in (\alpha, \beta). \end{cases}$$

From (4.8) and (4.17) we can easily see that $e_n(\lambda; G_n^\pm(\lambda)f, g) = e_0(\lambda; f, H_n^\pm(\lambda)g)$. Thus we have proved (4.23) completely. Q.E.D.

5. The orthogonality of \mathcal{F}^\pm

The purpose of this section is to prove the orthogonality of the operators \mathcal{F}^\pm which have been constructed in §3 under the assumption that \mathcal{F}_0^\pm are orthogonal. By the orthogonality of \mathcal{F}^\pm we mean the relation $\mathcal{F}^\pm(\mathcal{F}^\pm)^* = 1$ on $L^2((0, \infty), X, dk)$ or, equivalently, that \mathcal{F}^\pm transform $H^0(I, X)$ onto $L^2((0, \infty), X, dk)$. For this we shall make use of the spectral representation for the absolutely continuous part of a spectral measure with values in the set of orthogonal projections in a Hilbert space which has been given in Kato and Kuroda [6], §1.

Put $\mathcal{X} = H^{1+\epsilon}(I, X)$ and define the spectral form e for $E(\lambda)$, the resolution of the identity associated with A , by

$$(5.1) \quad e(\lambda; f, g) = \frac{1}{2\pi i} ((R(\lambda + i0, f) - R(\lambda - i0, f), g))_0$$

for each $\lambda > 0$ as in §4, where $R(\lambda \pm i0, f)$ is the radiative function for $\{L, \pm\sqrt{\lambda}, \iota[f], 0\}$. Let $\mathcal{N}(\lambda)$ be the set of all f with $e(\lambda; f) = e(\lambda; f, f) = 0$. Then the quotient space $\mathcal{X}/\mathcal{N}(\lambda)$ is a pre-Hilbert space with the inner product induced by $e(\lambda; \cdot, \cdot)$. Its completion is denoted by $\mathcal{X}(\lambda)$ and the inner product and norm in $\mathcal{X}(\lambda)$ are denoted by $(\cdot, \cdot)_\lambda$ and $\|\cdot\|_\lambda$. We denote by $J(\lambda)$ the canonical map of \mathcal{X} onto $\mathcal{X}/\mathcal{N}(\lambda) \subset \mathcal{X}(\lambda)$. Let Δ be a Borel set in $(0, \infty)$. Then a vector field $F = \{F(\lambda)\}_{\lambda \in \Delta} \in \mathcal{X}(\Lambda) = \prod_{\lambda \in \Delta} \mathcal{X}(\lambda)$ is said to be e -measurable if there is a sequence h_n of quasi-simple functions on Δ to \mathcal{X} such that

$$(5.2) \quad \lim_{n \rightarrow \infty} \|F(\lambda) - J(\lambda)h_n(\lambda)\|_\lambda = 0 \quad \text{for a.e. } \lambda \in \Delta.$$

Here we mean by a quasi-simple function h a function of the form (finite sum)

$$(5.3) \quad h(\lambda) = \sum \alpha_k(\lambda) f_k,$$

where $f_k \in \mathcal{X}$ and $\alpha_k(\lambda)$ is a measurable, bounded scalar function on Δ . $\mathcal{M}(\Delta)$

10) See, for example, Titchmarsh [8], p. 31.

is the set of all e -measurable elements $F \in \mathcal{X}(\Delta)$ such that $\|F\|_{\mathcal{M}(\Delta)}^2 = \int_{\Delta} \|F(\lambda)\|_{\lambda}^2 d\lambda < \infty$. $\mathcal{M}(\Delta)$ is known to be a Hilbert space with the inner product

$$(5.4) \quad (F_1, F_2)_{\mathcal{M}(\Delta)} = \int_{\Delta} (F_1(\lambda), F_2(\lambda))_{\lambda} d\lambda$$

(Proposition 1.9 in Kato and Kuroda [6]). Then we see that there is a unitary map $\Pi(\Delta)$ from $E(\Delta)H^0(I, X)$ onto $\mathcal{M}(\Delta)$ which satisfies the following (a) and (b) (Theorem 1.11 in Kato and Kuroda [6]):

(a) We have

$$(5.5) \quad \Pi(\Delta)\alpha(E)u = \alpha\Pi(\Delta)u,$$

for $u \in E(\Delta)H^0(I, X)$ and a measurable, bounded scalar function $\alpha(\cdot)$ on Δ , where

$$(5.6) \quad \alpha(E) = \int_{\Delta} \alpha(\lambda) dE(\lambda).$$

(b) For each $f \in \mathcal{X} = H^{1+\varepsilon}(I, X)$ we have

$$(5.7) \quad \Pi(\Delta)E(\Delta)f = \{J(\lambda)f\}.$$

Let $e_0(\lambda; \cdot, \cdot)$ be the spectral form for E_0 as in §4. In a similar way, starting with $e_0(\lambda, \cdot, \cdot)$ and $\mathcal{X}_0 = H^{1+\varepsilon}(I, X)$, we define the null set $\mathcal{N}_0(\lambda)$, the quotient space $\mathcal{X}_0/\mathcal{N}_0(\lambda)$, the Hilbert space $\mathcal{X}_0(\lambda)$ with the inner product $(\cdot, \cdot)_{\lambda, 0}$, the canonical map $J_0(\lambda)$, the e_0 -measurability, the Hilbert space $\mathcal{M}_0(\Delta)$ with the inner product $(\cdot, \cdot)_{\mathcal{M}_0(\Delta)}$ and the unitary map $\Pi_0(\Delta)$. Thus we have obtained spectral representations for $E(\lambda)$ and $E_0(\Delta)$. We denote by $\mathcal{M}'(\Delta)$ (or $\mathcal{M}'_0(\Delta)$) the set of all $F(\lambda) \in \mathcal{M}(\Delta)$ (or $\mathcal{M}_0(\Delta)$) of the form $F(\lambda) = \{J(\lambda)h\}_{\lambda \in \Delta}$ (or $F(\lambda) = \{J_0(\lambda)h\}_{\lambda \in \Delta}$), where h is a quasi-simple function. $\mathcal{M}'(\Delta)$ and $\mathcal{M}'_0(\Delta)$ are dense in $\mathcal{M}(\Delta)$ and $\mathcal{M}_0(\Delta)$, respectively (Proposition 1.10 in Kato and Kuroda [6]). We put $G^{\pm}(\lambda) = G_{\infty}^{\pm}(\lambda)$ and $H^{\pm}(\lambda) = H_{\infty}^{\pm}(\lambda)$, where $G_{\infty}^{\pm}(\lambda)$ and $H_{\infty}^{\pm}(\lambda)$ are as in §4. Define the operator $G^{\pm}(\Delta)$ on $\mathcal{M}'_0(\Delta)$ into $\mathcal{M}(\Delta)$ and the operator $H^{\pm}(\Delta)$ on $\mathcal{M}'(\Delta)$ into $\mathcal{M}_0(\Delta)$ as follows:

$$(5.8) \quad \begin{cases} G^{\pm}(\Delta): \{J_0(\lambda)h\}_{\lambda \in \Delta} \rightarrow \{J(\lambda)G^{\pm}(\lambda)h\}_{\lambda \in \Delta} \\ H^{\pm}(\Delta): \{J(\lambda)h\}_{\lambda \in \Delta} \rightarrow \{J_0(\lambda)H^{\pm}(\lambda)h\}_{\lambda \in \Delta}. \end{cases}$$

From (ii) of Lemma 4.4 we obtain

$$(5.9) \quad \begin{cases} \|J(\lambda)G^{\pm}(\lambda)h\|_{\lambda}^2 = e(\lambda; G^{\pm}(\lambda)h) = e_0(\lambda; h) = \|J_0(\lambda)h\|_{\lambda, 0}^2, \\ \|J_0(\lambda)H^{\pm}(\lambda)h\|_{\lambda, 0}^2 = e_0(\lambda; H^{\pm}(\lambda)h) = e(\lambda; h) = \|J(\lambda)h\|_{\lambda}^2, \end{cases}$$

and hence we have

$$(5.10) \quad \begin{cases} \|G^\pm(\Delta)J_0(\cdot)h\|_{\mathcal{M}(\Delta)} = \|J_0(\cdot)h\|_{\mathcal{M}_0(\Delta)}, \\ \|H^\pm(\Delta)J(\cdot)h\|_{\mathcal{M}_0(\Delta)} = \|J(\cdot)h\|_{\mathcal{M}(\Delta)}. \end{cases}$$

DEFINITION 5.1. We define the operators $\tilde{G}^\pm(\Delta)$ and $\tilde{H}^\pm(\Delta)$ as unique extensions $G^\pm(\Delta)$ and $H^\pm(\Delta)$, respectively. $\tilde{G}^\pm(\Delta)$ is an isometric operator on $\mathcal{M}_0(\Delta)$ into $\mathcal{M}(\Delta)$. $\tilde{H}^\pm(\Delta)$ is an isometric operator on $\mathcal{M}(\Delta)$ into $\mathcal{M}_0(\Delta)$.

We shall show that $\tilde{G}^\pm(\Delta)$ and $\tilde{H}^\pm(\Delta)$ are really unitary operators.

Lemma 5.2. *Let Δ be a Borel set in $(0, \infty)$. Then $\tilde{G}^\pm(\Delta)$ is a unitary operator from $\mathcal{M}_0(\Delta)$ onto $\mathcal{M}(\Delta)$ and $\tilde{H}^\pm(\Delta)$ is a unitary operator from $\mathcal{M}(\Delta)$ onto $\mathcal{M}_0(\Delta)$. Further, we have*

$$(5.11) \quad \begin{cases} \tilde{G}^\pm(\Delta)\tilde{H}^\pm(\Delta) = 1 & \text{on } \mathcal{M}(\Delta), \\ \tilde{H}^\pm(\Delta)\tilde{G}^\pm(\Delta) = 1 & \text{on } \mathcal{M}_0(\Delta). \end{cases}$$

Proof. It follows from (iii) of Lemma 4.3 that

$$(5.12) \quad \begin{aligned} G^\pm(\Delta)H^\pm(\Delta)\{J(\lambda)h\}_{\lambda \in \Delta} &= G^\pm(\Delta)\{J_0(\lambda)H^\pm(\lambda)h\}_{\lambda \in \Delta} \\ &= \{J(\lambda)G^\pm(\lambda)H^\pm(\lambda)h\}_{\lambda \in \Delta} = \{J(\lambda)h\}_{\lambda \in \Delta} \end{aligned}$$

holds for any quasi-simple function h and for any $\lambda > 0$. Similarly we have

$$(5.13) \quad H^\pm(\Delta)G^\pm(\Delta)\{J_0(\lambda)h\}_{\lambda \in \Delta} = \{J_0(\lambda)h\}_{\lambda \in \Delta}.$$

(5.11) is obtained from (5.12) and (5.13). Hence $\tilde{G}^\pm(\Delta)$ and $\tilde{H}^\pm(\Delta)$ are unitary operators. Q.E.D.

Lemma 5.3. *Let Δ be a Borel set in $(0, \infty)$ and let $\mathcal{M}_0(\Delta)$, $\mathcal{M}(\Delta)$, $\Pi_0(\Delta)$, $\Pi(\Delta)$ be as above. Then for $f, g \in H^0(I, X)$ we have*

$$(5.14) \quad \begin{aligned} (((\mathcal{F}^\pm)^*\chi_{\sqrt{\Delta}}\mathcal{F}_0^\pm f, g))_0 &= (\tilde{G}^\pm(\Delta)\Pi_0(\Delta)E_0(\Delta)f, \Pi(\Delta)E(\Delta)g)_{\mathcal{M}(\Delta)} \\ &= (\Pi_0(\Delta)E_0(\Delta)f, \tilde{H}^\pm(\Delta)\Pi(\Delta)E(\Delta)g)_{\mathcal{M}_0(\Delta)}. \end{aligned}$$

where $\chi_{\sqrt{\Delta}}$ is the characteristic function of $\sqrt{\Delta} = \{k > 0 / k^2 \in \Delta\}$, and \mathcal{F}^\pm and \mathcal{F}_0^\pm are as above.

Proof. Let f_m and g_m be sequences such that $f_m, g_m \in H^{1+\epsilon}(I, X)$ and, $f_m \rightarrow f, g_m \rightarrow g$ in $H^0(I, X)$. Then it follows from Lemma 4.5 that

$$(5.15) \quad \begin{aligned} (((\mathcal{F}^\pm)^*\chi_{\sqrt{\Delta}}\mathcal{F}_0^\pm f_m, g_m))_0 &= \int_{\Delta} e(\lambda; G^\pm(\lambda)f_m, g_m) d\lambda \\ &= \int_{\Delta} (J(\lambda)G^\pm(\lambda)f_m, J(\lambda)g_m) d\lambda \\ &= (G^\pm(\Delta)\{J_0(\lambda)f_m\}_{\lambda \in \Delta}, \{J(\lambda)g_m\}_{\lambda \in \Delta})_{\mathcal{M}(\Delta)} \\ &= (\tilde{G}^\pm(\Delta)\Pi_0(\Delta)E_0(\Delta)f_m, \Pi(\Delta)E(\Delta)g_m)_{\mathcal{M}(\Delta)}, \end{aligned}$$

where we have made use of (5.7). Let $m \rightarrow \infty$ in (5.15). Then the left-hand side of (5.15) tends to $((\mathcal{F}^\pm)^* \chi_{\sqrt{\Delta}} \mathcal{F}_0^\pm f, g)_0$ and the right-hand side tends to $(\tilde{G}^\pm(\Delta) \Pi_0(\Delta) E_0(\Delta) f, \Pi(\Delta) E(\Delta) g)_{\mathcal{M}(\Delta)}$. From the unitarity of the mappings $\tilde{G}^\pm(\Delta)$ and $\tilde{H}^\pm(\Delta)$ we obtain $(\tilde{G}^\pm(\Delta) \Pi_0(\Delta) E_0(\Delta) f, \Pi(\Delta) E(\Delta) g)_{\mathcal{M}(\Delta)} = (\Pi_0(\Delta) E_0(\Delta) f, \tilde{H}^\pm(\Delta) \Pi(\Delta) E(\Delta) g)_{\mathcal{M}(\Delta)}$. Q.E.D.

Now we can show the orthogonality of \mathcal{F}^\pm assuming the orthogonality of \mathcal{F}_0^\pm .

Theorem 5.4. *Let us assume Assumption 1.1. Let \mathcal{F}^\pm and \mathcal{F}_0^\pm be as above. Suppose that \mathcal{F}_0^\pm is orthogonal. Then \mathcal{F}^\pm is also orthogonal. Similar results hold for \mathcal{F}^- .*

Proof. Let $\Delta = (0, \infty)$ in Lemma 5.4. We put $\tilde{G}^+((0, \infty)) = \tilde{G}^\pm$, $\tilde{H}^+((0, \infty)) = \tilde{H}^\pm$, $\mathcal{M}((0, \infty)) = \mathcal{M}$, $\mathcal{M}_0((0, \infty)) = \mathcal{M}_0$, $\Pi((0, \infty)) = \Pi$, $\Pi_0((0, \infty)) = \Pi_0$ etc. It suffices to show that $(\mathcal{F}^+)^* \mathcal{F}^+ f_0 = 0$ and $f_0 \in E_0(0, \infty) H^0(I, X)$ imply $f_0 = 0$ in $H^0(I, X)$, because \mathcal{F}_0^+ transforms $H^0(I, X)$ onto $L_2((0, \infty); X, dk)$ and $\mathcal{F}_0^+ E_0(0, \infty) f = \mathcal{F}_0^+ f$. It follows from Lemma 5.4 that

$$(5.16) \quad (\tilde{G}^\pm \Pi_0 f_0, \Pi g)_{\mathcal{M}} = 0$$

for any $g \in E((0, \infty)) H^0(I, X)$. Since $\Pi(E((0, \infty)) H^0(I, X)) = \mathcal{M}$ we have $\tilde{G}^\pm \Pi_0 f_0 = 0$. Hence we obtain from the unitarity of $\tilde{G}^\pm \Pi_0 f_0 = 0$ in \mathcal{M}_0 , which implies $f_0 = 0$ in $H^0(I, X)$ by the unitarity of Π_0 . Q.E.D.

6. The wave operators $W^\pm(\Delta)$

In this section we shall investigate the wave operators $W^\pm(\Delta)$ for A which will be defined according to Kato and Kuroda [6]¹¹⁾. We shall see the ranges of $W^\pm(\Delta)$ to be equal to $E(\Delta) H^0(I, X)$. We also discuss the invariance of the wave operators $W^\pm(\Delta)$.

To make use of the results of Kato and Kuroda [6], §5 and §7, we consider the Cayley transforms U and U_0 of A and A_0 , respectively. Set

$$(6.1) \quad \begin{cases} U = (A - i)(A + i)^{-1} = \int_0^{2\pi} e^{i\theta} F(d\theta), \\ U_0 = (A_0 - i)(A_0 + i)^{-1} = \int_0^{2\pi} e^{i\theta} F_0(d\theta), \end{cases}$$

where $F(\theta)$ and $F_0(\theta)$ are the resolutions of the identity on $(0, 2\pi)$ associated with U and U_0 , and we have

11) In this section we assume Assumption 1.1 only. We do not assume the orthogonality of \mathcal{F}_0^\pm .

$$(6.2) \quad \begin{cases} F(\theta) = E(\lambda(\theta)), & \lambda(\theta) = \frac{-\sin \theta}{1-\cos \theta}, \\ F_0(\theta) = E_0(\lambda(\theta)), \end{cases}$$

$E(\lambda)$ and $E_0(\lambda)$ being the resolutions of the identity on $(-\infty, \infty)$ associated with A and A_0 . Let us define $R_U(\zeta)$ and $R_{U_0}(\zeta)$, $\zeta \in \mathbf{C}$, $|\zeta| \neq 1$ by

$$(6.3) \quad \begin{cases} R_U(\zeta) = (1 - \zeta U^*)^{-1} = U(U - \zeta)^{-1}, \\ R_{U_0}(\zeta) = (1 - \zeta U_0^*)^{-1} = U_0(U_0 - \zeta)^{-1}. \end{cases}$$

Then we have by simple calculations

$$(6.4) \quad \begin{cases} R_U(\zeta) = \frac{1}{1-\zeta} \left\{ 1 + \frac{2i\zeta}{1-\zeta} R\left(\frac{1+\zeta i}{1-\zeta}\right) \right\} \\ R_{U_0}(\zeta) = \frac{1}{1-\zeta} \left\{ 1 + \frac{2i\zeta}{1-\zeta} R_0\left(\frac{1+\zeta i}{1-\zeta}\right) \right\}, \end{cases}$$

and hence if we write $\zeta = re^{i\theta}$, $\zeta' = r^{-1}e^{i\theta}$, we obtain for $f \in H^{1+\epsilon}(I, X)$

$$(6.5) \quad \begin{cases} \lim_{r \uparrow 1} R_U(\zeta)f = \frac{1}{1-e^{i\theta}} f + \frac{1}{i(1-\cos \theta)} R(\lambda(\theta) + i0, f) \\ \lim_{r \uparrow 1} R_U(\zeta')f = \frac{1}{1-e^{i\theta}} f + \frac{1}{i(1-\cos \theta)} R(\lambda(\theta) - i0, f) \end{cases}$$

in $H^{-1-\epsilon}(I, X)$. We obtain quite similar relations for $R_{U_0}(\zeta)$ and $R_{U_0}(\zeta')$. Since we have

$$(6.6) \quad \begin{cases} \lim_{r \uparrow 1} \frac{1}{2\pi} \langle ([R_U(\zeta) - R_U(\zeta')]u, v) \rangle_0 = \frac{d}{d\theta} \langle (F(\theta)u, v) \rangle_0, \\ \lim_{r \uparrow 1} \frac{1}{2\pi} \langle ([R_{U_0}(\zeta) - R_{U_0}(\zeta')]u, v) \rangle_0 = \frac{d}{d\theta} \langle (F_0(\theta)u, v) \rangle_0 \end{cases}$$

for $u, v \in H^0(I, X)$ and a.e. $\theta \in (\pi, 2\pi)$ ((5.4) in Kato and Kuroda [6]), the bilinear forms e_U and e_{U_0} on $H^{1+\epsilon}(I, X) \times H^{1+\epsilon}(I, X)$ defined by

$$(6.7) \quad \begin{cases} e_U(\theta; f, g) = \frac{1}{2\pi} \lim_{r \uparrow 1} \langle ([R_U(\zeta) - R_U(\zeta')]f, g) \rangle_0 \\ \quad = \frac{1}{1-\cos \theta} e(\lambda(\theta); f, g) \\ e_{U_0}(\theta; f, g) = \frac{1}{2\pi} \lim_{r \uparrow 1} \langle ([R_{U_0}(\zeta) - R_{U_0}(\zeta')]f, g) \rangle_0 \\ \quad = \frac{1}{1-\cos \theta} e_0(\lambda(\theta); f, g) \end{cases}$$

12) Note that $\lambda(\theta)$ maps $(\pi, 2\pi)$ onto $(0, \infty)$.

are the spectral forms for $F(\theta)$ and $F_0(\theta)$ for each $\theta \in (\pi, 2\pi)$, respectively. Put $\mathcal{X}_U = \mathcal{X}_{U_0} = H^{1+\varepsilon}(I, X)$. Then we construct, starting with e_U (or e_{U_0}), the Hilbert space $\mathcal{X}_U(\theta)$ (or $\mathcal{X}_{U_0}(\theta)$) with the inner product $(\cdot, \cdot)_{\theta, U}$ (or $(\cdot, \cdot)_{\theta, U_0}$), the Hilbert space $\mathcal{M}_U(\Gamma)$ (or $\mathcal{M}_{U_0}(\Gamma)$) for a Borel set Γ in $(\pi, 2\pi)$ with the inner product $(\cdot, \cdot)_{\mathcal{M}_U(\Gamma)}$ (or $(\cdot, \cdot)_{\mathcal{M}_{U_0}(\Gamma)}$) and the unitary map $\Pi_U(\Gamma)$ (or $\Pi_{U_0}(\Gamma)$) as in §5.

For each $\zeta \in \mathcal{C}$, $|\zeta| \neq 1$, we put

$$(6.8) \quad \begin{cases} G_U(\zeta) = R_U(\zeta)^{-1} R_{U_0}(\zeta) \\ H_U(\zeta) = R_{U_0}(\zeta)^{-1} R_U(\zeta). \end{cases}$$

Then by the definition of $R_U(\zeta)$ and $R_{U_0}(\zeta)$ we have

$$(6.9) \quad \begin{cases} G_U(\zeta) = (1 - \zeta U^*)(1 - \zeta U_0^*)^{-1} = 1 - \zeta(U^* - U_0^*)R_{U_0}(\zeta) \\ \quad = 1 + \frac{2i\zeta}{1 - \zeta} R(i)CR_0\left(\frac{1 + \zeta}{1 - \zeta}i\right) \\ H_U(\zeta) = 1 - \frac{2i\zeta}{1 - \zeta} R_0(i)CR\left(\frac{1 + \zeta}{1 - \zeta}i\right), \end{cases}$$

where $R(z) = (A - z)^{-1}$, $R_0(z) = (A_0 - z)^{-1}$. We define the operators $G_U^\pm(\theta)$ and $H_U^\pm(\theta)$, $\theta \in (\pi, 2\pi)$ by

$$(6.10) \quad \begin{cases} G_U^\pm(\theta)f = f + \frac{2ie^{i\theta}}{1 - e^{i\theta}} R(i)CR_0(\lambda(\theta) \pm i0, f) \\ H_U^\pm(\theta)f = f - \frac{2ie^{i\theta}}{1 - e^{i\theta}} R_0(i)CR(\lambda(\theta) \pm i0, f), \end{cases}$$

which transform $H^{1+\varepsilon}(I, X)$ onto itself. In fact we can see from (6.8) that the relations

$$(6.11) \quad G_U^\pm(\theta)H_U^\pm(\theta) = H_U^\pm(\theta)G_U^\pm(\theta) = 1$$

hold on $H^{1+\varepsilon}(I, X)$. We can also see from (6.8) that we have the relations

$$(6.12) \quad \begin{cases} e_U(\theta; G_U^\pm(\theta)f, G_U^\pm(\theta)g) = e_{U_0}(\theta; f, g) \\ e_{U_0}(\theta; H_U^\pm(\theta)f, H_U^\pm(\theta)g) = e_U(\theta; f, g) \end{cases}$$

for $\theta \in (\pi, 2\pi)$ and $f, g \in H^{1+\varepsilon}(I, X)$. Thus, as in Definition 5.1, the unitary operators $\tilde{G}_U^\pm(\Gamma)$ and $\tilde{H}_U^\pm(\Gamma)$ are induced by $G_U^\pm(\theta)$ and $H_U^\pm(\theta)$, respectively, i.e., $\tilde{G}_U^\pm(\Gamma)$ (or $\tilde{H}_U^\pm(\Gamma)$) are defined by unique extensions of the operators

$$(6.13) \quad \begin{cases} G_U^\pm(\Gamma): \{J_{U_0}(\theta)f\}_{\theta \in \Gamma} \rightarrow \{J_U(\theta)G_U^\pm(\theta)f\}_{\theta \in \Gamma} \\ (\text{or } H_U^\pm(\Gamma)): \{J_U(\theta)f\}_{\theta \in \Gamma} \rightarrow \{J_{U_0}(\theta)H_U^\pm(\theta)f\}_{\theta \in \Gamma}, \end{cases}$$

$J_U(\theta)$ (or $J_{U_0}(\theta)$) being the canonical map on \mathcal{X}_U (or \mathcal{X}_{U_0}) into $\mathcal{X}_U(\theta)$ (or $\mathcal{X}_{U_0}(\theta)$). $\tilde{G}_U^\pm(\Gamma)$ transform $\mathcal{M}_{U_0}(\Gamma)$ onto $\mathcal{M}_U(\Gamma)$ and $\tilde{H}_U^\pm(\Gamma)$ transform $\mathcal{M}_U(\Gamma)$

onto $\mathcal{M}_{U_0}(\Gamma)$.

DEFINITION 6.1 Let Δ be a Borel set in $(0, \infty)$. Put $\Gamma(\Delta) = \{\theta \in (\pi, 2\pi) / \lambda(\theta) \in \Delta\}$. Then the *wave operators* $W^\pm(\Delta)$ on $H^0(I, X)$ are defined by

$$(6.14) \quad W^\pm(\Delta)u = \Pi_U^{-1}(\Gamma(\Delta))\tilde{G}_U^\pm(\Gamma(\Delta))\Pi_{U_0}(\Gamma(\Delta))F_0(\Gamma(\Delta))u,$$

where $\tilde{G}_U^\pm(\Gamma)$ are defined as above.

Using the unitarity of $\tilde{G}^\pm(\Gamma)$ and $\tilde{H}^\pm(\Gamma)$ we can show

Theorem 6.2. *Let us assume Assumption 1.1. Let $W^\pm(\Delta)$ be as above. Then $W^\pm(\Delta)$ are partial isometries with initial set $E_0(\Delta)H^0(I, X)$ and final set $E(\Delta)H^0(I, X)$. In particular $W^\pm(I)$ are complete. $W^\pm(\Delta)$ have the intertwining property $\alpha(E)W^\pm(\Delta) = W^\pm(\Delta)\alpha(E_0)$ for any bounded measurable function α on Δ .*

The proof is almost the same as the proofs of Theorems 2.3 and 2.4 in Kato and Kuroda [6], and hence we omit it.

Now we turn to the problem of the invariance of $W^\pm(\Delta)$. Since $\mathcal{X} = \mathcal{X}_0 = H^{1+\varepsilon}(I, X)$ is a Hilbert space, we can apply Theorem 7.1 in Kato and Kuroda [6] to our case.

Theorem 6.3. *Let us assume Assumption 1.1. Let α be a real-valued function on $(0, \infty)$ such that*

$$(6.15) \quad \lim_{t \rightarrow \infty} \sum_{k=0}^{\infty} \left| \int_0^{2\pi} e^{-ik\theta - it\alpha(\lambda(\theta))} \eta(\theta) d\theta \right|^2 = 0$$

for every $\eta \in L^2(0, 2\pi)$. Then we have

$$(6.16) \quad \text{s-lim}_{t \rightarrow \pm\infty} e^{it\alpha(E)} e^{-it\alpha(E_0)} E_0(\Delta)u = W^\pm(\Delta)u$$

for all $u \in H^0(I, X)$, where $\lambda(\theta) = -\sin \theta / (1 - \cos \theta)$ as given in (6.2). In particular, we have

$$(6.17) \quad W^\pm(\Delta) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itA} e^{-itA_0} E_0(\Delta)$$

on $H^0(I, X)$.

Finally we represent $W^\pm(\Delta)$ using \mathcal{F}^\pm and \mathcal{F}_0^\pm

Theorem 6.4. *Let us assume Assumption 1.1. Let Δ be a Borel set in $(0, \infty)$. Then we have on $H^0(I, X)$*

$$(6.18) \quad (\mathcal{F}^\pm)^* \chi_{\sqrt{\Delta}} \mathcal{F}_0^\pm = W^\pm(\Delta).$$

For the proof we need

Lemma 6.5. *Let $G_U^\pm(\theta)$, $\theta \in (\pi, 2\pi)$ be as in (6.10) and let $G^\pm(\lambda)$, $\lambda > 0$ be as in Definition 4.2. Then we have for $f \in C_0^{2,B}(I, X)$*

$$(6.19) \quad G_U^\pm(\theta)f = (A - i)^{-1}G^\pm(\lambda(\theta))(A_0 - i)f.$$

Proof. We have by an easy computation

$$(6.20) \quad \begin{aligned} G_U(\zeta)f &= U^{-1}(U - \zeta)(U_0 - \zeta)^{-1}U_0 f \\ &= (A - i)^{-1}G\left(\frac{1 + \zeta i}{1 - \zeta}\right)(A_0 - i)f, \end{aligned}$$

where $\zeta = re^{i\theta}$, $r \neq 1$. Since $(A_0 - i)f \in H^{1+\epsilon}(I, X)$, we obtain (6.19) from (6.20), letting $r \uparrow 1$ and $r \downarrow 1$. Q.E.D.

Proof of Theorem 6.4. Let $f, g \in C_0^{2,B}(I, X)$. Then it follows from Lemma 5.3 that

$$(6.21) \quad \begin{aligned} &(((\mathcal{F}^\pm)^* \chi_{\sqrt{\Delta}} \mathcal{F}_0^\pm f, g))_0 \\ &= (((\mathcal{F}^\pm)^* \chi_{\sqrt{\Delta}} \mathcal{F}_0^\pm (A_0 - i)f, (A + i)^{-1}g))_0 \\ &= (\tilde{G}^\pm(\Delta) \Pi_0(\Delta) E_0(\Delta) (A_0 - i)f, \Pi(\Delta) E(\Delta) (A + i)^{-1}g)_{\mathcal{M}(\Delta)} \\ &= \int_{\Delta} e(\lambda; G^\pm(\lambda)(A_0 - i)f, (A + i)^{-1}g) d\lambda \\ &= \int_{\Gamma} e(\lambda(\theta); G^\pm(\lambda(\theta))(A_0 - i)f, (A + i)^{-1}g) \frac{1}{1 - \cos \theta} d\theta \quad (\Gamma = \Gamma(\Delta)) \\ &= \int_{\Gamma} e_U(\theta; G^\pm(\lambda(\theta))(A_0 - i)f, (A + i)^{-1}g) d\theta, \end{aligned}$$

where we have made use of $\frac{d\lambda}{d\theta} = \frac{1}{1 - \cos \theta}$ and (6.7). On the other hand we see from (6.7) and Lemma 6.5

$$(6.22) \quad \begin{aligned} &e_U(\theta; G^\pm(\lambda(\theta))(A_0 - i)f, (A + i)^{-1}g) \\ &= \frac{1}{2\pi} \lim_{\substack{\theta \uparrow 1 \\ \theta \uparrow 2}} (((R_U(\zeta) - R_U(\zeta')) G^\pm(\lambda(\theta))(A_0 - i)f, (A + i)^{-1}g))_0 \\ &= \frac{1}{2\pi} \lim_{\substack{\theta \uparrow 1 \\ \theta \uparrow 2}} (((R_U(\zeta) - R_U(\zeta')) (A - i)^{-1} G^\pm(\lambda(\theta))(A_0 - i)f, g))_0 \\ &= \frac{1}{2\pi} \lim_{\substack{\theta \uparrow 1 \\ \theta \uparrow 2}} (((R_U(\zeta) - R_U(\zeta')) G_U^\pm(\theta)f, g))_0 \\ &= e_U(\theta; G_U^\pm(\theta)f, g). \end{aligned}$$

(6.21) and (6.22) are cooperated to give

$$(6.23) \quad \begin{aligned} &(((\mathcal{F}^\pm)^* \chi_{\sqrt{\Delta}} \mathcal{F}_0^\pm f, g))_0 \\ &= \int_{\Gamma} e_U(\theta; G_U^\pm(\theta)f, g) d\theta \end{aligned}$$

$$\begin{aligned}
&= \int_{\Gamma} (G_U^{\pm}(\Gamma) J_{U_0}(\theta) f, J_U(\theta) g)_{\lambda, U} d\theta \\
&= (\tilde{G}_U^{\pm}(\Gamma) \Pi_{U_0}(\Gamma) F_0(\Gamma) f, \Pi_U(\Gamma) F(\Gamma) g)_{\mathcal{M}_U(\Gamma)} \\
&= ((F(\Gamma) \Pi_U^{-1}(\Gamma) \tilde{G}_U^{\pm}(\Gamma) \Pi_{U_0}(\Gamma) F_0(\Gamma) f, g))_0 \\
&= ((W^{\pm}(\Delta) f, g))_0.
\end{aligned}$$

Here we used (6.13) and (6.14). Thus we have proved (6.23) on $C_0^{2,B}(I, X)$. Since both $(\mathcal{F}^{\pm})^* \chi_{\sqrt{\Delta}} \mathcal{F}_0^{\pm}$ and $W^{\pm}(\Delta)$ are bounded operators, (6.23) holds on the whole $H^0(I, X)$. Q.E.D.

7. The Schrödinger operator in \mathbf{R}^n ($n \geq 3$)

The results obtained in the preceding sections can be applied to the Schrödinger operator in the whole space \mathbf{R}^n , $n \geq 3$.

We set in this section

$$(7.1) \quad X = L^2(S^{n-1}),$$

where S^{n-1} is the $(n-1)$ -sphere. Then there is a unitary operator V from $L^2(\mathbf{R}^n)$ onto $H^0(I, X)$ defined by

$$(7.2) \quad V : L^2(\mathbf{R}^n) \ni F(y) \rightarrow r^{(n-1)/2} F(r\omega) \in H^0(I, X),$$

where $y \in \mathbf{R}^n$, $r = |y|$ and $\omega = \frac{y}{r} \in S^{n-1}$.

Let us consider the Laplace operator in \mathbf{R}^n

$$(7.3) \quad -\Delta = - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.$$

We obtain

$$(7.4) \quad V(-\Delta) V^{-1} = - \frac{d^2}{dr^2} + \frac{1}{r^2} \left(-\Lambda_n + \frac{(n-1)(n-3)}{4} \right),$$

where Λ_n is the Laplace-Beltrami operator on S^{n-1} . We put

$$(7.5) \quad \begin{cases} B(r) = \frac{1}{r^2} \left(-\Lambda_n + \frac{(n-1)(n-3)}{4} \right), \\ \mathcal{D}(B(r)) = \mathcal{D}(\Lambda_n) = D. \end{cases}$$

Then it follows from (7.4) that we have for $\varphi \in C_0^{2,B}(I, X)$

$$(7.6) \quad \|\varphi\|_B = \|V^{-1}\varphi\|_1,$$

$\|\cdot\|_1$ being the norm of $\mathcal{D}_{L^2}^1(\mathbf{R}^n)$, i.e.,

$$(7.7) \quad \|F\|_1 = \int_{\mathbf{R}^n} \left\{ \sum_{j=1}^n \left\| \frac{\partial F}{\partial x_j} \right\|^2 + |F(x)|^2 \right\} dx.^{13)}$$

Hence (7.6) holds on $H_0^{1, B}(I, X)$. It is easy to see that $B(r)$ satisfies (B.1)~(B.4) of Assumption 1.1. (B.4) is implied by the compactness of the imbedding $\mathcal{D}_{L^2}^1(\Omega) \rightarrow L^2(\Omega)$ for any bounded domain Ω in \mathbf{R}^n . As is well known, the spectrum of the non-negative self-adjoint operator $-\Lambda_n$ is discrete. Let λ_0 be an eigenvalue of $-\Lambda_n$ and let $x_0 = x_0(\omega)$ be a normalized eigenfunction associated with λ_0 . Then we have

$$(7.8) \quad B(r)x_0 = \frac{1}{r^2} \left\{ \lambda_0 + \frac{(n-1)(n-3)}{4} \right\} x_0 \rightarrow 0 \quad (r \rightarrow \infty)$$

in $L^2(S^{n-1})$. Thus we have seen that the condition (3.1) is satisfied.

Define a symmetric operator T_0 by

$$(7.9) \quad \begin{cases} \mathcal{D}(T_0) = C_0^\infty(\mathbf{R}^n), \\ T_0 F = -\Delta F. \end{cases}$$

It is well-known¹⁴⁾ that T_0 is essentially self-adjoint with a unique extension H_0 .

Lemma 7.1. *Let H_0 be as above and let A_0 be as in Definition 1.1. Then we have $VH_0V^{-1} = A_0$.*

Proof. Noting that $\mathcal{D}(VT_0V^{-1}) = VC_0^\infty(\mathbf{R}^n) \subset \mathfrak{A}_0$, we can easily see that $M_0 \supset VT_0V^{-1}$, where M_0 is as given in (1.9). Since $\tilde{T}_0 = H_0^{15)}$, VT_0V^{-1} is essentially self-adjoint in $H^0(I, X)$. Therefore M_0 is essentially self-adjoint in $H^0(I, X)$. Thus we obtain

$$(7.10) \quad VH_0V^{-1} = V\tilde{T}_0V^{-1} = \widetilde{VT_0V^{-1}} = \tilde{M}_0 = A_0,$$

which completes the proof. Q.E.D.

Denote by $q(y)$ a real-valued function on \mathbf{R}^n . $q(y)$ is assumed to satisfy the following conditions:

(Q) $q(y)$ is a real-valued function which is continuously differentiable. Further, $q(y)$ behaves like $O(|y|^{-1-\varepsilon})$ ($0 < \varepsilon < 1$) at infinity, i.e., there exist $\rho > 0$, $C > 0$ and $0 < \varepsilon < 1$ such that

$$(7.11) \quad |q(y)| \leq c|y|^{-1-\varepsilon} \quad (|y| \geq \rho).$$

We define a symmetric operator T by

13) As usual $\mathcal{D}_{L^2}^1(\mathbf{R}^n)$ denotes the Hilbert space obtained by the completion of $C_0^\infty(\mathbf{R}^n)$ in the norm $\|\cdot\|_1$.

14) See Kato [4].

15) \tilde{T} means the closure of T .

$$(7.12) \quad \begin{cases} \mathcal{D}(T) = C_0^\infty(\mathbf{R}^n) \\ TF = -\Delta F + q(y)F = T_0 F + q(y)F \end{cases}$$

Since $q(y) \times$ is a bounded, linear operator and T_0 is essentially self-adjoint, T is also essentially self-adjoint¹⁶⁾. We put $\tilde{T} = H$. On the other hand we define an operatorvalued function $C(r)$ on I by

$$(7.13) \quad \begin{cases} \mathcal{D}(C(r)) = C, \\ C(r)f = q(r\omega)f(r). \end{cases}$$

We can easily see from (Q) that $C(r)$ satisfies (C-1)~(C-3) in Assumption 1.1. Let M be as in (1.10). Then, proceeding as in the proof of Lemma 7.1, we have the following

Lemma 7.2. *Let A be as in Definition 1.2. Then we have $VHV^{-1} = A$.*

Let $G_0(r, s, k)$, $\Gamma_0^\pm(r, k)$ and \mathcal{F}_0^\pm be as in §4. In order to show the orthogonality of \mathcal{F}_0^\pm , we have to calculate $\Gamma_0^\pm(r, k)$. Denote by $g_0(y, y', z)(y, y' \in \mathbf{R}^n, z \in \mathbf{C} - \mathbf{R})$ the Green kernel for H_0 . Then we have

$$(7.14) \quad g_0(y, y', z) = \frac{iz^{(n/4)-1/2}}{4(2\pi|y-y'|)^{(n/2)-1}} H_{(n/2)-1}^{(1)}(\sqrt{|z|}|y-y'|)^{17)},$$

where $H_v^{(1)}(t)$ is the Hankel function of the first kind. $G_0(z, r, k)$ is represented by $g_0(y, y', z)$ as follows:

$$(7.15) \quad \begin{aligned} G_0(s, r, k)x(\omega) &= s^{(n-1)/2}r^{(n-1)/2} \int_{S^{n-1}} g_0(s\omega, r\omega', k^2 + i0)x(\omega')d\omega' \\ &= \frac{i}{4}s^{(n-1)/2}r^{(n-1)/2}k^{(n/2)-1} \int_{S^{n-1}} \frac{H_{(n/2)-1}^{(1)}(k|s\omega-r\omega'|)}{(2\pi|s\omega-r\omega'|)^{(n/2)-1}} x(\omega')d\omega', \end{aligned}$$

where $k > 0$ and $x(\omega) \in L^2(S^{n-1})$. Thus by an easy computation we obtain from (3.3)

$$(7.16) \quad \begin{aligned} \Gamma_0(r, k)x(\omega) &= \lim_{s \rightarrow \infty} e^{-iks} G_0(s, r, k) \\ &= -e^{-(\pi/4)(n-1)i} \sqrt{\frac{\pi}{2}} \frac{1}{ik} \frac{k^{(n-1)/2}r^{(n-1)/2}}{(2\pi)^{n/2}} \int_{S^{n-1}} e^{-ikr(\omega, \omega')} x(\omega')d\omega', \end{aligned}$$

where $k > 0$ and we used the asymptotic formula

$$(7.17) \quad H_v^{(1)}(t) \sim \sqrt{\frac{2}{\pi t}} e^{i(t(-\pi/4)(2v+1))} \quad (t \rightarrow \infty)^{18)}.$$

16) See Kato [5], p. 287 – p. 293.

17) See Titchmarsh [9], p. 79.

18) See, for example, Watson [10], p. 197.

Hence we have

$$(7.18) \quad \Gamma_0^+(r, k)x(\omega) = -e^{-(\pi/4)(n-1)i} \frac{k^{(n-1)/2} r^{(n-1)/2}}{(2\pi)^{n/2}} \int_{S^{n-1}} e^{-ikr(\omega, \omega')} x(\omega') d\omega'.$$

On the other hand we have

$$\begin{aligned} (\Gamma_0(r, -k)x_1(\cdot), x_2(\cdot)) &= \lim_{s \rightarrow \infty} (e^{is\mathbf{k}} G_0(s, r, -k)x_1(\cdot), x_2(\cdot)) \\ &= \lim_{s \rightarrow \infty} (x_1(\cdot), e^{-is\mathbf{k}} G_0(r, s, k)x_2(\cdot)). \end{aligned}$$

As in (7.16) we have

$$\begin{aligned} (7.19) \quad \lim_{s \rightarrow \infty} e^{-is\mathbf{k}} G_0(r, s, k)x_2(\omega) \\ &= -e^{-(\pi/4)(n-1)i} \sqrt{\frac{\pi}{2}} \frac{1}{ik} \frac{r^{(n-1)/2} k^{(n-1)/2}}{(2\pi)^{n/2}} \int_{S^{n-1}} e^{-ikr(\omega, \omega')} x_2(\omega') d\omega' \end{aligned}$$

whence follows for $k > 0$

$$\begin{aligned} (7.20) \quad \Gamma_0^-(r, k)x(\omega) &= -\sqrt{\frac{2}{\pi}} ik \Gamma_0(r, -k)x(\omega) \\ &= -e^{(\pi/4)(n-1)i} \frac{r^{(n-1)/2} k^{(n-1)/2}}{(2\pi)^{n/2}} \int_{S^{n-1}} e^{ikr(\omega, \omega')} x(\omega') d\omega'. \end{aligned}$$

We see from (7.16) and (7.20) that \mathcal{F}_0^\pm are essentially Fourier transforms, and hence we have

Lemma 7.3. \mathcal{F}_0^\pm are orthogonal transforms.

By Lemmas 7.1~7.3 we can apply to H the results obtained in the preceding sections.

Theorem 7.4. Let $q(x)$ be a real-valued function on \mathbf{R}^n , $n \geq 3$, satisfying the conditions (Q). Let H be as above.

(i) Then H is bounded below. We have $\sigma_e(H) = (0, \infty)$ and the negative eigenvalues, if they exist, are of finite multiplicity. The spectrum of H on $(0, \infty)$ is absolutely continuous.

(ii) Denote by $E_H(\lambda)$ the resolution of the identity associated with H . Then there exist unitary operators $\mathcal{F}_H^\pm (= \mathcal{F}^\pm V)$ from $E_H((0, \infty))L^2(\mathbf{R}^n)$ onto $L^2((0, \infty))$, $L^2(S^{n-1})$, dk such that we have for a Borel set Δ in $(0, \infty)$

$$(7.21) \quad \begin{cases} E_H(\Delta) = (\mathcal{F}_H^\pm)^* \chi_{\sqrt{\Delta}} \mathcal{F}_H^\pm, \\ \mathcal{F}_H^\pm (\mathcal{F}_H^\pm)^* = 1. \end{cases}$$

(iii) Let Δ be as above. Then there exist the wave operators $W_H^\pm(\Delta)$ which are partial isometries with initial set $E_{H_0}(\Delta)L^2(\mathbf{R}^n)$ and final set $E_H(\Delta)L^2(\mathbf{R}^n)$,

where $E_{H_0}(\Delta)$ denotes the resolution of the identity associated with H_0 . $W_H^\pm(\Delta)$ have the intertwining property. We have

$$(7.22) \quad W_H^\pm(\Delta) = (\mathcal{F}_H^\pm)^* \chi_{\sqrt{\Delta}} \mathcal{F}_{H_0}^\pm,$$

where $\mathcal{F}_{H_0}^\pm = \mathcal{F}_0^\pm V$ which are unitary operators from $E_{H_0}((0, \infty))L^2(\mathbf{R}^n)$ onto $L^2((0, \infty), L^2(S^{n-1}), dk)$. For a real valued function α on $(0, \infty)$ which satisfies (6.15) for every $\eta \in L^2(0, 2\pi)$ we have

$$(7.23) \quad W_H^\pm(\Delta)F = \text{s-lim}_{t \rightarrow \pm\infty} e^{it\alpha(E_H)} e^{-it\alpha(E_{H_0})} E_{H_0}(\Delta)F$$

for every $F \in L^2(\mathbf{R}^n)$. In particular we have

$$(7.24) \quad W_H^\pm(\Delta) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} E_{H_0}(\Delta).$$

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References

- [1] N. Dunford and J. Schwartz: *Linear Operators, Part II*, Interscience, New York, 1963.
- [2] D.M. Eidus: *The principle of limit amplitude*, Uspehi Mat. Nauk. **24** (1969), 91–156 (Russian). (Russian Math. Surveys. **24** (1969), 97–169).
- [3] W. Jäger: *Ein gewöhnlicher Differentialoperator zweiter Ordnung für Funktionen mit Werten in einem Hilbertraum*, Math. Z. **113** (1970), 68–98.
- [4] T. Kato: *Fundamental properties of Hamiltonian operators of Schrödinger type*, Trans. Amer. Math. Soc. **70** (1951) 195–211.
- [5] ———: *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.
- [6] T. Kato and S.T. Kuroda: *Theory of simple scattering and eigenfunction expansions*, Functional analysis and related fields, ed. by Browder, F., Springer, Berlin, 1970, 99–131.
- [7] Y. Saitō: *The principle of limiting absorption for second order differential equations with operator-valued coefficients*, Publ. Res. Inst. Math. Sci. **7** (1972), 581–619.
- [8] E.C. Titchmarsh: *Introduction to the Theory of Fourier Integrals*, Oxford, 1937.
- [9] ———: *Eigenfunction Expansions Associated with Second-Order Differential Equations, Part II*, Oxford, 1958.
- [10] G.N. Watson: *Theory of Bessel Functions*, Cambridge, 1922.