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LOCAL TRIANGULATION OF REAL ANALYTIC VARIETIES

Dedicated to Professor K. Shoda on his sixtieth birthbay

By

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Introduction

In the recent study of real analytic varieties, one of the main problems is to decompose a given variety into some reasonable subsets. H. Whitney proved that a real algebraic variety can be expressed as a union of mutually disjoint manifolds of various dimensions [6], while A. H. Wallace decomposed it into sheets (analytically connected sets) [4]. Later, Whitney and F. Bruhat extended Whitney's result to the case of so-called C-analytic varieties [7], and Wallace also generalized his result to real analytic varieties in somewhat milder form [5]. In these studies, local connectivity of real analytic varieties (see, for example, [7], Prop. 2) plays a fundamental role.

In our present paper, we first prove that a real analytic variety E is locally triangulable with given subvarieties as subcomplexes (Theorem 1), from which local connectivity follows immediately, and as a consequence of this Theorem, we show that a real algebraic variety is globally triangulable into a finite number of simplexes (Theorem 2). Next, in a global vein, we show that a real analytic variety admits, what we call, pseudo-cell docomposition (Corollary to Theorem 3).

When we carry out the proof by induction, the main difficulty lies in the fact that a (local) projection of E on a subspace (with respect to a coordinate system) is not necessarily a variety, even though a coordinate system is p-proper (see § 1) for the (local) complexification E^* of E. The most part of our proof is devoted to eliminate this difficulty. The idea of our proof is to get a (local) triangulation of E as a subcomplex of a bigger complex $\tilde{G}$ which has a more convenient form than E itself. To do so, we first construct two imbedding varieties $\tilde{E}^*$ and $\hat{E}^*$ which locally contain E in such a way that $\hat{E}^*$ contains the real part of $\tilde{E}^*$ (Lemma 1). Next we introduce the notions of p-proper simplex and p-proper complex (§ 4 and § 5) and show that the triangulation of p-proper complex can be extended to the whole neighborhood (Lemma 3). Taking the real part of $\tilde{E}^*$, we obtain a desired p-proper complex $\tilde{G}$ for E. Lemma 3 and the property of $\hat{E}^*$ along with existence of proper coordinate system (Lemma 2) for a family of varieties which appear in the course of the induction enable us to complete the induction, where E is automatically triangulated as a subcomplex of $\tilde{G}$ with given subvarieities of E as subcomplexes.

The finite global triangulation of a real algebraic variety A reduces to the above case by considering the local triangulation of the associated projective variety A^0 at the center of the projection. Here we again see the finiteness property of algebraic varieties which distinguishes them from general analytic varieties (cf., [6], Theorem 3, 4 and 5, and [4], Theorem 15).

After defining cell and pseudo-cell decompositions (§ 8), we observe that in the same way as we get local triangulation of E, we can obtain (local) cell decomposition of E such that each cell is determined by real analytic varieties (Lemma 4). We then consider some family of real analytic varieties which we call scattered family of varieties (§ 9) and show that it admits pseudo-cell decomposition (Theorem 3). In this proof by induction, Lemma 4 plays a substantial role, namely, the fact that each cell in the (local) cell docomposition is determined by real analytic varieties makes it possible to carry out the proof by induction. As a special case of Theorem 3, we see that a real analytic variety in R^n has a pseudo-cell decomposition (Corollary to Theorem 3).

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1. Preliminaries[1)]

Let M be a real analytic manifold of dimension n and let M^* be a complexification (see, for example, [7], Prop. 1) of M. A coordinate system $\varphi^*=(z_1, \cdots, z_n)$ in a neighborhood U^* (in M^*) of a point a of M is called real if every coordinate function $z_j(b)$ $(1 \leq j \leq n)$ takes on real values if and only if b belongs to $U=U^* \cap M$. Then U is a coordinate neighborhood (in M) with the restriction $\varphi=(x_1, \cdots, x_n)$ of φ^* to U as a coordinate system. Let E be a real analytic variety in M. Then, if U^* is sufficiently small, there exists the complexification E^* of E in U^*, that is to say, E^* is the smallest complex analytic variety in U^* such

1) For the definitions and fundamental properties with respect to complex varieties, see [1], [3]; for real varieties, see [2], [7]. Especially, [7] is our good guide.

that $E^* \cap M = E \cap U^*$, the dimension of E^* is equal to that of E at a, its germ at a is the complexification of that of E at a, and the conjugate variety of E^* coincides with E^*, provided that U^* is invariant under the conjugate operation.

Let $E^{*\prime}$ be an irreducible component of E^* in U^*. Then, for simplicity, we call $E^{*\prime} \cap M$ an *irreducible*[2] component of E in $U = U^* \cap M$. For sufficiently small U^*, we may consider that the germs of $E' = E^{*\prime} \cap M$ and $E^{*\prime}$ at a are irreducible components of those of E and E^* respectively and $E^{*\prime}$ is the complexification of E' in U^*.

For $1 \leq i_1 < \cdots < i_r \leq n$, if $j_1 < \cdots < j_{n-r}$ are the complement of $i_1, \cdots, i_r$, $C^{(i_1 \cdots i_r)}$ denotes the subset of all the elements of C^n (complex n-space) whose j_k-th coordinates are all zero ane $\pi_{(i_1 \cdots i_r)}$ denotes the canonical projection of C^n onto $C^{(i_1 \cdots i_r)}$. For $0 < p \leq n$, we identify $C^{(1 \cdots p)}$ with p-space and write C^p for $C^{(1 \cdots p)}$ and π_p for $\pi_{(1 \cdots p)}$. Finally, for $0 < j \leq n-p$, we write $C^{(p,j)}$ for $C^{(1 \cdots p p+j)}$ and $\pi_{(p,j)}$ for $\pi_{(1 \cdots p p+j)}$.

2. Imbedding varieties

We use the same notations as above, and assume that U^* is polycylindrical with radii $(\eta_1, \cdots, \eta_n)$ and that the coordinate system φ^* is zero at a. Suppose that φ^* is *p-proper*[3] for E^* at a (i.e., the first p coordinates of a point which belongs to E^* and distinct from a are not all zero). Then if the radii $\eta_i (1 \leq i \leq n)$ are sufficiently small, we know (see [3] and [7], pp. 137-138) that for $j = 1, \cdots, n-p$, there exist distinguished pseudo polynomials $Q_j(W; z_1, \cdots, z_p)$ with vertices at the origin and with coefficients defined in $\pi_p(U^*)$ such that

$$E^* \subset \tilde{E}^*$$

where $\tilde{E}^*$ is the imbedding variety of E^* which is defined in U^* by the equations:

$$Q_j(z_{p+j}; z_1, \cdots, z_p) = 0 \qquad j = 1, \cdots, n-p.$$

Now we assume that φ^* is real. Then, Q_j can be taken as real, i.e., all the coefficients are real analytic.

A polycylindrical neighborhood V^* of a with radii $(\varepsilon_1, \cdots, \varepsilon_n)$ such that $\varepsilon_i < \eta_i$ $(1 \leq i \leq n)$ is called *standardized for E^* with respect to φ^**, if for $(z_1, \cdots, z_n) \in \pi_p(\bar{V}^*)$, all the roots of Q_j are smaller than ε_j $(p < j \leq n)$ in their absolute values. We know that there exists a fundamental

2) In the terminology of [7], this is called C-irreducible. Notice that any real analytic variety is locally C-analytic.

3) See [7], p. 137.

system of neighborhoods of a each member of which is standardized[4].

Now, put $z_i=x_i+\sqrt{-1}y_i$ ($i=1, \cdots, n$) and consider $\pi_p^{-1}(\pi_p(U^*)\cap M)$ to be a real $\{p+2(n-p)\}$-space in $(x_1, \cdots, x_p, x_{p+1}, \cdots, x_n, y_{p+1}, \cdots, y_n)$. Then the equations:

$$Q_j(x_{p+j}+\sqrt{-1}y_{p+j}\,;\, x_1, \cdots, x_p) = 0 \quad (j = 1, \cdots, n-p)$$

define a real analytic variety in $\pi_p^{-1}(\pi_p(U^*)\cap M)$, which we denote by $\tilde{E}_r$. We have $\tilde{E}^*\cap\pi_p^{-1}(\pi_p(U^*)\cap M)=\tilde{E}_r$. Let $C^{p+2(n-p)}$ be a copy of complex $\{p+2(n-p)\}$-space whose points are written as $(z_1, \cdots, z_n, w_{p+1}, \cdots, w_n)$. We identify a point of $\pi_p^{-1}(\pi_p(U^*)\cap M)$ with a point in the real part of $C^{p+2(n-p)}$ and a point of U^* with a point in $\pi_n(C^{p+2(n-p)})$. Write $Q_j(X+\sqrt{-1}Y\,;\, x_1, \cdots, x_p)=Q_j'(x_1, \cdots, x_p, X, Y)+\sqrt{-1}Q_j''(x_1, \cdots, x_p, X, Y)$, where Q_j' and Q_j'' are real and imaginary part of Q_j respectively, and denote by $\tilde{E}_c^*$ the complex analytic variety at the origin of $C^{p+2(n-p)}$ defined by the equations:

$$\begin{aligned} &Q_j' \,(z_1, \cdots, z_p, z_{p+j}, w_{p+j}) = 0 \qquad \text{and} \\ &Q_j''(z_1, \cdots, z_p, z_{p+j}, w_{p+j}) = 0 \qquad j = 1, \cdots, n-p\,. \end{aligned}$$

Since all the coefficients of Q_j vanish at the origin, Q_j' and Q_j'' do not have common factors for $z_1=\cdots=z_p=0$. Because they are also factors of $(z_{p+j}+\sqrt{-1}w_{p+j})^{m_j}$, where m_j is the degree of Q_j. This shows that if we choose a sufficiently small polycylindrical neighborhood U_1^* in $C^{p+2(n-p)}$, the coordinate system in U_1^* is p-proper at the origin for $\tilde{E}_c^*$. Hence, noticing that $\tilde{E}_c^*$ is invariant under the conjugate operation, we see that there exist real distinguished pseudo polynomials $P_j^1(W\,;\, z_1, \cdots, z_p)$ and $P_j^2(W\,;\, z_1, \cdots, z_p)$ ($j=1, \cdots, n-p$) such that

$$\begin{aligned} &P_j^1(z_{p+j}\,;\, z_1, \cdots, z_p) = 0 \qquad \text{and} \\ &P_j^2(w_{p+j}\,;\, z_1, \cdots, z_p) = 0 \qquad \text{on } \tilde{E}_c^* \text{ in sufficiently} \end{aligned}$$

small U_1^*.

Let $\hat{E}^*$ denote the complex analytic variety of dimension p in $\pi_n(U^*)$ which is defined by the equations:

$$P_j^1(z_{p+j}\,;\, z_1, \cdots, z_p) = 0 \qquad j = 1, \cdots, n-p\,.$$

Then, since W is a factor of $Q_j''(z_1, \cdots, z_p, Z, W)$, $\tilde{E}^*\subset\hat{E}^*$ in $\pi_n(U_1^*)$ (we can, of course, assume that $\pi_n(U_1^*)\subset U^*$), and if $(x_1, \cdots, x_p, z_{p+1}, \cdots, z_n)$ belongs to $\tilde{E}^*$, $(x_1, \cdots, x_p, x_{p+1}, \cdots, z_n)$ belongs to $\hat{E}^*$. Summarizing the above results, we have

4) For details, see [7], § 2 and [3]. For later convenience (see § 5, (5.3)), in the statement of the definition, we take the closure $\pi_p(\bar{V}^*)$ of $\pi_p(V^*)$ instead of $\pi_p(V^*)$.

Lemma 1. *Consider a real analytic variety E in a real analytic manifold M of dimension n and take a complexification M^* of M. Let a be a point in M and let U^* be a polycylindrical neighborhood of a in M^* with a real coordinate system $\varphi^*=(z_1, \cdots, z_n)$. Suppose that φ^* is p-proper at a for the complexification E^* of E in U^*. Then, if U^* is sufficiently small, there exist complex analytic varieties $\tilde{E}^*$ and $\hat{E}^*$ in U^* which are defined by equations: $Q_j(z_{p+j}; z_1, \cdots, z_p)=0$ and $P_j(z_{p+j}; z_1, \cdots, z_p)=0$ $(j=1, \cdots, n-p)$ respectively, where $Q_j(W; z_1, \cdots, z_p)$ and $P_j(W; z_1, \cdots, z_p)$ are real distinguished pseudo polynomials in W with vertices at the origin and with coefficients defined in $\pi_p(U^*)$, and the following relations hold*[5].

(2.1) *$E \subset E^* \subset \tilde{E}^* \subset \hat{E}^*$ in U^*.*

(2.2) *If $(x_1, \cdots, x_p, z_{p+1}, \cdots, z_n)$ belongs to $\tilde{E}^*$, $(x_1, \cdots, x_p, x_{p+1}, \cdots, x_n)$ belongs to $\hat{E}^*$, where $x_i(i=1, \cdots, p)$ is real and $x_j(j=p+1, \cdots, n)$ is the real part of z_j.*

Let E''^* be the union of the irreducible components of E^* of dimension strictly less than p and let D^* be the set of zeros in $\pi_p(U^*)$ of the product of the discriminants of $P_j(j=1, \cdots, n-p)$. Put $D^*(E^*)=D^* \cup \pi_p(E''^*)$. Since we may assume that P_j does not have multiple factors, for sufficiently small U^*, $D^*(E^*)$ is a complex analytic variety of dimension $\leq p-1$ in U^*.

REMARK 1. Since $\tilde{E}^* \subset E^*$, the set of zeros of the product of the discriminants of Q_j is contained in D^*.

3. Proper coordinate system for a family of subvarieties

Suppose that the dimension of E is p in $U=U^* \cap M$. Let $\mathfrak{F}=(F_1, \cdots, F_m)$ be a (finite) family of subvarieties of E. For E and for each F_i, we consider the decompositions $E=E' \cup E''$ and $F_i=F'_i \cup F''_i$, where E' and F'_i are the unions of the irrsducible components of dimension p of E and F_i respectively and E'' and F''_i are the unions of the irreducible components of dimension strictly less than p of E and F_i respectively (see §1). Then an irreducible component of F'_i coincides with an irreducible component of E'. Put $F=\bigcup_{i=1}^{m} F''_i$. Then the dimension of F is strictly less than p. The couple (E, F) is called the *reduced couple associated with $\mathfrak{F}$*.

5) Since Q_j and P_j are not unique, in a strict sense, it is not proper to use such notations as $\hat{E}^*$, $\hat{E}^*$ and those which appear in later sections. But, since there are no confusions, we use them without refering to the polynomials.

We are goint to define *proper coordinate system* $\varphi^* = (z_1, \cdots, z_n)$ in a polycylindrical neighborhood U^* of a $(\in M)$ for the couple $(E; \mathcal{F})$. The definition is done by induction on p. For $p=0$, any real coordinate system is proper. Suppose that the notion is defined for $<p$. Then a real coordinate system φ^* is proper if it satisfies the following conditions:

(3.1) φ^* is p-proper for E at a[6]. Let (E, F) denote the reduced couple associated with $\mathcal{F}$ and put:

$$D(E, F) = (D^*(E^*) \cup \pi_p(F^*)) \cap M$$
$$T(E, F) = E \cap U^* \cap \pi_p^{-1}(D(E, F))$$
$$E^1(E, F) = \hat{E}^* \cap M \cap \pi_p^{-1}(D(E, F))$$

We take U^* so small that D, T and E^1 are real analytic varieties of dimension $\leq p-1$ in $U = U^* \cap M$. Hence the following second condition has meaning.

(3.2) φ^* is proper for $(E^1; (T, T \cap F_1, \cdots, T \cap F_m))$.

We notice that by the definition of $\hat{E}^*$, $\pi_p(E^1(E, F)) = D(E, F)$.

Now we prove the existence of proper coordinate system. Keeping the same notations as above, we have

Lemma 2. *Let $\mathcal{F} = (F_1, \cdots, F_m)$ be a family of subvarieties of E and suppose that E is of dimension p. Then for a point $a \in M$, a proper coordinate system for $(E; \mathcal{F})$ at a can be obtained by a linear transformation of the given real coordinate system $\varphi^* = (z_1, \cdots, z_n)$ and shrinking the given neighborhood of a, if necessary. If φ^* is p-proper for E at a, the proper coordinate system can be obtained by a linear transformation of the first p coordinates, leaving the others fixed.*

Proof[7]. We prove this Lemma by induction on p. If $p=0$, Lemma is trivial for any n. By a linear transformation of φ^*, we get[8] a real p-proper coordinate system φ'^* for E at a. Since the dimension p_1 of E^1 is $\leq p-1$, making a linear transformation of the first p coordinates

6) This means that φ^* is p-proper for the complexification E^* of E. We sometimes use the same expression later.

7) Cf., [7], Proof of Lemma 1.

8) If a complex analytic variety E^* is of dimension $\leq p$, by a linear transformation, we can always find a p-proper coordinate system at a point of complex manifold M^* where E^* is imbedded (see [3]). In fact, this property is taken as the definition of the dimension in [3], while, in [1], E^* is of dimension $\leq p$ if each connected component of the set of its regular points is a manifold of pimension $\leq p$. Furthermore, we can find a real coordinate system as a p-proper coordinate system at a real point, if M^* is a complexification of a real analytic manifold M.

of φ'^*, we can get a real coordinate system which is p-proper for E and $(p-1)$-proper for E^1 at a[9]. Repeating the same process, we get a real coordinate system which is p-proper for E and p_1-proper for E^1 at a. Lemma follows from the assumption of the induction.

By repeating the construction leading from a couple $(E; \mathcal{F})$ to another (see (3.1) and (3.2)) and by picking up the first member (namely variety) of each couple (E, E', etc.), we get a finite family $\mathcal{E}$ of real analytic varieties. Fix a proper coordinate system φ^* for $(E; \mathcal{F})$ at a. Let G be a real analytic variety of dimension r which belongs to $\mathcal{E}$. Then φ^* is r-proper for G at a. A polycylindrical neighborhood V^* in U^* is called *standardized for* $(E; \mathcal{F})$ *with respect to* φ^*, if it is standarized for each member of $\mathcal{E}$. Since $\mathcal{E}$ consists of a finite number of real analytic varieties, we see that there exists a standardized neighborhood and that such neighborhoods form a fundamental system of neighborhoods of a.

4. Proper simplexes

Let M be a real analytic manifold of dimension n imbedded in its complexification M^* and let Δ^r be a straight-linear r-simplex in a real N-space R^N with the dimension N sufficiently large whose points will be denoted by t. We denote by Δ_0^r the interior of Δ^r. Suppose that we have a continuous map f of Δ^r into M. Then the combination $\mathfrak{S}=(f(\Delta^r), f, \Delta^r)$ is a singular r-simplex in M. We call $f(\Delta^r)$ and Δ^r the image of $\mathfrak{S}$ (or Δ^r) and the model of $\mathfrak{S}$ (or $f(\Delta^r)$) respectively.

Let U^* be a real polycylindrical neighborhood in M^* with a coordinate system $\varphi^*=(z_1, \cdots, z_n)$ whose restriction to $U=U^*\cap M$ is $\varphi=(x_1, \cdots, x_n)$. Let $\mathfrak{S}=(S, f, \Delta^r)$ be a singular r-simplex in M such that $S=f(\Delta^r)$ is contained in U. Then f is represented by n continuous functions $x_1=f_1(t), \cdots, x_n=f_n(t)$ and we write $f=(f_1, \cdots, f_n)$. Suppose that $0\leq r\leq p\leq n$. Then $\mathfrak{S}=(S, f, \Delta^r)$ is called *p-proper r-simplex with respect to* φ, if the following conditions are satisfied:

(4.1) There exist r functions $f_{i_1}, \cdots, f_{i_r}$ with $1\leq i_1<\cdots<i_r\leq p$ such that Δ^r is homeomorphic onto the image $S^{i_1\cdots i_r}$ (in $\pi_{i_1\cdots i_r}(U)$) of Δ^r and Δ_0^r is analytically homeomorphic onto the interior $S_0^{i_1\cdots i_r}$ of $S^{i_1\cdots i_r}$ by $x_{i_1}=f_{i_1}(t), \cdots, x_{i_r}=f_{i_r}(t), x_{j_1}=0, \cdots, x_{j_{n-r}}=0$, where $\{j_1\cdots j_{n-r}\}$ is the complement of $\{i_1, \cdots, i_r\}$.

(4.2) Each $f_{j_l}(1\leq l\leq n-r)$ depends on $f_{i_1}, \cdots, f_{i_r}$ continuously on Δ^r and analytically on Δ_0^r, in other words, f_{j_l}, considered as a function on $S^{i_1\cdots i_r}$ by the homeomorphism mentioned above (4.1), is continuous

9) Cf., [7], p. 138.

on $S^{i_1\cdots i_r}$ and analytic on $S_0^{i_1\cdots i_r}$ in $(x_{i_1}, \cdots, x_{i_r})$.

REMARK 2. For $0 \leq j \leq n-p$, put $f^{p+j} = \pi_{p+j} \circ f$ and $S^{p+j} = f^{p+j}(\Delta^r)$, $S_0^{p+j} = f^{p+j}(\Delta_0^r)$. Then Δ^r is homeomorphic onto S^{p+j} and Δ_0^r is analytically homeomorphic onto S_0^{p+j} by f^{p+j}. Put $\mathfrak{S}_{p+j} = (S^{p+j}, f^{p+j}, \Delta^r)$. Then $\mathfrak{S}_{p+j}$ is a p-proper simplex. We call $\mathfrak{S}_{p+j}$ the $(p+j)$-*base* of p-proper simplex $\mathfrak{S} = (S, f, \Delta^r)$ and we say that $\mathfrak{S}$ *lies over* $\mathfrak{S}_{p+j}$. $f^{p+j} = (f_1, \cdots, f_{p+j}, 0, \cdots, 0)$ and $\mathfrak{S}_n = \mathfrak{S}$. If we restrict the projection π_{p+j} to S, it is a homeomorphism of S onto S^{p+j} and an analytic homeomorphism of S_0 onto S_0^{p+j}. We denote this homeomorphism by $\pi(S, S^{p+j})$.

REMARK 3. For $1 \leq j \leq n-p$, we put $f^{(p,j)} = \pi_{(p,j)} \circ f$ and also put $S^{(p,j)} = f^{(p,j)}(\Delta^r)$, $S_0^{(p,j)} = f^{(p,j)}(\Delta_0^r)$. Then the combination $(S^{(p,j)}, f^{(p,j)}, \Delta^r)$ is a p-proper simplex, which we denote by $\mathfrak{S}_{(p,j)}$. $\mathfrak{S}_{(p,j)}$ is called (p, j)-*base* of $\mathfrak{S} = (S, f, \Delta^r)$, $f^{(p,j)} = (f_1, \cdots, f_p, 0, \cdots, 0, f_{p+j}, 0, \cdots, 0)$ and $\mathfrak{S}_{(p,j)}$ lies over $\mathfrak{S}_p$ (Remark 2).

5. Proper Triangulations

We use the same notations as the previous section. Let K be a straight-linear complex in R^N and let f be a homeomorphism of K into M. Then the combination $\mathfrak{K} = (f(K), f, K)$ is a complex in M or a triangulation of $f(K)$. $f(K)$ and K are called the image of $\mathfrak{K}$ and the model of $\mathfrak{K}$ (or $f(K)$) respectively. If f is analytically homeomorphic on the interior of each simplex of each dimension which belongs to K, $\mathfrak{K}$ is called *analytic* or *analytic triangulation* of $f(K)$.

Let V^* be a polycylindrical neighborhood with radii $(\varepsilon_1, \cdots, \varepsilon_n)$ such that $V^* \subset U^*$. Put $V = V^* \cap M$. Let $\mathfrak{K} = (\tilde{G}, g, K)$ be a complex of dimension p in M. Then we say that $\mathfrak{K}$ is a *p-proper complex with respect to* φ *and* $\bar{V}$, if the following conditions are satisfies (5.1)-(5.6):

(5.1) $\tilde{G}$ is contained in $\bar{V}$ and all simplexes which belong to $\mathfrak{K}$ are p-proper with respect to φ.

(5.2) Each simplex of $\mathfrak{K}$ of dimension strictly less than p is a face of at least one p-simplex of $\mathfrak{K}$.

The *boundary* $\partial\mathfrak{K}$ of $\mathfrak{K}$ is the collection of all $(p-1)$-simplexes each of which is a face of only one p-simplex of $\mathfrak{K}$ and all their faces. $\partial\mathfrak{K}$ is a subcomplex of $\mathfrak{K}$ and is written as $(\partial\tilde{G}, g, \partial K)$.

(5.3) Let (S, g, Δ^r) $(0 \leq r \leq p-1)$ be a simplex of $\partial\mathfrak{K}$ and let $g = (g_1, \cdots, g_n)$. Then $S^p \subset \pi_p(\bar{V}) - \pi_p(V)$ (for S^p, see Remark 2) and for $p < j \leq n$, $-\varepsilon_j < g_j(t) < \varepsilon_j$ for $t \in \Delta^r$, where ε_j are radii of V^*.

(5.4) There exists a triangulation $\mathfrak{K}_p=(\pi_p(\bar{V}), g_p, K_p)$ of $\pi_p(\bar{V})$ in such a way that π_p gives rise to a non-degenerate simplicial map of $\mathfrak{K}$ onto $\mathfrak{K}_p$, that is to say, there exists a non-degenerate simplicial map h_p of K onto K_p such that $\pi_p\circ g=g_p\circ h_p$. We write $\pi_p\mathfrak{S}$ for $(\pi_p S, g_p, h_p\Delta)$, where $\mathfrak{S}=(S, g, \Delta)\in\mathfrak{K}$.

REMARK 4. If (S, g, Δ) is a simplex of $\mathfrak{K}$, it is equivalent to $(S, \pi^{-1}(S, S^p)\circ g_p, h_p(\Delta))$ and any simplex (S', g_p, Δ') of $\mathfrak{K}_p$ is equivalent to the p-base (S^p, g^p, Δ) of a simplex (S, g, Δ') of $\mathfrak{K}$ such that $S^p=S'$.

(5.5) Let $\mathfrak{S}$ be a $(p-1)$-simplex which does not belong to $\partial\mathfrak{K}$. Then there exist at least two p-simplexes $\mathfrak{S}'$ and $\mathfrak{S}''$ of $\mathfrak{K}$ which have $\mathfrak{S}$ as the common face, such that $\pi_p\mathfrak{S}'$ and $\pi_p\mathfrak{S}''$ are different.

In this case, of course, $\pi_p\mathfrak{S}'$ and $\pi_p\mathfrak{S}''$ have $\pi_p\mathfrak{S}$ as the common face.

(5.6) For $1\leq j\leq n-p$, there exists a triangulation $\mathfrak{K}_{(p,j)}=(\pi_{(p,j)}(\tilde{G}), g_{(p,j)}, K_{(p,j)})$ of $\pi_{(p,j)}(\tilde{G})$ in such a way that $\pi_{(p,j)}$ gives rise to a non-degenerate simplicial map of $\mathfrak{K}$ onto $\mathfrak{K}_{(p,j)}$, i.e., there exists a non-degenerate simplicial map $h_{(p,j)}$ of K onto $K_{(p,j)}$ such that $\pi_{(p,j)}\circ g=g_{(p,j)}\circ h_{(p,j)}$.

We call $\mathfrak{K}_p$ and $\mathfrak{K}_{(p,j)}$ *p-base* and *(p, j)-base of $\mathfrak{K}$ with respect to φ and $\bar{V}$* respectively.

REMAK 5. $\mathfrak{K}_p$ is p-proper with respect to φ and $\bar{V}$ with itself as the p-base and the (p, j)-base. And $\mathfrak{K}_{(p,j)}$ is also p-proper with respect to φ and $\bar{V}$ with itself as (p, j)-base and for $1\leq k\leq n-p$, $k\neq j$, $\mathfrak{K}_p$ is the (p, k)-base. The simplicial map $h^{(p,j)}_{(p,k)}$ of $K_{(p,j)}$ onto K_p is defined by $h_p=h^{(p,j)}_{(p,k)}\circ h_{(p,j)}$. $\mathfrak{K}_p$ is also p-base of $\mathfrak{K}_{(p,j)}$ with $h^{(p,j)}_p=h^{(p,j)}_{(p,k)}$ again as the simplicial map of $K_{(p,j)}$ onto K_p.

Let G be a subset in $\bar{V}$ and let $\mathfrak{K}=(\tilde{G}, g, K)$ be as above. Then we say that G is *p-properly triangulated in* $\mathfrak{K}$ (or in $\tilde{G}$), if $\tilde{G}$ contains G and $g^{-1}(G)$ is a subcomplex (of K) (i.e., $(G, g, g^{-1}(G))$ is a subcomplex of $\mathfrak{K}$).

REMARK 6. Let $\mathfrak{S}_i=(S_i, g_p, \Delta_i)$ be a simplex of $\mathfrak{K}_p=(\pi_p(\bar{V}), g_p, K_p)$. Then $\pi_p^{-1}((S_i)_0)\cap\pi_{(p,j)}(\tilde{G})$ is the union $\bigcup_{l=1}^{l=k(i,j)}(S^l_{(i,j)})_0$ of the interiors $(S^l_{(i,j)})_0$ of all the simplexes $(S^l_{(i,j)}, g_{(p,j)}, {}^l\Delta_{(i,j)})$ $(1\leq l\leq k(i, j))$ of $\mathfrak{K}_{(p,j)}$ which lies over $\mathfrak{S}_i$ (see Remark 5, (5.4) and (5.6)). Put $S^0_{(i,j)}=\{(x_1, \cdots, x_p, 0\cdots 0, -\varepsilon_{p+j}, 0\cdots 0)\,|\,(x_1, \cdots, x_p)\in S_i\}$ and $S^{k(i,j)+1}_{(i,j)}=\{(x_1, \cdots, x_p, 0\cdots 0, \varepsilon_{p+j}, 0\cdots 0)\,|\,(x_1, \cdots, x_p)\in S_i\}$, where $(\varepsilon_1, \cdots, \varepsilon_n)$ are radii of V^*. Denote by $(S^0_{(i,j)})_0$ and $(S^{k(i,j)+1}_{(i,j)})_0$ the interiors of $S^0_{(i,j)}$ and $S^{k(i,j)+1}_{(j,j)}$ respectively. Since $(S^l_{(i,j)})_0$ $(0\leq l\leq k(i, j)+1)$ are mutually disjoint, we can naturally introduce an order in $C_{(i,j)}=\{S^1_{(i,j)}\}$ $(0\leq 1\leq k(i,j)+1)$ by means of the $(p+j)$-th coordinates: $S^0_{(i,j)}<S^1_{(i,j)}<\cdots<S^{k(i,j)}_{(i,j)}<S^{k(i,j)+1}_{(i,j)}$.

Keeping the same notations as above, we prove

Lemma 3. *If $\mathfrak{K}=(\tilde{G}, g, K)$ is a p-proper complex with respect to φ and $\bar{V}$ such that the model K_p of its p-base $\mathfrak{K}_p=(\pi_p(\bar{V}), g_p, K_p)$ is contained in R^m, $\bar{V}$ has an analytic triangulation $\mathfrak{K}^n=(\bar{V}, f, K^n)$ which has the following properties* (5. a)-(5. c):

(5. a) *K^n is imbedded in R^{m+n-p} in such a way that π_n (acting on R^{m+n-p}) yields a simplicial map of K^n onto K_p.*

(5. b) *The model K of the complex $\mathfrak{K}=(\tilde{G}, g, K)$ can be taken to be a subcomplex of K^n, f is an extension of g, h_p is a restriction of π_m to K and $K_{(p,j)}=\pi_{(m,j)}(K)$ with $h_{(p,j)}=\pi_{(m,j)}$ (restricted to K).*

(5. c) *For $1\leq j\leq n-p$, $\pi_{(m,j)}(K^n)$has a triangulation in such a way that $\pi_{(m,j)}$ is a simplicial map of K^n onto $\pi_{(m,j)}(K^n)$ and $\pi_{(p,j)}(\bar{V})$ has a triangulation $\mathfrak{K}^n_{(p,j)}=(\pi_{(p,j)}(\bar{V}),\ f^n_{(p,j)},\ \pi_{(m,j)}(K^n))$ such that $\pi_{(p,j)}\circ f=f^n_{(p,j)}\circ\pi_{(m,j)}$ and $\mathfrak{K}^n_{(p,j)}$ is an extension of $\mathfrak{K}_{(p,j)}$ (i.e., $f^n_{(p,j)}$ is an extension of $g_{(p,j)}$ and $\mathfrak{K}_{(p,j)}$ is a subcomplex of $K^n_{(p,j)}=\pi_{(m,j)}(K^n)$).*

Proof. First we arbitrarily fix order in all the vertices of $\mathfrak{K}_p$: (v_1, g_p, α_1), $\cdots$, (v_B, g_p, α_B). Suspose that $v^l_{(i,j)}$ $(1\leq i\leq B,\ i\leq j\leq n-p,$ $0\leq l\leq k(i,j)+1)$ (see Remark 6) has coordinate $(x^i, \cdots, x^i_p, 0\cdots 0, x^l_{(i,j)}, 0\cdots 0)$, where $(x^i_1, \cdots, x^i_p, 0\cdots 0)$ is ths coordinate of v_i, and put $C'_{(i,j)}=\{\alpha^l_{(i,j)}\}$ where $\alpha^l_{(i,j)}$ is a point in $\pi_{(m,j)}(R^{m+n-p})$ which has the expression:

$$\alpha^l_{(i,j)}=(t^i_1,\ \cdots,\ t^i_m,\ 0\cdots 0,\ x^l_{(i,j)},\ 0\cdots 0)$$

where
$$\alpha_i=(t^i,\ \cdots,\ t^i_m,\ 0\cdots 0)\,.$$

Put

$$A=\{w\in\bar{V}\,|\,\pi_{(p,j)}(w)=v^l_{(i,j)},\ 1\leq i\leq B,\ 1\leq j\leq n-p,\ 0\leq l\leq k(i,j)+1\}$$

and

$$A'=\{\beta\in R^{m+n-p}\,|\,\pi_{(m,j)}(\beta)=\alpha^l_{(i,j)},\ 1\leq i\leq B,\ 1\leq j\leq n-p,\ 0\leq l\leq k\ (i,j)+1\}.$$

Let $\beta=(t^i_1, \cdots, t^i_m, x^{l_1}_{(i,1)}, \cdots, x^{l_{n-p}}_{(i,n-p)})\in A'$. Then the function f' which maps β to the point $(x^i_1, \cdots, x^i_p, x^{l_1}_{(i,1)}, \cdots, x^{l_{n-p}}_{(i,n-p)})$ of A is a one-to-one map of A' onto A. Let $f'_{(p,j)}$ be the one-to-one map of $C'_j=\bigcup_{i=1}^{B}C'_{(i,j)}$ onto $C_j=\bigcup_{i=1}^{B}C_{(i,j)}$ such that $\alpha^l_{(i,j)}$ corresponds to $v^l_{(i,j)}$. Then $\pi_{(p,j)}\circ f'=f'_{(p,j)}\circ\pi_{(m,j)}$. Let $\beta_0=(t^{i_0}_1, \cdots, t^{i_0}_m, x^{l^0_1}_{(i_0,1)}, \cdots, x^{l^0_{n-p}}_{(i_0,n-p)})$, $\cdots$, $\beta_r=(t^{i_r}_1, \cdots, t^{i_r}_m, x^{l^r_1}_{(i_r,1)}, \cdots, x^{l^r_{n-p}}_{(i_r,n-p)})$ be $r+1$ different points of A' such that

i) $v_{i_0}=\pi_m(\beta_0), \cdots, v_{i_r}=\pi_m(\beta_r)$ are contained in a simplex Δ of K_p and $i_0\leq\cdots\leq i_r$. Put $S=g_p(\Delta)$.

ii) For $1\leq j\leq n-p$, there exists a number q $(0\leq q\leq k(S, j))$ such that for $0\leq t\leq r$, $f'_{(p,j)}\circ\pi_{(m,j)}(\beta_t)$ belongs either $S^q_{(S,j)}$ or $S^{q+1}_{(S,j)}$ and if, for some t, $f'_{(p,j)}\circ\pi_{(m,j)}(\beta_t)$ belongs to $S^q_{(S,j)}$, $f'_{(p,j)}\circ\pi_{(m,j)}(\beta_{t-1})$ belongs to $S^q_{(S,j)}$ (if it belongs to both of them, we consider that it belongs to $S^q_{(S,j)}$). Then $\beta_0, \cdots, \beta_r$ span a r-simplex in R^{m+n-p} and all these simplexes form a complex in R^{m+n-p} which turns out to be the desired model of the triangulation of $\bar{V}$ mentioned in the Lemma, and hence is denoted by K^n. From the construction of K^n, we see that π_m yields a simplicial map of K^n onto K_p and that $\pi_{(m,j)}(K^n)$ has a triangulation so that $\pi_{(m,j)}$ yields a simplicial map of K^n onto it. $\pi_{(m,j)}(K^n)$ with this triangulation is to become the model of $\pi_{(p,j)}(\bar{V})$ mentioned in (5. c), and is denoted by $K^n_{(p,j)}$. Let Δ be a simplex of K^n with vertices $\beta_0, \cdots, \beta_r$ such that $f'(\beta_0), \cdots, f'(\beta_r)$ are the images of vertices of a simplex of complex $\mathfrak{K}$. Then all these simplexes Δ form a subcomplex of K^n which turns out to be equivalent to the model of the given complex $\mathfrak{K}$, and hence is denoted by K. Also from the construction of K^n, π_m is a simplicial map of K onto K_p, and $\pi_{(m,j)}(K)$ is considered a subcomplex of $K^n_{(p,j)}$ such that $\pi_{(m,j)}$ is a simplicial map of K onto it. $\pi_{(m,j)}(K)$ with this triangulation is naturally going to be the model of $\mathfrak{K}_{(p,j)}$ and is denoted by $K_{(p,j)}$. Now by (5. 4) and (5. 6), the homeomorphisms g and $g_{(p,j)}$ of K onto $\tilde{G}$ and of $K_{(p,j)}$ onto $\pi_{(p,j)}(\tilde{G})$ are naturally defined with the properties; $\pi_p\circ g=g_p\circ\pi_m$, $\pi_{(p,j)}\circ g=g_{(p,j)}\circ\pi_{(m,j)}$ and $\pi_p\circ g_{(p,j)}=g_p\circ\pi_m$ so that $(\tilde{G}, g, K)$ is equivalent to the given triangulation of $\tilde{G}$, the given $\mathfrak{K}_p$ and $(\pi_{(p,j)}(\tilde{G}), g_{(p,j)}, K_{(p,j)})$ are its p-base and (p, j)-base respectively.

Now we define a homeomorphism f of K^n onto $\bar{V}$ with the desired properties as an extension of g. Let $t=(t_1, \cdots, t_m, t_{m+1}, \cdots, t_{m+n-p})$ be a point of K^n and let Δ be the minimal simplex of K_p which contains $\pi_m(t)$. For $1\leq j\leq n-p$, let ${}_1{}^j\Delta$ and ${}_2{}^j\Delta$ be simplexes of $K_{(p,j)}$ such that $\pi_m({}_1{}^j\Delta)=\pi_m({}_2{}^j\Delta)=\Delta$ and such that if $t^1=(t_1, \cdots, t_m, 0\cdots0, t^1_{m+j}, 0\cdots0)$ and $t^2=(t_1, \cdots, t_m, 0\cdots0, t^2_{m+j}, 0\cdots0)$ are points of ${}_1{}^j\Delta$ and ${}_2{}^j\Delta$ respectively, $t^1_{m+j}\leq t_{m+j}\leq t^2_{m+j}$, and no $(m+j)$-th coordinate of any other simplex Δ' of $K_{(m,j)}$ with $\pi_m(\Delta')=\Delta$ comes between t^1_{m+j} and t^2_{m+j}. Define $f(t)=x=(x_1, \cdots, x_p, x_{p+1}, \cdots, x_n)\in\bar{V}$ as follows: Let x^1_{m+j} and x^2_{m+j} be the $(p+j)$-th coordinates of $g_{(p,j)}(t^1)$ and $g_{(p,j)}(t^2)$ respectively.

$$\pi_p(x) = g_p\circ\pi_m(t),$$

$$x_{p+j} = x_{m+j}+\frac{t_{m+j}-t^1_{m+j}}{t^2_{m+j}-t^1_{m+j}}(x^2_{m+j}-x^1_{m+j}), \quad \text{if} \quad t^2_{m+j}>t^1_{m+j},$$

and

$$x_{p+j} = x^1_{m+j} = x^2_{m+j}, \quad \text{if} \quad t^2_{m+j} = t^1_{m+j}.$$

It is now easy to check all the properties (5. 1), (5. 2) and (5. 3) with all we have constructed.

REMARK 7. For $0 \leq j \leq n-p$, $\pi_{m+j}(K^n)$ and $\pi_{m+j}(K)$ can be considered as complexes such that π_{p+j} is a simplicial map of K^n onto $\pi_{m+j}(K^n)$ and of K onto $\pi_{m+j}(K^p)$, and $\pi_{p+j}(\bar{V})$ and $\pi_{p+j}(\tilde{G})$ have the natural triangulations $(\pi_{p+j}(\bar{V}), f^n_{p+j}, \pi_{m+j}(K^n))$ and $(\pi_{p+j}(\tilde{G}), g_{p+j}, \pi_{m+j}(K^p))$ respectively, where f^n_{p+j} is defined as $\pi_{p+j} \circ f = f^n_{p+j} \circ \pi_{m+j}$ and g_{p+j} is the restriction of f^n_{p+j} to $\pi_{m+j}(K)$. Then $(\pi_{p+j}(\tilde{G}), g_{p+j}, \pi_{m+i}(k))$ is a subcomplex of $(\pi_{p+j}(\bar{V}), f^n_{p+j}, \pi_{m+j}(K^n))$ and consists of all $(p+j)$-bases of simplexes of $(\tilde{G}, g, K)$. If G is p-properly triangulated in $\tilde{G}$, $\pi_{p+j}(G)$ is also p-properly triangulated in $\pi_{p+j}(\tilde{G})$.

6. Local triangulation of real analytic verieties

Theorem 1. *Let E be a real analytic variety in a real analytic manifold M and let $F_1, \cdots, F_m$ be a finite number of subvarieties of E. Then for any point a of M, there exists a neighborhood V of a such that $E \cap \bar{V}$ is triangulable and each $F_i \cap \bar{V}\,(1 \leq i \leq m)$ is a subcomplex of $E \cap \bar{V}$ for the triangulation of $E \cap \bar{V}$. $E \cap \bar{V}$, considered as a complex, has a model $\hat{K}$ in the Euclidean space of the same dimension as that of M. The homeomorphism g of $\hat{K}$ onto $E \cap \bar{V}$ is analytically homeomorphic on the interior of each simplex of each dimension of $\hat{K}$.*

Proof. We may assume that a belongs to each $F_i\,(1 \leq i \leq m)$. Suppose that the dimension of E and M are $\leq p$ and n respectively, and M is imbedded in a complexification M^* of M. Let U^* be a polycylindrical neighborhood of a with real coordinate system $\varphi^* = (z_1, \cdots, z_n)$ whose restriction to $U = U^* \cap M$ is $\varphi = (x_1, \cdots, x_n)$. Suppose that φ^* is proper for $(E\,; (F_1, \cdots, F_m))$ at a and let V^* be a standardized neighborhood for $(E\,; (F_1, \cdots, F_m))$ such that $\bar{V}^* \subset U^*$. Put $V = V^* \cap M$. Then we are actually going to prove:

If such V is sufficiently small, there exists a p-proper complex $\mathfrak{K} = (\tilde{G}, g, K)$ with respect to φ and $\bar{V}$ such that $E \cap \bar{V}$ is p-properly triangulated in $\mathfrak{K}$, K is imbedded in R^n and the model K_p of p-base $\mathfrak{K}_p$ of $\mathfrak{K}$ is imbedded in $\pi_p(R^n)$.

The proof is carried out by induction: Since for $p=0$, Theorem is trivial for any n, we fix n arbitrarily and prove for p under the assumption that it is true for $<p$. Since φ^* is proper for $(E^1\,; (T, T \cap F_1, \cdots, T \cap F_m))$, V^* is standardized for it and since the dimension p_1 of E^1 is $\leq p-1$, by the assumption of the induction, there exists a p_1-proper

complex ${}^1\mathfrak{R}=({}^1\tilde{G}_1, {}^1g, {}^1K)$ with respect to φ and $\bar{V}$ such that $E_1\cap\bar{V}$ is p_1-properly triangulated in ${}^1\mathfrak{R}$ and $T\cap\bar{V}$, $T\cap F_1\cap\bar{V}$, $\cdots$, $T\cap F_m\cap\bar{V}$ are subcomplexs of $E^1\cap\bar{V}$, and such that the model ${}^1K_{p_1}$ of p_1-base ${}^1\mathfrak{R}_{p_1}$ of ${}^1\mathfrak{R}$ is imbedded in $\pi_{p_1}(R^n)$. Hence, by Lemma 3, there exists an analytic triangulation ${}^1\mathfrak{R}^n=(\bar{V}, {}^1f, {}^1K^n)$ of $\bar{V}$ with the properties (5. a), (5. b) and (5. c). We denote by ${}^1\mathfrak{R}^n_p=(\pi_p(\bar{V}), {}^1f_p, {}^1K^n_p)$ the p-base of ${}^1\mathfrak{R}^n$ (see Remark 7), which gives a triangulation of $\pi_p(\bar{V})$.

Let $\mathfrak{S}=(S, {}^1f_p, \Delta)$ be a simplex of dimension p of ${}^1\mathfrak{R}^n_p$ and let $(S_0, {}^1f_p, \Delta_0)$ be its interior. Since $\pi_p(E^1)=D(E, F)$ is of dimension $\leq p-1$, $\pi_p(E_1)\cap S_0=\emptyset$ and especially $S_0\cap D^*=\emptyset$. Furthermore, since S_0 is simply connected, $\pi_p^{-1}(S_0)\cap(\tilde{E}^*\cap\bar{V}^*)$ consists of connected components each of which is homeomorphic to S_0 by π_p (see Remark 1). Let $\tilde{S}_0$ be one of the connected components. Then if S_0 is represented by $x_1=u_1(t)$, $\cdots$, $x_p=u_p(t)$, $x_{p+1}=\cdots=x_n=0$, $\tilde{S}_0$ is represented by $x_1=u_1(t)$, $\cdots$, $x_p=u_p(t)$, $z_{p+1}=w_{p+1}(t)$, $\cdots$, $z_n=w_n(t)$, where $w_j\,(p<j\leq n)$ depends analytically on $u_1, \cdots, u_p$ on S_0. Write $w_j(t)=u_j(t)+\sqrt{-1}v_j(t)$ where $u_j(t)$ and $v_j(t)$ are real and imaginary part respectively and consider an open simplex $\hat{S}_0$ which is represented by $u_1(t), \cdots, u_p(t), u_{p+1}(t), \cdots, u_n(t)$. Then $\hat{S}_0$ is contained in $\hat{E}^*\cap\bar{V}$ (see (2. 2)). If $\hat{S}_0'$ is another open simplex derived from S_0 as above and is represented by $u_1(t), \cdots, u_p(t), u_{p+1}'(t), \cdots, u_n'(t)$, for $p<j\leq n$, either $u_j(t)=u_j'(t)$ on Δ_0 or $u_j(t)$ is distinct from $u_j'(t)$ at any point of Δ_0, because $S_0\cap D^*=\emptyset$.

Denote by $\hat{S}$ the closure of $\hat{S}_0$ in $\bar{V}$ (therefore also in $\hat{E}^*\cap\bar{V}$). Since $E^1\cap\bar{V}$ is p_1-properly $(p_1<p)$ triangulated, by the continuity of roots of the distinguished pseudo polynomials $P_j\,(1\leq j\leq n-p)$ which defines $\hat{E}^*$ and local connectivity of $\hat{E}^*$ (see [1], [3]), we see that all $u_j(t)\,(p<j\leq n)$ have unique continuous extensions to Δ. Since, for $1\leq i\leq p$, $u_i(t)$ has already the extension to Δ as a representation function of ${}^1f_p$, $u=(u_1, \cdots, u_n)$ defines a homeomorphism of Δ onto $\hat{S}$ and $\mathfrak{S}'=(\hat{S}, u, \Delta)$ is a p-proper p-simplex. Every face of $\mathfrak{S}'$ lies over a simplex of the same dimension of ${}^1\mathfrak{R}^n_p$ and analytic (notice that the image of its p-base is either contained in $\pi_p(E^1)$ or the interior is disjoint from D^*).

Now we denote by $\tilde{G}$ the union of all these $\hat{S}$ and consider the collection C of all simplexes obtained by the barycentric subdivision of all these $\mathfrak{S}'$. Then the collection C gives an analytic triangulation $\mathfrak{R}=(\tilde{G}, g, K)$ of $\tilde{G}$. Remembering that $\tilde{E}^*$ is defined by distinguished pseudo polynomials and that V^* is standardized, it is easy to see that $\mathfrak{R}$ satisfies the condition (5. 1)-(5. 6), i.e., $\mathfrak{R}$ is a p-proper complex with respect to φ and $\bar{V}$ and its p-base is the barycentric subdivision of ${}^1\mathfrak{R}^n_p=(\pi_p(\bar{V}), {}^1f_p, {}^1K^n_p)$.

From the construction of $\tilde{G}$, we see that $E\cap\bar{V}\subset\tilde{G}$ (see Lemma 1).

Suppose $x \ni F_i$ and $\pi_p(x) \notin \pi_p(E^1)$. Then $\pi_p(x) \notin D^*(E^*)$. Since, in this case, x is a regular point of both F_i^* and $\tilde{E}^*$ (see, for example, [7], p. 138), E^* is expressed locally by $n-p$ real analytic functions in $(z_1, \cdots, z_p)$ and E is locally real analytic manifold of dimension p expressed by the same functions restricted to real variables $(x_1, \cdots, x_p)$ (see, for example, [2], p. 92). Hence any simplex of $\tilde{G}$ which contains x in the interior is contained in F_i. This shows that each F_i is a subcomplex of $\tilde{G}$ and hence $E \cap \bar{V}$ is properly triangulated in $\mathfrak{R}=(\tilde{G}, g, K)$ in such a way as stated above.

REMARK 8. It is easy to see that we can reconstruct the model $\hat{K}$ of $E \cap \bar{V}$ in such a manner that for any vertex t of $\hat{K}$, $x=g(t)$ has the same coordinate as that of t, preserving all other properties.

REMARK 9. In the above triangulation $\hat{\mathfrak{R}} = (E \cap \bar{V}, g, K)$; $E \cap \partial \bar{V}$ is a subcomplex of $E \cap \bar{V}$, where $\partial \bar{V} = \bar{V} - V$.

7. Global triangulation of real algebraic varieties

We consider a real algebraic variety A in R^n with an usual coordinate system $\varphi=(x_1, \cdots, x_n)$. Suppose that A is defined by polynomials $f_i(X_1, \cdots, X_n)$ with degrees r_i $(1 \leq i \leq m < \infty)$. Let f_i^0 be a polynomial in $n+1$ variables defined by $f_i^0(X_0, X_1, \cdots, X_n) = X_0^{r_i} f_i\left(\frac{X_1}{X_0}, \cdots, \frac{X_n}{X_0}\right)$. Then $f_1^0, \cdots, f_m^0$ defines a real projective variety A^0 in R^{n+1} with a coordinate system $\varphi_0=(x_0, x_1, \cdots, x_n)$. We cell A^0, the projective variety associated with A. Let H_0 and H_1 be the hyper-planes of R^{n+1} defined by $X_0=0$ and $X_0=1$ respectively. Then A^0 consists of a projective subvariety A_0^0 in H_0 and all the lines which join the origin a of R^{n+1} and a point of $A_1^0 = A^0 \cap H_1$, where A_1^0 is isomorphic to A by the canonical map of H_1 to R^n: $(1, x_1, \cdots, x_n) \to (x_1, \cdots, x_n)$.

Let $F^1, \cdots, F^k$ be a finite number of algebraic subvarieties of A and let $(F^i)^0$ be the projective variety associated with F^i $(1 \leq i \leq k)$. By Lemma 2, there exists a proper coordinate system $\varphi_0' = (y_0, y_1, \cdots, y_n)$ for $(A_0; ((F^1)^0, \cdots, (F^k)^0, A_0^0))$ at a. Since φ_0' can be obtained from φ_0 by a linear transformation, φ_0' is also a global coordinate system. Then by Theorem 1, there exists a neighborhood V of a such that $A^0 \cap \bar{V}$ has a triangulation $\hat{\mathfrak{R}} = (A^0 \cap \bar{V}, g, \hat{K})$ and such $(F^i)^0 \cap \bar{V}$ and $A_0^0 \cap \bar{V}$ are subcomplexes of $A^0 \cap \bar{V}$. Since, in Lemma 1, $\tilde{E}^*$ and $\hat{E}^*$ can be taken to be symmetric with respect to the orign, and A^0 and each $(F^i)^0$ are symmetric with respect to the origin for φ_0' (hence also for φ_0), we may assume that the triangulation $\mathfrak{R}$ is symmetric with respect to a for φ_0' (hence also for φ_0). Therefore, considering the projection of $A^0 \cap \partial \bar{V}$ (see Remark 9) from a, we get a triangulation of A^0 (considered to be imbedded in projective n-space P^n) each $(F^1)^0$ and A^0_0 are subcomplexes of A^0. The model of $A^0 \cap \partial \bar{V}$ can also be assumed as symmetric and the model $K(A^0)$ is obtained from the symmetric model of $A^0 \cap \partial \bar{V}$ by the identification with respect to the symmetry, the homeomorphism is the projection preceded by g and by the form of $A^0 \cap \partial \bar{V}$, we see the triangulation is analytic.

Summarizing the above results, we have

Theorem 2. *Let A be a real algebraic variety in R^n and let $F^1, \cdots, F^k$ be a finite number of algebraic subvarieties of A. Let A^0 and $(F^i)^0$ be the projective subvarieties associated with A^0 and $(F^i)^0$ respectively. Suppose that A^0 and $(F^i)^0$ are imbedded in projective n-space P^n and consider that A is a subset of A^0. Then there exists a triangulation $\mathfrak{A}=(A^0, f, K(A^0))$ of A^0 such that each $(F^i)^0$ is a subcomplex of A^0 and A^0-A is also a subcomplex of A^0 imbedded in a hyperplane of P^n. The homeomorphism f is analytic in the interior of each simplex of each dimension of $K(A^0)$.*

8. Pseudo-cells and cells

If a real analytic manifold e is analytically homeomorphic onto a relatively compact domain D in R^r, e is called *analytic pseudo-cell* or simply *pseudo-cell* (since from now onwards, we are concerned only with analytic case). The dimension of e as a manifold is equal to r and we say that pseudo-cell e is of dimension r and write e^r if it is necessary to specify the dimension.

In the above definition if e, hence D, is homeomorphic to the interior of a sphere, we call *e cell.*

Let X be a topological space and suppose that X is a union of mutually disjoint pseudo-cells: $X=\bigcup_\alpha e_\alpha$. Then we say that X has a *pseude-cell decomposition* $\{e_\alpha\}$, or X is a *pseudo-cell complex*, if for each e^r_α, there exist a relatively compact domain D_α in R^r and a homeomorphism f_α of $\bar{D}_\alpha$ onto $\bar{e}_\alpha$, and if each combination $(e_\alpha, f_\alpha, D_\alpha)$ satisfies the following conditions (8.1)-(8.4):

(8.1) $\{e_\alpha\}$ is locally finite.

(8.2) $\bar{e}_\alpha$ is written as a union of a finite number of pseudo-cells in $\{e_\alpha\}$.

Each pseudo-cell of $\partial e_\alpha = \bar{e}_\alpha - e_\alpha$ is called a *boundary pseudo-cell* of e_α.

(8.3) f_α is analytic homeomorphism of D_α onto e_α. If e_β is a boundary pseudo-cell of e^r_α, $f^{-1}_\alpha(e_\beta)$ is an anlytic submanifold of the space

R^r which contains D_α and $f_\alpha^{-1}(e_\beta)$ is analytically homeomorphic onto e_β by f_α, and $\partial D=\bar{D}-D=\bigcup_\beta f_\alpha^{-1}(e_\beta)$, where the union is taken for all the boundary pseudo-cells e_β of e_α.

(8. 4) For any point a of ∂D, any neighborhood $V(a)$ of a in R^r contains a point of $R^r-\bar{D}$.

When X has a pseudo-cell decomposition $\{e_\alpha\}$, if each e_α is a cell, we say that X has a *cell decomposition* or X is a *cell complex*.

Let E, $(F_1, \cdots, F_m)$, φ, V be the same as appeared in the proof of Theorem 1. We may always assume that φ, E, F_k $(1\leq k\leq m)$ are defined in some rectangular neighborhood U which contains $\bar{V}$. Then just considering cells in place of simplexes, by the same method as used in the proof of Theorem 1, we can prove the following

Lemma 4. *Let E, $(F_1, \cdots, F_m)$, φ, V be as above and suppose that E is of dimension $\leq p$. Then, in addition to a triangulation $\hat{\mathfrak{K}}=(E\cap \bar{V}, g, \hat{K})$ in a p-proper complex $\mathfrak{K}=(\tilde{G}, g, K)$ which is constructed in the same way as in the proof of Theorem* 1, *there exists a cell decomposition $\{e_i\}$ of $\bar{V}$ which satisfies the following conditions* (8. a)-(8. e):

(8. a) *The triangulation of $\bar{V}$ constructed from $\mathfrak{K}=(\tilde{G}, g, K)$ by Lemma* 3 *is a subdecomposition of $\{e_i\}$.*

(8. b) *$\tilde{G}$, $E\cap\bar{V}$, and each $F_k\cap\bar{V}$ $(1\leq k\leq m)$ are subcomplexes of $\bar{V}$ as cell complex.*

(8. c) *$\{\pi_p(e_i)\}$ gives a cell decomposition of $\pi_p(\bar{V})$ and for each j $(0<j\leq n-p)$, $\{\pi_{(p,j)}(e_i)\}$ gives a cell decomposition of $\pi_{(p,j)}(\bar{V})$.*

(8. d) *For each cell $\pi_p(e_i)$ of $\pi_p(\bar{V})$, there exists a real analytic variety $H(\pi_p(e_i))$ defined in $\pi_p(U)$ $(\bar{V}\subset U)$ such that $\pi_p(e_i)\subset H(\pi_p(e_i))$ and each point of $\pi_p(e_i)$ is a simple point (i.e., in a neighborhood of the point, H is a real analytic manifold of the same dimension as H) of $H(\pi_p(e_i))$.*

We say that $H(\pi_p(e_i))$ is *attached* to $\pi_p(e_i)$. Put $\mathcal{H}_p=\{H(\pi_p(e_i))\}$ where $\pi_p(e_i)$ ranges over all the cells of $\pi_p(\bar{V})$. Take $H\in\mathcal{H}_p$. Then $\pi_p^{-1}(H)$ is a real analytic variety in U. Next, suppose that $\hat{E}^*$ is defined by $P_j=0$ $(j=1, \cdots, n-p)$, where each P_j is real (see Lemma 1). Then for each subset $\{j_1, \cdots, j_r\}$ of $\{1, \cdots, n-p\}$, we denote by $H_{j_1\cdots j_r}$ the real analytic variety defined in U by $P_{j_1}=0, \cdots, P_{j_r}=0$. Now put $\mathcal{H}=\{\pi_p^{-1}(H), \pi_p^{-1}(H)\cap H_{j_1\cdots j_r}, H_{j_1\cdots j_r}\}$, where H ranges over every element of $\mathcal{H}_p$ and $\{j_1, \cdots, j_r\}$ ranges over every subset of $\{1, \cdots, n-p\}$.

(8. e) *For each e_i, there exists a real analytic variety $H(e_i)$ in $\mathcal{H}$ such that $e_i \subset H(e_i)$ and each point of e_i is a simple point of $H(e_i)$.*

9. Scattered family of varieties in R^n

We consider a combination $(E, \mathcal{F}, D)$, where D is a relatively compact domain in R^n, E is a real analytic variety defined in a domain D' in R^n such that $\bar{D} \subset D'$ and $\mathcal{F}=(F_1, \cdots, F_m)$ is a finite set of subvarieties of E defined also in D'. Let $\{(E_\alpha, \mathcal{F}^\alpha, D_\alpha)\}$ be a set of these combinations. We call $\{(E_\alpha, \mathcal{F}^\alpha, D_\alpha)\}$ *scattered family of varieties*, if the set $\{D_\alpha\}$ is locally finite.

Let a be a point in R^n with usual coordinate system $\varphi=(x_1, \cdots, x_n)$ and let E be a real analytic variety E defined in a neighborhood U of a. Let $\mathcal{F}$ be a finite family of subvarieties $(F_1, \cdots, F_m)$ of E. R^n is considered to be contained in complex n-space C^n with coordinate system $\varphi^*=(z_1, \cdots, z_n)$ whose restriction to R^n is φ. U is also considered the intersection of R^n with a neighborhood U^* of a in C^n. Suppose that we have a proper coordinate system $\varphi'^*=(z'_1, \cdots, z'_n)$ for $(E; \mathcal{F})$ at a and let $\varphi'=(x'_1, \cdots, x'_n)$ be the restriction of φ'^* to U. Then we can consider that φ' is obtained by an affine transformation of φ. Finally let $V^*(\bar{V}^* \subset U^*)$ be a standardized neighborhood for $(E; \mathcal{F})$ with respect to φ'^* which is so small that triangulation of $E \cap \bar{V} (V=V^* \cap R^n)$ can be carried out in the same way as in the proof of Theorem 1. In this case, we call $(E, \mathcal{F}, \varphi', V)$ *normalized combination for triangulation at the center a* and if we wish to specify the center, we write $(E, \mathcal{F}, \varphi', V)_a$ for $(E, \mathcal{F}, \varphi', V)$.

Let $\{(E_\alpha, \mathcal{F}^\alpha, \varphi^\alpha, V_\alpha)\}$ be a set of normalized combinations $(E_\alpha, \mathcal{F}^\alpha, \varphi^\alpha, V_\alpha)_{a_\alpha}$ for triangulation at the center $a_\alpha \in R^n$. We call the set $\{(E_\alpha, \mathcal{F}^\alpha, \varphi^\alpha, V_\alpha)\}$ *normalized scattered family of varieties*, if the set $\{(E_\alpha, \mathcal{F}^\alpha, V_\alpha)\}$ is a scattered family of varieties (i.e. if $\{V_\alpha\}$ is locally finite).

Lemma 5. *Let $\{(E_\alpha, \mathcal{F}^\alpha, D_\alpha)\}$ $(\alpha \in A)$ be a scattered family of varieties, where by definition E_α and each element F_k^α $(1 \leq k \leq m_\alpha)$ of $\mathcal{F}^\alpha$ are defined in a domain D'_α such that $D'_\alpha \supset \bar{D}_\alpha$. Then there exists a normalized scattered family of varieties $\{(E_{(\alpha,j)}, \mathcal{F}^{(\alpha,j)}, \varphi^{(\alpha,j)}, V_{(\alpha,j)})_{a_{(\alpha,j)}}\}$ $(\alpha \in A$; for fixed α, $1 \leq j \leq j_\alpha < \infty)$ which satisfies the following conditions* (9.1)-(9.2):

(9. 1) $E_{(\alpha,j)}=E_\alpha$ *and* $\mathcal{F}^{(\alpha,j)}=\mathcal{F}^\alpha$ $(1 \leq j \leq j_\alpha)$.

(9. 2) $a_{(\alpha,j)} \in E_\alpha \cap \bar{D}_\alpha$ *and* $E_\alpha \cap \bar{D}_\alpha \subset \bigcup_{j=1}^{j_\alpha} V_{(\alpha,j)}$, $\bigcup_{j=1}^{j_\alpha} \bar{V}_{(\alpha,j)} \subset D'_\alpha$.

Proof. We have only to remember that $\{D_\alpha\}$ is locally finite and for each α, $E_\alpha \cap \bar{D}_\alpha$ is compact.

10. Pseudo-cell decomposition of a scattered family of varieties

Theorem 3. *Let* $\{(E_\alpha, \mathfrak{F}^\alpha, \varphi^\alpha, V_\alpha)\}$ $(\alpha \in A)$ *a normalized scattered family of varieties in* R^n. *Then* $E=\bigcup_{\alpha\in A}(E_\alpha \cap \bar{V}_\alpha)$ *has a pseudo-cell decomposition which satisfies the following conditions* (10.1)-(10.2):

(10.1) *For each* α *and each* $F_k^\alpha \in \mathfrak{F}^\alpha$ $(1\leq k\leq m_\alpha)$. $E_\alpha \cap \bar{V}_\alpha$ *and* $F_k^\alpha \cap \bar{V}_\alpha$ *are subcomplexes* (*as pseudo-cell complex*).

(10.2) *The decomposition of* $E_\alpha \cap \bar{V}_\alpha$ *is a subdivision of a cell decomposition of* $E_\alpha \cap \bar{V}_\alpha$ *obtained along the line of Lemma* 4.

Proof. By definition E_α is defined in a domain U_α such that $U_\alpha \supset \bar{V}_\alpha$. We may also assume that φ^α is defined on U_α and U_α is rectangular with the center at the same a_α as that of V_α. We define: $\operatorname{Dim} E = \operatorname{Max}_{\alpha\in A} \dim E_\alpha$ and prove this Theorem by imduction on the dimension p of E. For $p=0$, the Theorem is trivial. Then we prove for p assuming that the Theorem is true for $<p$. For each α, we have the imbedding varieties $\tilde{E}_\alpha^*$ and $\hat{E}_\alpha^*$ both of which are of dimension p (Lemma 1) and p-proper complex $\tilde{G}_\alpha$ which is obtained as the real part of $\tilde{E}_\alpha^*$ in $\bar{V}_\alpha$, and by Lemma 4, we have a cell decomposition of $\bar{V}_\alpha$ which satisfies the conditions (8. a)-(8. e).

Let e be a cell of $\pi_p(\bar{V}_\alpha)$ of dimension $\leq p-1$ (see (8. c)) and let $H(e)$ be the real analytic variety attached to e (see (8. d.)). Define $\hat{H}(e)=\hat{E}^* \cap U_\alpha \cap \pi_p^{-1}(H(e))$. Then $\hat{H}(e)$ is a real analytic variety of dimension $\leq p-1$ defined in U_α (notice $U_\alpha \supset \bar{V}_\alpha$). Put: $\hat{H}_\alpha=\bigcup_e H(e)$ and $\hat{\mathfrak{F}}^\alpha=\{\hat{H}(e)\}$, where e ranges over all the cells of $\pi_p(\bar{V}_\alpha)$ of dimension $\leq p-1$.

We fix a index α and for $\beta \in A$, we put $\hat{E}_{\alpha\beta}=(\hat{E}_\alpha^* \cap U_\alpha)\cap(\hat{E}_\beta^* \cap U_\beta)$. Then $\hat{E}_{\alpha\beta}$ is a real analytic variety defined in $U_\alpha \cap U_\beta$. Now suppose that $\hat{E}_{\alpha\beta}\cap(\bar{V}_\alpha \cap \bar{V}_\beta)\neq\emptyset$. In this case we construct a normalized scattered family $\{(\hat{E}_{\alpha\beta}^l, \hat{\mathfrak{F}}_l^{\alpha\beta}, \varphi_l^{\alpha\beta}, V_{\alpha\beta}^l)_{a_{\alpha\beta}^l}\}$ $(l \in I_{\alpha\beta})$ as follows:

1) $\hat{E}_{\alpha\beta}^l=\hat{E}_\alpha^*\cap(U_\alpha \cap U_\beta)$, $\hat{\mathfrak{F}}_l^{\alpha\beta}=\{\hat{E}_{\alpha\beta}\}$.

2) $I_{\alpha\beta}$ is a finite set, for any l, $a_{\alpha\beta}^l \in \hat{E}_{\alpha\beta}\cap(\bar{V}_\alpha \cap \bar{V}_\beta)$ and $\hat{E}_{\alpha\beta}\cap(\bar{V}_\alpha \cap \bar{V}_\beta)\subset \bigcup_{l\in I_{\alpha\beta}} V_{\alpha\beta}^l$, $\bigcup_{l\in I_{\alpha\beta}} \bar{V}_{\alpha\beta}^l \subset U_\alpha \cap U_\beta$.

3) $\varphi_l^{\alpha\beta}$ is obtained from φ^α, by making a linear transformation over the first p coordinates of φ_a and next, making a translation of origin from a_α to $a_{\alpha\beta}^l$, where a_α is the center of V_α. The existence of such a family follows directly from the fact that $\hat{E}_{\alpha\beta}\cap(\bar{V}_\alpha \cap \bar{V}_\beta)$ is compact and from Lemma 2.

Now, again by Lemma 4, $\bar{V}^l_{\alpha\beta}$ has a cell decomposition which satisfies the conditions (8.a)-(8.e). Fix indices α, β, l arbitrarily and let e' be a cell of $\pi_p(\bar{V}^l_{\alpha\beta})$ of dimension $\leq p-1$. Let $H(e')$ be the real analytic variety attached to e' and consider that $H(e')$ is defined in a rectangular domain $U^l_{\alpha\beta}$ such that $\bar{V}^l_{\alpha\beta} \subset U^l_{\alpha\beta} \subset U_\alpha \cap U_\beta$. Define $\hat{\hat{H}}(e') = E^l_{\alpha\beta} \cap U^l_{\alpha\beta} \cap \pi_p^{-1}(H(e'))$ and put $\hat{\hat{\mathfrak{F}}}^{\alpha\beta}_l = \{\hat{\hat{H}}(e')\}$ and $\hat{\hat{H}}^l_{\alpha\beta} = \bigcup_{e'} \hat{\hat{H}}(e')$, where e' ranges over all the cells of $\pi_p(\bar{V}^l_{\alpha\beta})$ of dimension $\leq p-1$.

We consider the set $\{(\hat{H}_\alpha, \hat{\mathfrak{F}}^\alpha, V_\alpha), (\hat{\hat{H}}^l_{\alpha\beta}, \hat{\hat{\mathfrak{F}}}^{\alpha\beta}_l, V^l_{\alpha\beta}) \mid \alpha \in A,\ \hat{E}_{\alpha\beta} \cap (\bar{V}_\alpha \cap \bar{V}_\beta) \neq \emptyset,\ l \in I_{\alpha\beta}\}$. As is easily seen, this set can be considered as a scattered family of varieties. Hence by Lemma 5 there exists a normarized scattered family of varieties $\{E^1_\gamma, \mathfrak{F}^1_\gamma, \varphi^1_\gamma, V^1_\gamma\}$ $(\gamma \in A_1)$ which satisfies the conditions (9.1) and (9.2). Since the dimension $E^1 = \bigcup_{\gamma \in A_1} E^1_\gamma$ is $\leq p-1$, by the assumption of the induction, E^1 has a pseudo-cell decomposition which satisfies the conditions (10.1) and (10.2).

For each $\alpha \in A$, $\tilde{G}_\alpha$ has a cell decomposition (see (8.b)). Let $\{e^p_{(\alpha,i)}\}$ $(1 \leq i \leq i_\alpha)$ be the set of p-cells $e^p_{(\alpha,i)}$ on $\tilde{G}_\alpha$ and let $\tilde{G}^1_\alpha$ denote the $(p-1)$-skelton $\tilde{G}_\alpha - \bigvee_{i=1}^{i_\alpha} e^p_{(\alpha,i)}$. Since, for each $\hat{H}(e) \in \hat{\mathfrak{F}}^\alpha$, E^1 gives a pseudo-cell decomposition to $\bar{V}_\alpha \cap \hat{H}(e)$, E^1 also gives psendo-cell decomposition to $\tilde{G}^1_\alpha$ which is finer than the original cell decomposition. Next, for each α, β, l, we also have a cell decomposition of $E^l_{\alpha\beta} \cap \bar{V}^l_{\alpha\beta} = \hat{E}^*_\alpha \cap \bar{V}^l_{\alpha\beta}$. Let $\{e^p_{(\alpha,\beta,l,j)}\}$ $(1 \leq j \leq j(\alpha, \beta, l))$ be the set of p-cells $e^p_{(\alpha,\beta,l,j)}$ on $\hat{E}^*_\alpha \cap \bar{V}^l_{\alpha\beta}$ such that $e^p_{(\alpha,\beta,l,j)} \cap \tilde{G}_\alpha \neq \emptyset$. Then by the condition 3) for $\varphi^{\alpha\beta}_l$, we see that $e^p_{(\alpha,\beta,l,j)} \cap \bar{V}_\alpha = \cap e^p_{(\alpha,\beta,l,j)} \cap \tilde{G}_\alpha$ and that if $\tilde{G}^1_{(\alpha,\beta,l)}$ is the intersection of $\tilde{G}_\alpha$ with the $(p-1)$-skelton of $\hat{E}^*_\alpha \cap \bar{V}^l_{\alpha\beta}$, E^1 gives a pseudo-cell decomposition to $\tilde{G}^1_{(\alpha,\beta,l)}$. Now put $\tilde{G}^2_\alpha = \tilde{G}^1_\alpha \cup (\bigcup_{\beta,l} G^1_{(\alpha,\beta,l)})$ $(\hat{E}_{\alpha\beta} \cap (\bar{V}_\alpha \cap \bar{V}_\beta) \neq \emptyset,\ l \in I_{\alpha\beta})$. Then $\tilde{G}^2_\alpha$ has a pseudo-cell decomposition given by that of E^1 and from the construction we see that each pseudo-cell of $\tilde{G}^2_\alpha$ is a boundary pseudo-cell of a p-cell $e^p_{(\alpha,i)}$ or $e^p_{(\alpha,\beta,l,j)}$ and any pseudo-cell of dimension $< p-1$ of $\tilde{G}^2_\alpha$ is a boundary cell of a $(p-1)$-pseudo-cell of $\tilde{G}^2_\alpha$. We consider the open set $\tilde{G}_\alpha - \tilde{G}^2_\alpha$ in $\tilde{G}_\alpha$ and express it as the union of the connected components $\hat{e}_{(\alpha,k)}$ $(k \in I_\alpha)$: $\tilde{G}_\alpha - \tilde{G} = \bigcup_{k \in I_\alpha} \hat{e}_{(\alpha,k)}$. Each $\hat{e}_{(\alpha,k)}$ is open in $\tilde{G}_\alpha$ and analytically homeomorphic to relatively compact domain $\pi_p(\hat{e}_{(\alpha,k)})$ in $\pi_p(U_\alpha)$ and hence is a p-pseudo-cell. Let e^{p-1} be a $(p-1)$-pseudo-cell of $\tilde{G}^2_\alpha$. Then e^{p-1} has a neighborhood W in $\tilde{G}_\alpha$ such that $W \cap (\tilde{G}^2_\alpha - e^{p-1}) = \emptyset$, and if $e^{p-1} \subset \bar{V}_\alpha - V_\alpha$, $W - e^{p-1}$ is connected and if $e^{p-1} \subset V_\alpha$, $W - e^{p-1}$ has a finite number of connected components. Hence we see that if $e^{p-1} \subset \bar{V}_\alpha - V_\alpha$, there exists one and only one $\hat{e}_{(\alpha,k)}$ whose boundary contains e^{p-1}, and if $e^{p-1} \subset V_\alpha$, there exist a finite number of connected components of $\tilde{G}_\alpha - \tilde{G}^2_\alpha$ which have e^{p-1} as a common boundary pseudo-cell. This shows especially that the boundary of each $\hat{e}_{(\alpha,k)}$ has a (finite) pseudo-cell decomposition with cells in $\tilde{G}_\alpha^2$ and that I_α is a finite set. Now it is easy to see that we get a pseudo-cell decomposition of $\tilde{G}_\alpha$ by attaching a finite number of p-pseudo-cells $\hat{e}_{(\alpha,k)}(k \in I_\alpha)$ to $\tilde{G}_\alpha^2$.

For each $\alpha \in A$, we consider a pseudo-cell decomposition of $\tilde{G}_\alpha$ obtained in the above manner. Let $\beta \in A$ be such that $\hat{E}_{\alpha\beta} \cap (\bar{V}_\alpha \cap \bar{V}_\beta) \neq \emptyset$. We consider again the cell decomposition of $\hat{E}_\alpha^* \cap \bar{V}_{\alpha\beta}^l$ given by Lemma 4. Then, by the condition 3) for $\varphi_l^{\alpha\beta}$, using the similar argument as above we can obtain a psendo-cell decomposition of $\hat{E}_\alpha^* \cap (\bigcup_{l \in I_{\alpha\beta}} \bar{V}_{\alpha\beta}^l \cap \bar{V}_\alpha)$ such that $\hat{E}_{\alpha\beta}$ and $\tilde{G}_\alpha$ are subcomplexes in $\bigcup_{l \in I_{\alpha\beta}} \bar{V}_{\alpha\beta}^l \cap \bar{V}_\alpha$ where $(p-1)$-skelton of $\tilde{G}_\alpha \cap (\bigcup_{l \in I_{\alpha\beta}} \bar{V}_{\alpha\beta}^l \cap V_\alpha)$ is $\bigcup_{l \in I_{\alpha\beta}} G_{(\alpha,\beta,l)}^1$. Applying the same discussion to β, we have a pseudo-cell decomposition of $\hat{E}_\beta^* \cap (\bigcup_{l \in I_{\beta\alpha}} \bar{V}_{\beta\alpha}^l \cap \bar{V}_\beta)$ in such a way that $\hat{E}_{\beta\alpha}$ $(=\hat{E}_{\alpha\beta})$ and $\tilde{G}_\beta$ are subcomplexes in $\bigcup_{l \in I_{\beta\alpha}} \bar{V}_{\beta\alpha}^l \cap \bar{V}_\beta$. Let $\hat{E}_{\alpha\beta}^1$ be the $(p-1)$-skelton of $\hat{E}_{\alpha\beta} \cap (\bigcup_{l \in I_{\alpha\beta}} \bar{V}_{\alpha\beta}^l \cap \bar{V}_\alpha)$. Then $(\hat{E}_{\alpha\beta}^1 \cup \hat{E}_{\beta\alpha}^1) \cap (\bar{V}_\alpha \cap \bar{V}_\beta)$ has a pseudo-cell decomposition given by E^1. Now attaching the connected components of $\hat{E}_{\alpha\beta} \cap (\bar{V}_\alpha \cap \bar{V}_\beta) - (\hat{E}_{\alpha\beta}^1 \cup \hat{E}_{\beta\alpha}^1) \cap (\bar{V}_\alpha \cap \bar{V}_\beta)$ to $(\hat{E}_{\alpha\beta}^1 \cap E_{\beta\alpha}^1) \cap (\bar{V}_\alpha \cap \bar{V}_\beta)$, we get a new pseudo-cell decomposition of $\hat{E}_{\alpha\beta} \cap (V_\alpha \cap V_\beta)$ and $\tilde{G}_\alpha \cap \tilde{G}_\beta$ is a subcomplex. We denote by $\tilde{G}_{\alpha\beta}^1$ the $(p-1)$-skelton of $\tilde{G}_\alpha \cap \tilde{G}_\beta$. Put $\tilde{G}_\alpha^3 = \tilde{G}_\alpha^2 \bigcup (\bigcup \tilde{G}_{\alpha\beta}^1)$. Then $\tilde{G}_\alpha^3$ has a pseudo-cell decomposition given by E^1 and we notice that any pseudo-cell of dimension $<p$ of $\tilde{G}_{\alpha\beta}^1$ which is not a boundary pseudo-cell of some p-pseudo-cell of $\tilde{G}_\alpha \cap \tilde{G}_\beta$ is already contained in $\tilde{G}_\alpha^2$. Hence, by attaching the connected components of $\tilde{G}_\alpha - \tilde{G}_\alpha^3$ to $\tilde{G}_\alpha^3$, we obtain a pseudo-cell decomposition of $\tilde{G}_\alpha$. Suppose that each $\tilde{G}_\alpha (\alpha \in A)$ has a pseudo-cell decomposition obtained in the above manner. Then $\bigcup_{\alpha \in A} \tilde{G}_\alpha$ has a pseudo-cell decomposition, where each $\tilde{G}_\alpha$ is a subcomplex with the decomposition just mentioned above and E is also a subcomplex with desired properties.

Let E be a real analytic variety in R^n and let $F_1, \cdots, F_m$ be subvarieties of E. Then there exists a normalized scattered family $\{(E_\alpha, \mathcal{F}^\alpha, \varphi^\alpha, V_\alpha)\}$ $(\alpha \in A)$ such that for any α, $E_\alpha \cap V_\alpha = E \cap V_\alpha$ and $\mathcal{F}^\alpha = \{E_1, \cdots, F_m\}$. Hence, as a direct consequence of the above Theorem, we have

Corollary. *Let E be a real analytic variety in R^n and let $F_1, \cdots, F_m$ be a finite number of subvarieties*[10] *of E. Then E has a pseudo-cell decomposition in such a way that each F_k $(1 \leq k \leq m)$ is a subcomplex.*

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10) Obviously, Corollary is also true for a locally finite family of subvarieties instead of a finite number of subvarieties.

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