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QF-3 AND SEMI-PRIMARY PP-RINGS I

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Recently the author has given a characterization of semi-primary hereditary ring in [4]. Furthermore, those results in [4] have been extended to a semi-primary PP-ring in [3], (a ring A is called a *left PP-ring* if every principal left ideal in A is A -projective).

This short note is a continuous work of [3] and [4]. Let K be a field and A an algebra over K with finite dimension. A is called a *QF-3 algebra* if A has a unique minimal faithful representation ([10]). Mochizuki has considered a hereditary QF-3 algebra in [6].

In this note we shall study a PP-ring with minimal condition or of semi-primary. To this purpose we generalize a notion of QF-3 algebra in a case of ring. We call A *left (resp. right) QF-3 ring* if A has a faithful, injective, projective left (resp. right) ideal, (cf. [5], Theorems 3.1 and 3.2).

Let $1 = \sum E_i$ be a decomposition of the identity element 1 of a semi-primary ring A into a sum of mutually orthogonal idempotents such that E_i modulo the radical N is the identity element of simple component of A/N . If Ax is A -projective for all $x \in E_i A E_j$, we call A a *partially PP-ring*, (see [3], §2). Such a class of rings contains properly classes of semi-primary hereditary rings and PP-rings.

Our main theorems are as follows: *Let A be directly indecomposable and a left QF-3 ring and semi-primary partially PP-ring. Then 1) there exists a unique primitive idempotent e in A (up to isomorphism) such that $eN = (0)$ and every indecomposable left injective ideal in A is faithful, projective and isomorphic to Ae . Furthermore, A is a right QF-3 ring. 2) Let $B = \text{Hom}_{eAe}(Ae, Ae)$, where Ae is regarded as a right eAe -module. Then eAe is a division ring and $B = (eAe)_n^{(1)}$. B is a left and right injective envelope of A as an A -module and B is A -projective. Furthermore, if A is hereditary, then A is a generalized uniserial ring whose basic ring is of triangular matrices over a division ring. (Mochizuki proved in [6] the above fact 2) in a case of hereditary algebra over a field with finite dimension).*

1) $(A)_n$ means a ring of matrices over a ring A with degree n .

We always consider a ring A with identity element 1 and every A -module is unitary.

1. Preliminary Lemmas.

In this paper we make use of some results in [3], [4] very often and we shall here summarize them.

Let $1 = \sum_{i=1}^t E_i$ be a decomposition of 1 into a sum of mutually orthogonal idempotents E_i . We assume that $E_i A E_j = (0)$ for $i < j$ and $E_i A E_i$ is semi-simple with minimal conditions. Then

$$(1) \quad \begin{aligned} A = S_1 & \\ & \oplus E_2 A E_1 \oplus S_2 \\ & \dots\dots\dots \\ & \dots\dots\dots \\ & \oplus E_t A E_1 \oplus \dots \oplus E_t A E_{t-1} \oplus S_t \end{aligned}$$

as a module, where $S_i = E_i A E_i$.

By $T_i(S_i; \mathfrak{M}_{i,j} \equiv E_i A E_j)$ we denote the above expression, and we call it a *generalized triangular matrix ring over S_i* (briefly g.t.a. matrix ring).

Let $S_i = \sum_{j=1}^{p(i)} \oplus T_{i,j}$: $T_{i,j}$ is a simple ring. Then we can easily check that

$$(2) \quad \mathfrak{M}_{p,q} \approx \begin{pmatrix} M_{1,1} & \dots & M_{1,p(q)} \\ M_{2,1} & \dots & M_{2,p(q)} \\ \dots\dots\dots \\ M_{p(p),1} & \dots & M_{p(p),p(q)} \end{pmatrix}$$

as a $S_p - S_q$ module, where $M_{i,s}$ is a $T_{p,i} - T_{q,s}$ module and the operations of S_p and S_q are naturally defined on the right side of (2).

From [3], p. 160 and the proof of [4], Proposition 10 we have

Lemma 1. *Let A be a semi-primary partially PP-ring. Then A is isomorphic to $T_i(S_i; \mathfrak{M}_{i,j})$ such that every row of (2) is non-zero and AE_1 is a faithful A -module. Furthermore, let $\{e_i\}$ be a set of non-isomorphic mutually orthogonal primitive idempotents e_i such that $e_i N = (0)$, then $E_1 \approx \sum e_i$ and every faithful projective A -module contains AE_1 as a direct summand, where $E_1 = T_i(1_1, 0, \dots, 0; 0)$ and 1_1 is the identity element in S_1 .*

If A is isomorphic to $T_i(S_i; \mathfrak{M}_{i,j})$ as in Lemma 1, we call $T_i(S_i; \mathfrak{M}_{i,j})$ a *normal right representation of A as a g.t.a. matrix ring*.

Lemma 2. *Let A be as in Lemma 1. Then $\mathfrak{M}_{i,j} \otimes_{S_j} S_j x \approx \mathfrak{M}_{i,j} x$ and $y S_i \otimes_{S_i} \mathfrak{M}_{i,k} \approx y \mathfrak{M}_{i,k}$ for $x \in \mathfrak{M}_{j,t}$ and $y \in \mathfrak{M}_{l,i}$.*

See [3], Lemma 5.

Let K be a field and A a K -algebra with finite dimension. Jans showed in [5] that A has a unique minimal faithful representation if and only if A has faithful, projective, injective left ideal L . Since L is projective, we know that $\text{Hom}_K(L, K)$ is faithful, projective, injective right A -module.

We are interested in a case of a triangular matrices with minimal conditions. We shall generalize the above fact in this case.

Now we assume that A is a g.t.a. matrix ring over semi-simple rings S_i ; $A = T_n(S_i; M_{i,j})$.

If e is a primitive idempotent, then eAe is division ring. By B we denote eAe . Since A satisfies the minimal conditions, $[Ae : B]_r^{(2)} < \infty$ by [4], § 5.

The following lemma is well known in a case of algebra over a field.

Lemma 3. *Let A, B and e be as above. If Ae is A -injective, then $\text{Hom}_B(Ae, B)$ is right A -projective and injective.*

Proof. For a finitely generated left A -module M we have $\text{Hom}_B(Ae, B) \otimes_A M \approx \text{Hom}_B(\text{Hom}_A(M, Ae), B)$ from [1], p. 120, Proposition 5.3. This isomorphism implies that $\text{Hom}_B(Ae, B)$ is right A -flat. Hence, $\text{Hom}_B(Ae, B)$ is A -projective by [2]. On the other hand, from an isomorphism: $\text{Hom}_A(N, \text{Hom}_B(Ae, B)) \approx \text{Hom}_B(N \otimes_A Ae, B)$ in [1], p. 120 for a right A -module N we know that $\text{Hom}_B(Ae, B)$ is A -injective, since Ae is A -flat.

Proposition 1³⁾. *Let A be a g.t.a. matrix ring over semi-simple rings with minimal conditions. If A has a faithful, injective, projective left ideal, then A has a faithful, injective, projective right ideal.*

Proof. Let L be a faithful, injective, projective left ideal $L = \sum \oplus Ae_i$; e_i primitive idempotent. Put $B_i = e_i Ae_i$ and $C_i = \text{Hom}_{B_i}(Ae_i, B_i)$. Then C_i is right A -projective and injective. Let $x \neq 0$ in A . Since L is faithful, $xAe_i \neq 0$ for some i . Since B_i is a division ring, there exists g in C_i such that $g(xAe_i) \neq (0)$. Therefore, if we put $R' = \sum \oplus C_i$, then R' is a faithful, projective, right A -module. Since $C_i \approx \sum \oplus e'_i A$, we have a faithful, projective, injective right ideal.

If A has a faithful, projective, injective left (resp. right) ideal, then we call A a *left (resp. right) QF-3 ring*.

If A is a g.t.a. matrix ring over semi-simple rings with minimal conditions, then a left QF-3 ring is a right QF-3 and conversely by

2) $[Ae : B]_r$ means the dimension of Ae as a right B -module.

3) Added in proof. We shall show in [12] that if A satisfies minimum conditions, then A is left QF-3 if and only if A is right QF-3.

Proposition 1. However, we do not know whether it is true in a general ring with minimal conditions.³⁾

We quote here the concept of basic ring following Osima [8].

Let

$$(3) \quad 1 = \sum_{i=1}^n \sum_{j=1}^{\rho(i)} e_{i,j}$$

be a decomposition of the identity element 1 of A into the sum of mutually orthogonal primitive idempotents such that $e_{i,j} \approx e_{h,k}$ if and only if $i=h$.

For each i we denote $e_{i,1}$ by e_i^* . Let $e^* = \sum_{i=1}^n e_i^* = \sum_{i=1}^n \sum_{j=1}^{\rho(i)} e_{i,j}$. We call $A^* = e^* A e^*$ the *basic ring* of A relative to the decomposition (3). We can find elements $c_{i,1j} \in e_{i,1} A e_{i,j}$ and $c_{i,j1} \in e_{i,j} A e_{i,1}$ such that $c_{i,1j} c_{i,j1} = e_{i,1}$ and $c_{i,j1} c_{i,1j} = e_{i,j}$. Put $c_{i,jk} = c_{i,j1} c_{i,1k}$. We may assume $e_{i,11} = e_{i,1}$. Then we have

$$c_{i,jk} c_{i',j'k'} = \delta_{i,i'} \delta_{k,j'} c_{i,jk'}.$$

A can be written

$$A = \sum_{i=1}^n \sum_{j=1}^{\rho(i)} \sum_{h=1}^n \sum_{k=1}^{\rho(h)} c_{i,j1} A^* c_{h,1k}.$$

The following observation is a direct proof of [7], Lemma 7.2. Let M^* be a left A^* -module. We put

$$M = E(M^*) = \sum_{i=1}^n \sum_{j=1}^{\rho(i)} \oplus c_{i,j1} e_i^* M^*,$$

where $c_{i,j1} e_i^* M^* \approx e_1^* M^*$ as a module. We can directly check that M is a left A -module and $e^* M = M^*$. Conversely, let M be a left A -module.

Then $M = \sum_{i=1}^n \sum_{j=1}^{\rho(i)} \oplus e_{i,j} M$ and $M^* = \sum_{i=1}^n \oplus e_{i,1} M$ is a left A^* -module. We define a mapping φ of M to $E(M^*)$ by setting

$$\varphi(e_{i,j} m_{i,j}) = c_{i,j1} e_{i,1} m_{i,j}.$$

Then we can easily check that $M \approx E(M^*)$ as a left A -module.

Let M and N be left A -modules. Then

$$\text{Hom}_A(N, M) = \text{Hom}_A(\sum c_{i,j1} N, \sum c_{i,j1} M).$$

For elements $f_{i,11} \in \text{Hom}_{e_i^* A e_i^*}(c_{i,11} N, c_{i,11} M)$ and $f_{i,j1} \in \text{Hom}_{e_{i,j} A e_{i,j}}(c_{i,j1} N, c_{i,j1} M)$ we consider a diagram:

$$(4) \quad \begin{array}{ccc} c_{i,11} N & \xrightarrow{f_{i,11}} & c_{i,11} M \\ \downarrow c_{i,j1} & f_{i,j1} & \downarrow c_{i,j1} \\ c_{i,j1} N & \xrightarrow{\quad} & c_{i,j1} M. \end{array}$$

Then we can easily see that the diagram (4) is commutative for $f_{i,j_1} = f|_{c_{i,j_1}N}$ and $f \in \text{Hom}_A(N, M)$. Conversely let M^* and N^* be left A^* -modules. For $f_i^* = f^*|_{e_i^*N}$ of f^* in $\text{Hom}_{A^*}(N^*, M^*)$ we define f_{i,j_1} such that $f_{i,11} = f_i^*$ and the diagram (4) is commutative. Then we can show that $f = \sum f_{i,j_1}$ is in $\text{Hom}_A(N, M)$. Thus we have

Lemma 4. *A is a left QF-3 ring if and only if so is a basic ring of A. (cf. [11], Proposition 5).*

2. Main theorems.

In this section we consider a semi-primary QF-3 partially PP-ring A.

From Lemma 4, [4], Corollary 1 and [3], Remark 1 and Lemma 4 we have

Proposition 2. *If A is a semi-primary left QF-3 and hereditary (resp. PP- or partially PP-) ring, then so is a basic ring of A. In the case of hereditary ring the converse is true.*

By N we denote the radical of A.

Proposition 3. *Let A be a left QF-3 and partially PP-ring and semi-primary. Let $\{e_i\}$ be a set of mutually orthogonal primitive non-isomorphic idempotents such that $e_iN = (0)$. Then $L = \sum \oplus Ae_i$ is a unique minimal left faithful, projective, injective A-module.*

Proof. It is clear from the definition and Lemma 1.

From Proposition 2 we may first restrict ourselves in a case where A coincides with its basic ring. Then $A/N = \sum \oplus \Delta_i$; Δ_i a division ring.

Let A be a g.t.a. matrix ring over division rings Δ_i ; $T_n(\Delta_i; M_{i,j})$. We put $C(i) = \{k | M_{k,i} \neq (0)\}$ and $R(j) = \{k | M_{j,k} \neq (0)\}$.

Lemma 5.⁴⁾ *Let A be as in Proposition 3 and $A = T_n(\Delta_i; M_{i,j})$. We assume Ae_i is A-injective. If t is the maximal index in $C(i)$, then $C(i) = R(t)$, where $e_i = T_n(o, o, 1_i, o, o; o)$ and 1_i is the identity element of Δ_i .*

Proof. Put $C(i) \equiv \{i(1) < i(2) < \dots < i(k) = t\}$. Then $M_{a,i} = (0)$ if $a \notin C(i)$. We first show that

$$(5) \quad M_{t,a} = (0) \quad a \notin C(i)$$

If $M_{t,a} \neq (0)$, we take $x \neq 0$ in $M_{t,a}$ and $y \neq 0$ in $M_{t,i}$. Since A is partially PP-ring, for any element z in A $zx = 0$ implies $z \in A(1 - e_t)$ by Lemma 2. Hence, $zy = 0$. Therefore, a mapping φ of Ax to $Ay \subseteq Ae_i$: $zx \rightarrow zy$ is homomorphism. Since Ae_i is A-injective, there exists an element w in Ae_i such that $y = xw$ by [1], p. 8, Theorem 3.2. Therefore, w might be

4) Added in proof. We shall give a simple proof in [12].

in $M_{a,i}$. Since φ is non-zero, w is not zero, which contradicts the fact $M_{a,i} = (0)$. We need a lemma to complete the proof.

Lemma 6. *Let A and $t = i(k)$ be as above. Then there exists an index $g = g(l)$ such that $M_{g,i(l)} \neq (0)$ for any l , $1 \leq l < k$.*

Proof. We assume $M_{g,i(l)} = (0)$ for all g and some l . Then $M_{i(l),i}$ is a non-zero left ideal contained in Ae_i . Furthermore, $M_{g',t} = (0)$ for all g' , because if $M_{g',t} \neq (0)$ (and hence $g' > t$), then $(0) \neq M_{g',t} M_{t,i} \subseteq M_{g',i}$. Hence, $Q = M_{i(l),i} \oplus M_{t,i}$ is a left ideal contained in Ae_i . Let $x \neq 0$ in Δ_i . Then a mapping ψ of Q to Ae_i defined by $\psi(n+m) = nx$ for $n \in M_{i(l),i}$, $m \in M_{t,i}$ is A -homomorphism. Since Ae_i is injective, there exists an element z in Ae_i such that $nz = nx$ and $mz = 0$. This is a contradiction, because $n = M_{i(l),i}$, $m \in M_{t,i}$. Q.E.D.

We continue the prove of Lemma 5. We shall show that $M_{t,i(s)} \neq (0)$ for $1 \leq s \leq k$. We have $M_{b,i} = (0)$ for $i(k-1) < b < t$, $t < b$ by the definition of $C(i)$ and t . If $M_{l,i(k-1)} \neq (0)$ for an integer l such that $i(k-1) < l \neq t = i(k)$ then $(0) \neq M_{l,i(k-1)} M_{i(k-1),i} \subseteq M_{l,i}$. Therefore, $M_{l,i(k-1)} = (0)$ for all $l \neq t$. Hence, we know $M_{t,i(k-1)} \neq (0)$ from Lemma 6. We assume $M_{t,i(c)} \neq (0)$ for integer $c > a$ fixed integer d . By the same argument as above we obtain $M_{q,i(d)} = (0)$ for $q \neq i(r)$; $d < r < k'$. Hence, we know by Lemma 6 that there exists an integer $f (> d)$ such that $M_{i(f),i(d)} \neq (0)$. Therefore, $(0) \neq M_{t,i(f)} M_{i(f),i(d)} \subseteq M_{t,i(d)}$. Thus we can prove Lemma 5 by induction.

Theorem 1.⁽⁴⁾ *Let A be a semi-primary, partially PP-ring. If A contains a finitely generated projective, injective left ideal L , then A is a directsum of two rings A_1, A_2 such that A_1 is a left QF-3 and L is a faithful, projective, injective left ideal in A_1 and A_2 is the annihilator ideal of L in A . In particular if A is a left QF-3, $A = \sum \oplus A_i$ as a ring and there exists a primitive idempotent e_i in A_i such that $A_i e_i$ is a unique minimal, faithful, projective injective ideal and e_i is uniquely determined up to isomorphism with property $e_i N = (0)$, where N is the radical of A .*

Proof. Since A is semi-primary, $L \approx \sum \oplus Ae_i$, e_i primitive idempotent. As before we may assume that A coincides with its basic ring. Let $T_n(\Delta_i; M_{i,j})$ be a normal right representation of A as a g.t.a. matrix ring. We assume $e_i = T_n(o, \dots, 1_i, o, \dots; o)$. Let $C^*(i) = i \cup C(i) \equiv \{i = i(o) < i(1) < \dots < i(k) = t\}$. For $j \in C^*(i)$ $(0) = M_{t,j} \supseteq M_{t,i(s)} M_{i(s),j}$ and $(0) = M_{j,i} \supseteq M_{j,i(p)} M_{i(p),i}$. Hence $M_{i(s),j} = M_{j,i(p)} = 0$ any $i(s) < j$ and $i(p) < j$, respectively. Put $E = \sum_{j \in C^*(i)} e_j$ and $E' = 1 - E$. Then the above facts imply

that $M_{k,k'} \subseteq EAE + E'AE'$ for all k, k' . Hence $A = EAE \oplus E'AE'$ as a ring and $EAE \supseteq Ae_i$. Furthermore, $EAE \approx T_n(\Delta_{i(j)}; M_{i(j), i(s)})$ and $M_{i(2), i(1)}, \dots, M_{i(n'), i(1)}$ are non-zero. Since Ae_i is EAE -injective, $Ae_i = T_n(\Delta_{i(1)}, 0, \dots, 0; M_{i(j), i(s)} = (0)$ if $s \neq 1$) by the fact (5) in the proof of Lemma 5. Hence, Ae_i is faithful. Therefore EAE is a left QF-3 ring. It is clear that $E'AE'$ is the annihilator of Ae_i . Repeating the above argument we have the first part of Theorem 1. The second one is an immediate consequence from the first part and Proposition 3.

REMARK 1. Let $A = T_n(\Delta_i; M_{i,j})$ be a partially PP-ring and indecomposable basic QF-3 ring. Then we have obtained in the above proof that $M_{i,1} \neq (0)$ for all i and hence, $M_{n,i} \neq (0)$ for all i by Lemma 5.

REMARK 2. We shall see later that the set of those indecomposable ideals Ae_i coincide with the set of indecomposable injective left ideals in A .

Next, we shall consider a QF-3 and semi-primary PP- (resp. hereditary) ring. We restrict ourselves again to a case of basic ring.

Lemma 7. *Let A be an indecomposable basic ring and semi-primary partially PP-ring. $A = T_n(\Delta_i; M_{i,j})$ be a normal right representation of A as a g.t.a. matrix ring. Then $[M_{n,i} : \Delta_n] = [M_{i,1} : \Delta_1] = 1$ for all i . Furthermore, if A is hereditary then $[M_{i,j} : \Delta_i] = [M_{i,j} : \Delta_1] = 1$ if $M_{i,j} \neq (0)$.*

Proof. We use the same notation as above. Since $T_n(\Delta_i; M_{i,j})$ is a normal representation, Ae_1 is A -injective. From Remark 1 we know $M_{n,i} \neq (0)$ and $M_{i,1} \neq (0)$ for all i . If $[M_{n,1} : \Delta_n] \geq 2$, then we have two independent elements x, y in $M_{n,1}$ over Δ_n . Let φ be a linear mapping of $M_{n,1}$ into itself such that $\varphi(x) = x$, $\varphi(y) = 0$. Then φ is A -homomorphism of $M_{n,1}$ to Ae_1 . Since Ae_1 is injective, this is a contradiction. If $[M_{n,1} : \Delta_1] \geq 2$, then there exist two independent elements x', y' in $M_{n,1}$ over Δ_1 . Let ψ be a linear mapping of $M_{n,1} = \Delta_n x'$ to itself such that $\psi(x') = y'$. Injectivity of Ae_1 implies that there exists an element z in Δ_1 such that $x'z = y'$. This contradicts a fact of independency. Since $M_{n,1} \supseteq M_{n,i}M_{i,1}$, $[M_{n,i} : \Delta_n] \leq [M_{n,i} : \Delta_n] = 1$ and $[M_{i,1} : \Delta_1] \leq [M_{n,1} : \Delta_1] = 1$. We assume that A is hereditary. Then $M_{n,i} \otimes_{\Delta_i} M_{i,1} \approx M_{n,i}M_{i,1}$ as $\Delta_n - \Delta_1$ module by [4], Theorem 1. Hence $1 = [M_{n,1} : \Delta_n] \geq [M_{i,1} : \Delta_i]$. If $M_{i,j} \neq (0)$, $(0) \neq M_{i,j}M_{j,1} \subseteq M_{i,1}$. Hence, $1 = [M_{i,1} : \Delta_i] \geq [M_{i,j} : \Delta_i]$. Similarly, we have $[M_{i,j} : \Delta_j] = 1$.

Theorem 2. *If A is a left QF-3 and semi-primary hereditary ring, then A is a directsum of rings whose basic ring is a ring of triangular matrices over division rings. And hence, A is right QF-3 and A satisfies*

minimal conditions. The converse is also true, (see Remark 3 below).

Proof. We assume that A is an indecomposable, basic ring. Then $A = T_n(\Delta_i; M_{i,j})$ and $M_{i,1} \neq (0)$ and $M_{n,i} \neq (0)$ for all i from Remark 1. We shall show that $M_{i,j} \neq (0)$ for all $i < j$. We quote the same notations of [4], Theorem 1. Since $M_{2,1} \neq (0)$, we assume that $M_{j,k} \neq (0)$ for any $j \leq i$. If $M_{i+1,i} = M_{i+1,i-1} = \dots = M_{i+1,t} = (0)$ and $M_{i+1,t-1} \neq (0)$, then $\bar{M}_{i+1,t-1} = M_{i+1,t-1} / \sum_{k=t}^i M_{i+1,k} M_{k,t-1} = M_{i+1,t-1}$. On the other hand, $\bar{M}_{t,t-1} = M_{t,t-1} \neq (0)$, since $t \leq i$. However, $M_{n,i+1} \bar{M}_{i+1,t-1} \neq (0)$, $M_{n,t} \bar{M}_{t,t-1} \neq (0)$ and $M_{n,i+1} \bar{M}_{i+1,t-1} \cap M_{n,t} \bar{M}_{t,t-1} = (0)$ by [4], Theorem 1. Which contradicts a fact $[M_{n,t-1} : \Delta_n] = 1$. Therefore, we know $M_{i+1,i} \neq (0)$. $M_{i+1,k} \supseteq M_{i+1,i} M_{i,i-1} \dots M_{k+1,k} \neq (0)$. Thus we can prove the fact $M_{i,j} \neq (0)$ for all $i > j$ by induction. Since $M_{i,j} \neq (0)$, $[M_{i,j} : \Delta_i] = [M_{i,j} : \Delta_j] = 1$ by Lemma 7. Therefore, A is isomorphic to a ring of triangular matrices by [4], Lemma 12. Thus, we have proved Theorem 2.

In the above proof if we replace $M_{i+1,t-1}$ by a non-zero element x in $M_{i+1,t-1}$ and $M_{t,t-1}$ by a non-zero element y in $M_{t,t-1}$, then $M_{n,i+1}x$ and $M_{n,t}y$ are not zero by Lemma 2, provided A is a PP-ring. Since $[M_{n,t-1} : \Delta_n] = 1$ by Lemma 7, $M_{n,i+1}x = M_{n,t}y$. This contradicts [3], Proposition 1. Hence, we have similarly

Proposition 4. *Let A be a left QF-3 and semi-primary PP-ring. We assume A is indecomposable. Then A is isomorphic to a g.t.a. matrix ring $T_n(S_i; \mathfrak{M}_{i,j})$ over simple ring S_i and each component of $\mathfrak{M}_{i,j}$ in (2) is non-zero. Therefore, $T_n(S_i; \mathfrak{M}_{i,j})$ is a right and left normal representation of A as a g.t.a. matrix ring and the nilpotency of the radical is equal to n . Let $S_i \approx (\Delta_i)_n$, Δ_i division ring. Then $\Delta_1 \approx \Delta_n$ and Δ_i is isomorphic into $\Delta_1 \approx \Delta_n$. Furthermore, we assume that A is K -algebra with finite dimension. Then A is hereditary if and only if $\Delta_i \approx \Delta_1$ for all i .*

REMARK 3. Theorem 2 says that the class of the QF-3 and semi-primary hereditary rings coincides with the class of the rings of directsum of g.t.a. matrix rings of the following form.

Let Δ be a division ring and $\Delta(n \times m)$ the module of rectangular matrices of $(n \times m)$ -form over Δ and it is regarded as $(\Delta)_n - (\Delta)_m$ module.

$$A = \begin{pmatrix} (\Delta)_{n_1} & 0 \\ \Delta(n_2 \times n_1) & (\Delta)_{n_2} \\ \dots\dots\dots & \dots\dots\dots \\ \Delta(n_r \times n_1) & \Delta(n_r \times n_2) \dots (\Delta)_{n_r} \end{pmatrix}$$

We consider the converse of the first half of Lemma 7.

Proposition 5. *Let $A = T_n(\Delta_i; M_{i,j})$ be a g.t.a. matrix ring over division ring Δ_i . If A is a partially PP-ring, then Ae_1 is A -injective and $M_{i,1} \neq (0)$ and $M_{n,i} \neq (0)$ for all i if and only if $[M_{i,1} : \Delta_1] = [M_{n,1} : \Delta] = 1$. Conversely if Ae_1 is faithful and $[M_{i,1} : \Delta_1] = 1$, then A is a partially PP-ring, where $e_1 = T_n(1, 0, \dots; 0)$.*

Proof. We assume that A is a partially PP-ring. We have proved "only if" part of the first half in the proof of Lemma 7. We shall prove "if" part. Since $[M_{i,1} : \Delta_1] = 1$, we put $M_{i,1} = x_i \Delta_1$ (x_i = the identity element of Δ_1). Since $[M_{n,1} : \Delta_1] = [M_{n,1} : \Delta_n] = 1$, there exists an isomorphism φ of Δ_1 to Δ_n such that $x_n \delta = \delta^\varphi x_n$ for $\delta \in \Delta_1$. It is clear that $\text{Hom}_{\Delta_1}(M_{i,1}, M_{n,1}) = \Delta_n f_i$, where $f_i \in \text{Hom}_{\Delta_1}(M_{i,1}, M_{n,1})$ such that $f_i(x_i) = x_n$, (for $f \in \text{Hom}_{\Delta_1}(M_{i,1}, M_{n,1})$ $f(x_i) = x_n \delta = \delta^\varphi x_n = (\delta^\varphi f_i)(x_i)$). On the other hand $M_{n,i} \approx M_{n,1} x_i = M_{n,1}$ by the assumption $[M_{n,1} : \Delta_n] = 1$ and Lemma 2. Hence, there exists a unique element g_i in $M_{n,i}$ such that $g_i x_i = x_n$, (g_n = the identity element in Δ_n). Therefore, $\text{Hom}_{\Delta_1}(M_{i,1}, M_{n,1})$ coincides with the multiplications of elements in $\Delta_n g_i$ from the left side. Let $M_{i,1}^* = \{f \in \text{Hom}_{\Delta_1}(Ae_1, \Delta_1) \mid f(M_{j,i}) = (0) \text{ for } j \neq i\}$. Then $\text{Hom}_{\Delta_1}(Ae_1, \Delta_1) = \sum_{i=1}^n \oplus M_{i,1}^*$ as a module. We have isomorphisms $\theta_i : M_{n,i} = \Delta_n g_i \rightarrow M_{i,1}^*$, by setting

$$\theta_i(\delta g_i)(x_i) = \delta^{\varphi^{-1}} \quad \text{and} \quad \theta_i(\delta g_i)(x_j) = 0 \quad \text{for } j \neq i.$$

Hence, we have an isomorphism Θ of $e_n A$ to $\text{Hom}_{\Delta_1}(Ae_1, \Delta_1)$ via θ_i as a module. We shall show that Θ is A -isomorphic. Let $\theta_i(\delta g_i) = f \in M_{i,1}^*$ and $m_{k,l} \in M_{k,l}$. Then $f m_{k,l} : M_{l,1} \xrightarrow{m_{k,l}} m_{k,l} M_{l,1} \xrightarrow{f} \Delta_1$. Hence if $k \neq i$, $f m_{k,l} = g_i m_{k,l} = 0$. Let $k = i$. Since $m_{i,l} x_l \in M_{i,1} = x_i \Delta_1$, $m_{i,l} x_l = x_i \delta$ for some $\delta_l \in \Delta_1$. Hence, $\theta_i^{-1}(f m_{i,l}) = \delta \delta_1^\varphi g_l$. On the other hand, $\delta g_i m_{i,l} x_l = \delta g_i x_i \delta_l = \delta x_n \delta_l = \delta \delta_1^\varphi x_n = \delta \delta_1^\varphi g_l x_l$. Hence $\delta g_i m_{i,l} = \delta \delta_1^\varphi g_l$ by Lemma 2. Therefore, Θ is A -isomorphic. Hence $e_n A$ is A -injective. It is clear that $\text{Hom}_{\Delta_1}(Ae_1, \Delta_1)$ is A -faithful (cf. the proof of Proposition 1). Thus, A has a faithful injective, projective right ideal $e_n A$. If we replace a position of $M_{i,1}$ by $M_{n,i}$ in the above, then we have similarly that A is a left QF-3 ring. Next, we assume that Ae_1 is faithful and $[M_{i,1} : \Delta_1] = 1$. Let $x_{i,j}, y_{j,k}$ be in $M_{i,j}, M_{j,k}$, respectively. If $x_{i,j} x_{j,k} = 0$, $(0) = x_{i,j} y_{j,k} M_{k,1} = x_{i,j} (y_{j,k} M_{k,1})$. Since Ae_1 is faithful, $y_{j,k} M_{k,1} \neq (0)$ if $y_{j,k} \neq 0$. Hence, $y_{j,k} M_{k,1} = M_{j,1}$. We have shown that $M_{i,k} \otimes_{\Delta_k} y_{k,j} \approx M_{i,k} y_{k,i}$. Therefore, A is a partially PP-ring by [3], Lemma 5.

Similarly to Theorem 2 we have

Theorem 3. *Let A be a semi-primally PP-ring. A is a left QF-3 ring if and only if its basic ring is of the form $T_n(\Delta_i; M_{i,j})$ such that*

$[M_{i,1} : \Delta_1] = [M_{n,i} : \Delta_n] = 1$. In this case A is also a right QF-3 ring.

Proof. It is clear from Theorem 1 and Proposition 5.

Finally, we shall generalize Mochizuki's result [6], Theorem 2.3 in a case of semi-primary partially PP-ring.

Let A be a basic QF-3 ring and semi-primary partially PP-ring. We assume that A is indecomposable. Then $A \approx T_n(\Delta_i ; M_{i,j})$ and $[M_{i,1} : \Delta_1] = [M_{n,i} : \Delta_n] = 1$ for all i by Lemma 7. Hence, we may assume that $\Delta_1 = \Delta_n = \Delta$ and Δ_i is contained in Δ . Let $L = T_n(\Delta, 0, \dots, 0 : M_{i,j} = (0)$ if $j \neq 1$). Then L is a unique minimal faithful projective, injective left A -module. Let $B = \text{Hom}_\Delta(L, L)$. Then $B = (\Delta)_n$. Let $B_{i,j} = \{f \in B, f(M_{j,1}) = M_{i,1}, f(M_{k,1}) = (0) \text{ for } k \neq j\}$. Then $B_{i,j} \cap A \supseteq M_{i,j}$, where A is regarded as a subring of B , since L is faithful. By virtue of this imbedding we can regard $M_{i,j}$ as a $\Delta_i - \Delta_j$ submodule in Δ . In such a setting, we have

$$B = \begin{pmatrix} \Delta & \Delta & \cdots & \Delta \\ \Delta & \Delta & \cdots & \Delta \\ \cdots & \cdots & \cdots & \cdots \\ \Delta & \Delta & \cdots & \Delta \end{pmatrix} \supseteq A = \begin{pmatrix} \Delta & & & \\ \Delta & \Delta_2 & & 0 \\ & \Delta & M_{3,2} & \Delta_3 \cdots \Delta_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta & \Delta & \cdots & \cdots & \Delta & \Delta \end{pmatrix} \supset L = \begin{pmatrix} \Delta & \\ \vdots & 0 \\ \vdots & \\ \vdots & \\ \Delta & \end{pmatrix},$$

where $M_{i,j}$ is a $\Delta_i - \Delta_j$ submodule in Δ and Δ_i is a subdivision ring of Δ . Since $B \approx L^{(n)}$ ⁵⁾ as a left A -module, B is left A -projective and injective.

Lemma 8. Let A and L be as above. Injective envelope of indecomposable left ideal Ae_i is isomorphic to L .

Proof. Since $M_{i,1} \neq (0)$, we can take $x \neq 0$ in $M_{i,1}$. Then $Ae_i x \approx Ae_i$ by Lemma 2. Since $Ae_i x \subseteq L$ and L is indecomposable, L is an injective envelope of $Ae_i x$.

We note that the double commutator ring of module which is a directsum of n -copies of a module M coincides with that ring of M up to isomorphism.

Summarizing the above we have

Theorem 4. Let A be a semi-primary partially PP-ring and e be an idempotent such that Ae is a faithful projective, injective left ideal. Then the following facts hold.

(1) Both the commutator ring eAe and the double commutator ring $B = \text{Hom}_{eAe}(Ae, Ae)$ of Ae are semi-simple.

5) $L^{(n)}$ means a directsum of n -copies of L .

- (2) B is an A - A module which is both the left and right injective envelope of A and left and right A -projective.
- (3) If A is hereditary, then A is a generalized uniserial ring with minimal conditions.

Corollary. *Let A be as above. If L is an indecomposable A -injective left ideal in A , then L is projective and $L \approx Ae$, $eN = (0)$.*

Proof. We may assume A is indecomposable. Let M be a minimal left ideal contained in L , since A is semi-primary, (see [5], p 1106). Then an injective envelope of M' is contained in L and hence L is isomorphic to an injective envelope of M' . Therefore, B in Theorem 4 contains an isomorphic image of L as direct summand by the proof of Theorem 3.2 in [5]. Hence, L is A -projective by Theorem 4. The second part is clear from Theorem 2.

We conclude this paper with the following examples.

EXAMPLE. Let K be a field and L proper extension of K . We put

$$A = \begin{pmatrix} L & 0 & 0 \\ L & K & 0 \\ L & L & L \end{pmatrix},$$

where L at $(2, 1)$ -component is regarded as K - L module and L at $(3, 2)$ -component as L - K module. Since a natural mapping $L \otimes_K L \rightarrow L$ is not monomorphic, A is not hereditary by [4], Theorem 1. It is clear that $\begin{pmatrix} L & 0 & 0 \\ L & 0 & 0 \end{pmatrix}$ is a faithful, projective, injective A -module and A is a PP-ring by Proposition 5 and [3], Proposition 1. Hence, A is a QF-3 and PP-ring and not hereditary. If $[L:K] = \infty$ A does not satisfy the minimal conditions.

Let

$$A = \begin{pmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & 0 & K & 0 \\ K & K & K & K \end{pmatrix},$$

then A is a QF-3 and partially PP-ring by [3], Lemm 5. However, A is not a PP-ring and hence, not hereditary.

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