

Title	Factorization of double transfer maps
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Citation	Osaka Journal of Mathematics. 1993, 30(4), p. 759–769
Version Type	VoR
URL	https://doi.org/10.18910/11492
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Imaoka, M. Osaka J. Math. 30 (1993), 759–769

# FACTORIZATION OF DOUBLE TRANSFER MAPS

Dedicated to Professor Seiya Sasao on his 60th birthday

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(Received July 30, 1992)

## 1. Introduction

In [6] and [4], the authors have studied a factorization of the double  $S^{1}$ -transfer map through the second stage of the chromatic filtration. In this paper, I show that such a factorization exists for other double transfer maps.

Let  $\alpha$  be an orientable vector bundle of fiber dimension a over a connected finite complex X, and X<sup>\*</sup> denote the Thom space of  $\alpha$ . Then we have a cofiber sequence

(1.1) 
$$S^{a} \xrightarrow{j} X^{a} \xrightarrow{j} X^{a}/S^{a} \xrightarrow{\tau} S^{a+1},$$

where *i* is the inclusion to the bottom sphere. Then, by [7], the S<sup>1</sup>-transfer map is stably homotopic to  $\tau$  when  $X=CP^*$  and  $\alpha=-\xi$  for the canonical C-line bundle  $\xi$  over the complex projective space  $CP^*$ . If  $X=\Sigma W$  a suspension of a space W, then  $\tau$  is stably homotopic to the stable J-map  $J(\alpha): X \rightarrow S^1$ . Thus, generalizing the original meaning of transfer maps, we call  $\tau$  in (1.1) a transfer map. Then the following stable map  $\tau_2$  is called to be a double transfer map.

(1.2) 
$$\tau_2 = \tau \wedge \tau \colon X^a / S^a \wedge Y^{\beta} / S^b \to S^{a+b+2}$$

where  $\beta$  is an orientable vector bundle of fiber dimension b over a connected finite complex Y.

By Ravenel [11] a geometric realization of the chromatic filtration has been given, and we shall denote the first two stages in it by

(1.3) 
$$\cdots \to \Sigma^{-2} N_2 \xrightarrow{\delta_2} \Sigma^{-1} N_1 \xrightarrow{\delta_1} S^0$$

Here, the spectra are localized at a prime p, and there is some difference in our treatment between the cases of an odd prime p and p=2. This difference is caused by the use of K-theory, and thus we treat the K-spectrum  $K_{\Delta}$  which denotes the complex K-spectrum  $K_{(p)}$  localized at p in case of an odd prime p and the real K-spectrum  $KO_{(2)}$  localized at 2 in case of p=2. Then we shall show the following:

**Theorem 1.4.** Let  $\tau_2$  be the double transfer map of (1.2), and  $N_2$  the second

stage of the chromatic filtration as in (1.3). If  $\alpha$  and  $\beta$  are  $K_{\Lambda}$ -orientable and  $K_{\Lambda}^{a-1}(X^{\alpha}/S^{0}; Q/Z)=0$ , then there is a factorization  $\tau_{2}\simeq\delta_{1}\delta_{2}\overline{u}_{2}$  by a map  $\overline{u}_{2}$ :  $X^{\alpha}/S^{a} \wedge Y^{\beta}/S^{b} \rightarrow \Sigma^{a+b}N_{2}$ .

For the important case that p is an odd prime,  $X=Y=CP^{N}$  and  $\alpha=\beta=-\xi$ , the theorem has been established in [6] and [4; Th. 5.2], and we show that their method can be extended to obtain the theorem. Theorem 1.4 is a corollary of Theorem 2.8 which makes a construction of  $u_{2}$  clear, and §2 is devoted to demonstrate Theorem 2.8.

Such a factorization as in Theorem 1.4 draws a clear strategy to understand the double transfer image, as seen in [6], and some detailed formuals for  $\overline{u}_2$  are required. In §3, we describe such formulas in the case of stunted projective spaces. When  $X=Y=CP^N$ ,  $\alpha=m\xi$  and  $\beta=n\xi$  for integers m and  $n, \tau_2$  of (1.2) is a double  $S^1$ -transfer map for stunted complex projective spaces. By Theorem 1.4, a factorization of such double  $S^1$ -transfer map exists if p is an odd prime. On the other hand, the double  $S^1$ -transfer map has no such factorization as in Theorem 1.4 if p=2 and both m and n are odd. In case of p=2, it might be natural to consider the quaternionic projective space  $HP^N$  instead of  $CP^N$ . Then  $\tau_2$  is called a double  $S^3$ -transfer map, and it always has a factorization by Theorem 1.4. For these  $S^1$  and  $S^3$ -transfer maps, formulas concerning  $\overline{u}_2$  are given in Theorem 3.5 and 3.13, (3.7) and (3.15). The method to obtain such formulas is attributed to Hilditch [6].

The author wishes ot express his heartfelt thanks to faculty members of the University of Manchester for their kind hospitality during his recent visit at the university, in particular, A. Baker, P. Eccles, N. Ray, G. Walker and R. Wood.

### 2. Factotization

Let S(G) be the Moore spectrum for a group G, and put  $E^{k}G = \Sigma^{k}E \wedge S(G)$ for a spectrum E. Then,  $E^{k}(-;G) = \{-, E^{k}G\}$  is the G-coefficient E-cohomology group. We have a cofiber sequence  $E^{k}Z \xrightarrow{l_{Q}} E^{k}Q \xrightarrow{\rho_{Z}} E^{k}Q/Z$ , where  $l_{Q}$  is induced from the inclusion of the ring Z of integers into the field Q of rational numbers and  $\rho_{Z}$  is induced from the mod Z reduction.

Now, let  $\alpha$  be an orientable vector bundle over a connected finite complex X. Since we work only in the stable category, it is convenient to assume that  $\alpha$  is a virtual vector bundle of dimension 0, and that cohomology groups are all assumed to be reduced. Then we have a Thom class  $U^H_{\alpha} \in H^0(X^{\alpha}; \mathbb{Z})$  of  $\alpha$  in the integral cohomology group. Let  $\pi^*_s(-)$  denote the stable cohomotopy group. Then, the Hurewicz map  $h^H: \pi^0_s(X^{\alpha}; \mathbb{Q}) \to H^0(X^{\alpha}; \mathbb{Q})$  is an isomorphism, and we can put  $u=(h^H)^{-1}(U^H_{\alpha}) \in \pi^0_s(X^{\alpha}; \mathbb{Q})$ . u yields an element  $\overline{u} \in \pi^0_s(X^{\alpha}/S^0; \mathbb{Q}/\mathbb{Z})$  which makes the following diagram stably homotopy commutative up to sign:

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(2.1)  
$$S^{0} \xrightarrow{i} X^{\alpha} \xrightarrow{j} X^{\alpha}/S^{0} \xrightarrow{\tau} S^{1}$$
$$\parallel u \downarrow u \downarrow u \downarrow \parallel$$
$$S^{0} \xrightarrow{l_{Q}} S^{0}Q \xrightarrow{\rho_{Z}} S^{0}Q/Z \xrightarrow{\delta_{1}} S^{1}.$$

This diagram generalizes the fundamental situation designed by Miller [8], and  $\tau$  represents a transfer map as in §1.  $\mathbf{u}$  is uniquely determined by the equation  $j^*(\mathbf{u}) = \rho_z(\mathbf{u})$ .

We denote by  $K_{\Lambda}$  the K-spectrum  $K_{(p)}$  for an odd prime p or  $KO_{(2)}$  for p=2, and we assume that  $\alpha$  is  $K_{\Lambda}$ -orientable. Then we have a  $K_{\Lambda}$ -theory Thom class  $U_{\alpha}^{K_{\Lambda}} \in K_{\Lambda}^{0}(X^{\alpha})$  of  $\alpha$ . Let  $ch_{\Lambda} \colon K_{\Lambda}^{0}(-) \to H^{*}(-; Q)$  be the Chern character, and  $h^{K_{\Lambda}} \colon \pi_{s}^{*}(-) \to K_{\Lambda}^{*}(-)$  the  $K_{\Lambda}$ -Hurewicz homomorphism. Then the characteristic class  $bh_{\Lambda}(\alpha) \in 1 + \sum_{i>0} H^{di}(X; Q)$  is defined by the equation  $ch_{\Lambda}(U_{\alpha}^{K_{\Lambda}}) =$  $U_{\alpha}^{H}bh_{\Lambda}(\alpha)$  (cf. [1]), where d=2 or 4 according as  $K_{\Lambda} = K_{(p)}$  or  $KO_{(2)}$ . We notice that  $ch_{\Lambda} \colon K_{\Lambda}^{0}(W; Q) \to \sum_{i\geq 0} H^{di}(W; Q)$  is an isomorphism for  $W = X_{+}$  or  $X^{\alpha}$ , since X is assumed to be a finite complex. Then the following is deduced from (2.1).

**Lemma 2.2.** For a  $K_{\Lambda}$ -orientable vector bundle  $\alpha$ ,

- (1)  $h^{\kappa}(u) = U^{\kappa}_{\alpha} \wedge ch^{-1}_{\Lambda}(bh_{\Lambda}(-\alpha))$  in  $K^{0}_{\Lambda}(X^{\alpha}; Q)$ , and
- (2) there is a unique element  $V_{\alpha} \in K^{0}_{\Lambda}(X^{\alpha}/S^{0}; Q)$  which satisfies

$$\rho_Z(V_{\alpha}) = h^{\kappa} \wedge (\overline{u}) \quad and \quad j^*(V_{\alpha}) = h^{\kappa} \wedge (u) - (l_Q)_*(U_{\alpha}^{\kappa} \wedge).$$

Proof. Apply  $ch_{\Lambda}$  on both sides of the equation in (1). Then they both become  $U^{H}_{\alpha}$ , since  $ch_{\Lambda}h^{K_{\Lambda}}(u) = h^{H}(u)$  for the left hand side. Since  $ch_{\Lambda}$  is an isomorphism over  $K^{0}_{\Lambda}(X^{\alpha}; Q)$ , we have (1). Let  $K^{0}_{\Lambda}(X^{\alpha}/S^{0}; G) \xrightarrow{j^{*}} K^{0}_{\Lambda}(X^{\alpha}; G) \xrightarrow{i^{*}} K^{0}_{\Lambda}(X^{\alpha}; G) \xrightarrow{i^{*}} K^{0}_{\Lambda}(X^{\alpha}; G) \xrightarrow{i^{*}} K^{0}_{\Lambda}(X^{\alpha}; G) \xrightarrow{j^{*}} K^{$ 

Let  $\psi = \psi^{\gamma} - 1: K_{\Delta} \rightarrow K_{\Delta}$  be the stable Adams operation for a generator  $\gamma$  of the unit group in  $Z/p^2$ , and Ad the fiber spectrum of  $\psi$ . We assume that  $\gamma = 3$  in cases of p = 2. Thus we have a cofiber sequence

(2.3) 
$$Ad^{0}G \xrightarrow{\kappa} K^{0}_{\Lambda}G \xrightarrow{\psi} K^{0}_{\Lambda}G$$

for  $G = Z_{(p)}$ , Q or  $Q/Z_{(p)}$ . The Ad-theory plays an important role later.

Now, let  $\beta$  be an orientable virtual vector bundle of dimension 0 over a connected finite complex Y, and  $1 \wedge i$ :  $X^{\alpha}/S^0 = X^{\alpha}/S^0 \wedge S^0 \rightarrow X^{\alpha}/S^0 \wedge Y^{\beta}$  the

inclusion. For the element  $V_{\omega}$  in Lemma 2.2, we have an extension  $\tilde{u}$  as follows:

**Proposition 2.4.** Assume that  $\alpha$  and  $\beta$  are  $K_{\Lambda}$ -orientable. Then, there is an element  $\tilde{u} \in K^{0}_{\Lambda}(X^{a}/S^{0} \wedge Y^{\beta}; Q)$  which satisfies

- (1)  $(1 \wedge i)^*(\overline{u}) = V_{\alpha}$ , and
- (2)  $\psi(\hat{u}) \in \operatorname{Im}[(l_q)_*: K^0_{\Lambda}(X^{\sigma}/S^0 \wedge Y^{\beta}) \to K^0_{\Lambda}(X^{\sigma}/S^0 \wedge Y^{\beta}; Q)].$

Proof. Since  $ch_{\Lambda}: K^{0}_{\Lambda}(X^{\sigma}/S^{0}; Q) \to \sum_{i>0} H^{di}(X^{\sigma}/S^{0}; Q)$  is an isomorphism, we can write  $ch_{\Lambda}(V_{\sigma}) = \sum_{i>0} a_{i}$  for some  $a_{i} \in H^{di}(X^{\sigma}/S^{0}; Q)$  and put  $A_{i} = (ch_{\Lambda})^{-1}(a_{i}) \in K^{0}_{\Lambda}(X^{\sigma}/S^{0}; Q)$ . Then  $V_{\sigma} = \sum_{i>0} A_{i}$ , and  $\psi^{\gamma}A_{i} = \gamma^{id/2}A_{i}$ . Similarly, regarding a Thom class  $U^{\kappa}_{\beta} \wedge \in K^{0}_{\Lambda}(Y^{\beta})$  as an element of  $K^{0}_{\Lambda}(Y^{\beta}; Q)$ , we have  $U^{\kappa}_{\beta} \wedge = \sum_{j\geq 0} B_{j}$  for some  $B_{j} \in K^{0}_{\Lambda}(Y^{\beta}; Q)$  with  $\psi^{\gamma}B_{j} = \gamma^{id/2}B_{j}$ . We put

(2.5) 
$$\tilde{u} = V_{\sigma} \otimes U_{\beta}^{\kappa} - \sum_{k,l \geq 0} \tilde{\Gamma}_{k,l} A_k \otimes B_l \in K^0_{\Lambda}(X^{\sigma} / S^0 \wedge Y^{\beta}; Q) ,$$

where  $\tilde{\Gamma}_{k,l} = (\gamma^{ld/2} - 1)/(\gamma^{(k+l)d/2} - 1)$ . Then,  $\tilde{u}$  satisfies (1), since  $i^*(U_{\beta}^K \Delta) = 1$  and  $i^*(B_l) = 0$ . Using the definitions of  $A_i$  and  $B_j$ , it follows that

(2.6) 
$$\psi(\tilde{u}) = \psi(V_{\mathfrak{s}})\psi^{\gamma}(U_{\beta}^{\kappa}\Lambda).$$

By the second equation in Lemma 2.2 (2), we have  $j^*(\psi(V_{\omega})) = h^{K_{\Delta}}(u) - \psi^{\gamma}((l_{Q})_*(U_{\omega}^{K_{\Delta}})) - j^*(V_{\omega}) = -(l_{Q})_*(\psi(U_{\omega}^{K_{\Delta}}))$ , where  $j: X^{\omega} \to X^{\omega}/S^0$  and  $l_{Q}: K_{\Delta}^0 Z \to K_{\Delta}^0 Q$ . But, there is an element  $w \in K_{\Delta}^0(X^{\omega}/S^0)$  with  $j^*(w) = -\psi(U_{\omega}^{K_{\Delta}})$ , and thus  $j^*(l_{Q})_*(w) = j^*(\psi(V_{\omega}))$  in  $K_{\Delta}^0(X^{\omega}; Q)$ . Since  $j^*: K_{\Delta}^0(X^{\omega}/S^0; Q) \to K_{\Delta}^0(X^{\omega}; Q)$  is a monomorphism, we have  $\psi(V_{\omega}) = (l_{Q})_*(w)$ , and thus  $\tilde{u}$  satisfies (2) by (2.6), which completes the proof.

We need to recall the geometric realization [11] of the chromatic filtration as in (1.3). Let  $l_i: E \to L_i E$  be the Bousfield localization [5] with respect to the  $v_i^{-1}BP_*$ -homology for a prime p. Then the *i*-stage of the filtration is realized by a spectrum  $N_i$  which is defined inductively, starting with  $N_0 = S^0$ , by the cofiber sequence

(2.7) 
$$N_i \xrightarrow{l_i} M_i = L_i N_i \xrightarrow{\rho_i} N_{i+1} \xrightarrow{\delta_{i+1}} \Sigma N_i .$$

In particular,  $M_0 = S(Q)$  and  $N_1 = S(Q/Z)$ . Furthermore, by [5] or [12], it is shown that there is a homotopy equivalence  $M_1 \simeq Ad^0 Q/Z$  through which  $l_1: N_1 \rightarrow M_1$  is identified with the Ad-theory Hurewicz homomorphism  $h^{Ad}$ :  $S^0 Q/Z \rightarrow Ad^0 Q/Z$ . Here, spectra are assumed to be localized at p, and Ad is the fiber spectrum of the stable Adams operation  $\psi = \psi^{\gamma} - 1$  defined on  $K_{(p)}$  if p is odd and on  $KO_{(2)}$  if p=2. Thus,  $\rho_1: M_1 \rightarrow N_2$  is identified with  $\overline{p}: Ad^0 Q/Z$  $\rightarrow \overline{Ad^0}Q/Z$  for  $\overline{Ad} = Ad/S_{(p)}^0$ , and we have maps  $\kappa: M_1 \rightarrow K_{\Lambda}^0 Q/Z$  and  $\overline{\kappa}: N_2 \rightarrow K_{\Lambda}^0 Q/Z$  induced from  $\kappa: Ad^0 Q/Z \rightarrow K_{\Lambda}^0 Q/Z$  as in (2.3). Then Theorem 5.2 in [4] is extended to the following form.

**Theorem 2.8.** Assume that  $\alpha$  and  $\beta$  are  $K_{\Delta}$ -orientable and  $K_{\Delta}^{-1}(X^{\omega}/S^0; Q/Z) = 0$ . Then, we have elements  $u_2 \in (M_1)^0(X^{\omega}/S^0 \wedge Y^{\beta})$  and  $u_2 \in (N_2)^0(X^{\omega}/S^0 \wedge Y^{\beta}/S^0)$  which make the following diagram stably homotopy commutative up to sign:

Here,  $u_2$  can be taken to satisfy  $\kappa_*(u_2) = \rho_z(\tilde{u})$  for  $\tilde{u}$  of Proposition 2.4.

Proof. We put  $W = X^{\alpha}/S^0 \wedge Y^{\beta}$ . Then by Proposition 2.4 (2),  $\psi(\rho_Z(\tilde{u})) = 0$  in  $K^0_{\Lambda}(W; Q/Z)$ , and thus we have an element  $u_2 \in (M_1)^0(W)$  satisfying  $\kappa_*(u_2) = \rho_Z(\tilde{u})$ . By Proposition 2.4 (1) and Lemma 2.2 (2),  $\kappa_*(1 \wedge i)^*(u_2) = (1 \wedge i)^*\rho_Z(\tilde{u}) = \rho_Z(V_{\alpha}) = h^{\kappa_{\Lambda}}(\tilde{u}) = \kappa_*(l_1)_*(\tilde{u})$ , where  $l_1: N_1 \rightarrow M_1$  is the map as in (2.7). Since  $\kappa_*: (M_1)^0(X^{\alpha}/S^0) \rightarrow K^0_{\Lambda}(X^{\alpha}/S^0; Q/Z)$  is a monomorphism by the assumption that  $K^{-1}_{\Lambda}(X^{\alpha}/S^0; Q/Z) = 0$ , we have

$$(1 \wedge i)^*(u_2) = (l_1)_*(\overline{u})$$
 in  $(M_1)^0(X^{\sigma}/S^0)$ .

Then,  $\overline{u}$  and  $u_2$  produce maps from the upper cofiber sequence in the diagram to the second cofiber sequence  $N_1 \rightarrow M_1 \rightarrow N_2 \rightarrow \Sigma N_1$ , and thus we have the required elements  $u_2$  and  $\overline{u}_2$  which make the diagram commutative up to sign.

We notice that the assumption  $K_{\Lambda}^{-1}(X^{\bullet}/S^0; Q/Z)=0$  in the theorem is satisfied if  $K_{\Lambda}^0(X)$  is torsion free and  $K_{\Lambda}^{-1}(X)$  is a torsion group. From (2.1) and the commutativity of the upper right square in the diagram of Theorem 2.8, it follows that the double transfer  $\tau_2: X^{\bullet}/S^0 \wedge Y^{\beta}/S^0 \rightarrow S^2$  is factored through the second stage  $N_2$  as  $\tau_2 \simeq \delta_1 \delta_2 u_2$ , and we have Theorem 1.4.

REMARK 2.9. For the canonical complex line bundle  $\xi$  over  $CP^N$ ,  $(2m+1)\xi$  is not KO-orientable for any integer m. By the same reason as in [6: Remark 3.2], there is no such factorization as in Theorem 1.4 in case of p=2,  $X=Y=CP^N$ ,  $\alpha=(2m+1)\xi$  and  $\beta=(2n+1)\xi$ .

#### 3. Stunted projective spaces

Let C and H be the field of the complex and quaternionic numbers, and put (F, d) = (C, 2) or (H, 4), respectively. We denote the N-th projective space over F by  $FP^N$  for  $N \ge 0$ , and the canonical F-line bundle over  $FP^N$  by  $\xi$ . Then,

for a positive integer k, the Thom space of  $k\xi$  is homeomorphic to the stunted projective space  $FP_k^{N+k} = FP^{N+k}/FP^{k-1}$  by [2]. Thus, for any integer k, we denote the Thom space of  $k\xi$  over  $FP^N$  simply by  $FP_k$ , since our results are valid for any N and compatible with each N. Then, in the cofiber sequence  $S^{dk} \xrightarrow{i} FP_k \xrightarrow{j} FP_{k+1} \xrightarrow{\tau} S^{dk+1}$ ,  $\tau$  represents a transfer map for  $k\xi$ , and we call this  $\tau a S^{d-1}$ -transfer map. Thus, a double  $S^{d-1}$ -transfer map is given by

(3.1) 
$$\tau_2 = \tau \wedge \tau \colon FP_{m+1} \wedge FP_{n+1} \to S^{d(m+n)+2}.$$

In this section, we are concerned with this  $\tau_2$ .

In Theorem 2.8,  $K_{\Lambda} = K_{(p)}$  or  $KO_{(2)}$  according as the spectra are assumed to be localized at an odd prime p or 2. Hereafter, we assume that p is odd whenever we discuss  $S^1$ -transfer maps, and that p=2 for  $S^3$ -transfer maps. Thus,  $(K_{\Lambda}, FP^N) = (K_{(p)}, CP^N)$  or  $(KO_{(2)}, HP^N)$  according as p is an odd prime or p=2. Then  $k\xi$  over  $FP^N$  is always  $K_{\Lambda}$ -orientable for any integer k. In the below, we denote the coefficient group  $\pi_i(K_{\Lambda})$  by  $(K_{\Lambda})_i$ , and the Bott generators by  $t \in K_2$  and  $g_i \in KO_{4i}$  respectively.

In order to express a formula for  $u_2$  of Theorem 2.8 with respect to  $\tau_2$  in (3.1), the  $K_{\Lambda}$ -Bernoulli numbers are necessary. Let  $e^T$  be the formal power expansion of the exponential function on T, and  $\sinh(T)$  that of the hyperbolic sin function on T. We put  $(2\sinh(\sqrt{T}/2))^2 = \sum_{j\geq 0} s_j T^{j+1}$ , where all  $s_j$  are rational numbers and  $s_0=1$ . Using these notations, we define the following:

DEFINITION 3.2. (1) 
$$\operatorname{Exp}^{K_{\Delta}}(-)$$
 and  $\operatorname{Log}^{K_{\Delta}}(-)$ :  
 $\operatorname{Exp}^{K}(T) = t^{-1}(1 - e^{-tT}) \in (K_* \otimes Q)[[T]],$   
 $\operatorname{Exp}^{KO}(T) = \sum_{j \ge 0} (-1)^j s_j(g_j/a(j))T^{j+1} \in (KO_* \otimes Q)[[T]],$   
 $\operatorname{Log}^{K_{\Delta}}(T) = (\operatorname{Exp}^{K_{\Delta}})^{-1}(T) \in ((K_{\Delta})_* \otimes Q)[[T]],$   
where  $a(j) = 1$  (resp. 2) if j is even (resp. odd).

(2) The  $K_{\Lambda}$ -Bernoulli numbers  $\tilde{B}^{K_{\Lambda}}(m, k) \in (K_{\Lambda})_{dk} \otimes Q$ :

$$\left(\frac{T}{\operatorname{Exp}^{K_{\Delta}}(T)}\right)^{m} = \sum_{k\geq 0} \tilde{B}^{K_{\Delta}}(m,k)T^{k}.$$

Let  $X^{\kappa} = t^{-1}[1-\xi] \in K^2(\mathbb{CP}^N)$  and  $X^{\kappa o} = [1-\xi] \in KO^4(\mathbb{HP}^N)$  be the  $K_{\Lambda}$ theory Euler classes of  $\xi$ , and  $x \in H^d(\mathbb{FP}^N; \mathbb{Z})$  the Euler class which satisfies  $ch_{\Lambda}(\xi) = e^x$  or  $e^{\sqrt{x}} + e^{-\sqrt{x}}$  for  $\mathbb{CP}^N$  or  $\mathbb{HP}^N$  respectively. Then, for  $(E, x^E) = (K_{\Lambda}, X^{\kappa}_{\Lambda})$  or (H, x), we have an isomorphism  $E^*(\mathbb{FP}^N) \cong E_*[[x^E]]/((x^E)^{N+1})$ , and  $E^*(\mathbb{FP}_k)$  is a free  $E^*(\mathbb{FP}^N)$  module with a Thom class  $U^E_{k\xi}$  as a generator. As in [8], we can put  $U^E_{k\xi} = (x^E)^k$  and  $(x^E)^i(x^E)^j = (x^E)^{i+j}$  for  $i \ge k$  and  $j \ge 0$ .

Let  $f_{\Delta}(x)=1-e^x$  or  $-(2\sinh\sqrt{x}/2)^2$  in  $H^*(FP^N;Q)$  according as  $FP^N=CP^N$  or  $HP^N$ . Then, we have the following:

**Lemma 3.3.**  $ch_{\Lambda}(X^{\kappa_{\Lambda}}) = f_{\Lambda}(x)$  and  $ch_{\Lambda}(\operatorname{Log}^{\kappa_{\Lambda}}(X^{\kappa_{\Lambda}})) = -x$ .

Proof. Since  $ch_{\Delta}\xi = d/2 - f_{\Delta}(x)$ , the first equation is clear. Let  $\log(T)$  be the power series exapansion of the logarithm function on T, and put  $(2\sinh^{-1}(\sqrt{T}/2))^2 = \sum_{j\geq 0} r_j T^{j+1}$ . Then,  $\log^{\kappa}(T) = -t^{-1}\log(1-tT)$  and  $\log^{\kappa o}(T) = \sum_{j\geq 0} (-1)^j r_j$   $(g_j/a(j))T^{j+1}$ . Since  $ch_{\Delta}$  is a ring homomorphism, we have the second required equation.

Let  $u \in \pi_s^{dm}(FP_m; Q)$  and  $V_{m\xi} \in K_{\Lambda}^{dm}(FP_{m+1}; Q)$  be the elements as in (2.1) and Lemma 2.2 respectively. Then, the following is a corollary of Lemmas 2.2 and 3.3.

Corollary 3.4. For any integer m,

$$h^{\kappa} \Lambda(u) = (\mathrm{Log}^{\kappa} \Lambda(X^{\kappa} \Lambda))^m$$
 and  $j^*(V_{m\xi}) = (\mathrm{Log}^{\kappa} \Lambda(X^{\kappa} \Lambda))^m - (X^{\kappa} \Lambda)^m$ 

where  $j^*: K^{dm}_{\Lambda}(FP_{m+1}; Q) \rightarrow K^{dm}_{\Lambda}(FP_m; Q)$  is a monomorphism.

Proof. As above,  $U_{m\xi}^{K}$  is taken to be  $(X^{K}\Lambda)^{m}$ . In order to satisfy  $ch_{\Lambda}(U_{\xi}^{K}\Lambda) = U_{\xi}^{H}bh_{\Lambda}(\xi)$  and  $bh_{\Lambda}(\xi) \in 1 + \sum_{i>0} H^{di}(FP^{N}; Q)$ , we must take  $U_{\xi}^{H} = -x$  instead of x, because  $ch_{\Lambda}(X^{K}\Lambda) = f_{\Lambda}(x) = (-x)(f_{\Lambda}(x)/(-x))$  by Lemma 3.3. Hence,  $U_{m\xi}^{H} = (-x)^{m}$  and  $bh_{\Lambda}(m\xi) = (-f_{\Lambda}(x)/x)^{m}$ . Then, it follows from Lemma 3.3 that

$$ch_{\Lambda}^{-1}(bh_{\Lambda}(-m\xi)) = \left(\frac{\mathrm{Log}^{\kappa}\Lambda(X^{\kappa}\Lambda)}{X^{\kappa}\Lambda}\right)^{m}.$$

Thus we have the first required equation by Lemma 2.2(1), and the second required equation by the first equation and Lemma 2.2 (2).

Now, we can show a formula for an element  $u_2 \in (M_1)^{d(m+n)}(FP_{m+1} \wedge FP_n)$  as in Theorem 2.8. For a while, we put  $FP(k, l) = FP_k \wedge FP_l$  for brevity. Since  $K_{\Lambda}^{d(m+n)-1}(FP(m+1, n); Q/Z) = 0$  and  $K_{\Lambda}^{d(n-1)}(FP_n; Q/Z) = 0$ , both  $\kappa_* : (M_1)^{d(m+n)}$  $(FP(m+1, n); Q/Z) \rightarrow K_{\Lambda}^{d(m+n)}(FP(m+1, n); Q/Z)$  and  $(j \wedge 1)^* : K_{\Lambda}^{d(m+n)}(FP(m+1, n); Q/Z) \rightarrow K_{\Lambda}^{d(m+n)}(FP(m, n); Q/Z)$  are monomorphisms. Hence we shall describe a formula for  $\kappa_*(u_2) \in K_{\Lambda}^{d(m+n)}(FP(m+1, n); Q/Z)$ , regarding it as an element of  $K_{\Lambda}^{d(m+n)}(FP(m, n); Q/Z)$  through  $(j \wedge 1)^*$ . We shall represent  $K_{\Lambda}^*(FP(m, n); Q/Z)$  as  $R\{(X^{K_{\Lambda}})^m\} \otimes R\{(Y^{K_{\Lambda}})^n\}$  for  $R = K_{\Lambda}^*(FP^N; Q/Z)$ , using  $Y^{K_{\Lambda}}$  to denote the  $K_{\Lambda}$ -theory Euler class of  $\xi$  for the second factor. Let  $\gamma$  be a generator of the unit group in  $Z/p^2$ , which is used in the definition of Ad before (2.3). Then we have the following formula.

Theorem 3.5. In 
$$K_{\Lambda}^{d(m+n)}(FP_{m+1} \wedge FP_n; Q/Z)$$
,  
 $\kappa_*(u_2) = ((\operatorname{Log}^{K_{\Lambda}}(X^{K_{\Lambda}}))^m - (X^{K_{\Lambda}})^m) \otimes (Y^{K_{\Lambda}})^n$   
 $+ \sum_{k,l>0} \widetilde{\Gamma}_{k,l} \widetilde{B}^{K_{\Lambda}}(-m, k) \widetilde{B}^{K_{\Lambda}}(-n, l) (\operatorname{Log}^{K_{\Lambda}}(X^{K_{\Lambda}}))^{m+k} \otimes (\operatorname{Log}^{K_{\Lambda}}(Y^{K_{\Lambda}}))^{n+l}$ ,  
where  $\widetilde{\Gamma}_{k,l} = (\gamma^{dl/2} - 1)/(\gamma^{d(k+l)/2} - 1)$ .

Proof. By Theorem 2.8, we take  $u_2$  to satisfy  $\kappa_*(u_2) = \rho_Z(\tilde{u})$  for  $\tilde{u}$  given by (2.5). Since  $(j \wedge 1)^* (V_{m\xi} \otimes U_{n\xi}^K) = ((\text{Log}^{K_{\Delta}}(X^{K_{\Delta}}))^m - (X^{K_{\Delta}})^m) \otimes (Y^{K_{\Delta}})^n$  by Corollary 3.4, all we need is formulas for  $A_k$  and  $B_l$  in (2.5). By Lemma 3.3 and Corollary 3.4, we have

$$ch_{\Delta}(j^{*}(V_{m\xi})) = ch_{\Delta}(\operatorname{Log}^{K} \Delta(X^{K} \Delta))^{m} - ch_{\Delta}(X^{K} \Delta)^{m} = -\sum_{i>0} [f_{\Delta}(x)^{m}]_{m+i},$$

where  $[f_{\Delta}(x)^{m}]_{j}$  denotes the *dj*-dimensional part of  $f_{\Delta}(x)^{m}$ . On the other hand, from Definition 3.2, it follows that

$$\left(\frac{X^{K_{\Delta}}}{\log^{K_{\Delta}}(X^{K_{\Delta}})}\right)^{m} = \sum_{i \geq 0} \tilde{B}^{K_{\Delta}}(-m, i) (\log^{K_{\Delta}}(X^{K_{\Delta}}))^{i}$$

Applying  $ch_{\Delta}$  on both sides of this equation and using Lemma 3.3, we have

$$f_{\Delta}(x)^{m} = (-x)^{m} + \sum_{k>0} ch_{\Delta}(\tilde{B}^{K}\Delta(-m,k))(-x)^{m+k}.$$

Then, we obtain

$$A_{\mathbf{k}} = ch_{\Delta}^{-1}(-[f_{\Delta}(x)^{m}]_{m+k}) = -\tilde{B}^{K_{\Delta}}(-m,k)(\mathrm{Log}^{K_{\Delta}}(X^{K_{\Delta}}))^{m+k}.$$

Similarly,  $B_l = \tilde{B}^{\kappa} (-n, l) (\text{Log}^{\kappa} (Y^{\kappa} ))^{n+l}$ . Thus, by (2.5), we have the required formula.

We have not got any explicit formula for  $\bar{\kappa}_*(\bar{u}_2) \in \bar{K}_{\Lambda}^{d(m+n)}(FP_{m+1} \wedge FP_{n+1}; Q/Z)$ . However, Theorem 2.8 shows

$$(3.7) (1 \wedge j)^* \bar{\kappa}_*(\bar{u}_2) = \bar{\rho}_* \kappa_*(u_2) ,$$

and thus the formula for  $\kappa_*(u_2)$  in Theorem 3.5 describes  $\bar{\kappa}_*(\bar{u}_2)$  with indeterminacy  $\operatorname{Ker}(1 \wedge j)^* = (1 \wedge \tau)^*(\bar{K}^{d(m+n)-1}_{\Delta}(FP_{m+1}; Q/Z))$  and  $\operatorname{Ker}(\bar{\rho}_*) = h^{K_{\Delta}}(\pi^{d(m+n)}_s, (FP_{m+1} \wedge FP_n; Q/Z))$ .

Let MG be the Thom spectrum MU or MSp for the complex or symplectic cobordism theory, respectively. We only consider these spectra in the case that  $(MG, K_{\Delta}, FP_k) = (MU, K_{(p)}, CP_k)$  or  $(MSp, KO_{(2)}, HP_k)$  according as p is an odd prime or 2. Let  $p_{k,l}$  be a generator of the primitive part  $PMG_{dk}(FP_l) \cong Z$ for  $k \ge l$ . The rest of this section is devoted to obtain a formula for  $\kappa_*(u_2)_*(p_{i,j} \otimes p_{k,l})$  using Theorem 3.5. Then it gives a formula for  $\bar{\kappa}_*(\bar{u}_2)_*(p_{i,j} \otimes p_{k,l})$  by (3.7).

Let  $\beta_i \in H_{di}(FP^{\infty}; Z)$  be the dual of  $x^i$ , and  $b_i^{MG} \in H_{di}(MG)$  be the image of  $\beta_{i+1}$  under the canonical homomorphism  $H_{d(i+1)}(FP^{\infty}; Z) \to H_{di}(MG; Z)$ , for  $i \ge 0$ . We define a ring spectrum E to be F-oriented if there is an element  $x^E \in E^d(FP^{\infty})$  with  $E^*(S^d) \simeq E_*\{i^*(x^E)\}$ , where F = C or H and  $i: S^d \to FP^{\infty}$  is the inclusion map. Then, as is well known, there is a map  $\Phi^E: MG \to E$  associated with  $x^E$  such that  $\iota^*(\Phi^E)$  is a unit of  $\pi_0(E)$  for the unit  $\iota: S^0 \to MG$ . Then we have an element  $b_i^E = \Phi_*^E(b_i^{MG}) \in H_{di}(E; Z)$ , and also an element  $\beta_i^E \in E_{di}(FP^{\infty})$  which is the dual of  $(x^E)^i$ . For an F-oriented spectrum E, the E-theory Bernoulli

numbers as in [8] are defined as follows:

**D**EFINITION 3.8.

- (1)  $\operatorname{Exp}^{E}(T) = \sum_{i \geq 0} b_{i}^{E} T^{i+1} \in (H \wedge E)_{*}[[T]] \text{ and } \operatorname{Log}^{E}(T) = (\operatorname{Exp}^{E})^{-1}(T).$
- (2) The *E*-theory Bernoulli number  $\tilde{B}^{E}(m, k) \in (E_{dk} \otimes Q)[[T]];$

$$\left(\frac{T}{\operatorname{Exp}^{E}(T)}\right)^{m} = \sum_{i\geq 0} \tilde{B}^{E}(m,k)T^{k}$$

In case of a C-oriented E,  $\operatorname{Exp}^{E}$  is the exponential sequence related to the formal group law over  $E_{*}$  induced from  $\Phi^{E}$ . Definition 3.2 coincides with this definition if  $(E, x^{E}) = (K_{\Delta}, X^{K_{\Delta}})$ . For later use, we put

$$(3.9) \quad b^{\mathcal{E}} = \sum_{i \ge 0} b^{\mathcal{E}}_i \in H_*(E; \mathbb{Z}) \quad \text{and} \quad \hat{\beta}^{\mathcal{E}}_k(T) = \sum_{i \ge k} \beta^{\mathcal{E}}_i T^i \in E_*(FP_k)[[T]].$$

As for a generator  $p_{n,0}$ , of the primitive part  $PMG_{dn}(FP_0)$ , an explicit formula is given for MU by Segal [13] and for MSp by Baker [3]. They have described a generator  $p_{n,0}^H \in PH_{dn}(FP^\infty; Z) \subset P(H \land MG)_{dn}(FP^\infty)$ , and their methods are immediately applicable to stunted projective spaces. Let c(k, l) be the positive minimal integer c which makes  $c \cdot [b^{MG}]_{k-i}^i$  an element of  $h^H(MG_{d(k-i)})$  in  $H_{d(k-i)}(MG; Q)$  for any i with  $l \leq i \leq k$ . Here  $[b^{MG}]_{k-i}^i$  is the d(k-i)-dimensional part of  $(b^{MG})^i$ . Then, using the methods in [13] and [3], we have the following:

Lemma 3.10. Let  $k \ge l$ .

(1)  $p_{k,l} = c(k, l) \sum_{i=1}^{k} [b^{MG}]_{k-i}^{i} \beta_{i}^{MG}$  is a generator of  $PMG_{dk}(FP_{l}) \simeq Z$ .

(2) When  $(MG, FP_l) = (MU, CP_l)$ , c(k, l) is equal to the K-codegree  $cd^{\kappa}(k, l)$  which is cited below.

REMARK 3.11. The  $K_{\Delta}$ -codegree  $cd^{K_{\Delta}}(k, l)$  is defined as the minimal positive integer c such that the d(k-j)-dimensional part of  $c \cdot bh_{\Delta}(j\xi)$  is in  $H^{k-j}(FP_0; Z)$ for  $l \leq j \leq k$ , that is,  $c \cdot bh_{\Delta}(j\xi)$  is integral. Thus  $K_{\Delta}$ -codegrees are computable. If the mod torsion Hattori-Stong conjecture for MSp (cf. [10], [9]) holds, then we also have  $c(k, l) = cd^{KO}(k, l)$  in the case of  $(MG, FP_l) = (MSp, HP_l)$ . This can be seen by the method in [3]. In general,  $cd^{KO}(k, l)$  is a factor of c(k, l).

Put  $p_{i,j}^{E} = (\Phi^{E})_{*}(p_{i,j}) \in PE_{di}(FP_{j})$  for a *F*-oriented spectrum *E*. Then, by Definition 3.8 (1), (3.9) and Lemma 3.10 (1), we have the following corollary.

Corollary 3.12. Let E be F-oriented. Then

$$\hat{\beta}_k^E(\operatorname{Exp}^E(T)) = \sum_{i\geq k} \frac{p_{i,k}^E}{c(i,k)} T^i.$$

We obtain the following formula, using the technique due to Miller[8] and Hilditch[6].

**Theorem 3.13.** Let  $k, l \ge 1$ . Then, as an element of  $(K_{\Lambda})_{d(k+l)}(E; \mathbb{Q}/\mathbb{Z})$ ,

$$\kappa_*(u_2)_*(p^{\mathbb{E}}_{m+k,m+1}\otimes p^{\mathbb{E}}_{n+l,n+1}) = c(m+k,m+1)c(n+l,n+1)\cdot (\tilde{B}^{K}(-m,k)\tilde{B}^{\mathbb{E}}(-n,l)-\Gamma_{k,l}\tilde{B}^{K}(-m,k)\tilde{B}^{K}(-n,l))$$

for  $\Gamma_{k,l} = \gamma^{dl/2} (\gamma^{dk/2} - 1) / (\gamma^{d(k+l)/2} - 1).$ 

Proof. Let  $g(X^{\kappa_{\Delta}}) = \sum_{i \ge n} a_i (X^{\kappa_{\Delta}})^i$  be an element of  $K^*_{\Delta}(FP_n; Q)$ , and put  $b(T) = \operatorname{Exp}^{\kappa_{\Delta}}(\operatorname{Log}^{\varepsilon}(T))$ . Then, by [8] or [6], it is shown that

(3.14) 
$$g(X^{K_{\Delta}})_{*}(\hat{\beta}_{n}^{E}(T)) = g(b(T)) \in ((K_{\Delta} \wedge E)_{*} \otimes Q)[[T]].$$

Hence, it follows that  $((X^{\kappa_{\Lambda}})^{j})_{*}(\hat{\beta}_{l}^{E}(T))=b(T)^{j}$  (resp. 0) if  $j \ge l$  (resp. j < l), and  $((\operatorname{Log}^{\kappa_{\Lambda}}(X^{\kappa_{\Lambda}}))^{m}-(X^{\kappa_{\Lambda}})^{m})_{*}(\hat{\beta}_{m+1}^{E}(T))=(\operatorname{Log}^{E}(T))^{m}-b(T)^{m}$ . Also, by Proposition 2.4 (1), Theorem 2.8 and Corollary 3.4,  $\kappa_{*}(u_{2})_{*}(\hat{\beta}_{m+1}^{E}(T)\otimes S^{n})=((\operatorname{Log}^{E}(T))^{m}-(b(T))^{m})\otimes S^{n}$ . Thus, we have

$$\kappa_*(u_2)_*(\hat{\beta}_{m+1}^E(\operatorname{Exp}^E(T))\otimes\hat{\beta}_{n+1}^E(\operatorname{Exp}^E(S))) = \sum_{k,l>0} (\tilde{B}^{K_{\Delta}}(-m,k)\tilde{B}^E(-n,l)-\Gamma_{k,l}\tilde{B}^{K_{\Delta}}(-m,k)\tilde{B}^{K_{\Delta}}(-n,l))T^{m+k}S^{n+l},$$

and the required equation by Corollary 3.12.

By (3.7), we have

$$(3.15) \qquad \bar{\kappa}_{*}(\bar{u}_{2})_{*}(p^{E}_{m+k,m+1} \otimes p^{E}_{n+l,n+1}) = \bar{\rho}_{*}\kappa_{*}(\bar{u}_{2})_{*}(p^{E}_{m+k,m+1} \otimes p^{E}_{n+l,n+1})$$

and Theorem 3.13 gives a formula for  $\bar{\kappa}_*(\bar{u}_2)_*(p_{m+k,m+1}^E \otimes p_{n+l,n+1}^E)$  with indeterminacy  $\operatorname{Ker}(\bar{\rho}_*) = h^{K_{\Delta}}(\pi_{d(k+l)}(E; Q/Z)).$ 

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