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QF-3 AND SEMI-PRIMARY PP-RINGS I

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Recently the author has given a characterization of semi-primary hereditary ring in [4]. Furthermore, those results in [4] have been extended to a semi-primary PP-ring in [3], (a ring A is called a left PP-ring if every principal left ideal in A is A-projective).

This short note is a continuous work of [3] and [4]. Let $K$ be a field and $A$ an algebra over $K$ with finite dimension. $A$ is called a QF-3 algebra if $A$ has a unique minimal faithful representation ([10]). Mochizuki has considered a hereditary QF-3 algebra in [6].

In this note we shall study a PP-ring with minimal condition or of semi-primary. To this purpose we generalize a notion of QF-3 algebra in a case of ring. We call $A$ left (resp. right) QF-3 ring if $A$ has a faithful, injective, projective left (resp. right) ideal, (cf. [5], Theorems 3.1 and 3.2).

Let $1 = \sum E_i$ be a decomposition of the identity element 1 of a semi-primary ring $A$ into a sum of mutually orthogonal idempotents such that $E_i$ modulo the radical $N$ is the identity element of simple component of $A/N$. If $Ax$ is $A$-projective for all $x \in E_i A E_j$, we call $A$ a partially PP-ring, (see [3], §2). Such a class of rings contains properly classes of semi-primary hereditary rings and PP-rings.

Our main theorems are as follows: Let $A$ be directly indecomposable and a left QF-3 ring and semi-primary partially PP-ring. Then 1) there exists a unique primitive idempotent $e$ in $A$ (up to isomorphism) such that $eN = (0)$ and every indecomposable left injective ideal in $A$ is faithful, projective and isomorphic to $eAe$. Furthermore, $A$ is a right QF-3 ring. 2) Let $B = \text{Hom}_{eAe}(Ae, Ae)$, where $Ae$ is regarded as a right $eAe$-module. Then $eAe$ is a division ring and $B = (eAe)_n$. $B$ is a left and right injective envelope of $A$ as an $A$-module and $B$ is $A$-projective. Furthermore, if $A$ is hereditary, then $A$ is a generalized uniserial ring whose basic ring is of triangular matrices over a division ring. (Mochizuki proved in [6] the above fact 2) in a case of hereditary algebra over a field with finite dimension).

1) $(A)_n$ means a ring of matrices over a ring $A$ with degree $n$. 
We always consider a ring $A$ with identity element $1$ and every $A$-module is unitary.

1. Preliminary Lemmas.

In this paper we make use of some results in [3], [4] very often and we shall here summarize them.

Let $1 = \sum_{i=1}^{t} E_i$ be a decomposition of 1 into a sum of mutually orthogonal idempotents $E_i$. We assume that $E_iAE_j = (0)$ for $i < j$ and $E_iAE_i$ is semi-simple with minimal conditions. Then

$$A = S_1 \oplus E_2 AE_1 \oplus S_2 \oplus \cdots \oplus E_t AE_{t-1} \oplus S_t$$

as a module, where $S_i = E_i AE_i$.

By $T_i(S_i; \mathcal{M}_{i,j})$ we denote the above expression, and we call it a \textit{generalized triangular matrix ring over} $S_i$ (briefly g.t.a. matrix ring).

Let $S_i = \sum_{j=1}^{p(q)} T_{i,j}$; $T_{i,j}$ is a simple ring. Then we can easily check that

$$\mathcal{M}_{p,q} \approx \begin{pmatrix} M_{1,1} & \cdots & M_{1, p(q)} \\ M_{2,1} & \cdots & M_{2, p(q)} \\ \vdots & \cdots & \vdots \\ M_{p,1} & \cdots & M_{p, p(q)} \end{pmatrix}$$

as a $S_p - S_q$ module, where $M_{i,z}$ is a $T_{p,1} - T_{q,z}$ module and the operations of $S_p$ and $S_q$ are naturally defined on the right side of (2).

From [3], p. 160 and the proof of [4], Proposition 10 we have

\textbf{Lemma 1.} Let $A$ be a semi-primary partially PP-ring. Then $A$ is isomorphic to $T_i(S_i; \mathcal{M}_{i,j})$ such that every row of (2) is non-zero and $AE_i$ is a faithful $A$-module. Furthermore, let $\{e_i\}$ be a set of non-isomorphic mutually orthogonal primitive idempotents $e_i$ such that $e_iN = (0)$, then $E_i \approx \sum e_i$ and every faithful projective $A$-module contains $AE_i$ as a direct summand, where $E_i = T_i(1, o, \cdots, o; o)$ and $1_i$ is the identity element in $S_i$.

If $A$ is isomorphic to $T_i(S_i; \mathcal{M}_{i,j})$ as in Lemma 1, we call $T_i(S_i; \mathcal{M}_{i,j})$ a normal right representation of $A$ as a g.t.a. matrix ring.

\textbf{Lemma 2.} Let $A$ be as in Lemma 1. Then $\mathcal{M}_{i,j} \otimes S_jx \approx \mathcal{M}_{i,j}x$ and $yS_j \otimes \mathcal{M}_{i,k} \approx y\mathcal{M}_{i,k}$ for $x \in \mathcal{M}_{i,j}$ and $y \in \mathcal{M}_{i,k}$.

See [3], Lemma 5.
Let $K$ be a field and $A$ a $K$-algebra with finite dimension. Jans showed in [5] that $A$ has a unique minimal faithful representation if and only if $A$ has faithful, projective, injective left ideal $L$. Since $L$ is projective, we know that $\text{Hom}_K(L, K)$ is faithful, projective, injective right $A$-module.

We are interested in a case of a triangular matrices with minimal conditions. We shall generalize the above fact in this case.

Now we assume that $A$ is a g.t.a. matrix ring over semi-simple rings $S_i, i = T_i(S_i; M_{i,j})$.

If $e$ is a primitive idempotent, then $eAe$ is division ring. By $B$ we denote $eAe$. Since $A$ satisfies the minimal conditions, $[\text{Ae} : B] < \infty$ by [4], §5.

The following lemma is well known in a case of algebra over a field.

**Lemma 3.** Let $A$, $B$ and $e$ be as above. If $Ae$ is $A$-injective, then $\text{Hom}_B(Ae, B)$ is right $A$-projective and injective.

Proof. For a finitely generated left $A$-module $M$ we have $\text{Hom}_B(Ae, B) \otimes_A M \approx \text{Hom}_B(\text{Hom}_A(M, Ae), B)$ from [1], p. 120, Proposition 5.3. This isomorphism implies that $\text{Hom}_B(Ae, B)$ is right $A$-flat. Hence, $\text{Hom}_B(Ae, B)$ is $A$-projective by [2]. On the other hand, from an isomorphism : $\text{Hom}_A(N, \text{Hom}_B(Ae, B)) \approx \text{Hom}_B(N \otimes_A Ae, B)$ in [1], p. 120 for a right $A$-module $N$ we know that $\text{Hom}_B(Ae, B)$ is $A$-injective, since $Ae$ is $A$-flat.

**Proposition 1.** Let $A$ be a g.t.a. matrix ring over semi-simple rings with minimal conditions. If $A$ has a faithful, injective, projective left ideal, then $A$ has a faithful, injective, projective right ideal.

Proof. Let $L$ be a faithful, injective, projective left ideal $L = \sum \oplus Ae_i; e_i$ primitive idempotent. Put $B_i = e_iAe_i$ and $C_i = \text{Hom}_B(Ae_i, B_i)$. Then $C_i$ is right $A$-projective and injective. Let $x \neq 0$ in $A$. Since $L$ is faithful, $xAe_i \neq 0$ for some $i$. Since $B_i$ is a division ring, there exists $g$ in $C_i$ such that $g(xAe_i) = (0)$. Therefore, if we put $R' = \sum \oplus C_i$, then $R'$ is a faithful, projective, right $A$-module. Since $C_i \approx \sum \oplus e_iA$, we have a faithful, projective, injective right ideal.

If $A$ has a faithful, projective, injective left (resp. right) ideal, then we call $A$ a left (resp. right) QF-3 ring.

If $A$ is a g.t.a. matrix ring over semi-simple rings with minimal conditions, then a left QF-3 ring is a right QF-3 and conversely by

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2) $[\text{Ae} : B]$, means the dimension of $Ae$ as a right $B$-module.

3) Added in proof. We shall show in [12] that if $A$ satisfies minimum conditions, then $A$ is left QF-3 if and only if $A$ is right QF-3.
Proposition 1. However, we do not know whether it is true in a general ring with minimal conditions.\textsuperscript{3}

We quote here the concept of basic ring following Osima \cite{8}.

Let

\[ 1 = \sum_{i=1}^{n} \sum_{j=1}^{n} e_{i,j} \]

be a decomposition of the identity element 1 of \( A \) into the sum of mutually orthogonal primitive idempotents such that \( e_{i,j} = e_{h,k} \) if and only if \( i = h \).

For each \( i \) we denote \( e_{i,1} \) by \( e_{i}^{*} \). Let \( e^{*} = \sum_{i=1}^{n} e_{i}^{*} = \sum_{i=1}^{n} e_{i,1} \). We call \( A^{*} = e^{*}Ae^{*} \) the basic ring of \( A \) relative to the decomposition (3). We can find elements \( c_{i,j} \in e_{i,j}Ae_{i,j} \) and \( c_{i,j} \in e_{i,j}Ae_{i,1} \) such that \( c_{i,j}c_{i,j} = e_{i,1} \) and \( c_{i,j}c_{i,j} = e_{i,j} \). Put \( c_{i,j} = c_{i,j}c_{i,j} \). We may assume \( e_{i,1} = e_{i,1} \). Then we have

\[ c_{i,j}c_{i',j'} = \delta_{i,i'}\delta_{j,j'}c_{i,j} \]

\( A \) can be written

\[ A = \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{i,i'}\delta_{j,j'}c_{i,j}^{*}c_{i,j} \]

The following observation is a direct proof of \cite{7}, Lemma 7.2. Let \( M^{*} \) be a left \( A^{*} \)-module. We put

\[ M = E(M^{*}) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j}^{*}e_{i,j}^{*}M^{*} \]

where \( c_{i,j}e_{i,j}^{*}M^{*} \approx e_{i,j}^{*}M^{*} \) as a module. We can directly check that \( M \) is a left \( A \)-module and \( e^{*}M = M^{*} \). Conversely, let \( M \) be a left \( A \)-module. Then \( M = \sum_{i=1}^{n} \sum_{j=1}^{n} e_{i,j}M \) and \( M^{*} = \sum_{i=1}^{n} e_{i,1}M \) is a left \( A^{*} \)-module. We define a mapping \( \varphi \) of \( M \) to \( E(M^{*}) \) by setting

\[ \varphi(e_{i,j}m_{i,j}) = c_{i,j}e_{i,j}m_{i,j} \]

Then we can easily check that \( M \approx E(M^{*}) \) as a left \( A \)-module.

Let \( M \) and \( N \) be left \( A \)-modules. Then

\[ \text{Hom}_{A}(N, M) = \text{Hom}_{A}(\sum c_{i,j}N, \sum c_{i,j}M) \]

For elements \( f_{i,j} \in \text{Hom}_{e^{*}Ae^{*}}(c_{i,j}N, c_{i,j}M) \) and \( f_{i,j} \in \text{Hom}_{e^{*}Ae^{*}}(c_{i,j}N, c_{i,j}M) \) we consider a diagram:

\[ \begin{array}{ccc}
\sum_{i,j} N & \overset{f_{i,j}}{\longrightarrow} & \sum_{i,j} M \\
\downarrow c_{i,j} & & \downarrow c_{i,j} \\
\sum_{i,j} N & \overset{f_{i,j}}{\longrightarrow} & \sum_{i,j} M.
\end{array} \]
Then we can easily see that the diagram (4) is commutative for $f_{i,j} = f|_{c_{i,j}N}$ and $f \in \text{Hom}_A(N, M)$. Conversely let $M^*$ and $N^*$ be left $A^*$-modules. For $f^* = f^*|_{e^*N}$ of $f^*$ in $\text{Hom}_{A^*}(N^*, M^*)$ we define $f_{i,j}$ such that $f_{i,j} = f^*$ and the diagram (4) is commutative. Then we can show that $f = \sum f_{i,j}$ is in $\text{Hom}_A(N, M)$. Thus we have

**Lemma 4.** $A$ is a left QF–3 ring if and only if so is a basic ring of $A$. (cf. [11], Proposition 5).

2. **Main theorems.**

In this section we consider a semi-primary QF–3 partially PP–ring $A$. From Lemma 4, [4], Corollary 1 and [3], Remark 1 and Lemma 4 we have

**Proposition 2.** If $A$ is a semi-primary left QF–3 and hereditary (resp. PP– or partially PP–) ring, then so is a basic ring of $A$. In the case of hereditary ring the converse is true.

By $N$ we denote the radical of $A$.

**Proposition 3.** Let $A$ be a left QF–3 and partially PP–ring and semi-primary. Let $\{e_i\}$ be a set of mutually orthogonal primitive non-isomorphic idempotents such that $e_iN = (0)$. Then $L = \sum \oplus Ae_i$ is a unique minimal left faithful, projective, injective $A$–module.

Proof. It is clear from the definition and Lemma 1.

From Proposition 2 we may first restrict ourselves in a case where $A$ coincides with its basic ring. Then $A/N = \sum \oplus \Delta_i$; $\Delta_i$ a division ring.

Let $A$ be a g.t.a. matrix ring over division rings $\Delta_i$; $T_n(\Delta_i; M_{i,j})$. We put $C(i) = \{k | M_{k,i} = (0)\}$ and $R(j) = \{k | M_{j,k} = (0)\}$.

**Lemma 5.** Let $A$ be as in Proposition 3 and $A = T_n(\Delta_i; M_{i,j})$. We assume $Ae_i$ is $A$–injective. If $i$ is the maximal index in $C(i)$, then $C(i) = R(t)$, where $e_t = T_n(o, o, 1_i, o, o; o)$ and $1_i$ is the identity element of $\Delta_i$.

Proof. Put $C(i) = \{i(1) < i(2) < \cdots < i(k) = t\}$. Then $M_{a,i} = (0)$ if $a \notin C(i)$. We first show that

(5) \[ M_{t,a} = (0) \quad a \notin C(i) \]

If $M_{t,a} = (0)$, we take $x \neq 0$ in $M_{t,a}$ and $y \neq 0$ in $M_{t,i}$. Since $A$ is partially PP–ring, for any element $z$ in $A$ $zx = 0$ implies $z \in A(1 - e_i)$ by Lemma 2. Hence, $zy = 0$. Therefore, a mapping $\phi$ of $Ax$ to $Ay \subseteq Ae_i$: $zx \rightarrow zy$ is homomorphism. Since $Ae_i$ is $A$–injective, there exists an element $w$ in $Ae_i$ such that $y = zw$ by [1], p. 8, Theorem 3.2. Therefore, $w$ might be

4) Added in proof. We shall give a simple proof in [12].
in $M_{a,i}$. Since $\varphi$ is non-zero, $w$ is not zero, which contradicts the fact $M_{a,i} = (0)$. We need a lemma to complete the proof.

**Lemma 6.** Let $A$ and $t = i(k)$ be as above. Then there exists an index $g = g(l)$ such that $M_{g,k(l)} = (0)$ for any $l$, $1 \leq l < k$.

Proof. We assume $M_{g,k(i)} = (0)$ for all $g$ and some $l$. Then $M_{g,k(l)} = (0)$ is a non-zero left ideal contained in $Ae_i$. Furthermore, $M_{g',t} = (0)$ for all $g'$, because if $M_{g',t} = (0)$ (and hence $g' > t$), then $(0) = M_{g',t}M_{t,i} = M_{g',i}$. Hence, $Q = M_{k(l)} = M_{g,k(l)}$ is a left ideal contained in $Ae_i$. Let $x \neq 0$ in $\Delta_i$. Then a mapping $\psi$ of $Q$ to $Ae_i$ defined by $\psi(n + m) = nx$ for $n \in M_{k(l)}, m \in M_{t,i}$ is $A$-homomorphism. Since $Ae_i$ is injective, there exists an element $z$ in $Ae_i$ such that $nz = nx$ and $mz = 0$. This is a contradiction, because $n = M_{k(l)}, m \in M_{t,i}$. Q.E.D.

We continue the prove of Lemma 5. We shall show that $M_{i,k(d)} = (0)$ for $1 \leq s < k$. We have $M_{s,i} = (0)$ for $i(k-1) < b < t$, $b < t$ by the definition of $C(i)$ and $t$. If $M_{i,k(d-1)} = (0)$ for an integer $l$ such that $i(k-1) < l < l + t = i(k)$ then $(0) = M_{i,k(d-1)}M_{i,k-1} = M_{i,i}$. Therefore, $M_{i,k(d-1)} = (0)$ for all $l < t$. Hence, we know $M_{i,k(d-1)} = (0)$ from Lemma 6. We assume $M_{i,k(c)} = (0)$ for integer $c > a$ fixed integer $d$. By the same argument as above we obtain $M_{i,k(d)} = (0)$ for $q = i(r); d < r < k$. Hence, we know by Lemma 6 that there exists an integer $f(>0)$ such that $M_{i<k,f}= (0)$. Therefore, $(0) = M_{i<k,f}, M_{i<k,d} = M_{i,k(d)}$. Thus we can prove Lemma 5 by induction.

**Theorem 1.** Let $A$ be a semi-primary, partially PP-ring. If $A$ contains a finitely generated projective, injective left ideal $L$, then $A$ is a direct sum of two rings $A_1$, $A_2$ such that $A_1$ is a left QF-3 and $L$ is a faithful, projective, injective left ideal in $A_1$ and $A_2$ is the annihilator ideal of $L$ in $A$. In particular if $A$ is a left QF-3, $A = \bigoplus A_i$ as a ring and there exists a primitive idempotent $e_i$ in $A_i$ such that $A_e$ is a unique minimal, faithful, projective injective ideal and $e_i$ is uniquely determined up to isomorphism with property $e_iN = (0)$, where $N$ is the radical of $A$.

Proof. Since $A$ is semi-primary, $L \cong \bigoplus A_1, e_i$ primitive idempotent. As before we may assume that $A$ coincides with its basic ring. Let $T_n(A_1; M_{i,j})$ be a normal right representation of $A$ as a g.t.a. matrix ring. We assume $e_i = T_n(o, \ldots, 1_i, o, \ldots; o)$. Let $C^*(i) = i \cup C(i) = \{1 = i(o) < i(1) < \cdots < i(k) = l\}$. For $j \in C^*(i)$ $(0) = M_{i,j} \supseteq M_{i,j}M_{i,j}$ and $0 = M_{i,j} \supseteq M_{i,j}M_{i,j}$. Hence $M_{i,j} = M_{i,j} = 0$ any $i(s) < j$ and $i(p) < j$, respectively. Put $E = \sum e_j$ and $E' = 1 - E$. Then the above facts imply
that $M_{k,k'} \subseteq \text{EAE} + E' \text{AE}'$ for all $k, k'$. Hence $A = \text{EAE} \oplus E' \text{AE}'$ as a ring and $\text{EAE} \supseteq A e_i$. Furthermore, $\text{EAE} \cong T_n(\Delta_{k,j}; M_{k,j'; k',j})$ and $M_{k,j'; k',j} = 0$ for all $k, k'$. Hence, $\text{EAE}$ is a left QF-3 ring. It is clear that $E' \text{AE}'$ is the annihilator of $A e_i$. Repeating the above argument we have the first part of Theorem 1. The second one is an immediate consequence from the first part and Proposition 3.

**Remark 1.** Let $A = T_n(\Delta_i; M_{i,j})$ be a partially PP-ring and indecomposable basic QF-3 ring. Then we have obtained in the above proof that $M_{i,j} \neq (0)$ for all $i$ and hence, $M_{n,i} \neq (0)$ for all $i$ by Lemma 5.

**Remark 2.** We shall see later that the set of those indecomposable ideals $A e_i$ coincide with the set of indecomposable injective left ideals in $A$.

Next, we shall consider a QF-3 and semi-primary PP- (resp. hereditary) ring. We restrict ourselves again to a case of basic ring.

**Lemma 7.** Let $A$ be an indecomposable basic ring and semi-primary partially PP-ring. $A = T_n(\Delta_i; M_{i,j})$ be a normal right representation of $A$ as a g.t.a. matrix ring. Then $[M_{n,i} : \Delta_n] = [M_{i,j} : \Delta_i] = 1$ for all $i$. Furthermore, if $A$ is hereditary then $[M_{i,j} : \Delta_i] = [M_{i,j} : \Delta_i] = 1$ if $M_{i,j} \neq (0)$.

Proof. We use the same notation as above. Since $T_n(\Delta_i; M_{i,j})$ is a normal representation, $A e_i$ is $A$-injective. From Remark 1 we know $M_{n,i} \neq (0)$ and $M_{n,i} \neq (0)$ for all $i$. If $[M_{n,i} : \Delta_n] \geq 2$, then we have two independent elements $x, y$ in $M_{n,i}$ over $\Delta_n$. Let $\varphi$ be a linear mapping of $M_{n,i}$ into itself such that $\varphi(x) = x, \varphi(y) = 0$. Then $\varphi$ is $A$-homomorphism of $M_{n,i}$ to $A e_i$. Since $A e_i$ is injective, this is a contradiction. If $[M_{n,i} : \Delta_i] \geq 2$, then there exist two independent elements $x', y'$ in $M_{n,1}$ over $\Delta_i$. Let $\varphi$ be a linear mapping of $M_{n,i} = \Delta_n x'$ to itself such that $\varphi(x') = y'$. Injectivity of $A e_i$ implies that there exists an element $z$ in $\Delta_i$ such that $x' z = y'$. This contradicts a fact of independedency. Since $M_{n,i} \cong M_{n,i} M_{i,1}$, $[M_{n,i} : \Delta_n] \leq [M_{n,i} : \Delta_n] = 1$ and $[M_{i,1} : \Delta_i] \leq [M_{n,i} : \Delta_i] = 1$. We assume that $A$ is hereditary. Then $M_{n,i} \otimes M_{i,1} \cong M_{n,i} M_{i,1}$ as $\Delta_n - \Delta_i$ module by [4], Theorem 1. Hence $1 = [M_{n,i} : \Delta_n] \geq [M_{i,1} : \Delta_i]$.

**Theorem 2.** If $A$ is a left QF-3 and semi-primary hereditary ring, then $A$ is a direct sum of rings whose basic ring is a ring of triangular matrices over division rings. And hence, $A$ is right QF-3 and $A$ satisfies
minimal conditions. The converse is also true, (see Remark 3 below).

Proof. We assume that \( A \) is an indecomposable, basic ring. Then \( A = T_n(\Delta_i; M_{i,j}) \) and \( M_{i,j} \neq (0) \) and \( M_{n,i} \neq (0) \) for all \( i \) from Remark 1. We shall show that \( M_{i,j} \neq (0) \) for all \( i < j \). We quote the same notations of [4], Theorem 1. Since \( M_{i,j} \neq (0) \), we assume that \( M_{j,k} \neq (0) \) for any \( j \leq i \). If \( M_{i+1,i} = M_{i+1,i-1} = \ldots = M_{i+1,i-1} = (0) \) and \( M_{i+1,i-1} \neq (0) \), then \( M_{i+1,i} = (0) \). On the other hand, \( M_{i+1,i} = M_{i+1,i-1} \neq (0) \) since \( t < i \). However, \( M_{n,i} \neq (0) \) and \( M_{n,i} \neq (0) \) by [4], Theorem 1. Which contradicts a fact \( [M_{n,i} : \Delta_n] = 1 \). Therefore, we know \( M_{i+1,i} \neq (0) \). \( M_{i+1,k} \subseteq M_{i+1,i} \cup M_{i+1,i-1} \cup M_{i+1,i-2} \cup \ldots \cup M_{k+1,k} \neq (0) \). Thus we can prove the fact \( M_{i,j} \neq (0) \) for all \( i > j \) by induction. Since \( M_{i,j} \neq (0) \), \( [M_{i,j} : \Delta_j] = [M_{i,j} : \Delta_j] = 1 \) by Lemma 7. Therefore, \( A \) is isomorphic to a ring of triangular matrices by [4], Lemma 12. Thus, we have proved Theorem 2.

In the above proof if we replace \( M_{i+1,i-1} \) by a non-zero element \( x \) in \( M_{i+1,i-1} \) and \( M_{i+1,i-1} \) by a non-zero element \( y \) in \( M_{i+1,i-1} \), then \( M_{n,i} \neq (0) \) and \( M_{n,i} \neq (0) \) are not zero by Lemma 2, provided \( A \) is a PP-ring. Since \( [M_{n,i} : \Delta_n] = 1 \) by Lemma 7, \( M_{n,i} = M_{n,i} \). This contradicts [3], Proposition 1. Hence, we have similarly

**Proposition 4.** Let \( A \) be a left QF-3 and semi-primary PP-ring. We assume \( A \) is indecomposable. Then \( A \) is isomorphic to a g.t.a. matrix ring \( T_n(S_i; \mathfrak{M}_{i,j}) \) over simple ring \( S_i \) and each component of \( \mathfrak{M}_{i,j} \) in (2) is non-zero. Therefore, \( T_n(S_i; \mathfrak{M}_{i,j}) \) is a right and left normal representation of \( A \) as a g.t.a. matrix ring and the nilpotency of the radical is equal to \( n \). Let \( S_i = (\Delta_i)_n \), \( \Delta_i \) division ring. Then \( \Delta_i \approx \Delta_n \) and \( \Delta_i \) is isomorphic into \( \Delta_i \approx \Delta_n \). Furthermore, we assume that \( A \) is K-algebra with finite dimension. Then \( A \) is hereditary if and only if \( \Delta_i \approx \Delta_i \) for all \( i \).

**Remark 3.** Theorem 2 says that the class of the QF-3 and semi-primary hereditary rings coincides with the class of the rings of direct sum of g.t.a. matrix rings of the following form.

Let \( \Delta \) be a division ring and \( \Delta(n \times m) \) the module of rectangular matrices of \((n \times m)\)-form over \( \Delta \) and it is regarded as \( (\Delta)_n - (\Delta)_m \) module.

\[
A = \begin{pmatrix}
(\Delta)_n^i & 0 \\
\Delta(n_2 \times n_1) (\Delta)_n^2 \\
\ldots & \\
\Delta(n_r \times n_1) \Delta(n_r \times n_1) \Delta(n_r \times n_2) \ldots (\Delta)_n^r
\end{pmatrix}
\]

We consider the converse of the first half of Lemma 7.
Proposition 5. Let \( A = T_\alpha(\Delta; M_{i,j}) \) be a g.t.a. matrix ring over division ring \( \Delta \). If \( A \) is a partially PP-ring, then \( Ae_1 \) is \( A \)-injective and \( M_{n,i} = 0 \) for all \( i \) and only if \( [M_{i,1}:\Delta] = [M_{n,1}:\Delta] = 1 \). Conversely if \( Ae_1 \) is faithful and \( [M_{i,1}:\Delta] = 1 \), then \( A \) is a partially PP-ring, where \( e_1 = T_\alpha(1,0,\cdots;0) \).

Proof. We assume that \( A \) is a partially PP-ring. We have proved “only if” part of the first half in the proof of Lemma 7. We shall prove “if” part. Since \([M_{n,1}:\Delta] = 1\) and \( \{M_{i,1}:\Delta\} \), we put \( M_{n,1} = 0 \) and \( \{M_{i,1}:\Delta\} = 1 \) for all \( i \). Hence, there exists an unique element \( g_i \) in \( M_{n,i} \) such that \( g_i x_i = x_n \), \( (g_n = \text{the identity element in } \Delta_n) \). Therefore, \( \text{Hom}_{\Delta_1}(M_{i,1}, M_{n,1}) = 0 \) for \( i \neq j \). Then \( \text{Hom}_{\Delta_1}(Ae_1, \Delta) \) is a module. We have isomorphisms \( \theta_i : M_{n,i} = \Delta_n g_i \rightarrow M_{i,1}^k \), by setting

\[
\theta_i(\delta g_i)(x_i) = \delta^{n-1} \quad \text{and} \quad \theta_i(\delta g_i)(x_j) = 0 \quad \text{for } j \neq i.
\]

Hence, we have an isomorphism \( \Theta \) of \( e_nA \) to \( \text{Hom}_{\Delta_1}(Ae_1, \Delta) \) via \( \theta_i \) as a module. We shall show that \( \Theta \) is \( A \)-isomorphic. Let \( \theta_i(\delta g_i) = f \in M_{i,1}^k \), and \( m_{k,l} \in M_{k,l} \). Then \( \theta_i(\delta g_i) = f \) as a module. Hence we have similarto Theorem 2 we have

**Theorem 3.** Let \( A \) be a semi-primally PP-ring. \( A \) is a left QF-3 ring if and only if its basic ring is of the form \( T_\alpha(\Delta; M_{i,j}) \) such that
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$[M_{i,j} : \Delta_i] = [M_{n,i} : \Delta_n] = 1$. In this case $A$ is also a right QF-3 ring.

Proof. It is clear from Theorem 1 and Proposition 5.

Finally, we shall generalize Mochizuki’s result [6], Theorem 2.3 in a case of semi-primary partially PP-ring.

Let $A$ be a basic QF-3 ring and semi-primary partially PP-ring. We assume that $A$ is indecomposable. Then $A \cong T_n(\Delta ; M_{i,j})$ and $[M_{i,j} : \Delta_i] = [M_{n,i} : \Delta_n] = 1$ for all $i$ by Lemma 7. Hence, we may assume that $\Delta_i = \Delta_n \equiv \Delta$ and $\Delta_i$ is contained in $\Delta$. Let $L = T_n(\Delta, 0, \ldots, 0 : M_{i,j}(0)$ if $j \neq 1)$. Then $L$ is a unique minimal faithful projective, injective left $A$-module. Let $B = \text{Hom}_\Delta(L, L)$. Then $B = (\Delta)$. Let $B_{i,j} = \{ f \in B, f(M_{i,j}) = M_{i,j}, f(M_{k,j}) = (0)$ for $k \neq j \}$. Then $B_{i,j} \cap A \cong M_{i,j}$, where $A$ is regarded as a subring of $B$, since $L$ is faithful. By virtue of this imbedding we can regard $M_{i,j}$ as a $\Delta_i - \Delta_j$ submodule in $\Delta$. In such a setting, we have

\[
B = \begin{pmatrix}
\Delta & \Delta & \cdots & \Delta \\
\Delta & \Delta & \cdots & \Delta \\
\vdots & \vdots & \ddots & \vdots \\
\Delta & \Delta & \cdots & \Delta 
\end{pmatrix} \supseteq A = \begin{pmatrix}
\Delta & 0 & 0 & \cdots & 0 \\
0 & \Delta_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \Delta_{n-1}
\end{pmatrix} \supseteq L = \begin{pmatrix}
\Delta & 0 & 0 & \cdots & 0 \\
0 & \Delta & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \Delta & 0
\end{pmatrix},
\]

where $M_{i,j}$ is a $\Delta_i - \Delta_j$ submodule in $\Delta$ and $\Delta_i$ is a subdivision ring of $\Delta$. Since $B \cong L^{(\infty, 0)}$ as a left $A$-module, $B$ is left $A$-projective and injective.

Lemma 8. Let $A$ and $L$ be as above. Injective envelope of indecomposable left ideal $Ae_i$ is isomorphic to $L$.

Proof. Since $M_{i,1}(0)$, we can take $x \neq 0$ in $M_{i,1}$. Then $Ae_i x \neq Ae_i$ by Lemma 2. Since $Ae_i x \subseteq L$ and $L$ is indecomposable, $L$ is an injective envelope of $Ae_i x$.

We note that the double commutator ring of module which is a directsum of $n$-copies of a module $M$ coincides with that ring of $M$ up to isomorphism.

Summarizing the above we have

Theorem 4. Let $A$ be a semi-primary partially PP-ring and $e$ be an idempotent such that $Ae$ is a faithful projective, injective left ideal. Then the following facts hold.

(1) Both the commutator ring $eAe$ and the double commutator ring $B = \text{Hom}_{eAe}(Ae, Ae)$ of $Ae$ are semi-simple.

5) $L^{(\infty)}$ means a directsum of $n$-copies of $L$. 

(2) \( B \) is an \( A - A \) module which is both the left and right injective envelope of \( A \) and left and right \( A \)-projective.

(3) If \( A \) is hereditary, then \( A \) is a generalized uniserial ring with minimal conditions.

**Corollary.** Let \( A \) be as above. If \( L \) is an indecomposable \( A \)-injective left ideal in \( A \), then \( L \) is projective and \( L \cong A e, eN=(0) \).

Proof. We may assume \( A \) is indecomposable. Let \( M \) be a minimal left ideal contained in \( L \), since \( A \) is semi-primary, (see [5], p 1106). Then an injective envelope of \( M' \) is contained in \( L \) and hence \( L \) is isomorphic to an injective envelope of \( M' \). Therefore, \( B \) in Theorem 4 contains an isomorphic image of \( L \) as direct summand by the proof of Theorem 3.2 in [5]. Hence, \( L \) is \( A \)-projective by Theorem 4. The second part is clear from Theorem 2.

We conclude this paper with the following examples.

**Example.** Let \( K \) be a field and \( L \) proper extension of \( K \). We put

\[
A = \begin{pmatrix}
L & 0 & 0 \\
L & K & 0 \\
L & L & L
\end{pmatrix},
\]

where \( L \) at \((2, 1)\)-component is regarded as \( K-L \) module and \( L \) at \((3, 2)\)-component as \( L-K \) module. Since a natural mapping \( L \otimes L \rightarrow L \) is not monomorphic, \( A \) is not hereditary by [4], Theorem 1. It is clear that \( \begin{pmatrix} L00 \\ L00 \end{pmatrix} \) is a faithful, projective, injective \( A \)-module and \( A \) is a PP-ring by Proposition 5 and [3], Proposition 1. Hence, \( A \) is a QF-3 and PP-ring and not hereditary. If \( [L:K]=\infty \) \( A \) does not satisfies the minimal conditions.

Let

\[
A = \begin{pmatrix}
K & 0 & 0 & 0 \\
K & K & 0 & 0 \\
K & 0 & K & 0 \\
K & K & K & K
\end{pmatrix},
\]

then \( A \) is a QF-3 and partially PP-ring by [3], Lemm 5. However, \( A \) is not a PP-ring and hence, not hereditary.

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References