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QF-3 AND SEMI-PRIMARY PP-RINGS I

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Recently the author has given a characterization of semi-primary hereditary ring in [4]. Furthermore, those results in [4] have been extended to a semi-primary PP-ring in [3], (a ring A is called a *left PP-ring if every principal left ideal in A is A-projective*).

This short note is a continuous work of [3] and [4]. Let K be a field and A an algebra over K with finite dimension. A is called a *QF-3 algebra if A has a unique minimal faithful representation* ([10]). Mochizuki has considered a hereditary QF-3 algebra in [6].

In this note we shall study a PP-ring with minimal condition or of semi-primary. To this purpose we generalize a notion of QF-3 algebra in a case of ring. We call *A left* (*resp. right*) *QF-3 ring if A has a faithful, injective, projective left* (*resp. right*) *ideal*, (cf. [5], Theorems 3.1 and 3.2).

Let $1=\sum E_i$ be a decomposition of the identity element 1 of a semi-primary ring A into a sum of mutually orthogonal idempotents such that E_i modulo the radical N is the identity element of simple component of A/N. *If Ax is A-projective for all $x \in E_iAE_j$*, we call A a *partially PP-ring*, (see [3], §2). Such a class of rings contains properly classes of semi-primary hereditary rings and PP-rings.

Our main theorems are as follows: *Let A be directly indecomposable and a left QF-3 ring and semi-primary partially PP-ring. Then* 1) *there exists a unique primitive idempotent e in A* (*up to isomorphism*) *such that $eN=(0)$ and every indecomposable left injective ideal in A is faithful, projective and isomorphic to Ae. Furthermore, A is a right QF-3 ring.* 2) *Let $B=\mathrm{Hom}_{eAe}(Ae, Ae)$, where Ae is regarded as a right eAe-module. Then eAe is a division ring and $B=(eAe)_n$*[1]. *B is a left and right injective envelope of A as an A-module and B is A-projective. Furthermore, if A is hereditary, then A is a generalized uniserial ring whose basic ring is of triangular matrices over a division ring.* (Mochizuki proved in [6] the above fact 2) in a case of hereditary algebra over a field with finite dimension).

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1) $(A)_n$ means a ring of matrices over a ring A with degree n.

We always consider a ring A with identity element 1 and every A-module is unitary.

1. Preliminary Lemmas.

In this paper we make use of some results in [3], [4] very often and we shall here summarize them.

Let $1=\sum_{i=1}^{t}E_i$ be a decomposition of 1 into a sum of mutually orthogonal idempotents E_i. We assume that $E_iAE_j=(0)$ for $i<j$ and E_iAE_i is semi-simple with minimal conditions. Then

$$(1)\qquad \begin{aligned} A = {} & S_1 \\ & \oplus E_2AE_1 \oplus S_2 \\ & \cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots \\ & \oplus E_tAE_1 \oplus \cdots \oplus E_tAE_{t-1} \oplus S_t \end{aligned}$$

as a module, where $S_i=E_iAE_i$.

By $T_t(S_i\,;\,\mathfrak{M}_{i,j}\equiv E_iAE_j)$ we denote the above expression, and we call it a *generalized triangular matrix ring over* S_i (briefly g.t.a. matrix ring).

Let $S_i=\sum_{j=1}^{\rho(i)}\oplus T_{i,j}$: $T_{i,j}$ is a simple ring. Then we can easily check that

$$(2)\qquad \mathfrak{M}_{p,q} \approx \begin{pmatrix} M_{1,1} & \cdots M_{1,\rho(q)} \\ M_{2,1} & \cdots M_{2,\rho(q)} \\ \cdots\cdots & \cdots\cdots \\ M_{\rho(p),1} & \cdots M_{\rho(p),\rho(q)} \end{pmatrix}$$

as a S_p-S_q module, where $M_{l,s}$ is a $T_{p,l}-T_{q,s}$ module and the operations of S_p and S_q are naturally defined on the right side of (2).

From [3], p. 160 and the proof of [4], Proposition 10 we have

Lemma 1. *Let A be a semi-primary partially PP-ring. Then A is isomorphic to $T_t(S_i\,;\,\mathfrak{M}_{i,j})$ such that every row of (2) is non-zero and AE_1 is a faithful A-module. Furthermore, let $\{e_i\}$ be a set of non-isomorphic mutually orthogonal primitive idempotents e_i such that $e_iN=(0)$, then $E_1\approx\sum e_i$ and every faithful projective A-module contains AE_1 as a direct summand, where $E_1=T_t(1_1, o, \cdots, o\,;\,o)$ and 1_1 is the identity element in S_1.*

If A is isomorphic to $T_t(S_i\,;\,\mathfrak{M}_{i,j})$ as in Lemma 1, we call $T_t(S_i\,;\,\mathfrak{M}_{i,j})$ *a normal right representation of A as a g.t.a. matrix ring.*

Lemma 2. *Let A be as in Lemma 1. Then $\mathfrak{M}_{i,j}\otimes_{S_j}S_jx\approx\mathfrak{M}_{i,j}x$ and $yS_i\otimes_{S_i}\mathfrak{M}_{i,k}\approx y\mathfrak{M}_{i,k}$ for $x\in\mathfrak{M}_{j,t}$ and $y\in\mathfrak{M}_{l,i}$.*

See [3], Lemma 5.

Let K be a field and A a K-algebra with finite dimension. Jans showed in [5] that A has a unique minimal faithful representation if and only if A has faithful, projective, injective left ideal L. Since L is projective, we know that $\mathrm{Hom}_K(L, K)$ is faithful, projective, injective right A-module.

We are interested in a case of a triangular matrices with minimal conditions. We shall generalize the above fact in this case.

Now we assume that A is a g.t.a. matrix ring over semi-simple rings S_i; $A=T_n(S_i; M_{i,j})$.

If e is a primitive idempotent, then eAe is division ring. By B we denote eAe. Since A satisfies the minimal conditions, $[Ae:B]_r$[2] $< \infty$ by [4], §5.

The following lemma is well known in a case of algebra over a field.

Lemma 3. *Let A, B and e be as above. If Ae is A-injective, then* $\mathrm{Hom}_B(Ae, B)$ *is right A-projective and injective.*

Proof. For a finitely generated left A-module M we have $\mathrm{Hom}_B(Ae, B)\otimes_A M \approx \mathrm{Hom}_B(\mathrm{Hom}_A(M, Ae), B)$ from [1], p. 120, Proposition 5.3. This isomorphism implies that $\mathrm{Hom}_B(Ae, B)$ is right A-flat. Hence, $\mathrm{Hom}_B(Ae, B)$ is A-projective by [2]. On the other hand, from an isomorphism: $\mathrm{Hom}_A(N, \mathrm{Hom}_B(Ae, B)) \approx \mathrm{Hom}_B(N\otimes_A Ae, B)$ in [1], p. 120 for a right A-module N we know that $\mathrm{Hom}_B(Ae, B)$ is A-injective, since Ae is A-flat.

Proposition 1[3]**.** *Let A be a g.t.a. matrix ring over semi-simple rings with minimal conditions. If A has a faithful, injective, projective left ideal, then A has a faithful, injective, projectve right ideal.*

Proof. Let L be a faithful, injective, projective left ideal $L=\sum\oplus Ae_i$; e_i primitive idempotent. Put $B_i=e_iAe_i$ and $C_i=\mathrm{Hom}_{B_i}(Ae_i, B_i)$. Then C_i is right A-projective and injective. Let $x\neq 0$ in A. Since L is faithful, $xAe_i\neq 0$ for some i. Since B_i is a division ring, there exists g in C_i such that $g(xAe_i)\neq(0)$. Therefore, if we put $R'=\sum\oplus C_i$, then R' is a faithful, projective, right A-module. Since $C_i\approx\sum\oplus e_i'A$, we have a faithful, projective, injective right ideal.

If A has a faithful, projective, injective left (resp. right) ideal, then we call A a *left* (*resp. right*) *QF*-3 *ring.*

If A is a g.t.a. matrix ring over semi-simple rings with minimal conditions, then a left QF-3 ring is a right QF-3 and conversely by

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2) $[Ae : B]_r$ means the dimension of Ae as a right B-module.

3) Added in proof. We shall show in [12] that if A satisfies minimum conditions, then A is left QF-3 if and only if A is right QF-3.

Proposition 1. However, we do not know whether it is true in a general ring with minimal conditions.[3)]

We quote here the concept of basic ring following Osima [8].

Let

$$(3) \qquad 1=\sum_{i=1}^{n}\sum_{j=1}^{\rho(i)}e_{i,j}$$

be a decomposition of the identity element 1 of A into the sum of mutually orthogonal primitive idempotents such that $e_{i,j}\approx e_{h,k}$ if and only if $i=h$.

For each i we denote $e_{i,1}$ by e_1^*. Let $e^*=\sum_{i=1}^{n}e_1^*=\sum_{i=1}^{n}e_{i,1}$. We call $A^*=e^*Ae^*$ the *basic ring* of A relative to the decomposition (3). We can find elements $c_{i,1j}\in e_{i,1}Ae_{i,j}$ and $c_{i,j1}\in e_{i,j}Ae_{i,1}$ such that $c_{i,1j}c_{i,j1}=e_{i,1}$ and $c_{i,j1}c_{i,1j}=e_{i,j}$. Put $c_{i,jk}=c_{i,j1}c_{i,1k}$. We may assume $e_{i,11}=e_{i,1}$. Then we have

$$c_{i,jk}c_{i',j'k'}=\delta_{i,i'}\delta_{k,j'}c_{i,jk'}\,.$$

A can be written

$$A=\sum_{i=1}^{n}\sum_{j=1}^{\rho(i)}\sum_{h=1}^{n}\sum_{k=1}^{\rho(h)}c_{i,j1}A^*c_{h,1k}\,.$$

The following observation is a direct proof of [7], Lemma 7.2. Let M^* be a left A^*-module. We put

$$M=E(M^*)=\sum_{i=1}^{n}\sum_{j=1}^{\rho(i)}\oplus c_{i,j1}e_i^*M^*\,,$$

where $c_{i,j1}e_i^*M^*\approx e_1^*M^*$ as a module. We can directly check that M is a left A-module and $e^*M=M^*$. Conversely, let M be a left A-module. Then $M=\sum_{i=1}^{n}\sum_{j=1}^{\rho(i)}\oplus e_{i,j}M$ and $M^*=\sum_{i=1}^{n}\oplus e_{i,1}M$ is a left A^*-module. We define a mapping φ of M to $E(M^*)$ by setting

$$\varphi(e_{i,j}m_{i,j})=c_{i,j1}e_{i,1}m_{i,j}\,.$$

Then we can easily check that $M\approx E(M^*)$ as a left A-module.

Let M and N be left A-modules. Then

$$\mathrm{Hom}_A(N, M)=\mathrm{Hom}_A(\sum c_{i,j1}N, \sum c_{i,j1}M)\,.$$

For elements $f_{i,11}\in\mathrm{Hom}_{e_i^*Ae_i^*}(c_{i,11}N, c_{i,11}M)$ and $f_{i,j1}\in\mathrm{Hom}_{e_{i,j}Ae_{i,j}}(c_{i,j1}N, c_{i,j1}M)$ we consider a diagram:

$$(4) \qquad \begin{array}{ccc} c_{i,11}N & \xrightarrow{f_{i,11}} & c_{i,11}M \\ \downarrow{\scriptstyle c_{i,j1}} & f_{i,j1} & \downarrow{\scriptstyle c_{i,j1}}\,. \\ c_{i,j1}N & \longrightarrow & c_{i,j1}M\,. \end{array}$$

Then we can easily see that the diagram (4) is commutative for $f_{i,j1}=f|c_{i,j1}N$ *and* $f\in\mathrm{Hom}_A(N, M)$. Converely let M^* and N^* be left A^*-modules. For $f_i^*=f^*|e_i^*N$ of f^* in $\mathrm{Hom}_{A^*}(N^*, M^*)$ we define $f_{i,j1}$ such that $f_{i,11}=f_i^*$ and the diagram (4) is commutative. Then we can show that $f=\sum f_{i,j1}$ is in $\mathrm{Hom}_A(N, M)$. Thus we have

Lemma 4. *A is a left QF-3 ring if and only if so is a basic ring of A.* (cf. [11], Proposition 5).

2. Main theorems.

In this section we consider a semi-primary QF-3 partially PP-ring A.

From Lemma 4, [4], Corollary 1 and [3], Remark 1 and Lemma 4 we have

Proposition 2. *If A is a semi-primary left QF-3 and hereditary (resp. PP- or partially PP-) ring, then so is a basic ring of A. In the case of hereditary ring the converse is true.*

By N we denote the redical of A.

Proposition 3. *Let A be a left QF-3 and partially PP-ring and semi-primary. Let $\{e_i\}$ be a set of mutually orthogonal primitive non-isomorphic idempotents such that $e_iN=(0)$. Then $L=\sum\oplus Ae_i$ is a unique minimal left faithful, projective, injective A-module.*

Proof. It is clear from the definition and Lemma 1.

From Proposition 2 we may first restrict ourselves in a case where A coincides with its basic ring. Then $A/N=\sum\oplus\Delta_i$; Δ_i a division ring.

Let A be a g.t.a. matrix ring over division rings Δ_i; $T_n(\Delta_i; M_{i,j})$. We put $C(i)=\{k\,|\,M_{k,i}\neq(0)\}$ and $R(j)=\{k\,|\,M_{j,k}\neq(0)\}$.

Lemma 5.[4)] *Let A be as in Proposition 3 and $A=T_n(\Delta_i; M_{i,j})$. We assume Ae_i is A-injective. If t is the maximal index in $C(i)$, then $C(i)=R(t)$, where $e_i=T_n(o, o, 1_i, o, o; o)$ and 1_i is the identity element of Δ_i.*

Proof. Put $C(i)\equiv\{i(1)<i(2)<\cdots<i(k)=t\}$. Then $M_{a,i}=(0)$ if $a\notin C(i)$. We first show that

$$(5)\qquad M_{t,a}=(0)\qquad a\notin C(i)$$

If $M_{t,a}\neq(0)$, we take $x\neq0$ in $M_{t,a}$ and $y\neq0$ in $M_{t,i}$. Since A is partially PP-ring, for any element z in A $zx=0$ impleis $z\in A(1-e_t)$ by Lemma 2. Hence, $zy=0$. Therefore, a mapping φ of Ax to $Ay\subseteq Ae_i$: $zx\to zy$ is homomorphism. Since Ae_i is A-injective, there exists an element w in Ae_i such that $y=xw$ by [1], p. 8, Theorem 3.2. Therefore, w might be

4) Added in proof. We shall give a simple proof in [12].

in $M_{a,i}$. Since φ is non-zero, w is not zero, which contradicts the fact $M_{a,i}=(0)$. We need a lemma to complete the proof.

Lemma 6. *Let A and $t=i(k)$ be as above. Then there exists an index $g=g(l)$ such that $M_{g\ i(l)}\neq(0)$ for any l, $1\leqslant l<k$.*

Proof. We assume $M_{g,i(l)}=(0)$ for all g and some l. Then $M_{i(l),i}$ is a non-zero left ideal contained in Ae_i. Furthermore, $M_{g',t}=(0)$ for all g', because if $M_{g',t}\neq(0)$ (and hence $g'>t$), then $(0)\neq M_{g'\ t}M_{t,i}\subseteq M_{g',i}$. Hence, $Q=M_{i(l),i}\oplus M_{t,i}$ is a left ideal contained in Ae_i. Let $x\neq 0$ in Δ_i. Then a mapping ψ of Q to Ae_i defined by $\psi(n+m)=nx$ for $n\in M_{i(l),i}$, $m\in M_{t,i}$ is A-homomorphism. Since Ae_i is injective, there exists an element z in Ae_i such that $nz=nx$ and $mz=0$. This is a contradiction, because $n=M_{i(1),i}$, $m\in M_{t,i}$. Q.E.D.

We continue the prove of Lemma 5. We shall show that $M_{t,i(s)}\neq(0)$ for $1\leq s\leq k$. We have $M_{b,i}=(0)$ for $i(k-1)<b<t$, $t<b$ by the definition of $C(i)$ and t. If $M_{l,i(k-1)}\neq(0)$ for an integer l such that $i(k-1)<l\neq t$ $=i(k)$ then $(0)\neq M_{l,i(k-1)}M_{i(k-1),i}\subseteq M_{l,i}$. Therefore, $M_{l,i(k-1)}=(0)$ for all $l\neq t$. Hence, we know $M_{t,i(k-1)}\neq(0)$ from Lemma 6. We assume $M_{t,i(c)}\neq(0)$ for integer $c>$ a fixed integer d. By the same argument as above we obtain $M_{q,i(d)}=(0)$ for $q\neq i(r)$; $d<r<k'$. Hence, we know by Lemma 6 that there exists an integer $f\,(>d)$ such that $M_{i(f),i(d)}\neq(0)$. Therefore, $(0)\neq M_{t,i(f)}M_{i(f),i(d)}\subseteq M_{t,i(d)}$. Thus we can prove Lemma 5 by induction.

Theorem 1.[4] *Let A be a semi-primary, partially PP-ring. If A contains a finitely generated projective, injective left ideal L, then A is a directsum of two rings A_1, A_2 such that A_1 is a left QF-3 and L is a faithful, projective, injective left ideal in A_1 and A_2 is the annihirator ideal of L in A. In particular if A is a left QF-3, $A=\sum\oplus A_i$ as a ring and there exists a primitive idempotent e_i in A_i such that A_ie_i is a unique minimal, faithful, projective injective ideal and e_i is uniquely determined up to isomorphism with property $e_iN=(0)$, where N is the radial of A.*

Proof. Since A is semi-primary, $L\approx\sum\oplus Ae_i$, e_i primitive idempotent. As before we may assume that A coincides with its basic ring. Let $T_n(\Delta_i\,;\,M_{i,j})$ be a normal right representation of A as a g.t.a. matrix ring. We assume $e_i=T_n(o,\cdots,1_i,o,\cdots;o)$. Let $C^*(i)=i\cup C(i)\equiv\{i=i(o)$ $<i(1)<\cdots<i(k)=t\}$. For $j\notin C^*(i)$ $(0)=M_{t,j}\supseteq M_{t,i(s)}M_{i(s),j}$ and $(0)=M_{j,i}$ $\supseteq M_{j,i(p)}M_{i(p),i}$. Hence $M_{i(s),j}=M_{j,i(p)}=0$ any $i(s)<j$ and $i(p)<j$, respectively. Put $E=\sum_{j\in C^*(i)}e_j$ and $E'=1-E$. Then the above facts imply

that $M_{k,k'}\subseteq EAE+E'AE'$ for all k, k'. Hence $A=EAE\oplus E'AE'$ as a ring and $EAE\supseteq Ae_i$. Furthermore, $EAE\approx T_{n'}(\Delta_{i(j)}; M_{i(j),i(s)})$ and $M_{i(2),i(1)}, \cdots, M_{i(n'),i(1)}$ are non-zero. Since Ae_i is EAE-injective, $Ae_i=T_{n'}(\Delta_{i(1)}, 0, \cdots, 0; M_{i(j),i(s)}=(0)$ if $s\neq 1)$ by the fact (5) in the proof of Lemma 5. Hence, Ae_i is faithful. Therefore EAE is a left QF-3 ring. It is lear that $E'AE'$ is the annihitator of Ae_i. Repeating the above argument we have the first part of Theorem 1. The second one is an immediate consequence from the first part and Proposition 3.

REMARK 1. Let $A=T_n(\Delta_i; M_{i,j})$ be a partially PP-ring and indecomposable basic QF-3 ring. Then we have obtained in the above proof that $M_{i,1}\neq(0)$ for all i and hence, $M_{n,i}\neq(0)$ for all i by Lemma 5.

REMARK 2. We shall see later that the set of those indecomposable ideals A_ie_i coincide with the set of indecomposable injective left ideals in A.

Next, we shall consider a QF-3 and semi-primary PP- (resp. hereditary) ring. We restrict ourselves again to a case of basic ring.

Lemma 7. *Let A be an indecomposable basic ring and semi-primary partially PP-ring. $A=T_n(\Delta_i; M_{i,j})$ be a normal right representation of A as a g.t.a. matrix ring. Then $[M_{n,i}:\Delta_n]=[M_{i,1}:\Delta_1]=1$ for all i. Furthermore, if A is hereditary then $[M_{i,j}:\Delta_i]=[M_{i,j}:\Delta_1]=1$ if $M_{i,j}\neq(0)$.*

Proof. We use the same notation as above. Since $T_n(\Delta_i; M_{i,j})$ is a normal representation, Ae_1 is A-injective. From Remark 1 we know $M_{n,i}\neq(0)$ and $M_{i,1}\neq(0)$ for all i. If $[M_{n,1}:\Delta_n]\geqslant 2$, then we have two independent elements x, y in $M_{n,1}$ over Δ_n. Let φ be a linear mapping of $M_{n,1}$ into itself such that $\varphi(x)=x$, $\varphi(y)=0$. Then φ is A-homomorphism of $M_{n,1}$ to Ae_1. Since Ae_1 is injective, this is a contradiction. If $[M_{n,1}:\Delta_1]\geqslant 2$, then there exist two independent elements x', y' in $M_{n,1}$ over Δ_1. Let ψ be a linear mapping of $M_{n,1}=\Delta_n x'$ to itself such that $\psi(x')=y'$. Injectivity of Ae_1 implies that there exists an element z in Δ_1 such that $x'z=y'$. This contradicts a fact of independency. Since $M_{n,1}\supseteq M_{n,i}M_{i,1}$, $[M_{n,i}:\Delta_n]\leqslant[M_{n,i}:\Delta_n]=1$ and $[M_{i,1}:\Delta_1]\leqslant[M_{n,1}:\Delta_1]=1$. We assume that A is hereditary. Then $M_{n,i}\underset{\Delta_i}{\otimes}M_{i,1}\approx M_{n,i}M_{i,1}$ as $\Delta_n-\Delta_1$ module by [4], Theorem 1. Hence $1=[M_{n,1}:\Delta_n]\geqslant[M_{i,1}:\Delta_i]$. If $M_{i,j}\neq(0)$, $(0)\neq M_{i,j}M_{j,1}\subseteq M_{i,1}$. Hence, $1=[M_{i,1}:\Delta_i]\geqslant[M_{i,j}:\Delta_i]$. Similarly, we have $[M_{i,j}:\Delta_j]=1$.

Theorem 2. *If A is a left QF-3 and semi-primary hereditary ring, then A is a directsum of rings whose basic ring is a ring of triangular matrices over division rings. And hence, A is right QF-3 and A satisfies* *minimal conditions. The converse is also true*, (see Remark 3 below).

Proof. We assume that A is an indecomposable, basic ring. Then $A=T_n(\Delta_i; M_{i,j})$ and $M_{i,1}\neq(0)$ and $M_{n,i}\neq(0)$ for all i from Remark 1. We shall show that $M_{i,j}\neq(0)$ for all $i<j$. We quote the same notations of [4], Theorem 1. Since $M_{2,1}\neq(0)$, we assume that $M_{j,k}\neq(0)$ for any $j\leqslant i$. If $M_{i+1,i}=M_{i+1,i-1}=\cdots=M_{i+1,t}=(0)$ and $M_{i+1,t-1}\neq(0)$, then $\bar{M}_{i+1,t-1}=M_{i+1,t-1}/\sum_{k=t}^{i}M_{i+1,k}M_{k,t-1}=M_{i+1,t-1}$. On the other hand, $\bar{M}_{t,t-1}=M_{t,t-1}\neq(0)$, since $t\leqslant i$. However, $M_{n,i+1}\bar{M}_{i+1,t-1}\neq(0)$, $M_{n,t}\bar{M}_{t,t-1}\neq(0)$ and $M_{n,i+1}\bar{M}_{i+1,t-1}\cap M_{n,t}\bar{M}_{t,t-1}=(0)$ by [4], Theoreme 1. Which contradicts a fact $[M_{n,t-1}:\Delta_n]=1$. Therefore, we know $M_{i+1,i}\neq(0)$. $M_{i+1,k}\supseteq M_{i+1,i}M_{i,i-1}\cdots M_{k+1,k}\neq(0)$. Thus we can prove the fact $M_{i,j}\neq(0)$ for all $i>j$ by induction. Since $M_{i,j}\neq(0)$, $[M_{i,j}:\Delta_i]=[M_{i,j}:\Delta_j]=1$ by Lemma 7. Therefore, A is isomorphic to a ring of triangular matrices by [4], Lemma 12. Thus, we have proved Theorem 2.

In the above proof if we replace $M_{i+1,t-1}$ by a non-zero element x in $M_{i+1,t-1}$ and $M_{t,t-1}$ by a non-zero element y in $M_{t,t-1}$, then $M_{n,i+1}x$ and $M_{n,t}y$ are not zero by Lemma 2, provided A is a PP-ring. Since $[M_{n,t-1}:\Delta_n]=1$ by Lemma 7, $M_{n,i+1}x=M_{n,t}y$. This contradicts [3], Proposition 1. Hence, we have similarly

Proposition 4. *Let A be a left QF-3 and semi-primary PP-ring. We assume A is indecomposable. Then A is isomorphic to a g.t.a. matrix ring $T_n(S_i;\mathfrak{M}_{i,j})$ over simple ring S_i and each component of $\mathfrak{M}_{i,j}$ in* (2) *is non-zero. Therefore, $T_n(S_i;\mathfrak{M}_{i,j})$ is a right and left normal representation of A as a g.t.a. matrix ring and the nilpotency of the radical is equal to n. Let $S_i\approx(\Delta_i)_n$, Δ_i division ring. Then $\Delta_1\approx\Delta_n$ and Δ_i is isomorphic into $\Delta_1\approx\Delta_n$. Furthermore, we assume that A is K-algebra with finite dimension. Then A is hereditary if and only if $\Delta_i\approx\Delta_1$ for all i.*

Remark 3. Theorem 2 says that the class of the QF-3 and semi-primary hereditary rings coincides with the class of the rings of directsum of g.t.a. matrix rings of the following form.

Let Δ be a division ring and $\Delta(n\times m)$ the module of rectangular matrices of $(n\times m)$-form over Δ and it is regarded as $(\Delta)_n-(\Delta)_m$ module.

$$A=\begin{pmatrix} (\Delta)_{n_1} & & 0 & \\ \Delta(n_2\times n_1) & (\Delta)_{n_2} & & \\ \cdots\cdots\cdots & & & \\ \cdots\cdots\cdots & & & \\ \Delta(n_r\times n_1) & \Delta(n_r\times n_2) & \cdots & (\Delta)_{n_r}\end{pmatrix}$$

We consider the converse of the first half of Lemma 7.

Proposition 5. *Let $A=T_n(\Delta_i\,;M_{i,j})$ be a g.t.a. matrix ring over division ring Δ_i. If A is a partially PP-ring, then Ae_1 is A-injective and $M_{i,1}\neq(0)$ and $M_{n,i}\neq(0)$ for all i if and only if $[M_{i,1}:\Delta_1]=[M_{n,1}:\Delta]=1$. Conversely if Ae_1 is faithful and $[M_{i,1}:\Delta_1]=1$, then A is a partially PP-ring, where $e_1=T_n(1_1, 0, \cdots; 0)$.*

Proof. We assume that A is a partially PP-ring. We have proved "only if" part of the first half in the proof of Lemma 7. We shall prove "if" part. Since $[M_{i,1}:\Delta_1]=1$, we put $M_{i,1}=x_i\Delta_1$ ($x_1=$ the identity element of Δ_1). Since $[M_{n,1}:\Delta_1]=[M_{n,1}:\Delta_n]=1$, there exists an isomorphism φ of Δ_1 to Δ_n such that $x_n\delta=\delta^{\varphi}x_n$ for $\delta\in\Delta_1$. It is clear that $\mathrm{Hom}_{\Delta_1}(M_{i,1}, M_{n,1})=\Delta_n f_i$, where $f_i\in\mathrm{Hom}_{\Delta_1}(M_{i,1}, M_{n,1})$ such that $f_i(x_i)=x_n$, (for $f\in\mathrm{Hom}_{\Delta_1}(M_{i,1}, M_{n,1})f(x_i)=x_n\delta=\delta^{\varphi}x_n=(\delta^{\varphi}f_i)(x_i)$). On the other hand $M_{n,i}\approx M_{n,i}x_i=M_{n,1}$ by the assumption $[M_{n,1}:\Delta_n]=1$ and Lemma 2. Hence, there exists a unique element g_i in $M_{n,i}$ such that $g_ix_i=x_n$, ($g_n=$ the identity element in Δ_n). Therefore, $\mathrm{Hom}_{\Delta_1}(M_{i,1}, M_{n,1})$ coincides with the multiplications of elements in $\Delta_n g_i$ from the left side. Let $M^*_{i,1}=\{f\in\mathrm{Hom}_{\Delta_1}(Ae_1, \Delta_1)\,|\,f(M_{j,i})=(0) \text{ for } j\neq i\}$. Then $\mathrm{Hom}_{\Delta_1}(Ae_1, \Delta_1)=\sum_{i=1}^{n}\oplus M^*_{i,1}$ as a module. We have isomorphisms $\theta_i\colon M_{n,i}=\Delta_n g_i\to M^*_{i,1}$ by setting

$$\theta_i(\delta g_i)(x_i)=\delta^{\varphi^{-1}} \quad\text{and}\quad \theta_i(\delta g_i)(x_j)=0 \qquad \text{for } j\neq i\,.$$

Hence, we have an isomorphism Θ of e_nA to $\mathrm{Hom}_{\Delta_1}(Ae_1, \Delta_1)$ via θ_i as a module. We shall show that Θ is A-isomorphic. Let $\theta_i(\delta g_i)=f\in M^*_{i,1}$ and $m_{k,l}\in M_{k,l}$. Then $fm_{k,l}\colon M_{l,1}\xrightarrow{m_{k,l}} m_{k,l}M_{l,1}\xrightarrow{f}\Delta_1$. Hence if $k\neq i$, $fm_{k,l}=g_im_{k,l}=0$. Let $k=i$. Since $m_{i,l}x_l\in M_{i,1}=x_i\Delta_1$, $m_{i,l}x_l=x_i\delta$ for some $\delta_1\in\Delta_1$. Hence, $\theta_l^{-1}(fm_{i,l})=\delta\delta_1^{\varphi}g_l$. On the other hand, $\delta g_im_{i,l}x_l=\delta g_ix_i\delta_1=\delta x_n\delta_1=\delta\delta_1^{\varphi}x_n=\delta\delta_1^{\varphi}g_lx_l$. Hence $\delta g_im_{i,l}=\delta\delta_1^{\varphi}g_l$ by Lemma 2. Therefore, Θ is A-isomorphic Hence e_nA is A-injective. It is clear that $\mathrm{Hom}_{\Delta_1}(Ae_1, \Delta_1)$ is A-faithful (cf. the proof of Proposition 1). Thus, A has a faithful injective, projective right ideal e_nA. If we replace a position of $M_{i,1}$ by $M_{n,i}$ in the above, then we have similarly that A is a left QF-3 ring. Next, we assume that Ae_1 is faithful and $[M_{i,1}:\Delta_1]=1$. Let $x_{i,j}$, $y_{j,k}$ be in $M_{i,j}$, $M_{j,k}$, respectively. If $x_{i,j}x_{j,k}=0$, $(0)=x_{i,j}y_{j,k}M_{k,1}=x_{i,j}(y_{j,k}M_{k,1})$. Since Ae_1 is faithful, $y_{j,k}M_{k,1}\neq(0)$ if $y_{j,k}\neq0$. Hence, $y_{j,k}M_{k,1}=M_{j,1}$. We have shown that $M_{i,k}\underset{\Delta_k}{\otimes}y_{k,j}\approx M_{i,k}y_{k,i}$. Therefore, A is a partially PP-ring by [3], Lemma 5.

Similarly to Theorem 2 we have

Theorem 3. *Let A be a semi-primally PP-ring. A is a left QF-3 ring if and only if its basic ring is of the form $T_n(\Delta_i\,;M_{i,j})$ such that*

$[M_{i,1}:\Delta_1]=[M_{n,i}:\Delta_n]=1$. *In this case A is also a right QF-3 ring.*

Proof. It is clear from Theorem 1 and Proposition 5.

Finally, we shall generalize Mochizuki's result [6], Theorem 2.3 in a case of semi-primary partially PP-ring.

Let A be a basic QF-3 ring and semi-primary partially PP-ring. We assume that A is indecomposable. Then $A\approx T_n(\Delta_i\,;M_{i,j})$ and $[M_{i,1}:\Delta_1]=[M_{n,i}:\Delta_n]=1$ for all i by Lemma 7. Hence, we may asume that $\Delta_1=\Delta_n\equiv\Delta$ and Δ_i is contained in Δ. Let $L=T_n(\Delta, 0, \cdots, 0:M_{i,j}=(0)$ if $j\neq 1)$. Then L is a unique minimal faithful projective, injective left A-module. Let $B=\mathrm{Hom}_\Delta(L, L)$. Then $B=(\Delta)_n$. Let $B_{i,j}=\{f|\in B, f(M_{j,1})=M_{i,1}, f(M_{k,1})=(0)$ for $k\neq j\}$. Then $B_{i,j}\cap A\supseteq M_{i,j}$, where A is regarded as a subring of B, since L is faithful. By virtue of this imbedding we can regard $M_{i,j}$ as a $\Delta_i-\Delta_j$ submodule in Δ. In such a setting, we have

$$B=\begin{pmatrix}\Delta & \Delta & \cdots & \Delta\\ \Delta & \Delta & \cdots & \Delta\\ \cdots & \cdots & \cdots & \cdots\\ \Delta & \Delta & \cdots & \Delta\end{pmatrix}\supseteq A=\begin{pmatrix}\Delta & & & & \\ \Delta & \Delta_2 & & 0 & \\ \Delta & M_{3,2} & \Delta_3 & & \\ \cdots & \cdots & \cdots & \Delta_{n-1} & \\ \Delta & \Delta & \cdots & \Delta & \Delta\end{pmatrix}\supset L=\begin{pmatrix}\Delta & & 0\\ \vdots & & \\ \vdots & & \\ \Delta & & \end{pmatrix},$$

where $M_{i,j}$ is a $\Delta_i-\Delta_j$ submodule in Δ and Δ_i is a subdivision ring of Δ. Since $B\approx L^{(n)}$[5] as a left A-module, B is left A-projective and injective.

Lemma 8. *Let A and L be as above. Injective envelope of indecomposable left ideal Ae_i is isomorphic to L.*

Proof. Since $M_{i,1}\neq(0)$, we can take $x\neq 0$ in $M_{i,1}$. Then $Ae_ix\approx Ae_i$ by Lemma 2. Since $Ae_ix\subseteq L$ and L is indecomposable, L is an injective envelope of Ae_ix.

We note that the double commutator ring of module which is a directsum of n-copies of a module M coincides with that ring of M up to isomorphism.

Summarizing the above we have

Theorem 4. *Let A be a semi-primary partially PP-ring and e be an idempotent such that Ae is a faithful projective, injective left ideal. Then the following facts hold.*

(1) *Both the commutator ring eAe and the double commutor ring* $B=\mathrm{Hom}_{eAe}(Ae, Ae)$ *of Ae are semi-simple.*

5) $L^{(n)}$ means a directsum of n-copies of L.

(2) *B is an $A-A$ module which is both the left and right injective envelope of A and left and right A-projective.*
(3) *If A is hereditary, then A is a generalized uniserial ring with minimal conditions.*

Corollary. *Let A be as above. If L is an indecomposable A-injective left ideal in A, then L is projective and $L \approx Ae$, $eN=(0)$.*

Proof. We may assume A is indecomposable. Let M be a minimal left ideal contained in L, since A is semi-primary, (see [5], p 1106). Then an injective envelope of M' is contained in L and hence L is isomorphic to an injective envelope of M'. Therefore, B in Theorem 4 contains an isomorphic image of L as direct summand by the proof of Theorem 3.2 in [5]. Hence, L is A-projective by Theorem 4. The second part is clear from Theorem 2.

We conclude this paper with the following examples.

EXAMPLE. Let K be a field and L proper extension of K. We put

$$A = \begin{pmatrix} L & 0 & 0 \\ L & K & 0 \\ L & L & L \end{pmatrix},$$

where L at (2, 1)-component is regarded as $K-L$ module and L at (3, 2)-component as $L-K$ module. Since a natural mapping $L \underset{K}{\otimes} L \to L$ is not monomorphic, A is not hereditary by [4], Theorem 1. It is clear that $\begin{pmatrix} L00 \\ L00 \end{pmatrix}$ is a faithful, projective, injective A-module and A is a PP-ring by Proposition 5 and [3], Proposition 1. Hence, A is a QF-3 and PP-ring and not hereditary. If $[L:K]=\infty$ A does not satisfies the minimal conditions.

Let

$$A = \begin{pmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & 0 & K & 0 \\ K & K & K & K \end{pmatrix},$$

then A is a QF-3 and partially PP-ring by [3], Lemm 5. However, A is not a PP-ring and hence, not hereditary.

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