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NOTE ON SINGULAR PERTURBATION FOR ABSTRACT DIFFERENTIAL EQUATIONS

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The present paper is concerned with abstract differential equations in a Banach space containing a small parameter in its coefficient

$$du_\varepsilon(t)/dt+A_\varepsilon(t)u_\varepsilon(t)=f_\varepsilon(t)\,. \tag{0.1}$$

As $\varepsilon\downarrow 0$ (0.1) degenerates to

$$du_0(t)/dt+A_0(t)u_0(t)=f_0(t)\,, \tag{0.2}$$

where $A_0(t)$ is weaker than $A_\varepsilon(t)$, $\varepsilon>0$, in the sense usually employed. We shall be interested in the behaviour of the solution $u_\varepsilon(t)$ of (0.1) as $\varepsilon\downarrow 0$, chiefly in the pointwise convergence of $u_\varepsilon(t)$ to the solution $u_0(t)$ of (0.2). The main theorem of section 2 is concerned with a sufficient condition in order that not only $u_\varepsilon(t)$ but also $A_\varepsilon(t)u_\varepsilon(t)$ and $du_\varepsilon(t)/dt$ converge to their corresponding limits in the weak topology for each fixed t. It is almost essential that the limit equation (0.2) is well posed, which should be admitted to be a restrictive assumption.

In section 3 an example to which the above theorem can be applied is considered making frequent use of T. Kato's results on maximal accretive operators ([1], [2], [3]). This example is the initial-boundary value problem for the equation with coefficients having a singularity along $x=t$

$$\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}+\frac{u}{(x-t)^2}=f\,,\qquad a<x<b$$

or

$$\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}+\frac{\varepsilon}{x-t}\frac{\partial u}{\partial x}+\frac{u}{(x-t)^2}=f\,,\qquad a<x<b$$

with the boundary condition $u(t,a)=u(t,b)=0$, and was first motivated

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by the construction of an example to which the main result of [4] on the initial value problem for the evolution equation

$$du(t)/dt+A(t)u(t)=f(t) \tag{0.3}$$

can be applied although $A(t)^{\alpha}$ has a variable domain whenever $\alpha>0$.

As a preparation a theorem on the unique solvability of the initial value problem for (0.3) is given in section 1 assuming among other things that

$$A(t)^{\rho}\cdot dA(t)^{-1}/dt \quad \text{is bounded and continuous in } t \tag{0.4}$$

for some $\rho>0$. This hypothesis which implies

$$\left\|\frac{\partial}{\partial t}(\lambda-A(t))^{-1}\right\|\leq\frac{C}{|\lambda|^{\rho}},$$

makes it possible to weaken the smoothness assumption of $A(t)$ as was made in [4], namely it enables us to remove the Hoelder continuity of $dA(t)^{-1}/dt$. It is a little interesting to note that (0.4) with $\rho=1$ implies the independence of the domain of $A(t)$ while if $\rho<1$ there exists a simple example for which $D(A(t)^{\alpha})$ is not independent of t whenever $\alpha>0$ although (0.4) is true.

1. We begin with a variant of the main theorem of [4]. By $D(A)$ and $R(A)$ we denote the domain and the range of an operator A.

Theorem 1.1. *For each $t\in[0, T]$ $A(t)$ is a densely defined, closed linear operator in a Banach space X. Let $A(t)$ satisfy the following assumptions:*

(I) *For each $t\in[0, T]$ the resolvent set of $A(t)$ contains a fixed closed angular domain*

$$\Sigma=\{\lambda: \arg\lambda\notin(-\theta_0, \theta_0)\}$$

where θ_0 is a positive number satisfying $0<\theta_0<\pi/2$. The resolvent of $A(t)$ satisfies

$$\|(\lambda-A(t))^{-1}\|\leq M/|\lambda| \tag{1.1}$$

for any $t\in[0, T]$ and $\lambda\in\Sigma$, where M is a constant which is independent of λ and t;

(II) *$A(t)^{-1}$, which is bounded by* (I), *is continuously differentiable in t in the uniform operator topology;*

(III) *There exists a positive number $\rho\leq 1$ such that $R(dA(t)^{-1}/dt)\subset D(A(t)^{\rho})$ and $A(t)^{\rho}\cdot dA(t)^{-1}/dt$ is strongly continuous in $t\in[0, T]$. Hence* *with some positive constant N independent of t we have*

$$\left\| A(t)^\rho \frac{d}{dt} A(t)^{-1} \right\| \leqq N . \tag{1.2}$$

Then there exists a fundamental solution $U(t, s)$, $0 \leqq s \leqq t \leqq T$, to the equation

$$du(t)/dt + A(t)u(t) = f(t) ; \tag{1.3}$$

if $s < t$, $R(U(t, s)) \subset D(A(t))$ and $U(t, s)$ satisfies

$$(\partial/\partial t)U(t, s) + A(t)U(t, s) = 0 , \qquad 0 \leqq s < t \leqq T , \tag{1.4}$$

$$U(s, s) = I . \tag{1.5}$$

There exists a positive constant C_0 such that

$$\left\| \frac{\partial}{\partial t} U(t, s) \right\| = \|A(t)U(t, s)\| \leqq \frac{C_0}{t-s} . \tag{1.6}$$

If $f(t)$ is strongly Hoelder continuous, then the unique solution of (1.3) *in $s < t \leqq T$ satisfying the initial condition $u(s) = u$ is given by*

$$u(t) = U(t, s)u + \int_s^t U(t, \sigma) f(\sigma) d\sigma . \tag{1.7}$$

Proof. In what follows $C_1, C_2, \cdots, C_8$ denote constants which depend only on θ_0, M, ρ, N and T. First we note that (III) implies

$$\left\| \frac{\partial}{\partial t} (\lambda - A(t))^{-1} \right\| \leqq \frac{C_1}{|\lambda|^\rho} , \tag{1.8}$$

which is a consequence of the formula

$$\begin{aligned} &(\partial/\partial t)(\lambda - A(t))^{-1} \\ &= -A(t)(\lambda - A(t))^{-1} \cdot dA(t)^{-1}/dt \cdot A(t)(\lambda - A(t))^{-1} \\ &= -A(t)^{1-\rho}(\lambda - A(t))^{-1} \cdot A(t)^\rho dA(t)^{-1}/dt \cdot A(t)(\lambda - A(t))^{-1} \end{aligned} \tag{1.9}$$

and the inequality

$$\|A(t)^{1-\rho}(\lambda - A(t))^{-1}\| \leqq C_2/|\lambda|^\rho . \tag{1.10}$$

Hence just as in [4], it is possible to construct the fundamental solution by means of E. E. Levi's method:

$$U(t, s) = \exp(-(t-s)A(t)) + \int_s^t \exp(-(t-\tau)A(t)) R(\tau, s) d\tau , \tag{1.11}$$

$$R_1(t, s) = -(\partial/\partial t + \partial/\partial s) \exp(-(t-s)A(t)) ;$$

$$R(t, s) = \sum_{m=1}^{\infty} R_m(t, s);$$

$$R_m(t, s) = \int_s^t R_1(t, \sigma) R_{m-1}(\sigma, s) d\sigma, \qquad m = 2, 3, \cdots.$$

$R_1(t, s)$ may also be expressed as

$$R_1(t, s) = -\frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)} \frac{\partial}{\partial t}(\lambda - A(t))^{-1} d\lambda,$$

where Γ is a smooth path running in $\sum$ from $\infty e^{-\theta_0 i}$ to $\infty e^{\theta_0 i}$. We have

$$||R_1(t, s)|| \leqq \frac{C_3}{(t-s)^{1-\rho}}, \qquad ||R(t, s)|| \leqq \frac{C_4}{(t-s)^{1-\rho}}. \tag{1.12}$$

Lemma 1.1. *If $s < t$, we have*

$$||A(t) \exp(-(t-s)A(t)) - A(s) \exp(-(t-s)A(s))|| \leqq \frac{C_5}{(t-s)^{1-\rho}}. \tag{1.13}$$

Proof. By (1.8)

$$||(\lambda - A(t))^{-1} - (\lambda - A(s))^{-1}|| \leqq \frac{C_1(t-s)}{|\lambda|^\rho},$$

hence the right member of (1.13) which is equal to

$$\left\| \frac{-1}{2\pi i}\int_\Gamma \lambda e^{-\lambda(t-s)}\{(\lambda - A(t))^{-1} - (\lambda - A(s))^{-1}\} d\lambda \right\|$$

is dominated by

$$\frac{C_1(t-s)}{2\pi}\int_\Gamma |\lambda|^{1-\rho} e^{-(t-s)Re\lambda} |d\lambda| \leqq \frac{C_5}{(t-s)^{1-\rho}}.$$

Lemma 1.2. *If $0 < \beta < \rho$ and $s < t$, then $A(t)^\beta R(t, s)$ is bounded and*

$$||A(t)^\beta R(t, s)|| \leqq \frac{C_6}{(t-s)^{1-\rho+\beta}}. \tag{1.14}$$

Proof. First we show that

$$||A(t)^\beta R_1(t, s)|| \leqq \frac{C_7}{(t-s)^{1-\rho+\beta}}, \tag{1.15}$$

(1.15) is a consequence of

$$A(t)^\beta R_1(t, s) = -\frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)} A(t)^\beta \frac{\partial}{\partial t}(\lambda - A(t))^{-1} d\lambda,$$

$$\|A(t)^{\beta}(\partial/\partial t)(\lambda-A(t))^{-1}\|$$
$$=\|A(t)^{1+\beta-\rho}(\lambda-A(t))^{-1}\cdot A(t)^{\rho}dA(t)^{-1}/dt\cdot A(t)(\lambda-A(t))^{-1}\|\leqq C_8/|\lambda|^{\rho-\beta}.$$

(1.14) follows from (1.15), (1.12) and

$$A(t)^{\beta}R(t,s)=A(t)^{\beta}R_1(t,s)+\int_s^t A(t)^{\beta}R_1(t,\tau)R(\tau,s)d\tau.$$

According to the above two lemmas we may write

$$A(t)U(t,s)=A(t)\exp(-(t-s)A(t))$$
$$+\int_s^t\{A(t)\exp(-(t-\tau)A(t))-A(\tau)\exp(-(t-\tau)A(\tau))\}R(\tau,s)d\tau$$
$$+\int_s^t A(\tau)^{1-\beta}\exp(-(t-\tau)A(\tau))A(\tau)^{\beta}R(\tau,s)d\tau.$$

The inequality (1.6) is a simple consequence of (1.13), (1.14) as well as the above formula. The remaining part of the proof is the same as the argument of [4].

REMARK. The assumption (III) enables us to remove the Hoelder continuity of $dA(t)^{-1}/dt$ in t which was used in [4] when we proved that $U(t,s)$ satisfies (1.4) in the strict sense.

2. Singular perturbation. Letting $A(t)$, $0\leqq t\leqq T$, be a family of linear closed operators in X,

DEFINITION 1. $u(t)$ is called a strict solution of

$$du(t)/dt+A(t)u(t)=f(t),\qquad s<t\leqq T, \tag{2.1}$$
$$u(s)=u \tag{2.2}$$

in $(s, T]$ if

(1) $u(t)$ is strongly continuous in the closed interval $[s, T]$ and strongly continuously differentiable in the left open interval $(s, T]$,

(2) for each $t\in(s, T]$, $u(t)\in D(A(t))$,

(3) $u(t)$ satisfies (2.1)-(2.2);

DEFINITION 2. $u(t)$ is called a weak solution of (2.1)-(2.2) in $(s, T]$ if

(1) $u(t)$ is weakly continuous in $[s, T]$,

(2) $u(t)$ satisfies

$$\int_s^T(u(t),\varphi'(t)-A^*(t)\varphi(t))dt+\int_s^T(f(t),\varphi(t))dt+(u,\varphi(s))=0$$

for any function $\varphi(t)$ with values in X^* satisfying

(i) for each t, $\varphi(t)\in D(A^*(t))$,

(ii) $\varphi(t)$, $\varphi'(t)=d\varphi(t)/dt$ and $A^*(t)\varphi(t)$ are strongly continuous in $[s, T]$,

(iii) $\varphi(T)=0$.

The above definition of weak solution is slightly different from the one given in [4] where a weak solution was assumed to be strongly continuous.

Theorem 2.1. *Suppose that X be reflexive. Let $A_\varepsilon(t)$, $0\leqq t\leqq T$, $0\leqq\varepsilon\leqq\varepsilon_0$, be a family of closed linear operators in X. Suppose that the assumptions of Theorem 1.1 are satisfied by $A_\varepsilon(t)$, $0\leqq t\leqq T$, $0<\varepsilon\leqq\varepsilon_0$, with constants θ_0, M, ρ and N which are independent of t and ε. We assume also that letting $A(t)$ stand for $A_{\varepsilon_0}(t)$*

(*a*) *$D(A_\varepsilon(t))\equiv D(A(t))$ and $D(A_\varepsilon^*(t))\equiv D(A^*(t))$ do not depend on ε if $0<\varepsilon\leqq\varepsilon_0$;*

(*b*) *$D(A_0(t))\supset D(A(t))$ and $D(A_0^*(t))\supset D(A^*(t))$;*

(*c*) *for each $\varphi\in D(A^*(t))$ $A_\varepsilon^*(t)\varphi\rightarrow A_0^*(t)\varphi$ strongly in X^* as $\varepsilon\downarrow 0$;*

(*d*) *$A_\varepsilon(t)A(t)^{-1}$, $A_0(t)A_\varepsilon(t)^{-1}$, $A_\varepsilon^*(t)A^*(t)^{-1}$ and $A_0^*(t)A_\varepsilon^*(t)^{-1}$ are all uniformly bounded with respect to ε and t and are continuous in t for each fixed ε in the strong operator topology in X or X^*.*

If the initial value problem

$$du(t)/dt+A_0(t)u(t) = f(t), \tag{2.4}$$

$$u(t) = u_0 \tag{2.5}$$

has only one weak solution which is also a strict solution when $f(t)$ is strongly Hoelder continuous in t, then the solution $u_\varepsilon(t)$ of the equation

$$du_\varepsilon(t)/dt+A_\varepsilon(t)u_\varepsilon(t) = f_\varepsilon(t) \tag{2.6}$$

converges to the solution of (2.4)-(2.5) in the following sense:

$$\begin{aligned}&\text{for each}\quad t\in(0, T]\ u_\varepsilon(t)\rightarrow u(t),\quad A_\varepsilon(t)u_\varepsilon(t)\rightarrow A_0(t)u(t),\\ &du_\varepsilon(t)/dt\rightarrow du(t)/dt\quad \text{all in the weak topology,}\end{aligned}$$

provided that

(i) *$u_\varepsilon(0)\rightarrow u_0$ weakly,*

(ii) *$f_\varepsilon(t)$ is uniformly Hoelder continuous:*

$$\|f_\varepsilon(t)-f_\varepsilon(s)\|\leqq F|t-s|^\alpha,\qquad F>0,\quad \alpha>0, \tag{2.7}$$

where F and α are independent of ε,

(iii) *$f_\varepsilon(t)$ converges to a strongly Hoelder continuous function $f(t)$ uniformly in the weak topology.*

In this section we denote by $C_9, C_{10}, \cdots$ constants which are dependent only on θ_0, M, ρ, N, T, F, α, $\sup_\varepsilon \|u_\varepsilon(0)\|$ and $\sup_{\varepsilon,t} \|f_\varepsilon(t)\|$.

Proof. If

$$U_\varepsilon(t, s) = \exp(-(t-s)A_\varepsilon(t)) + W_\varepsilon(t, s) \tag{2.8}$$

is the fundamental solution of (2.6), then by Theorem 1.1 there exists a constant C which is independent to t, s and ε such that

$$\left\|\frac{\partial}{\partial t} U_\varepsilon(t, s)\right\| = \|A_\varepsilon(t) U_\varepsilon(t, s)\| \leqq \frac{C}{t-s}, \tag{2.9}$$

$$\left\|\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right) \exp(-(t-s)A_\varepsilon(t))\right\| \leqq \frac{C}{(t-s)^{1-\rho}}, \tag{2.10}$$

$$\left\|\frac{\partial}{\partial t} \exp(-(t-s)A_\varepsilon(t))\right\| \leqq \frac{C}{t-s}, \tag{2.11}$$

$$\left\|\frac{\partial}{\partial t} W_\varepsilon(t, s)\right\| \leqq \frac{C}{(t-s)^{1-\rho}}, \quad \|A_\varepsilon(t) W_\varepsilon(t, s)\| \leqq \frac{C}{(t-s)^{1-\rho}}, \tag{2.12}$$

$$\|A_\varepsilon(t)^\gamma \exp(-(t-s)A_\varepsilon(t))\| \leqq \frac{C}{(t-s)^\gamma}, \quad 0 \leqq \gamma \leqq 1. \tag{2.13}$$

By the formula

$$u_\varepsilon(t) = U_\varepsilon(t, 0)u_\varepsilon(0) + \int_0^t U_\varepsilon(t, \sigma) f_\varepsilon(\sigma) d\sigma, \tag{2.14}$$

$$\begin{aligned} \frac{\partial}{\partial t} u_\varepsilon(t) = {} & \frac{\partial}{\partial t} U_\varepsilon(t, 0)u_\varepsilon(0) + \int_0^t \frac{\partial}{\partial t} \exp(-(t-\sigma)A_\varepsilon(t))(f_\varepsilon(\sigma) - f_\varepsilon(t)) d\sigma \\ & + \int_0^t \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \sigma}\right) \exp(-(t-\sigma)A_\varepsilon(t)) d\sigma \cdot f_\varepsilon(t) \qquad (2.15) \\ & + \exp(-(t-s)A_\varepsilon(t)) f_\varepsilon(t) + \int_0^t \frac{\partial}{\partial t} W_\varepsilon(t, \sigma) f_\varepsilon(\sigma) d\sigma \end{aligned}$$

as well as (2.7)$\sim$(2.13) we immediately see that

$$\|u_\varepsilon(t)\| \leqq C_9, \quad \left\|\frac{\partial}{\partial t} u_\varepsilon(t)\right\| \leqq \frac{C_9}{t}, \quad \|A_\varepsilon(t) u_\varepsilon(t)\| \leqq \frac{C_9}{t}. \tag{2.16}$$

If $\varphi(s)$ is an arbitrary function with values in X^* such that $\varphi(s) \in D(A^*(s))$ for each s (recall that $A(s) = A_{\varepsilon_0}(s)$) and $\varphi(s)$, $d\varphi(s)/ds = \varphi'(s)$, $A^*(s)\varphi(s)$ are all strongly continuous in $0 \leqq s \leqq T$, then $A_\varepsilon^*(s)\varphi(s) = A_\varepsilon^*(s)A^*(s)^{-1} \times A^*(s)\varphi(s)$ is also strongly continuous and for each t

$$
\begin{aligned}
&(u_\varepsilon(t),\ \varphi(t))-(u_\varepsilon(0),\ \varphi(0))-\int_0^t (u_\varepsilon(s),\ \varphi'(s))ds \\
&\quad +\int_0^t (u_\varepsilon(s),\ A_\varepsilon^*(s)\varphi(s))ds = \int_0^t (f_\varepsilon(s),\ \varphi(s))ds\,.
\end{aligned} \tag{2.17}
$$

For $1<p<\infty$ let $L^p(0,\ T;X)$ be the space of all measurable functions with values in X in $0<t<T$ for which $\|u(t)\|\in L^p(0,\ T)$. By Theorem 5.7 of [6], $(L^p(0,\ T;X))^*=L^{p'}(0,\ T;X^*)$ where $p^{-1}+p'^{-1}=1$, hence $L^p(0,\ T;X)$ is reflexive. Since $\{u_\varepsilon\}$ is a bounded sequence in $L^p(0,\ T;X)$ by (2.16), it contains a subsequence $\{u_{\varepsilon_i}\}$ which converges weakly to some function $u\in L^p(0,\ T;X)$. Replacing ε by ε_i in (2.17) and then letting $i\to\infty$, we get

$$
\begin{aligned}
&\lim_{i\to\infty}(u_{\varepsilon_i}(t),\ \varphi(t))-(u_0,\ \varphi(0))-\int_0^t (u(s),\ \varphi'(s))ds \\
&\quad +\int_0^t (u(s),\ A_0^*(s)\varphi(s))ds = \int_0^t (f(s),\ \varphi(s))ds\,.
\end{aligned} \tag{2.18}
$$

Choosing $A^*(s)^{-1}\varphi$ as $\varphi(s)$ in (2.18) with an arbitrary $\varphi\in X^*$, which is possible by the assumptions, we conclude that $\lim_{i\to\infty}(u_{\varepsilon_i}(t),\ A^*(t)^{-1}\varphi)$ exists. Since $A^*(t)^{-1}\varphi$ is an arbitrary element of $D(A^*(t))$ which is dense in X^* and $u_{\varepsilon_i}(t)$ is bounded by (2.16), it follows that $u_{\varepsilon_i}(t)$ converges weakly to some element $v(t)$ satisfying

$$
\begin{aligned}
&\|v(t)\|\leqq C_9\,, \\
&(v(t),\ \varphi(t))-(u_0,\ \varphi(0))-\int_0^t (u(s),\ \varphi'(s))ds \\
&\quad +\int_0^t (u(s),\ A_0^*(s)\varphi(s))ds = \int_0^t (f(s),\ \varphi(s))ds\,.
\end{aligned} \tag{2.19}
$$

Clearly

$$
(v(t),\ \varphi(t))-(v(\tau),\ \varphi(\tau))\to 0 \quad \text{as} \quad t-\tau\to 0\,. \tag{2.20}
$$

Choosing again $\varphi(s)=A^*(s)^{-1}\varphi$ and using (2.20)

$$
\begin{aligned}
&(v(t)-v(\tau),\ A^*(\tau)^{-1}\varphi) = (v(t),\ A^*(t)^{-1}\varphi) \\
&-(v(\tau),\ A^*(\tau)^{-1}\varphi)-(v(t),\ A^*(t)^{-1}\varphi-A^*(\tau)^{-1}\varphi)\to 0 \quad \text{as} \quad t\to\tau\,.
\end{aligned} \tag{2.21}
$$

Using (2.19) and (2.21) and noting that $D(A^*(\tau))$ is dense we conclude that $v(t)$ is weakly continuous in $0\leqq t\leqq T$. If φ is an arbitrary element of $L^{p'}(0,\ T;X^*)$, then

$$
\int_0^T (u_{\varepsilon_i}(t)-v(t),\ \varphi(t))dt\to 0 \quad \text{as} \quad i\to\infty\,,
$$

because of the measurability of the integrand and of the well known theorem on dominated convergence sequences. Thus u and v are both a weak limit of $\{u_{\varepsilon_i}\}$, which implies that $u(t)\equiv v(t)$ is weakly continuous in $0\leqq s\leqq T$ and $u(0)=u_0$. When $s>0$, $A_\varepsilon(s)u_\varepsilon(s)$ is bounded by (2.16), and so is $A_0(s)u_\varepsilon(s)$ due to the assumed uniform boundedness of $A_0(s)A_\varepsilon(s)^{-1}$. It follows consequently that $u(s)\in D(A_0(s))$ and

$$A_0(s)u_{\varepsilon_i}(s)\to A_0(s)u(s)\,, \tag{2.22}$$

$$A_{\varepsilon_i}(s)u_{\varepsilon_i}(s)\to A_0(s)u(s) \tag{2.23}$$

both in the weak topology. Next if $\varphi(s)$ is an arbitrary function with values in X^* such that $\varphi(s)\in D(A_0^*(s))$ and $\varphi(s)$, $\varphi'(s)$, $A_0^*(s)\varphi(s)$ are all strongly continuous in $0\leqq s\leqq T$, then for any $\delta>0$, we have

$$\begin{aligned}(u_{\varepsilon_i}(t),\ \varphi(t))-(u_{\varepsilon_i}(\delta),\ \varphi(\delta))-\int_\delta^t (u_{\varepsilon_i}(s),\ \varphi'(s))ds\\ +\int_\delta^t (A_{\varepsilon_i}(s)u_{\varepsilon_i}(s),\ \varphi(s))ds = \int_\delta^t (f_{\varepsilon_i}(s),\ \varphi(s))ds\,.\end{aligned}$$

Letting $i\to\infty$ and then $\delta\downarrow 0$, we get noting (2.23)

$$\begin{aligned}(u(t),\ \varphi(t))-(u_0,\ \varphi(0))-\int_0^t (u(s),\ \varphi'(s))ds\\ +\int_0^t (u(s),\ A_0^*(s)\varphi(s))ds = \int_0^t (f(s),\ \varphi(s))ds\,,\end{aligned}$$

which shows that $u(t)$ is a weak solution of (2.4)-(2.5), hence by the assumption it is the unique strict solution of the same problem. Therefore it follows that the original sequence $\{u_\varepsilon\}$ itself converges to u weakly in $L^p(0,\ T;\ X)$. We furthermore conclude that for each $t\in(0,\ T]$ $A_\varepsilon(t)u_\varepsilon(t)\to A_0(t)u(t)$ in the weak topology of X, and hence also that $du_\varepsilon(t)/dt\to du(t)/dt$ in the same sense.

3. Example. As an application of Theorem 2.1 we consider the following example. Let $-\infty<a<0<T<b<\infty$ and for each $t\in[0,\ T]$

$$V(t)=\left\{u\in L^2(a,\ b):\frac{du}{dx},\ \frac{u}{x-t}\in L^2(a,\ b),\ u(a)=u(b)=0\right\}$$

where the derivatives in the above as well as in what follows are interpreted in the distribution sense. $a_\varepsilon(t\,;\ u,\ v)$ denotes a family of sesquilinear forms on $V(t)\times V(t)$ defined by either of

(1) $$a_\varepsilon(t\,;\ u,\ v)=\int_a^b\left\{\varepsilon\frac{du}{dx}\frac{\overline{dv}}{dx}+\frac{u\bar v}{(x-t)^2}\right\}dx\,,$$

(2) $$a_\varepsilon(t\,;\ u,\ v)=\int_a^b\left\{\varepsilon\frac{du}{dx}\frac{\overline{dv}}{dx}+\varepsilon\frac{du}{dx}\frac{\bar v}{x-t}+\frac{u\bar v}{(x-t)^2}\right\}dx\,.$$

$A_\varepsilon(t)$ is an operator corresponding to $a_\varepsilon(t; u, v)$ which is defined in the following usual manner:

$$\begin{aligned} &u \in V(t) \quad \text{belongs to } D(A_\varepsilon(t)) \quad \text{and} \quad A_\varepsilon(t)u = f \in L^2(a, b) \\ &\text{if} \quad a_\varepsilon(t; u, v) = (f, v) \quad \text{for every} \quad v \in V(t). \end{aligned}$$

Thus $A_\varepsilon(t)$ is a differential operator

$$(A_\varepsilon(t)u)(x) = -\varepsilon\frac{d^2u}{dx^2} + \frac{u}{(x-t)^2} \qquad \text{or}$$

$$(A_\varepsilon(t)u)(x) = -\varepsilon\frac{d^2u}{dx^2} + \frac{\varepsilon}{x-t}\frac{du}{dx} + \frac{u}{(x-t)^2}$$

restricted to some class of functions satisfying $u(a)=u(b)=0$. In the first case $A_\varepsilon(t)$ is positive definite while in the second case $A_\varepsilon(t)$ is not self-adjoint although it is regularly accretive in the terminology of Kato [1]. These two cases can be treated quite similarly and we shall confine ourselves to the second case in what follows. The adjoint form $a_\varepsilon^*(t; u, v)$ of $a_\varepsilon(t; u, v)$ is

$$a_\varepsilon^*(t; u, v) = \int_a^b \left\{\varepsilon\frac{du}{dx}\frac{d\bar{v}}{dx} - \varepsilon\frac{du}{dx}\frac{\bar{v}}{x-t} + (1+\varepsilon)\frac{u\bar{v}}{(x-t)^2}\right\} dx$$

which is also defined on $V(t)\times V(t)$. As is easily seen we have

$$|\mathrm{Im}\, a_\varepsilon(t; u, u)| \leqq \frac{1}{2} Re\, a_\varepsilon(t; u, u),$$

namely the index (Kato [1]) of $a_\varepsilon(t; u, v)$ does not exceed $\frac{1}{2}$. Hence by Theorem 2.2 of [1] any complex number λ with $|\arg\lambda| > \tan^{-1}\frac{1}{2}$ belongs to the resolvent set of $A_\varepsilon(t)$ and

$$\|(\lambda - A_\varepsilon(t))^{-1}\| \leqq \begin{cases} (|\lambda| \sin(|\arg\lambda| - \tan^{-1}\frac{1}{2}))^{-1}, \\ \qquad\qquad\qquad \tan^{-1}\frac{1}{2} < |\arg\lambda| \leqq \pi/2 + \tan^{-1}\frac{1}{2}, \\ |\lambda|^{-1}, \qquad |\arg\lambda| > \pi/2 + \tan^{-1}\frac{1}{2}. \end{cases}$$

Thus (I) of Theorem 1.1 is satisfied by $\{A_\varepsilon(t)\}$ uniformly with respect to t and ε. The real part of $a_\varepsilon(t; u, v)$ is

$$Re\, a_\varepsilon(t; u, v) = \int_a^b \left\{\varepsilon\frac{du}{dx}\frac{\overline{dv}}{dx} + \left(1+\frac{\varepsilon}{2}\right)\frac{u\bar{v}}{(x-t)^2}\right\} dx,$$

its corresponding operator being denoted by $H_\varepsilon(t)$. It is also possible to express the solutions of $A_\varepsilon(t)u_\varepsilon(t)=g$, $H_\varepsilon(t)v_\varepsilon(t)=g$ and $A_\varepsilon^*(t)w_\varepsilon(t)=g$ for given $g\in L^2(a, b)$ explicitly all of which may be written below for the sake of convenience:

$$\left.\begin{aligned}
u_\varepsilon(t,x) &= \frac{1}{2\varepsilon\sqrt{1+\varepsilon^{-1}}}\left\{\int_x^t (t-y)^{\sqrt{1+\varepsilon^{-1}}} g(y)dy(t-x)^{1-\sqrt{1+\varepsilon^{-1}}}\right.\\
&\quad -\int_a^t (t-y)^{\sqrt{1+\varepsilon^{-1}}} g(y)dy(t-a)^{-2\sqrt{1+\varepsilon^{-1}}}(t-x)^{1+\sqrt{1+\varepsilon^{-1}}}\\
&\quad \left.+\int_a^x (t-y)^{-\sqrt{1+\varepsilon^{-1}}} g(y)dy(t-x)^{1+\sqrt{1+\varepsilon^{-1}}}\right\} \quad \text{if } a\leqq x<t,\\
u_\varepsilon(t,x) &= \frac{1}{2\varepsilon\sqrt{1+\varepsilon^{-1}}}\left\{\int_t^x (y-t)^{\sqrt{1+\varepsilon^{-1}}} g(y)dy(x-t)^{1-\sqrt{1+\varepsilon^{-1}}}\right.\\
&\quad -\int_t^b (y-t)^{\sqrt{1+\varepsilon^{-1}}} g(y)dy(b-t)^{-2\sqrt{1+\varepsilon^{-1}}}(x-t)^{1+\sqrt{1+\varepsilon^{-1}}}\\
&\quad \left.+\int_x^b (y-t)^{-\sqrt{1+\varepsilon^{-1}}} g(y)dy(x-t)^{1+\sqrt{1+\varepsilon^{-1}}}\right\} \quad \text{if } t<x\leqq b;
\end{aligned}\right\} \tag{3.1}$$

$$\left.\begin{aligned}
v_\varepsilon(t,x) &= \frac{1}{\varepsilon\sqrt{3+4\varepsilon^{-1}}}\left\{\int_x^t (t-y)^{\frac{\sqrt{3+4\varepsilon^{-1}}+1}{2}} g(y)dy(t-x)^{\frac{1-\sqrt{3+4\varepsilon^{-1}}}{2}}\right.\\
&\quad -\int_a^t (t-y)^{\frac{\sqrt{3+4\varepsilon^{-1}}+1}{2}} g(y)dy(t-a)^{-\sqrt{3+4\varepsilon^{-1}}}(t-x)^{\frac{1+\sqrt{3+4\varepsilon^{-1}}}{2}}\\
&\quad \left.+\int_a^x (t-y)^{\frac{1-\sqrt{3+4\varepsilon^{-1}}}{2}} g(y)dy(t-x)^{\frac{\sqrt{3+4\varepsilon^{-1}}+1}{2}}\right\} \quad \text{if } a\leqq x<t,\\
v_\varepsilon(t,x) &= \frac{1}{\varepsilon\sqrt{3+4\varepsilon^{-1}}}\left\{\int_x^t (y-t)^{\frac{\sqrt{3+4\varepsilon^{-1}}+1}{2}} g(y)dy(x-t)^{\frac{1-\sqrt{3+4\varepsilon^{-1}}}{2}}\right.\\
&\quad -\int_t^b (y-t)^{\frac{\sqrt{3+4\varepsilon^{-1}}+1}{2}} g(y)dy(b-t)^{-\sqrt{3+4\varepsilon^{-1}}}(x-t)^{\frac{1+\sqrt{3+4\varepsilon^{-1}}}{2}}\\
&\quad \left.+\int_x^b (y-t)^{\frac{1-\sqrt{3+4\varepsilon^{-1}}}{2}} g(y)dy(x-t)^{\frac{\sqrt{3+4\varepsilon^{-1}}+1}{2}}\right\} \quad \text{if } t<x\leqq b;
\end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned}
w_\varepsilon(t,x) &= \frac{1}{2\varepsilon\sqrt{1+\varepsilon^{-1}}}\left\{\int_x^t (t-y)^{\sqrt{1+\varepsilon^{-1}}+1} g(y)dy(t-x)^{-\sqrt{1+\varepsilon^{-1}}}\right.\\
&\quad -\int_a^t (t-y)^{\sqrt{1+\varepsilon^{-1}}+1} g(y)dy(t-a)^{-2\sqrt{1+\varepsilon^{-1}}}(t-x)^{\sqrt{1+\varepsilon^{-1}}}\\
&\quad \left.+\int_a^x (t-y)^{1-\sqrt{1+\varepsilon^{-1}}} g(y)dy(t-x)^{\sqrt{1+\varepsilon^{-1}}}\right\} \quad \text{if } a\leqq x<t,\\
w_\varepsilon(t,x) &= \frac{1}{2\varepsilon\sqrt{1+\varepsilon^{-1}}}\left\{\int_t^x (y-t)^{\sqrt{1+\varepsilon^{-1}}+1} g(y)dy(x-t)^{-\sqrt{1+\varepsilon^{-1}}}\right.\\
&\quad -\int_t^b (y-t)^{\sqrt{1+\varepsilon^{-1}}+1} g(y)\,dy(b-t)^{-2\sqrt{1+\varepsilon^{-1}}}(x-t)^{\sqrt{1+\varepsilon^{-1}}}\\
&\quad \left.+\int_x^b (y-t)^{1-\sqrt{1+\varepsilon^{-1}}} g(y)dy(x-t)^{\sqrt{1+\varepsilon^{-1}}}\right\} \quad \text{if } t<x\leqq b.
\end{aligned}\right\} \tag{3.3}$$

Using (3.1), (3.2) and (3.3) as well as the following two inequalities in p. 245 of [5]:

$$\int_0^\infty y^{p(\alpha-1)}\left(\int_0^y x^{-\alpha}f(x)dx\right)^p dy < \left(\frac{p}{p-\alpha p-1}\right)^p \int_0^\infty f^p dx\,, \tag{3.4}$$

$$\int_0^\infty x^{-\alpha p'}\left(\int_x^\infty y^{\alpha-1}g(y)dy\right)^{p'} dx < \left(\frac{p'}{1-\alpha p'}\right)^{p'} \int_0^\infty g^{p'}dx\,, \tag{3.5}$$

where $\alpha<1/p'$ we can prove that if $0<\varepsilon<4/5$

$$\begin{aligned} &D(A_\varepsilon(t)) = D(H_\varepsilon(t)) = D(A_\varepsilon^*(t)) \\ &= \left\{u\in L^2(a,b) : \frac{d^2u}{dx^2},\ \frac{1}{x-t}\frac{du}{dx},\ \frac{u}{(x-t)^2}\in L^2(a,b),\ u(a)=u(b)=0\right\}, \end{aligned}$$

which shows that (a) and (b) of the assumptions of Theorem 2.1 are satisfied where $(A_0(t)u)(x)=u(x)/(x-t)^2$ in the present case. Similarly we can show that

$$\frac{du_\varepsilon(t)}{dt}\in V(t)\,,$$

$$\varepsilon\sqrt{\int_a^b \left|\frac{1}{x-t}\frac{\partial}{\partial x}u_\varepsilon(t,x)\right|^2 dx}\leqq K\sqrt{\varepsilon}\,||g||\,, \tag{3.6}$$

$$\sqrt{\int_a^b \left|\frac{u_\varepsilon(t,x)}{(x-t)^2}\right|^2 dx}\leqq K||g||\,, \tag{3.7}$$

$$||A_\varepsilon(t)H_\varepsilon(t)^{-1}-I||\leqq K\sqrt{\varepsilon}\,, \tag{3.8}$$

where K is a constant which does not depend on t and ε. By a general result on sesquilinear forms ([1]) we get after an integration by part

$$\begin{aligned} \left\|H_\varepsilon(t)^{\frac{1}{2}}\frac{d}{dt}u_\varepsilon(t)\right\|^2 &= Re\, a_\varepsilon\left(t\,;\ \frac{d}{dt}u_\varepsilon(t),\ \frac{d}{dt}u_\varepsilon(t)\right) \\ &= \varepsilon\int_a^b\left|\frac{\partial}{\partial x}\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx + \left(1+\frac{\varepsilon}{2}\right)\int_a^b\left|\frac{1}{x-t}\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx\,. \end{aligned} \tag{3.9}$$

Differentiating both sides of

$$-\varepsilon\frac{\partial^2}{\partial x^2}u_\varepsilon(t,x)+\frac{\varepsilon}{x-t}\frac{\partial}{\partial x}u_\varepsilon(t,x)+\frac{u_\varepsilon(t,x)}{(x-t)^2}=g(x)$$

in t we obtain

$$-\varepsilon\frac{\partial^2}{\partial x^2}\frac{\partial u_\varepsilon}{\partial t}+\frac{\varepsilon}{x-t}\frac{\partial}{\partial x}\frac{\partial u_\varepsilon}{\partial t}+\frac{1}{(x-t)^2}\frac{\partial u_\varepsilon}{\partial t} = -\frac{\varepsilon}{(x-t)^2}\frac{\partial u_\varepsilon}{\partial x}-\frac{2u_\varepsilon}{(x-t)^3}\,.$$

Multiplying both members of the above relation by $\overline{\partial u_\varepsilon/\partial t}$, and integrating the resulting equality by part over (a,b), and using the formula

$$Re\int_a^b \frac{1}{x-t}\frac{du}{dx}\bar{u}dx = \frac{1}{2}\int_a^b \left|\frac{u}{x-t}\right|^2 dx$$

which holds for $u \in V(t)$ and may be proved by integration by part, and finally comparing the real parts of both sides of the relation thus derived, we get

$$\begin{aligned} &\varepsilon\int_a^b \left|\frac{\partial}{\partial x}\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx + \left(1+\frac{\varepsilon}{2}\right)\int_a^b \left|\frac{1}{x-t}\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx \\ &= -\varepsilon Re\int_a^b \frac{1}{(x-t)^2}\frac{\partial u_\varepsilon}{\partial x}\overline{\frac{\partial u_\varepsilon}{\partial t}}dx - 2Re\int_a^b \frac{u_\varepsilon}{(x-t)^3}\overline{\frac{\partial u_\varepsilon}{\partial t}}dx\,. \end{aligned} \tag{3.10}$$

It is not difficult to prove the above procedure rigourously noting for example that if $u \in V(t)$ we have

$$|u(x)| = \left|\int_t^x \frac{du}{dy}dy\right| \leqq \sqrt{|x-t|}\sqrt{\int_t^x \left|\frac{du}{dy}\right|^2 dy}\,.$$

Applying Schwarz inequality to the right of (3.10) and recalling (3.9) we obtain

$$\begin{aligned} &\int_a^b \left|\frac{1}{x-t}\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx \leqq \left\|H_\varepsilon(t)^{1/2}\frac{d}{dt}u_\varepsilon(t)\right\|^2 \\ &\leqq \left(\varepsilon\sqrt{\int_a^b \left|\frac{1}{x-t}\frac{\partial u_\varepsilon}{\partial x}\right|^2 dx} + 2\sqrt{\int_a^b \left|\frac{u_\varepsilon}{(x-t)^2}\right|^2 dx}\right)\sqrt{\int_a^b \left|\frac{1}{x-t}\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx}\,. \end{aligned}$$

(3.6), (3.7) and (3.11) implies

$$\sqrt{\int_a^b \left|\frac{1}{x-t}\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx} \leqq K(2+\sqrt{\varepsilon})\|g\|\,. \tag{3.12}$$

Combining (3.11), (3.12), (3.16) and (3.17) we get

$$\left\|H_\varepsilon(t)^{1/2}\frac{d}{dt}A_\varepsilon(t)^{-1}\right\| \leqq K(2+\sqrt{\varepsilon})\,. \tag{3.13}$$

(3.7) implies $\|A_\varepsilon(t)H_\varepsilon(t)^{-1}\| \leqq 1+K\sqrt{\varepsilon}$, and hence by the generalization of Heinz inequality by Kato [3] we conclude

$$\|A_\varepsilon(t)^{1/2}H_\varepsilon(t)^{-1/2}\| \leqq e^{\pi^2/8}(1+K\sqrt{\varepsilon})^{1/2}\,.$$

It follows from (3.13) and (3.14) that (3.14)

$$\left\|A_\varepsilon(t)^{1/2}\frac{d}{dt}A_\varepsilon(t)^{-1}\right\| \leqq e^{\pi^2/8}(1+K\sqrt{\varepsilon})^{1/2}K(2+\sqrt{\varepsilon})\,,$$

which states that (III) of Theorem 1.1 holds for $A_\varepsilon(t)$, $0<\varepsilon\leqq 4/5$, with constants independent of ε and t. It is not difficult to prove that the remaining part of the assumptions of Theorem 2.1 is satisfied by $A_\varepsilon(t)$, $0<\varepsilon\leqq 4/5$.

REMARK. If $B(t)$ is the multiplication operator

$$(B(t)u)(x)=\frac{u(x)}{(x-t)^2},$$

then we have $D(A(t))\subset D(B(t))$. Application of T. Kato's generalization of Heinz's theorem ([3]) shows that $D(A(t)^\rho)\subset D(B(t)^\rho)$ for any ρ with $0<\rho<1$, therefore any function belonging to $D(A(t)^\rho)$ must vanish at t in some sense. Thus we conclude that $D(A(t)^\rho)$ is not independent of t whenever $\rho>0$. The same thing remains true in the first case as a consequence of Heinz's theorem itself.

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