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# FACTORIZATION OF DOUBLE TRANSFER MAPS

Dedicated to Professor Seiya Sasao on his 60th birthday

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## 1. Introduction

In [6] and [4], the authors have studied a factorization of the double $S^1$-transfer map through the second stage of the chromatic filtration. In this paper, I show that such a factorization exists for other double transfer maps.

Let $\alpha$ be an orientable vector bundle of fiber dimension $a$ over a connected finite complex $X$, and $X^\alpha$ denote the Thom space of $\alpha$. Then we have a cofiber sequence

$$S^a \xrightarrow{i} X^\alpha \xrightarrow{j} X^\alpha/S^a \xrightarrow{\tau} S^{a+1}, \tag{1.1}$$

where $i$ is the inclusion to the bottom sphere. Then, by [7], the $S^1$-transfer map is stably homotopic to $\tau$ when $X=CP^n$ and $\alpha=-\xi$ for the canonical $C$-line bundle $\xi$ over the complex projective space $CP^n$. If $X=\Sigma W$ a suspension of a space $W$, then $\tau$ is stably homotopic to the stable $J$-map $J(\alpha): X\to S^1$. Thus, generalizing the original meaning of transfer maps, we call $\tau$ in (1.1) a transfer map. Then the following stable map $\tau_2$ is called to be a double transfer map.

$$\tau_2=\tau\wedge\tau: X^\alpha/S^a\wedge Y^\beta/S^b\to S^{a+b+2}, \tag{1.2}$$

where $\beta$ is an orientable vector bundle of fiber dimension $b$ over a connected finite complex $Y$.

By Ravenel [11] a geometric realization of the chromatic filtration has been given, and we shall denote the first two stages in it by

$$\cdots\to\Sigma^{-2}N_2\xrightarrow{\delta_2}\Sigma^{-1}N_1\xrightarrow{\delta_1}S^0. \tag{1.3}$$

Here, the spectra are localized at a prime $p$, and there is some difference in our treatment between the cases of an odd prime $p$ and $p=2$. This difference is caused by the use of $K$-theory, and thus we treat the $K$-spectrum $K_\Lambda$ which denotes the complex $K$-spectrum $K_{(p)}$ localized at $p$ in case of an odd prime $p$ and the real $K$-spectrum $KO_{(2)}$ localized at 2 in case of $p=2$. Then we shall show the following:

**Theorem 1.4.** *Let $\tau_2$ be the double transfer map of* (1.2), *and $N_2$ the second* *stage of the chromatic filtration as in* (1.3). *If* $\alpha$ *and* $\beta$ *are* $K_\Lambda$*-orientable and* $K_\Lambda^{a-1}(X^\alpha/S^0; Q/Z)=0$, *then there is a factorization* $\tau_2 \simeq \delta_1\delta_2\bar{u}_2$ *by a map* $\bar{u}_2$: $X^\alpha/S^a \wedge Y^\beta/S^b \to \Sigma^{a+b}N_2$.

For the important case that $p$ is an odd prime, $X=Y=CP^N$ and $\alpha=\beta=-\xi$, the theorem has been established in [6] and [4; Th. 5.2], and we show that their method can be extended to obtain the theorem. Theorem 1.4 is a corollary of Theorem 2.8 which makes a construction of $\bar{u}_2$ clear, and §2 is devoted to demonstrate Theorem 2.8.

Such a factorization as in Theorem 1.4 draws a clear strategy to understand the double transfer image, as seen in [6], and some detailed formuals for $\bar{u}_2$ are required. In §3, we describe such formulas in the case of stunted projective spaces. When $X=Y=CP^N$, $\alpha=m\xi$ and $\beta=n\xi$ for integers $m$ and $n$, $\tau_2$ of (1.2) is a double $S^1$-transfer map for stunted complex projective spaces. By Theorem 1.4, a factorization of such double $S^1$-transfer map exists if $p$ is an odd prime. On the other hand, the double $S^1$-transfer map has no such factorization as in Theorem 1.4 if $p=2$ and both $m$ and $n$ are odd. In case of $p=2$, it might be natural to consider the quaternionic projective space $HP^N$ instead of $CP^N$. Then $\tau_2$ is called a double $S^3$-transfer map, and it always has a factorization by Theorem 1.4. For these $S^1$ and $S^3$-transfer maps, formulas concerning $\bar{u}_2$ are given in Theorem 3.5 and 3.13, (3.7) and (3.15). The method to obtain such formulas is attributed to Hilditch [6].

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## 2. Factotization

Let $S(G)$ be the Moore spectrum for a group $G$, and put $E^kG=\Sigma^kE\wedge S(G)$ for a spectrum $E$. Then, $E^k(-; G)=\{-, E^kG\}$ is the $G$-coefficient $E$-cohomology group. We have a cofiber sequence $E^kZ \xrightarrow{l_Q} E^kQ \xrightarrow{\rho_Z} E^kQ/Z$, where $l_Q$ is induced from the inclusion of the ring $Z$ of integers into the field $Q$ of rational numbers and $\rho_Z$ is induced from the mod $Z$ reduction.

Now, let $\alpha$ be an orientable vector bundle over a connected finite complex $X$. Since we work only in the stable category, it is convenient to assume that $\alpha$ is a virtual vector bundle of dimension 0, and that cohomology groups are all assumed to be reduced. Then we have a Thom class $U_\alpha^H \in H^0(X^\alpha; Z)$ of $\alpha$ in the integral cohomology group. Let $\pi_s^*(-)$ denote the stable cohomotopy group. Then, the Hurewicz map $h^H: \pi_s^0(X^\alpha; Q) \to H^0(X^\alpha; Q)$ is an isomorphism, and we can put $u=(h^H)^{-1}(U_\alpha^H) \in \pi_s^0(X^\alpha; Q)$. $u$ yields an element $\bar{u} \in \pi_s^0(X^\alpha/S^0; Q/Z)$ which makes the following diagram stably homotopy commutative up to sign:

$$
\begin{array}{ccccccc}
S^0 & \xrightarrow{i} & X^\alpha & \xrightarrow{j} & X^\alpha/S^0 & \xrightarrow{\tau} & S^1 \\
\| & & u\downarrow & & \bar{u}\downarrow & & \| \\
S^0 & \xrightarrow{l_Q} & S^0Q & \xrightarrow{\rho_Z} & S^0Q/Z & \xrightarrow{\delta_1} & S^1 .
\end{array} \tag{2.1}
$$

This diagram generalizes the fundamental situation designed by Miller [8], and $\tau$ represents a transfer map as in §1. $\bar{u}$ is uniquely determined by the equation $j^*(\bar{u})=\rho_Z(u)$.

We denote by $K_\Lambda$ the $K$-spectrum $K_{(p)}$ for an odd prime $p$ or $KO_{(2)}$ for $p=2$, and we assume that $\alpha$ is $K_\Lambda$-orientable. Then we have a $K_\Lambda$-theory Thom class $U_\alpha^{K_\Lambda}\in K_\Lambda^0(X^\alpha)$ of $\alpha$. Let $ch_\Lambda\colon K_\Lambda^0(-)\to H^*(-;Q)$ be the Chern character, and $h^{K_\Lambda}\colon \pi_s^*(-)\to K_\Lambda^*(-)$ the $K_\Lambda$-Hurewicz homomorphism. Then the characteristic class $bh_\Lambda(\alpha)\in 1+\sum_{i>0}H^{di}(X;Q)$ is defined by the equation $ch_\Lambda(U_\alpha^{K_\Lambda})=U_\alpha^H bh_\Lambda(\alpha)$ (cf. [1]), where $d=2$ or 4 according as $K_\Lambda=K_{(p)}$ or $KO_{(2)}$. We notice that $ch_\Lambda\colon K_\Lambda^0(W;Q)\to\sum_{i\geq 0}H^{di}(W;Q)$ is an isomorphism for $W=X_+$ or $X^\alpha$, since $X$ is assumed to be a finite complex. Then the following is deduced from (2.1).

**Lemma 2.2.** *For a $K_\Lambda$-orientable vector bundle $\alpha$,*
(1) $h^{K_\Lambda}(u)=U_\alpha^{K_\Lambda}ch_\Lambda^{-1}(bh_\Lambda(-\alpha))$ *in* $K_\Lambda^0(X^\alpha;Q)$, *and*
(2) *there is a unique element* $V_\alpha\in K_\Lambda^0(X^\alpha/S^0;Q)$ *which satisfies*

$$\rho_Z(V_\alpha)=h^{K_\Lambda}(\bar{u}) \quad \textit{and} \quad j^*(V_\alpha)=h^{K_\Lambda}(u)-(l_Q)_*(U_\alpha^{K_\Lambda}) .$$

Proof. Apply $ch_\Lambda$ on both sides of the equation in (1). Then they both become $U_\alpha^H$, since $ch_\Lambda h^{K_\Lambda}(u)=h^H(u)$ for the left hand side. Since $ch_\Lambda$ is an isomorphism over $K_\Lambda^0(X^\alpha;Q)$, we have (1). Let $K_\Lambda^0(X^\alpha/S^0;G)\xrightarrow{j^*}K_\Lambda^0(X^\alpha;G)\xrightarrow{i^*}K_\Lambda^0(S^0;G)$ for $G=Q$ or $Q/Z$ be the exact sequence induced from the cofiber sequence as in (1.1). Then $j^*$ is a monomorphism, since $K_\Lambda^{-1}(S^0;G)=0$. We put $z=h^{K_\Lambda}(u)-(l_Q)_*(U_\alpha^{K_\Lambda})\in K_\Lambda^0(X^\alpha;Q)$. Then $i^*(z)=0$, and there is a unique element $V_\alpha\in K_\Lambda^0(X^\alpha/S^0;Q)$ with $j^*(V_\alpha)=z$. $V_\alpha$ is the required element of (2), because $j^*(\rho_Z(V_\alpha))=\rho_Z(z)=j^*(h^{K_\Lambda}(\bar{u}))$.

Let $\psi=\psi^\gamma-1\colon K_\Lambda\to K_\Lambda$ be the stable Adams operation for a generator $\gamma$ of the unit group in $Z/p^2$, and $Ad$ the fiber spectrum of $\psi$. We assume that $\gamma=3$ in cases of $p=2$. Thus we have a cofiber sequence

$$Ad^0G\xrightarrow{\kappa}K_\Lambda^0G\xrightarrow{\psi}K_\Lambda^0G \tag{2.3}$$

for $G=Z_{(p)}$, $Q$ or $Q/Z_{(p)}$. The $Ad$-theory plays an important role later.

Now, let $\beta$ be an orientable virtual vector bundle of dimension 0 over a connected finite complex $Y$, and $1\wedge i\colon X^\alpha/S^0=X^\alpha/S^0\wedge S^0\to X^\alpha/S^0\wedge Y^\beta$ the inclusion. For the element $V_\alpha$ in Lemma 2.2, we have an extension $\tilde{u}$ as follows:

**Proposition 2.4.** *Assume that $\alpha$ and $\beta$ are $K_\Lambda$-orientable. Then, there is an element $\tilde{u}\in K^0_\Lambda(X^\alpha/S^0\wedge Y^\beta; Q)$ which satisfies*

(1) $(1\wedge i)^*(\tilde{u})=V_\alpha$, *and*

(2) $\psi(\tilde{u})\in \mathrm{Im}[(l_Q)_*: K^0_\Lambda(X^\alpha/S^0\wedge Y^\beta)\to K^0_\Lambda(X^\alpha/S^0\wedge Y^\beta; Q)]$.

Proof. Since $ch_\Lambda: K^0_\Lambda(X^\alpha/S^0; Q)\to\sum_{i>0}H^{di}(X^\alpha/S^0; Q)$ is an isomorphism, we can write $ch_\Lambda(V_\alpha)=\sum_{i>0}a_i$ for some $a_i\in H^{di}(X^\alpha/S^0; Q)$ and put $A_i=(ch_\Lambda)^{-1}(a_i)\in K^0_\Lambda(X^\alpha/S^0; Q)$. Then $V_\alpha=\sum_{i>0}A_i$, and $\psi^\gamma A_i=\gamma^{id/2}A_i$. Similarly, regarding a Thom class $U^{K_\Lambda}_\beta\in K^0_\Lambda(Y^\beta)$ as an element of $K^0_\Lambda(Y^\beta; Q)$, we have $U^{K_\Lambda}_\beta=\sum_{j\geq 0}B_j$ for some $B_j\in K^0_\Lambda(Y^\beta; Q)$ with $\psi^\gamma B_j=\gamma^{jd/2}B_j$. We put

$$\tilde{u} = V_\alpha\otimes U^{K_\Lambda}_\beta-\sum_{k,l>0}\tilde{\Gamma}_{k,l}A_k\otimes B_l\in K^0_\Lambda(X^\alpha/S^0\wedge Y^\beta; Q)\,, \tag{2.5}$$

where $\tilde{\Gamma}_{k,l}=(\gamma^{ld/2}-1)/(\gamma^{(k+l)d/2}-1)$. Then, $\tilde{u}$ satisfies (1), since $i^*(U^{K_\Lambda}_\beta)=1$ and $i^*(B_l)=0$. Using the definitions of $A_i$ and $B_j$, it follows that

$$\psi(\tilde{u}) = \psi(V_\alpha)\psi^\gamma(U^{K_\Lambda}_\beta)\,. \tag{2.6}$$

By the second equation in Lemma 2.2 (2), we have $j^*(\psi(V_\alpha))=h^{K_\Lambda}(u)-\psi^\gamma((l_Q)_*(U^{K_\Lambda}_\alpha))-j^*(V_\alpha)=-(l_Q)_*(\psi(U^{K_\Lambda}_\alpha))$, where $j: X^\alpha\to X^\alpha/S^0$ and $l_Q: K^0_\Lambda Z\to K^0_\Lambda Q$. But, there is an element $w\in K^0_\Lambda(X^\alpha/S^0)$ with $j^*(w)=-\psi(U^{K_\Lambda}_\alpha)$, and thus $j^*(l_Q)_*(w)=j^*(\psi(V_\alpha))$ in $K^0_\Lambda(X^\alpha; Q)$. Since $j^*: K^0_\Lambda(X^\alpha/S^0; Q)\to K^0_\Lambda(X^\alpha; Q)$ is a monomorphism, we have $\psi(V_\alpha)=(l_Q)_*(w)$, and thus $\tilde{u}$ satisfies (2) by (2.6), which completes the proof.

We need to recall the geometric realization [11] of the chromatic filtration as in (1.3). Let $l_i: E\to L_iE$ be the Bousfield localization [5] with respect to the $v_i^{-1}BP_*$-homology for a prime $p$. Then the $i$-stage of the filtration is realized by a spectrum $N_i$ which is defined inductively, starting with $N_0=S^0$, by the cofiber sequence

$$N_i \xrightarrow{l_i} M_i = L_iN_i \xrightarrow{\rho_i} N_{i+1} \xrightarrow{\delta_{i+1}} \Sigma N_i\,. \tag{2.7}$$

In particular, $M_0=S(Q)$ and $N_1=S(Q/Z)$. Furthermore, by [5] or [12], it is shown that there is a homotopy equivalence $M_1\simeq Ad^0Q/Z$ through which $l_1: N_1\to M_1$ is identified with the $Ad$-theory Hurewicz homomorphism $h^{Ad}$: $S^0Q/Z\to Ad^0Q/Z$. Here, spectra are assumed to be localized at $p$, and $Ad$ is the fiber spectrum of the stable Adams operation $\psi=\psi^\gamma-1$ defined on $K_{(p)}$ if $p$ is odd and on $KO_{(2)}$ if $p=2$. Thus, $\rho_1: M_1\to N_2$ is identified with $\bar{\rho}: Ad^0Q/Z\to\overline{Ad}^0Q/Z$ for $\overline{Ad}=Ad/S^0_{(p)}$, and we have maps $\kappa: M_1\to K^0_\Lambda Q/Z$ and $\bar{\kappa}: N_2\to \bar{K}^0_\Lambda Q/Z$ induced from $\kappa: Ad^0Q/Z\to K^0_\Lambda Q/Z$ as in (2.3). Then Theorem 5.2 in [4] is extended to the following form.

**Theorem 2.8.** *Assume that $\alpha$ and $\beta$ are $K_\Lambda$-orientable and $K_\Lambda^{-1}(X^\alpha/S^0; Q/Z)=0$. Then, we have elements $u_2 \in (M_1)^0(X^\alpha/S^0 \wedge Y^\beta)$ and $\bar{u}_2 \in (N_2)^0(X^\alpha/S^0 \wedge Y^\beta/S^0)$ which make the following diagram stably homotopy commutative up to sign:*

$$\begin{array}{ccccccc}
X^\alpha/S^0 & \xrightarrow{1\wedge i} & X^\alpha/S^0\wedge Y^\beta & \xrightarrow{1\wedge j} & X^\alpha/S^0\wedge Y^\beta/S^0 & \xrightarrow{1\wedge\tau} & \Sigma X^\alpha/S^0 \\
{\scriptstyle V_\alpha}\downarrow & & {\scriptstyle u_2}\downarrow & & {\scriptstyle \bar{u}_2}\downarrow & & {\scriptstyle \bar{u}}\downarrow \\
K_\Lambda^0 Q & & M_1 & \xrightarrow{\rho_1} & N_2 & \xrightarrow{\delta_2} & \Sigma N_1 \\
\| & & {\scriptstyle \kappa}\downarrow & & {\scriptstyle \bar{\kappa}}\downarrow & & \\
K_\Lambda^0 Q & \xrightarrow{\rho_Z} & K_\Lambda^0 Q/Z & \xrightarrow{\bar{\rho}} & \bar{K}_\Lambda^0 Q/Z & &
\end{array}$$

*Here, $u_2$ can be taken to satisfy $\kappa_*(u_2)=\rho_Z(\tilde{u})$ for $\tilde{u}$ of Proposition 2.4.*

Proof. We put $W=X^\alpha/S^0\wedge Y^\beta$. Then by Proposition 2.4 (2), $\psi(\rho_Z(\tilde{u}))=0$ in $K_\Lambda^0(W; Q/Z)$, and thus we have an element $u_2\in(M_1)^0(W)$ satisfying $\kappa_*(u_2)=\rho_Z(\tilde{u})$. By Proposition 2.4 (1) and Lemma 2.2 (2), $\kappa_*(1\wedge i)^*(u_2)=(1\wedge i)^*\rho_Z(\tilde{u})=\rho_Z(V_\alpha)=h^{K_\Lambda}(\bar{u})=\kappa_*(l_l)_*(\bar{u})$, where $l_1\colon N_1\to M_1$ is the map as in (2.7). Since $\kappa_*\colon (M_1)^0(X^\alpha/S^0)\to K_\Lambda^0(X^\alpha/S^0; Q/Z)$ is a monomorphism by the assumption that $K_\Lambda^{-1}(X^\alpha/S^0; Q/Z)=0$, we have

$$(1\wedge i)^*(u_2)=(l_1)_*(\bar{u}) \qquad \text{in} \quad (M_1)^0(X^\alpha/S^0)\,.$$

Then, $\bar{u}$ and $u_2$ produce maps from the upper cofiber sequence in the diagram to the second cofiber sequence $N_1\to M_1\to N_2\to\Sigma N_1$, and thus we have the required elements $u_2$ and $\bar{u}_2$ which make the diagram commutative up to sign.

We notice that the assumption $K_\Lambda^{-1}(X^\alpha/S^0; Q/Z)=0$ in the theorem is satisfied if $K_\Lambda^0(X)$ is torsion free and $K_\Lambda^{-1}(X)$ is a torsion group. From (2.1) and the commutativity of the upper right square in the diagram of Theorem 2.8, it follows that the double transfer $\tau_2\colon X^\alpha/S^0\wedge Y^\beta/S^0\to S^2$ is factored through the second stage $N_2$ as $\tau_2\simeq\delta_1\delta_2\bar{u}_2$, and we have Theorem 1.4.

REMARK 2.9. For the canonical complex line bundle $\xi$ over $CP^N$, $(2m+1)\xi$ is not $KO$-orientable for any integer $m$. By the same reason as in [6: Remark 3.2], there is no such factorization as in Theorem 1.4 in case of $p=2$, $X=Y=CP^N$, $\alpha=(2m+1)\xi$ and $\beta=(2n+1)\xi$.

## 3. Stunted projective spaces

Let $C$ and $H$ be the field of the complex and quaternionic numbers, and put $(F, d)=(C, 2)$ or $(H, 4)$, respectively. We denote the $N$-th projective space over $F$ by $FP^N$ for $N\geq 0$, and the canonical $F$-line bundle over $FP^N$ by $\xi$. Then, for a positive integer $k$, the Thom space of $k\xi$ is homeomorphic to the stunted projective space $FP_k^{N+k}=FP^{N+k}/FP^{k-1}$ by [2]. Thus, for any integer $k$, we denote the Thom space of $k\xi$ over $FP^N$ simply by $FP_k$, since our results are valid for any $N$ and compatible with each $N$. Then, in the cofiber sequence $S^{dk}\xrightarrow{i}FP_k\xrightarrow{j}FP_{k+1}\xrightarrow{\tau}S^{dk+1}$, $\tau$ represents a transfer map for $k\xi$, and we call this $\tau$ a $S^{d-1}$-transfer map. Thus, a double $S^{d-1}$-transfer map is given by

$$\tau_2=\tau\wedge\tau\colon FP_{m+1}\wedge FP_{n+1}\to S^{d(m+n)+2}\,. \tag{3.1}$$

In this section, we are concerned with this $\tau_2$.

In Theorem 2.8, $K_\Lambda=K_{(p)}$ or $KO_{(2)}$ according as the spectra are assumed to be localized at an odd prime $p$ or 2. Hereafter, we assume that $p$ is odd whenever we discuss $S^1$-transfer maps, and that $p=2$ for $S^3$-transfer maps. Thus, $(K_\Lambda, FP^N)=(K_{(p)}, CP^N)$ or $(KO_{(2)}, HP^N)$ according as $p$ is an odd prime or $p=2$. Then $k\xi$ over $FP^N$ is always $K_\Lambda$-orientable for any integer $k$. In the below, we denote the coefficient group $\pi_i(K_\Lambda)$ by $(K_\Lambda)_i$, and the Bott generators by $t\in K_2$ and $g_i\in KO_{4i}$ respectively.

In order to express a formula for $u_2$ of Theorem 2.8 with respect to $\tau_2$ in (3.1), the $K_\Lambda$-Bernoulli numbers are necessary. Let $e^T$ be the formal power expansion of the exponential function on $T$, and $\sinh(T)$ that of the hyperbolic sin function on $T$. We put $(2\sinh(\sqrt{T}/2))^2=\sum_{j\geq 0}s_jT^{j+1}$, where all $s_j$ are rational numbers and $s_0=1$. Using these notations, we define the following:

DEFINITION 3.2. (1) $\mathrm{Exp}^{K_\Lambda}(-)$ and $\mathrm{Log}^{K_\Lambda}(-)$:

$$\mathrm{Exp}^K(T)=t^{-1}(1-e^{-tT})\in(K_*\otimes Q)[[T]],$$

$$\mathrm{Exp}^{KO}(T)=\sum_{j\geq 0}(-1)^j s_j(g_j/a(j))T^{j+1}\in(KO_*\otimes Q)[[T]],$$

$$\mathrm{Log}^{K_\Lambda}(T)=(\mathrm{Exp}^{K_\Lambda})^{-1}(T)\in((K_\Lambda)_*\otimes Q)[[T]],$$

where $a(j)=1$ (resp. 2) if $j$ is even (resp. odd).

(2) The $K_\Lambda$-Bernoulli numbers $\tilde{B}^{K_\Lambda}(m,k)\in(K_\Lambda)_{dk}\otimes Q$:

$$\left(\frac{T}{\mathrm{Exp}^{K_\Lambda}(T)}\right)^m=\sum_{k\geq 0}\tilde{B}^{K_\Lambda}(m,k)T^k\,.$$

Let $X^K=t^{-1}[1-\xi]\in K^2(CP^N)$ and $X^{KO}=[1-\xi]\in KO^4(HP^N)$ be the $K_\Lambda$-theory Euler classes of $\xi$, and $x\in H^d(FP^N;Z)$ the Euler class which satisfies $ch_\Lambda(\xi)=e^x$ or $e^{\sqrt{x}}+e^{-\sqrt{x}}$ for $CP^N$ or $HP^N$ respectively. Then, for $(E,x^E)=(K_\Lambda, X^{K_\Lambda})$ or $(H,x)$, we have an isomorphism $E^*(FP^N)\cong E_*[[x^E]]/((x^E)^{N+1})$, and $E^*(FP_k)$ is a free $E^*(FP^N)$ module with a Thom class $U^E_{k\xi}$ as a generator. As in [8], we can put $U^E_{k\xi}=(x^E)^k$ and $(x^E)^i(x^E)^j=(x^E)^{i+j}$ for $i\geq k$ and $j\geq 0$.

Let $f_\Lambda(x)=1-e^x$ or $-(2\sinh\sqrt{x}/2)^2$ in $H^*(FP^N;Q)$ according as $FP^N=CP^N$ or $HP^N$. Then, we have the following:

**Lemma 3.3.** *$ch_\Lambda(X^{K_\Lambda})=f_\Lambda(x)$ and $ch_\Lambda(\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda}))=-x$.*

Proof. Since $ch_\Lambda\xi=d/2-f_\Lambda(x)$, the first equation is clear. Let $\log(T)$ be the power series exapansion of the logarithm function on $T$, and put $(2\sinh^{-1}(\sqrt{T}/2))^2=\sum_{j\geq 0} r_j T^{j+1}$. Then, $\mathrm{Log}^K(T)=-t^{-1}\log(1-tT)$ and $\mathrm{Log}^{KO}(T)=\sum_{j\geq 0}(-1)^j r_j (g_j/a(j))T^{j+1}$. Since $ch_\Lambda$ is a ring homomorphism, we have the second required equation.

Let $u\in\pi_s^{dm}(FP_m; Q)$ and $V_{m\xi}\in K_\Lambda^{dm}(FP_{m+1}; Q)$ be the elements as in (2.1) and Lemma 2.2 respectively. Then, the following is a corollary of Lemmas 2.2 and 3.3.

**Corollary 3.4.** *For any integer m,*

$$h^{K_\Lambda}(u)=(\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda}))^m \quad \textit{and} \quad j^*(V_{m\xi})=(\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda}))^m-(X^{K_\Lambda})^m,$$

*where* $j^*: K_\Lambda^{dm}(FP_{m+1}; Q)\to K_\Lambda^{dm}(FP_m; Q)$ *is a monomorphism.*

Proof. As above, $U_{m\xi}^{K_\Lambda}$ is taken to be $(X^{K_\Lambda})^m$. In order to satisfy $ch_\Lambda(U_\xi^{K_\Lambda})=U_\xi^H bh_\Lambda(\xi)$ and $bh_\Lambda(\xi)\in 1+\sum_{i>0}H^{di}(FP^N; Q)$, we must take $U_\xi^H=-x$ instead of $x$, because $ch_\Lambda(X^{K_\Lambda})=f_\Lambda(x)=(-x)(f_\Lambda(x)/(-x))$ by Lemma 3.3. Hence, $U_{m\xi}^H=(-x)^m$ and $bh_\Lambda(m\xi)=(-f_\Lambda(x)/x)^m$. Then, it follows from Lemma 3.3 that

$$ch_\Lambda^{-1}(bh_\Lambda(-m\xi))=\left(\frac{\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda})}{X^{K_\Lambda}}\right)^m.$$

Thus we have the first required equation by Lemma 2.2(1), and the second required equation by the first equation and Lemma 2.2 (2).

Now, we can show a formula for an element $u_2\in(M_1)^{d(m+n)}(FP_{m+1}\wedge FP_n)$ as in Theorem 2.8. For a while, we put $FP(k, l)=FP_k\wedge FP_l$ for brevity. Since $K_\Lambda^{d(m+n)-1}(FP(m+1, n); Q/Z)=0$ and $K_\Lambda^{dn-1}(FP_n; Q/Z)=0$, both $\kappa_*: (M_1)^{d(m+n)}(FP(m+1, n); Q/Z)\to K_\Lambda^{d(m+n)}(FP(m+1, n); Q/Z)$ and $(j\wedge 1)^*: K_\Lambda^{d(m+n)}(FP(m+1, n); Q/Z)\to K_\Lambda^{d(m+n)}(FP(m, n); Q/Z)$ are monomorphisms. Hence we shall describe a formula for $\kappa_*(u_2)\in K_\Lambda^{d(m+n)}(FP(m+1, n); Q/Z)$, regarding it as an element of $K_\Lambda^{d(m+n)}(FP(m, n); Q/Z)$ through $(j\wedge 1)^*$. We shall represent $K_\Lambda^*(FP(m, n); Q/Z)$ as $R\{(X^{K_\Lambda})^m\}\otimes R\{(Y^{K_\Lambda})^n\}$ for $R=K_\Lambda^*(FP^N; Q/Z)$, using $Y^{K_\Lambda}$ to denote the $K_\Lambda$-theory Euler class of $\xi$ for the second factor. Let $\gamma$ be a generator of the unit group in $Z/p^2$, which is used in the definition of $Ad$ before (2.3). Then we have the following formula.

**Theorem 3.5.** *In* $K_\Lambda^{d(m+n)}(FP_{m+1}\wedge FP_n; Q/Z)$,

$$\kappa_*(u_2)=((\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda}))^m-(X^{K_\Lambda})^m)\otimes(Y^{K_\Lambda})^n + \sum_{k,l>0}\tilde{\Gamma}_{k,l}\tilde{B}^{K_\Lambda}(-m, k)\tilde{B}^{K_\Lambda}(-n, l)(\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda}))^{m+k}\otimes(\mathrm{Log}^{K_\Lambda}(Y^{K_\Lambda}))^{n+l},$$

*where* $\tilde{\Gamma}_{k,l}=(\gamma^{dl/2}-1)/(\gamma^{d(k+l)/2}-1)$.

Proof. By Theorem 2.8, we take $u_2$ to satisfy $\kappa_*(u_2)=\rho_Z(\tilde{u})$ for $\tilde{u}$ given by (2.5). Since $(j\wedge 1)^*(V_{m\xi}\otimes U^{K_\Lambda}_{n\xi})=((\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda}))^m-(X^{K_\Lambda})^m)\otimes(Y^{K_\Lambda})^n$ by Corollary 3.4, all we need is formulas for $A_k$ and $B_l$ in (2.5). By Lemma 3.3 and Corollary 3.4, we have

$$ch_\Lambda(j^*(V_{m\xi}))=ch_\Lambda(\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda}))^m-ch_\Lambda(X^{K_\Lambda})^m=-\sum_{i>0}[f_\Lambda(x)^m]_{m+i},$$

where $[f_\Lambda(x)^m]_j$ denotes the $dj$-dimensional part of $f_\Lambda(x)^m$. On the other hand, from Definition 3.2 ,it follows that

$$\left(\frac{X^{K_\Lambda}}{\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda})}\right)^m=\sum_{i\geq 0}\tilde{B}^{K_\Lambda}(-m,i)(\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda}))^i.$$

Applying $ch_\Lambda$ on both sides of this equation and using Lemma 3.3, we have

$$f_\Lambda(x)^m=(-x)^m+\sum_{k>0}ch_\Lambda(\tilde{B}^{K_\Lambda}(-m,k))(-x)^{m+k}.$$

Then, we obtain

$$A_k=ch_\Lambda^{-1}(-[f_\Lambda(x)^m]_{m+k})=-\tilde{B}^{K_\Lambda}(-m,k)(\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda}))^{m+k}.$$

Similarly, $B_l=\tilde{B}^{K_\Lambda}(-n,l)(\mathrm{Log}^{K_\Lambda}(Y^{K_\Lambda}))^{n+l}$. Thus, by (2.5), we have the required formula.

We have not got any explicit formula for $\bar{\kappa}_*(\bar{u}_2)\in\bar{K}^{d(m+n)}_\Lambda(FP_{m+1}\wedge FP_{n+1};Q/Z)$. However, Theorem 2.8 shows

$$(1\wedge j)^*\bar{\kappa}_*(\bar{u}_2)=\bar{p}_*\kappa_*(u_2),\tag{3.7}$$

and thus the formula for $\kappa_*(u_2)$ in Theorem 3.5 describes $\bar{\kappa}_*(\bar{u}_2)$ with indeterminacy $\mathrm{Ker}(1\wedge j)^*=(1\wedge\tau)^*(\bar{K}^{d(m+n)-1}_\Lambda(FP_{m+1};Q/Z))$ and $\mathrm{Ker}(\bar{p}_*)=h^{K_\Lambda}(\pi_s^{d(m+n)}(FP_{m+1}\wedge FP_n;Q/Z))$.

Let $MG$ be the Thom spectrum $MU$ or $MSp$ for the complex or symplectic cobordism theory, respectively. We only consider these spectra in the case that $(MG,K_\Lambda,FP_k)=(MU,K_{(p)},CP_k)$ or $(MSp,KO_{(2)},HP_k)$ according as $p$ is an odd prime or 2. Let $p_{k,l}$ be a generator of the primitive part $PMG_{dk}(FP_l)\cong Z$ for $k\geq l$. The rest of this section is devoted to obtain a formula for $\kappa_*(u_2)_*(p_{i,j}\otimes p_{k,l})$ using Theorem 3.5. Then it gives a formula for $\bar{\kappa}_*(\bar{u}_2)_*(p_{i,j}\otimes p_{k,l})$ by (3.7).

Let $\beta_i\in H_{di}(FP^\infty;Z)$ be the dual of $x^i$, and $b_i^{MG}\in H_{di}(MG)$ be the image of $\beta_{i+1}$ under the canonical homomorphism $H_{d(i+1)}(FP^\infty;Z)\to H_{di}(MG;Z)$, for $i\geq 0$. We define a ring spectrum $E$ to be $F$-oriented if there is an element $x^E\in E^d(FP^\infty)$ with $E^*(S^d)\cong E_*\{i^*(x^E)\}$, where $F=C$ or $H$ and $i\colon S^d\to FP^\infty$ is the inclusion map. Then, as is well known, there is a map $\Phi^E\colon MG\to E$ associated with $x^E$ such that $\iota^*(\Phi^E)$ is a unit of $\pi_0(E)$ for the unit $\iota\colon S^0\to MG$. Then we have an element $b_i^E=\Phi_*^E(b_i^{MG})\in H_{di}(E;Z)$, and also an element $\beta_i^E\in E_{di}(FP^\infty)$ which is the dual of $(x^E)^i$. For an $F$-oriented spectrum $E$, the $E$-theory Bernoulli numbers as in [8] are defined as follows:

DEFINITION 3.8.
(1) $\mathrm{Exp}^E(T)=\sum_{i\geq 0} b_i^E T^{i+1}\in(H\wedge E)_*[[T]]$ and $\mathrm{Log}^E(T)=(\mathrm{Exp}^E)^{-1}(T)$.
(2) The $E$-theory Bernoulli number $\tilde{B}^E(m,k)\in(E_{dk}\otimes Q)[[T]]$;

$$\left(\frac{T}{\mathrm{Exp}^E(T)}\right)^m=\sum_{i\geq 0}\tilde{B}^E(m,k)T^k .$$

In case of a $C$-oriented $E$, $\mathrm{Exp}^E$ is the exponential sequence related to the formal group law over $E_*$ induced from $\Phi^E$. Definition 3.2 coincides with this definition if $(E, x^E)=(K_\Lambda, X^{K_\Lambda})$. For later use, we put

$$(3.9)\quad b^E=\sum_{i\geq 0} b_i^E\in H_*(E;Z)\quad\text{and}\quad \hat{\beta}_k^E(T)=\sum_{i\geq k}\beta_i^E T^i\in E_*(FP_k)[[T]] .$$

As for a generator $p_{n,0}$, of the primitive part $PMG_{dn}(FP_0)$, an explicit formula is given for $MU$ by Segal [13] and for $MSp$ by Baker [3]. They have described a generator $p_{n,0}^H\in PH_{dn}(FP^\infty;Z)\subset P(H\wedge MG)_{dn}(FP^\infty)$, and their methods are immediately applicable to stunted projective spaces. Let $c(k,l)$ be the positive minimal integer $c$ which makes $c\cdot[b^{MG}]_{k-i}^i$ an element of $h^H(MG_{d(k-i)})$ in $H_{d(k-i)}(MG;Q)$ for any $i$ with $l\leq i\leq k$. Here $[b^{MG}]_{k-i}^i$ is the $d(k-i)$-dimensional part of $(b^{MG})^i$. Then, using the methods in [13] and [3], we have the following:

**Lemma 3.10.** *Let $k\geq l$.*
(1) $p_{k,l}=c(k,l)\sum_{i=l}^k[b^{MG}]_{k-i}^i\beta_i^{MG}$ *is a generator of* $PMG_{dk}(FP_l)\cong Z$.
(2) *When $(MG,FP_l)=(MU,CP_l)$, $c(k,l)$ is equal to the $K$-codegree $cd^K(k,l)$ which is cited below.*

REMARK 3.11. The $K_\Lambda$-codegree $cd^{K_\Lambda}(k,l)$ is defined as the minimal positive integer $c$ such that the $d(k-j)$-dimensional part of $c\cdot bh_\Lambda(j\xi)$ is in $H^{k-j}(FP_0;Z)$ for $l\leq j\leq k$, that is, $c\cdot bh_\Lambda(j\xi)$ is integral. Thus $K_\Lambda$-codegrees are computable. If the mod torsion Hattori-Stong conjecture for $MSp$ (cf. [10], [9]) holds, then we also have $c(k,l)=cd^{KO}(k,l)$ in the case of $(MG,FP_l)=(MSp,HP_l)$. This can be seen by the method in [3]. In general, $cd^{KO}(k,l)$ is a factor of $c(k,l)$.

Put $p_{i,j}^E=(\Phi^E)_*(p_{i,j})\in PE_{di}(FP_j)$ for a $F$-oriented spectrum $E$. Then, by Definition 3.8 (1), (3.9) and Lemma 3.10 (1), we have the following corollary.

**Corollary 3.12.** *Let $E$ be $F$-oriented. Then*

$$\hat{\beta}_k^E(\mathrm{Exp}^E(T))=\sum_{i\geq k}\frac{p_{i,k}^E}{c(i,k)}T^i .$$

We obtain the following formula, using the technique due to Miller[8] and Hilditch[6].

**Theorem 3.13.** *Let $k, l \geq 1$. Then, as an element of $(K_\Lambda)_{d(k+l)}(E; Q/Z)$,*

$$\kappa_*(u_2)_*(p^E_{m+k,m+1} \otimes p^E_{n+l,n+1}) = c(m+k, m+1)c(n+l, n+1)\cdot (\tilde{B}^{K_\Lambda}(-m, k)\tilde{B}^E(-n, l) - \Gamma_{k,l}\tilde{B}^{K_\Lambda}(-m, k)\tilde{B}^{K_\Lambda}(-n, l))$$

*for $\Gamma_{k,l} = \gamma^{dl/2}(\gamma^{dk/2}-1)/(\gamma^{d(k+l)/2}-1)$.*

Proof. Let $g(X^{K_\Lambda}) = \sum_{i \geq n} a_i(X^{K_\Lambda})^i$ be an element of $K^*_\Lambda(FP_n; Q)$, and put $b(T) = \mathrm{Exp}^{K_\Lambda}(\mathrm{Log}^E(T))$. Then, by [8] or [6], it is shown that

$$g(X^{K_\Lambda})_*(\hat{\beta}^E_n(T)) = g(b(T)) \in ((K_\Lambda \wedge E)_* \otimes Q)[[T]]\,. \tag{3.14}$$

Hence, it follows that $((X^{K_\Lambda})^j)_*(\hat{\beta}^E_l(T)) = b(T)^j$ (resp. 0) if $j \geq l$ (resp. $j < l$), and $((\mathrm{Log}^{K_\Lambda}(X^{K_\Lambda}))^m - (X^{K_\Lambda})^m)_*(\hat{\beta}^E_{m+1}(T)) = (\mathrm{Log}^E(T))^m - b(T)^m$. Also, by Proposition 2.4 (1), Theorem 2.8 and Corollary 3.4, $\kappa_*(u_2)_*(\hat{\beta}^E_{m+1}(T) \otimes S^n) = ((\mathrm{Log}^E(T))^m - (b(T))^m) \otimes S^n$. Thus, we have

$$\kappa_*(u_2)_*(\hat{\beta}^E_{m+1}(\mathrm{Exp}^E(T)) \otimes \hat{\beta}^E_{n+1}(\mathrm{Exp}^E(S))) = \sum_{k,l>0} (\tilde{B}^{K_\Lambda}(-m, k)\tilde{B}^E(-n, l) - \Gamma_{k,l}\tilde{B}^{K_\Lambda}(-m, k)\tilde{B}^{K_\Lambda}(-n, l))T^{m+k}S^{n+l}\,,$$

and the required equation by Corollary 3.12.

By (3.7), we have

$$\bar{\kappa}_*(\bar{u}_2)_*(p^E_{m+k,m+1} \otimes p^E_{n+l,n+1}) = \bar{p}_*\kappa_*(u_2)_*(p^E_{m+k,m+1} \otimes p^E_{n+l,n+1})\,, \tag{3.15}$$

and Theorem 3.13 gives a formula for $\bar{\kappa}_*(\bar{u}_2)_*(p^E_{m+k,m+1} \otimes p^E_{n+l,n+1})$ with indeterminacy $\mathrm{Ker}(\bar{p}_*) = h^{K_\Lambda}(\pi_{d(k+l)}(E; Q/Z))$.

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