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On Algebras of Bounded Representation Type[1]

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Let A be an associative algebra with a unit element over a field K and let $A=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}Ae_{\kappa i}$ be a direct decomposition of A into directly indecomposable left ideals where $Ae_{\kappa i}\cong Ae_{\kappa 1}=Ae_{\kappa}$. It is well known that if A is generalised uniserial a directly indecomposable left module is homomorphic to one of $Ae_{\kappa}(\kappa=1,\cdots\cdots,n)$[2]. But in general a directly indecomposable left (or right) module of A is not neceassarily homomorphic to Ae_{κ} and there may exist directly indecomposable left modules of arbitrary high degrees[3]. In his paper [2] T. Nakayama propounded the problem to determine the general type of rings which possess arbitrary large directly indecomposable left (or right) modules and in [3] D. G. Higmann proved that every group has not indecomposable modular representations of arbitrary high degrees of characteristic p if and only if it has cyclic p-sylow subgroups.

Now in this paper we shall study necessary and sufficient conditions for an algebra to be of bounded representation type in a special case where $N^2=0$ (N is the radical of A)[4] and K is algebraically closed.

First the *chain* $\{Ne_1,\cdots\cdots,Ne_s\}$ means that $Ne_{i+1}\neq Ne_i$ and Ne_{i+1} and Ne_i contain simple left ideals isomorphic to each other, namely $Ne_{i+1}\supset Au_{i+1}^{(\kappa)}$, $Ne_i\supset Au_{i+1}^{(\kappa)}$ and $Au_{i+1}^{(\kappa)}\cong Au_i^{(\kappa)}$. If Ne_1, Ne_2 and Ne_3 contain simple left ideals isomorphic to each other, namely $Ne_1\supset Au_1^{(\lambda)}$, $Ne_2\supset Au_2^{(\lambda)}$, $Ne_3\supset Au_3^{(\lambda)}$ and $Au_1^{(\lambda)}\cong Au_2^{(\lambda)}\cong Au_3^{(\lambda)}$, we define Ne_1 to be *divided* into Ne_2 and Ne_3. Then we shall prove that if $N^2=0$ and K is algebraically closed A has not directly indecomposable left (or right) modules of arbitrary high degrees if and only if the following conditions are satisfied;

(1) $Ne_{\lambda}(e_{\lambda}N)$ $(\lambda=1,\cdots\cdots,n)$ do not contain at least two simple

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1) This means that the degree of the directly indecomposable representation is bounded. Cf. James P. Jans [4].

2) T. Nakayama [1].

3) H. Brummund [6].

4) If $N^2\neq 0$, it is very difficult and our conditions are extended as necessary conditions to the case where $N^2\neq 0$. But it is not proved yet that these conditions are sufficient for an algebra to be of bounded representation type. Cf. James P. Jans [4].

components isomorphic to each other.

(2) $Ne_\lambda(e_\lambda N)$ ($\lambda=1, \cdots\cdots, n$) are the direct sums of at most three simple components.

(3) There is no chain such that $\{Ne_{\kappa_1}=Ne_\kappa, \cdots\cdots, Ne_{\kappa m}, Ne_{\kappa m+1}=Ne_\kappa, \cdots\cdots\}$.

(4) If $Ne_\kappa(e_\kappa N)$ is the direct sum of three simple components and $Ne_\lambda(e_\lambda N)$ is the direct sum of three simple components or divided, there is no chain which connects Ne_κ and $Ne_\lambda(e_\kappa N$ and $e_\lambda N)$.

(5) If Ne_1, Ne_2 and Ne_3 (e_1N, e_2N and e_3N) are direct sums of two simple components, $Ne_1(e_1N)$ is not divided into Ne_2 and $Ne_3(e_2N$ and $e_3N)$.

(6) Suppose that $\{Ne_1, \cdots\cdots, Ne_r\}$ ($\{e_1N, \cdots\cdots, e_rN\}$) is a chain. Then Ne_1 or $Ne_r(e_1N$ or $e_rN)$ is the direct sum of three simple components or, if $Ne_j(e_jN)$ ($j\neq 1, r$) is the direct sum of three simple components, the chain is $\{Ne_1, Ne_2, Ne_3\}$, or $\{Ne_4, Ne_2, Ne_5, Ne_6\}$ where Ne_2 is the direct sum of three simple components and Ne_4 and Ne_6 are simple.

Last autumn Professor Brauer informed me that he and Professor Thrall obtained almost the same results as ours but their works are not published[5]. The author expresses here his hearty thanks to Professor Brauer and Professor Thrall for their valuable advices.

1. G. Köthe[6] and H. Brummund[7] showed that, when A is commutative or A is the group ring of a p-group over a field with characteristic p, A has directly indecomposable left (or right) modules of arbitrary high degrees if $N^ie/N^{i+1}e$ contains at least two simple left ideals isomorphic to each other. In general if the ground field is algebraically closed, this is true. This is proved by the same method as Brummund's because $(\bar{e}_\kappa\bar{A}\bar{e}_\kappa : K)=1$. But if the ground field is not algebraically closed, it is shown by the following example that Brummund's method is not used and it is possible that even if $N^ie/N^{i+1}e$ contains two simple left ideals isomorphic to each other we have not a directly indecomposable left module of arbitrary high degrees.

Example:

	e_1	u	e_2	u_1	u_2
e_1	e_1	u	0	0	0
u	u	αe_1	0	0	0
e_2	0	0	e_2	u_1	u_2
u_1	u_1	u_2	0	0	0
u_2	u_2	αu_1	0	0	0

where $\alpha \in K$ and $K \subsetneqq K(\sqrt{\alpha}.)$

5) See [4] for the outline of their works.

6) G. Köthe [5].

7) H. Brummund [6].

Then $A=e_1A+e_2A=Ae_1+Ae_2$, $e_1N=0$, $e_2N=u_1A$, $Ne_1=Au_1\oplus Au_2$, $Ne_2=0$ and $Au_1\cong Au_2\cong\bar{A}\bar{e}_2$. If we construct an A-left module $\mathfrak{M}=Ae_1m_1+Ae_1m_2$ where $u_1m_1=u_2m_2$, it is easily shown that $\mathfrak{M}$ is directly decomposable. Thus we have

Lemma 1. *Let the ground field K be algebraically closed. If $N^ie/N^{i+1}e$ contains at least two simple left ideals isomorphic to each other, A has directly indecomposable left modules of arbitrary high degrees.*

2. From this chapter we assume that $N^2=0$ and K is algebraically closed and each $Ne_\kappa\,(e_\kappa N)$ is a direct sum of simple left ideals not isomorphic to each other. Moreover it is clear that A is of bounded representation type if and only if the basic algebra $\hat{A}$ of A is of bounded representation type. Hence from now on we shall assume that A is the basic algebra.

Then we have the following

Lemma 2. *If Ne is a direct sum of at least four simpl left ideals, there exists a directly indecomposable left module of arbitrary high degrees.*

Proof. Suppose that $Ne=Au_1\oplus Au_2\oplus Au_3\oplus Au_4$ and $Au_i\cong\bar{A}\bar{e}_i$. Then we construct an A-left module $\mathfrak{M}$ as follows;

$$\mathfrak{M}=\sum_{i=1}^{2s}Aem_i \text{ where } u_1m_1\neq 0,\ u_2m_1=0,\ u_3m_1\neq 0,\ u_4m_1=u_4m_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$u_1m_{2j}=u_1m_{2j+1},\ u_2m_{2j+1}=0,\ u_3m_{2j+1}\neq 0,$$

$$u_4m_{2j+1}=u_4m_{2j+2},\ u_2m_{2j+2}\neq 0,\ u_3m_{2j+2}=0,$$

$$u_1m_{2j+2}=u_1m_{2j+3}\ \cdots\cdots\cdots\cdots\cdots$$

$$u_2m_{2s}\neq 0,\ u_3m_{2s}\neq 0,\ u_1m_{2s}\neq 0\,.$$

Now if it is proved that $\mathfrak{M}$ is directly indecomposable, this lemma follows immediately. Hence we shall prove that $\mathfrak{M}$ is directly indecomposable.

First the representation $R(a)$ by $\mathfrak{M}$ has the following form:

$$R(a)=\begin{pmatrix} I_{2s}\times y & 0 & 0 & 0 & 0 \\ Q_{11}\times z_{11} & I_{s+1}\times x_1 & 0 & 0 & 0 \\ Q_{21}\times z_{21} & 0 & I_s\times x_2 & 0 & 0 \\ Q_{31}\times z_{31} & 0 & 0 & I_s\times x_3 & 0 \\ Q_{41}\times z_{41} & 0 & 0 & 0 & I_s\times x_4 \end{pmatrix}^{8)}$$

8) $I_s=\overbrace{\begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix}}^{s}$ and $x_i,\ y,\ z_{ij}\in k$.

where $a=ye+x_1e_1+x_2e_2+x_3e_3+x_4e_4+z_{11}u_1+z_{21}u_2+z_{31}u_3+z_{41}u_4+\cdots\cdots$

and
$$R(u_1)=\begin{pmatrix} 0 & & & & \\ Q_{11} & 0 & & & \\ 0 & & 0 & & \\ 0 & & & 0 & \\ 0 & & & & 0 \end{pmatrix}^{9)}=\left(\begin{array}{c} \overbrace{}^{s}\ \overbrace{}^{s} \\ \begin{pmatrix} 1\cdots\cdots0 & 0\cdots\cdots0 \\ \vdots\ \ddots\ \vdots & 1\cdots\cdots0 \\ \ddots\ \vdots & \vdots\ \ddots\ \vdots \\ 0\cdots\cdots1 & \vdots\ \ \ddots\ \vdots \\ 0\cdots\cdots0 & 0\cdots\cdots1 \end{pmatrix} \end{array}\right),$$

$$R(u_2)=\begin{pmatrix} \\ Q_{21} \\ \\ \end{pmatrix}=\left(\begin{array}{c} \overbrace{}^{s}\ \overbrace{}^{s} \\ \begin{pmatrix} 0\cdots\cdots0 & 1\cdots\cdots0 \\ \cdots\cdots\cdots & \vdots\ \ddots\ \vdots \\ & \vdots\ \ \ddots\ \vdots \\ 0\cdots\cdots0 & 0\cdots\cdots1 \end{pmatrix} \end{array}\right),$$

$$R(u_3)=\begin{pmatrix} \\ Q_{31} \\ \\ \end{pmatrix}=\left(\begin{array}{c} \overbrace{}^{s}\ \overbrace{}^{s} \\ \begin{pmatrix} 1\cdots\cdots0 & 0\cdots\cdots0 \\ \vdots\ \ddots\ \vdots & \vdots\ \ddots\ \vdots \\ \vdots\ \ \ddots\ \vdots & \vdots\ \ \ddots\ \vdots \\ 0\cdots\cdots1 & 0\cdots\cdots0 \end{pmatrix} \end{array}\right),$$

$$R(u_4)=\begin{pmatrix} \\ Q_{41} \\ \\ \end{pmatrix}=\left(\begin{array}{c} \overbrace{}^{s}\ \overbrace{}^{s} \\ \begin{pmatrix} 1\cdots\cdots0 & 1\cdots\cdots0 \\ \vdots\ \ddots\ \vdots & \vdots\ \ddots\ \vdots \\ \vdots\ \ \ddots\ \vdots & \vdots\ \ \ddots\ \vdots \\ 0\cdots\cdots1 & 0\cdots\cdots1 \end{pmatrix} \end{array}\right),$$

where

$$Q_{11}=\begin{pmatrix} 1\cdots0 & 0\cdots0 \\ \vdots\ \ddots & 1\cdots0 \\ 0\cdots1 & \vdots\ \ddots\ \vdots \\ 0\cdots0 & 0\cdots1 \end{pmatrix},$$

$$Q_{21}=\begin{pmatrix} 0, & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{pmatrix},$$

$$Q_{31}=\begin{pmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \end{pmatrix},$$

$$Q_{41}=\begin{pmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{pmatrix},$$

and s is an arbitrary integer. Then any commutator B of $R(a)$ has the following form:

$$B=\begin{pmatrix} B_1 & & & & \\ & B'_1 & & & \\ & & B'_2 & & \\ & * & & B'_3 & \\ & & & & B'_4 \end{pmatrix}^{10)}, \quad \text{where} \quad \begin{aligned} &Q_{11}B_1=B'_1Q_{11},\ Q_{21}B_1=B'_2Q_{21}, \\ &Q_{31}B_1=B'_3Q_{31},\ Q_{41}B_1=B'_4Q_{41}. \end{aligned}$$

This is easily obtained from $BR(a)=R(a)B$ if $a=u_1, u_2, u_3, u_4$. Now from $Q_{21}B_1=B'_2Q_{21}$ and $Q_{31}B_1=B'_3Q_{31}$, we have $B_1=\begin{pmatrix} B'_2 & \\ & B'_3 \end{pmatrix}$ and next from

9) From now the empty place means zero, namely $x_i=y_j=z_{\kappa\lambda}=0$ if $z_{11}\neq z_{\kappa\lambda}$, x_i, y_j.

10) This is broken up into submatrices to correspond to the divisions of $R(a)$.

$Q_{41}B_1 = B'_4 Q_{41}$ we have $B'_2 = B'_3 = B'_4$ and moreover from $Q_{11}B_1 = B'_1 Q_{11}$ we have

$$B_1 = \begin{pmatrix} b & & & & \\ & \ddots & & & \\ & & b & & \\ & & & b & \\ & & & & \ddots \\ & & & & & b \end{pmatrix}, \quad B'_1 = \begin{pmatrix} b & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ & & & & b \end{pmatrix}.$$

Thus any commutator of $R(a)$ has just one eigenvalue and therefore $R(a)$ is directly indecomposable[11]. Hence $\mathfrak{M}$ is directly indecomposable.

3. Here we have the following

Lemma 3. *Suppose that there is a chain such that* $Ne_{\kappa_1} = Ne_{\kappa}, \cdots\cdots,$ $Ne_{\kappa_{t-1}}, Ne_{\kappa_t} = Ne_{\kappa}, \cdots\cdots$. *Then there is a directly indecomposable A-left module of arbitrary high degrees.*

Proof. Put $Ne_{\kappa_1} = A\bar{u}^{(1)} \oplus Au^{(2)}$, $Ne_{\kappa_2} = A\bar{u}^{(2)} \oplus Au^{(3)}$, $\cdots\cdots$, $Ne_{\kappa_{t-1}} = A\bar{u}^{(t-1)} \oplus Au^{(1)}$ where $Au^{(i)} \cong A\bar{u}^{(i)}$ [12]. Then we construct an A-left module $\mathfrak{M}$ as follows,

$$\begin{aligned} \mathfrak{M} = {} & Ae_{\kappa_1}m_{1,1} + Ae_{\kappa_2}m_{2,1} + \cdots\cdots + Ae_{\kappa_{t-1}}m_{t-1,1} \\ & + Ae_{\kappa_1}m_{1,2} + Ae_{\kappa_2}m_{2,2} + \cdots\cdots + Ae_{\kappa_{t-1}}m_{t-1,2} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + Ae_{\kappa_1}m_{1,s} + Ae_{\kappa_2}m_{2,s} + \cdots\cdots + Ae_{\kappa_{t-1}}m_{t-1,s} \end{aligned}$$

where $\bar{u}^{(1)}m_{1,1} \neq 0$, $u^{(2)}m_{1,1} = \bar{u}^{(2)}m_{2,2}$, $\cdots\cdots$, $u^{(t-1)}m_{t-2,1} = \bar{u}^{(t-1)}m_{t-1,1}$,
$u^{(1)}m_{t-1,1} = \bar{u}^{(1)}m_{1,2}$, $u^{(2)}m_{1,2} = \bar{u}^{(2)}m_{2,1}$, $\cdots\cdots$, $u^{(t-1)}m_{t-2,s} = \bar{u}^{(t-1)}m_{t-1,s}$,
$u^{(1)}m_{t-1,s} \neq 0$

and s is an arbitrary integer.

Then if we prove that $\mathfrak{M}$ is directly indecomposable, this lemma follows immediately. Hence we shall prove that $\mathfrak{M}$ is directly indecomposable.

Now the representation $R(a)$ by $\mathfrak{M}$ has the following form:

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11) R. Brauer [7] or Brauer-Schur [8].

12) $Au^{(i)}$ means that $Au^{(i)} \cong A\bar{e}_i$.

$$R(a)=\begin{pmatrix} I_s\times y_1 & & & & & & & \\ 0 & \ddots & & & & & & \\ \vdots & & \ddots & & & & & \\ 0 & \cdots & 0 & I_s\times y_{t-1} & & & & \\ Q_{11}\times z_{11} & 0 & \cdots\cdots 0 & Q_{1,t-1}\times z_{1,t-1}, & I_{s+1}\times x_1 & & & \\ Q_{21}\times z_{21} & Q_{22}\times z_{22} & 0\cdots\cdots & 0 & 0 & I_s\times x_2 & & \\ & & \ddots & \vdots & \vdots & & \ddots & \\ & & Q_{t-1,t-2}\times z_{t-1,t-2} & Q_{t-1,t-1}\times z_{t-1,t-1} & 0 & \cdots & 0 & I_s\times x_{t-1} \end{pmatrix}$$

where

$$a=y_1e_{\kappa_1}+\cdots\cdots+y_{t-1}e_{\kappa_{t-1}}+x_1e_1+\cdots\cdots+x_{t-1}e_{t-1}+z_{1,1}\bar{u}^{(1)}+\cdots\cdots+z_{1,t-1}u^{(1)}+\cdots\cdots$$

and

$$R(\bar{u}^{(1)})=\begin{pmatrix} \\ Q_{11} \\ \\ \end{pmatrix}=\begin{pmatrix} \overbrace{\begin{pmatrix} 1&\cdots\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots\cdots&1\\ 0&\cdots\cdots&0 \end{pmatrix}}^{s} \end{pmatrix}, \qquad R(u^{(2)})=\begin{pmatrix} \\ Q_{21} \\ \\ \end{pmatrix}=\begin{pmatrix} \overbrace{\begin{pmatrix} 1&&\\ &\ddots&\\ &&1 \end{pmatrix}}^{s} \end{pmatrix},$$

$$R(\bar{u}^{(2)})=\begin{pmatrix} \\ Q_{22} \\ \\ \end{pmatrix}=\begin{pmatrix} \overbrace{\begin{pmatrix} 1&&\\ &\ddots&\\ &&1 \end{pmatrix}}^{s} \end{pmatrix}, \qquad R(u^{(3)})=\begin{pmatrix} \\ Q_{32} \\ \\ \end{pmatrix}=\begin{pmatrix} \overbrace{\begin{pmatrix} 1&&\\ &\ddots&\\ &&1 \end{pmatrix}}^{s} \end{pmatrix},$$

....

$$R(\bar{u}^{(t-1)})=\begin{pmatrix} \\ Q_{t-1,t-1} \\ \\ \end{pmatrix}=\begin{pmatrix} \overbrace{\begin{pmatrix} 1&&\\ &\ddots&\\ &&1 \end{pmatrix}}^{s} \end{pmatrix}, \qquad R(u^{(1)})=\begin{pmatrix} \\ Q_{1,t-1} \\ \\ \end{pmatrix}=\begin{pmatrix} \overbrace{\begin{pmatrix} 0&\cdots\cdots&0\\ 1&\cdots\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots\cdots&1 \end{pmatrix}}^{s} \end{pmatrix}.$$

Then by the same way as lemma 2, any commutator B of $R(a)$ has the following form:

$$B = \begin{pmatrix} B_1 & & & & & \\ & \ddots & & & & \\ & & B_{t-1} & & & \\ & & & B'_1 & & \\ & * & & & \ddots & \\ & & & & & B'_{t-1} \end{pmatrix}$$

where $Q_{11}B_1 = B'_1Q_{11}$, $Q_{21}B_1 = B'_2Q_{21}$, $Q_{22}B_2 = B'_2Q_{22}$, $Q_{32}B_2 = B'_3Q_{32}$, $\cdots\cdots$, $Q_{t-1,t-2}B_{t-2} = B'_{t-1}Q_{t-1,t-2}$, $Q_{t-1,t-1}B_{t-1} = B'_{t-1}Q_{t-1,t-1}$, $Q_{1,t-1}B_{t-1} = B'_1Q_{1,t-1}$.

Now from $Q_{2,1}B_1 = B'_2Q_{2,1}$, $\cdots\cdots$ and $Q_{t-1,t-1}B_{t-1} = B'_{t-1}Q_{t-1,t-1}$, we have $B_1 = B_2 = B'_2 = \cdots\cdots\cdots = B'_{t-1} = B_{t-1}$. Next from $Q_{11}B_1 = B'_1Q_{11}$, and $Q_{1,t-1}B_{t-1} = B'_1Q_{1,t-1}$, we have

$$B_{t-1} = B_1 = \begin{pmatrix} b & & \\ & \ddots & \\ & & b \end{pmatrix} \quad \text{and} \quad B'_1 = \begin{pmatrix} b & & \\ & \ddots & \\ & & b \end{pmatrix}.$$

Thus B has just one eigenvalue and $R(a)$ is directly indecomposable. Hence $\mathfrak{M}$ is directly indecomposable.

4. Lemma 4. *Suppose that Ne_κ is the direct sum of three simple left ideals and Ne_λ is the direct sum of three simple left ideals or divided. Then if there is a chain which connects Ne_κ and Ne_λ, there is a directly indecomposable left module of arbitrary high degrees.*

Proof. In order to prove this lemma we must consider two cases depending on whether Ne_λ is the direct sum of three simple components or divided.

(i) Suppose that $Ne_\kappa = Au^{(\alpha)} \oplus Au^{(\beta)} \oplus Au^{(1)}$, $Ne_\lambda = A\bar{u}^{(t)} \oplus Au^{(\xi)} \oplus Au^{(\eta)}$ and $Ne_{\kappa_1} = A\bar{u}^{(1)} \oplus Au^{(2)}$, $Ne_{\kappa_2} = A\bar{u}^{(2)} \oplus Au^{(3)}$, $\cdots\cdots$, $Ne_{\kappa_{t-1}} = A\bar{u}^{(t-1)} \oplus Au^{(t)}$ where $A\bar{u}^{(i)} \simeq Au^{(i)} \simeq \bar{A}\bar{e}_i$.

Then we construct an A-left module $\mathfrak{M}$ as follows;

$$\begin{aligned} \mathfrak{M} = {} & Ae_\kappa m_{\kappa,1} + Ae_{\kappa_1} m_{\kappa_1,1} + \cdots\cdots\cdots\cdots + Ae_{\kappa_{t-1}} m_{\kappa_{t-1},1} + Ae_\lambda m_{\lambda,1} \\ & + Ae_\lambda m_{\lambda,2} + Ae_{\kappa_{t-1}} m_{\kappa_{t-1},2} + \cdots\cdots + Ae_{\kappa_1} m_{\kappa_1,2} + Ae_\kappa m_{\kappa,2} \\ & + Ae_\kappa m_{\kappa,3} + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + Ae_{\kappa_1} m_{\kappa_1,2s} + Ae_\kappa m_{\kappa,2s}, \end{aligned}$$

where $u^{(\alpha)}m_{\kappa,1} \neq 0$, $u^{(\beta)}m_{\kappa,1} = 0$, $u^{(1)}m_{\kappa,1} = \bar{u}^{(1)}m_{\kappa_1,1}$, $\cdots\cdots$, $u^{(t)}m_{\kappa_{t-1},1} = \bar{u}^{(t)}m_{\lambda,1}$, $u^{(\xi)}m_{\lambda,1} \neq 0$, $u^{(\eta)}m_{\lambda,1} = u^{(\eta)}m_{\lambda,2}$, $u^{(\xi)}m_{\lambda,2} = 0$, $\bar{u}^{(t)}m_{\lambda,2} = u^{(t)}m_{\kappa_{t-1},2}$, $\cdots\cdots$, $\bar{u}^{(1)}m_{\kappa_1,2s} = u^{(1)}m_{\kappa_1,2s}$, $u^{(\alpha)}m_{\kappa,2s} \neq 0$, $u^{(\beta)}m_{\kappa,2s} \neq 0$.

Then the representation $R(a)$ by $\mathfrak{M}$ has the following form;

$$
R(a)=\left(\begin{array}{cccccccccc}
I_{2s}\times y_\kappa & & & & & & & & & \\
0 & I_{2s}\times y_1 & & & & & & & & \\
\vdots & 0 & \ddots & & & & & & & \\
\vdots & \vdots & & I_{2s}\times y_{t-1} & & & & & & \\
0 & \cdots & \cdots & 0 & I_{2s}\times y_\lambda & & & & & \\
Q_{\alpha\kappa}\times z_{\alpha\kappa} & 0 & \cdots & \cdots & 0 & I_{s+1}\times x_\alpha & & & & \\
Q_{\beta\kappa}\times z_{\beta\kappa} & 0 & \cdots & \cdots & \cdots & 0 & I_s\times x_\beta & & & \\
Q_{1\kappa}\times z_{1\kappa}, & Q_{11}\times z_{11} & 0 & \cdots & \cdots & \cdots & 0 & I_s\times x_1 & & \\
 & & \ddots & 0 & \vdots & \vdots & & & \ddots & \\
 & & & Q_{t\lambda}\times z_{t\lambda} & 0 & & & & & \\
 & & & Q_{\xi\lambda}\times z_{\xi\lambda} & 0 & & & & & \\
 & & & Q_{\eta\lambda}\times z_{\eta\lambda} & 0 & \cdots & \cdots & \cdots & 0 & I_s\times x_\eta
\end{array}\right)
$$

where $a=y_\kappa e_\kappa+y_1e_{\kappa_1}+\cdots\cdots+y_{t-1}e_{\kappa_{t-1}}+y_\lambda e_\lambda+x_\alpha e_\alpha+\cdots\cdots+x_\eta e_\eta+z_{\alpha\kappa}u^{(\alpha)}+\cdots\cdots$
$+z_{\eta\lambda}u^{(\eta)}+\cdots\cdots$

and

$$
R(u^{(\alpha)})=\begin{pmatrix} \\ Q_{\alpha\kappa} \\ \\ \end{pmatrix}=\left(\begin{array}{c}\overbrace{}^{s}\ \overbrace{}^{s}\\ \begin{pmatrix} 1\cdots\cdots0 & 0\cdots\cdots0 \\ \vdots\ \ddots\ \ \vdots & 1\cdots\cdots0 \\ \ddots\ \vdots & \vdots\ \ddots\ \vdots \\ 0\cdots\cdots1 & \vdots\ \ \ddots\ \vdots \\ 0\cdots\cdots0 & 0\cdots\cdots1 \end{pmatrix}\end{array}\right),
$$

$$
R(u^{(\beta)})=\begin{pmatrix} \\ Q_{\beta\kappa} \\ \\ \end{pmatrix}=\left(\begin{array}{c}\overbrace{}^{s}\ \overbrace{}^{s}\\ \begin{pmatrix} 0\cdots\cdots0 & 1\cdots\cdots0 \\ \cdots\cdots\cdots & \vdots\ \ddots\ \vdots \\ 0\cdots\cdots0 & 0\cdots\cdots1 \end{pmatrix}\end{array}\right),\quad
R(u^{(1)})=\begin{pmatrix} \\ Q_{1\kappa} \\ \\ \end{pmatrix}=\left(\begin{array}{c}\overbrace{}^{2s}\\ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\end{array}\right),
$$

$$
R(\bar{u}^{(1)})=\begin{pmatrix} \\ Q_{11} \\ \\ \end{pmatrix}=\left(\begin{array}{c}\overbrace{}^{2s}\\ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\end{array}\right),\qquad
R(u^{(2)})=\begin{pmatrix} \\ Q_{21} \\ \\ \end{pmatrix}=\left(\begin{array}{c}\overbrace{}^{2s}\\ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\end{array}\right),
$$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$$
R(u^{(t)})=\begin{pmatrix} \\ Q_{t,t-1} \\ \\ \end{pmatrix}=\left(\begin{array}{c}\overbrace{}^{2s}\\ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\end{array}\right),\qquad
R(\bar{u}^{(t)})=\begin{pmatrix} \\ Q_{t,\lambda} \\ \\ \end{pmatrix}=\left(\begin{array}{c}\overbrace{}^{2s}\\ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\end{array}\right),
$$

$$R(u^{(\xi)})=\begin{pmatrix} \\ Q_{\xi,\lambda} \\ \\ \end{pmatrix}=\left(\begin{array}{c} \overbrace{\qquad\qquad}^{s}\ \overbrace{\qquad\qquad}^{s} \\ \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \cdots\cdots & \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}\end{array}\right),\quad R(u^{(\eta)})=\begin{pmatrix} \\ Q_{\eta,\lambda} \\ \\ \end{pmatrix}=\left(\begin{array}{c} \overbrace{\qquad\qquad}^{s}\ \overbrace{\qquad\qquad}^{s} \\ \begin{pmatrix} 1 & 0\cdots & 0 & 1 & 0\cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 1 \end{pmatrix}\end{array}\right)$$

and any commutator B of $R(a)$ has the following form:

$$B=\begin{pmatrix} B_\kappa & & & & & & & & & & & & \\ & B_1 & & & & & & & & & & & \\ & & \ddots & & & & & & & 0 & & & \\ & & & B_{t-1}, & & & & & & & & & \\ & & & & B_\lambda, & & & & & & & & \\ & & & & & B'_\alpha & & & & & & & \\ & & & & & & B'_\beta & & & & & & \\ & & & & & & & B'_1 & & & & & \\ & & * & & & & & & \ddots & & & & \\ & & & & & & & & & B'_t & & & \\ & & & & & & & & & & B'_\xi & & \\ & & & & & & & & & & & B'_\eta \end{pmatrix}$$

where

$$Q_{\alpha\kappa}B_\kappa=B'_\alpha Q_{\alpha\kappa},\quad Q_{\beta\kappa}B_\kappa=B'_\beta Q_{\beta\kappa},\quad Q_{1\kappa}B_\kappa=B'_1 Q_{1\kappa},\quad Q_{11}B_1=B'_1 Q_{11},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Q_{t,t-1}B_{t-1}=B'_t Q_{t,t-1},\ Q_{t,\lambda}B_\lambda=B'_t Q_{t,\lambda},\ Q_{\xi,\lambda}B_\lambda=B'_\xi Q_{\xi,\lambda},\ Q_{\eta,\lambda}B_\lambda=B'_\eta Q_{\eta,\lambda}.$$

From $Q_{1\kappa}B_\kappa=B'_1 Q_{1\kappa}$, $Q_{11}B_1=B'_1 Q_{11}$, $\cdots\cdots$ and $Q_{t\lambda}B_\lambda=B'_t Q_{t\lambda}$, we have $B_\kappa=B_1=\cdots\cdots=B_\lambda=B'_1=\cdots\cdots=B'_t$. Next from $Q_{\beta\kappa}B_\kappa=B'_\beta Q_{\beta\kappa}$, we have $B_\kappa=\begin{pmatrix} * & * \\ 0 & B'_\beta \end{pmatrix}$ and from $Q_{\xi\lambda}B_\lambda=B'_\xi Q_{\xi\lambda}$, we have $B_\kappa=\begin{pmatrix} B'_\xi & 0 \\ 0 & B'_\beta \end{pmatrix}$. Next from $Q_{\eta\lambda}B_\lambda=B'_\eta Q_{\eta\lambda}$, we have $B'_\eta=B'_\xi=B'_\beta$ and from $Q_{\alpha\kappa}B_\kappa=B'_\alpha Q_{\alpha\kappa}$, we have $B_\kappa=\begin{pmatrix} b & & & & \\ & \ddots & & & \\ & & b_b & & \\ & & & \ddots & \\ & & & & b \end{pmatrix}$ and $B'_\alpha=\begin{pmatrix} b & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & b \end{pmatrix}$. Thus B has just one eigenvalue and $R(a)$ is directly indecomposable.

(ii) Now suppose that $Ne_\kappa=Au^{(\alpha)}\oplus Au^{(\beta)}\oplus Au^{(1)}$, $Ne_{\kappa_1}=A\bar{u}^{(1)}\oplus Au^{(2)}$, $\cdots\cdots$, $Ne_{\kappa_{t-1}}=A\bar{u}^{(t-1)}\oplus Au^{(t)}$, $Ne_{\lambda_1}=Au_1^{(t)}\oplus Au_1^{(\xi)}$, $Ne_{\lambda_2}=Au_2^{(t)}\oplus Au_2^{(\eta)}$ where $Au^{(i)}\cong A\bar{u}^{(i)}$ and $Au^{(t)}\cong Au_1^{(t)}\cong Au_2^{(t)}$. Then we construct $\mathfrak{M}$ as follows;

$$\begin{aligned}\mathfrak{M} = {} & Ae_{\lambda_1}m_{\lambda_{1},1} + Ae_{\kappa_{t-1}}m_{\kappa_{t-1},1} + \cdots\cdots + Ae_{\kappa_1}m_{\kappa_1,1} + Ae_{\kappa}m_{\kappa,1} \\ & + Ae_{\kappa}m_{\kappa,2} + Ae_{\kappa_1}m_{\kappa_1,2} + \cdots\cdots + Ae_{\kappa_{t-1}}m_{\kappa_{t-1},2} + Ae_{\lambda_2}m_{\lambda_2,1}, \\ & + Ae_{\lambda_2}m_{\lambda_2,2} + Ae_{\kappa_{t-1}}m_{\kappa_{t-1},3} + \cdots\cdots + Ae_{\kappa_1}m_{\kappa_1,3} + Ae_{\kappa}m_{\kappa,3} \\ & + Ae_{\kappa}m_{\kappa,4} + Ae_{\kappa_1}m_{\kappa_1,4} + \cdots\cdots + Ae_{\kappa_{t-1}}m_{\kappa_{t-1},4} + Ae_{\lambda_1}m_{\lambda_1,2} \\ & + Ae_{\lambda_1}m_{\lambda_1,3} + \cdots\cdots + Ae_{\kappa_{t-1}}m_{\kappa_{t-1},4s} + A_{\lambda_1}m_{\lambda_1,2s},\end{aligned}$$

where

$$\begin{aligned}& u_1^{(\xi)}m_{\lambda_1,1} \neq 0,\ u_1^{(t)}m_{\lambda_1,1} = u^{(t)}m_{\kappa_{t-1},1},\ \cdots\cdots,\ \bar{u}^{(1)}m_{\kappa_1,1} = u^{(1)}m_{\kappa,1},\ u^{(\beta)}m_{\kappa,1} \neq 0, \\ & u^{(\alpha)}m_{\kappa,1} = u^{(\alpha)}m_{\kappa,2},\quad u^{(\beta)}m_{\kappa,2} = 0,\quad u^{(1)}m_{\kappa,2} = \bar{u}^{(1)}m_{\kappa_1,2},\quad \cdots\cdots,\quad u^{(t)}m_{\kappa_{t-1},2} \\ & = u_2^{(t)}m_{\lambda_2,1},\ u_2^{(\eta)}m_{\lambda_2,1} = u_2^{(\eta)}m_{\lambda_2,2},\ u_2^{(t)}m_{\lambda_2,2} = u^{(t)}m_{\kappa_{t-1},3},\ \cdots\cdots,\ u^{(t)}m_{\kappa_{t-1},4s} \\ & = u_1^{(t)}m_{\lambda_1,2s},\ u_1^{(\xi)}m_{\lambda_1,2s} \neq 0.\end{aligned}$$

Then the representation $R(a)$ by $\mathfrak{M}$ has the following form;

$R(a) =$

$$\left(\begin{array}{cccccccc}
I_{2s}\times y_{\lambda_1} & & & & & & & \\
0 & \ddots & & & & & & \\
0 & \cdots & I_{4s}\times y_{\kappa} & & & & & \\
Q_{\xi\lambda_1}\times z_{\xi\lambda_1} & 0 \cdots 0 & I_{s+1}\times x_{\xi} & & & & & \\
0 & Q_{\eta\lambda_2}\times z_{\eta\lambda_2} & 0 \cdots 0 & I_s\times x_{\eta} & & & & \\
Q_{t\lambda_1}\times z_{t\lambda_1}, & Q_{t\lambda_2}\times z_{t\lambda_2} & 0 \cdots 0 & 0 & I_{4s}\times x_t & & & \\
 & \ddots & \vdots & \vdots & & \ddots & & \\
 & & 0 & & & & & \\
 & Q_{11}\times z_{11}, & Q_{1\kappa}\times z_{1\kappa} & 0 \cdots 0 & & I_{4s}\times x_1 & & \\
 & & Q_{\beta\kappa}\times z_{\beta\kappa} & 0 \cdots\cdots 0 & & & I_{2s}\times x_{\beta} & \\
 & & Q_{\alpha\kappa}\times z_{\alpha\kappa} & 0 \cdots\cdots\cdots 0 & & & & I_{2s}\times x_{\alpha}
\end{array}\right)$$

where

$$a = y_{\lambda_1}e_{\lambda_1} + \cdots\cdots + y_{\kappa}e_{\kappa} + x_{\xi}e_{\xi} + \cdots\cdots + x_{\alpha}e_{\alpha} + z_{\xi\lambda_1}u_1^{(\xi)} + \cdots\cdots + z_{\alpha\kappa}u^{(\alpha)} + \cdots\cdots$$

and

$$R(u_1^{(\xi)}) = \left(\begin{array}{c} \\ Q_{\xi\lambda_1} \\ \\ \end{array}\right) = \left(\begin{array}{c} \overbrace{}^{s}\ \overbrace{}^{s} \\ \begin{pmatrix} 1\cdots\cdots0 & 0\cdots\cdots0 \\ \vdots\ \ddots\ \vdots & 1\cdots\cdots0 \\ \vdots\ \ \ \ddots\ \vdots & \vdots\ \ddots\ \vdots \\ 0\cdots\cdots1 & \vdots\ \ \ \ddots\ \vdots \\ 0\cdots\cdots0 & 0\cdots\cdots1 \end{pmatrix} \end{array}\right),$$

$$R(u_1^{(t)}) = \left(\quad Q_{t\lambda_1} \quad \right) = \left(\begin{array}{l} 1\ 0 \cdots\cdots\cdots 0 \cdots\cdots\cdots \\ 0\ 0 \cdots\cdots\cdots 0 \cdots\cdots\cdots \\ 0\ 0 \cdots\cdots\cdots 0 \cdots\cdots\cdots \\ 0\ 0 \cdots\cdots 0\ 1\ 0 \cdots\cdots \\ 0\ 1 \cdots\cdots\cdots 0\ 0 \cdots\cdots \\ 0\ 0 \cdots\cdots\cdots\cdots 0 \cdots\cdots \\ 0\ 0 \cdots\cdots\cdots 0\ 0 \cdots\cdots \\ 0\ 0 \cdots\cdots\cdots 0\ 1 \cdots\cdots \\ \quad \ddots \qquad\qquad \ddots \\ \qquad\quad \ddots \qquad\qquad \ddots \\ \underbrace{\qquad\qquad\qquad}_{s} \ \underbrace{\qquad\qquad\qquad}_{s} \end{array} \right),$$

$$R(u_2^{(\eta)}) = \left(\quad Q_{\eta,\lambda_2} \quad \right) = \left(\overbrace{\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}}^{s} \overbrace{\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}}^{s} \right),$$

$$R(u_2^{(t)}) = \left(\quad Q_{t,\lambda_2} \quad \right) = \left(\begin{pmatrix} 0 \cdots\cdots\cdots\cdots 0 \cdots\cdots\cdots \\ 1\ 0 \cdots\cdots\cdots 0 \cdots\cdots\cdots \\ 0 \cdots\cdots\cdots 0\ 1\ 0 \cdots\cdots \\ 0\ 0 \cdots\cdots\cdots 0\ 0 \cdots\cdots \\ 0\ 0 \cdots\cdots\cdots\cdots 0 \cdots\cdots \\ 0\ 1\ 0 \cdots\cdots\cdots 0 \cdots\cdots \\ 0\ 0 \cdots\cdots\cdots 0\ 1\ 0 \cdots \\ 0\ 0 \cdots\cdots\cdots\cdots 0 \cdots\cdots \\ \quad \ddots \qquad\qquad \ddots \\ \qquad\quad \ddots \qquad\qquad \ddots \\ \underbrace{\qquad\qquad\qquad}_{s} \ \underbrace{\qquad\qquad\qquad}_{s} \end{pmatrix} \right),$$

$$R(u^{(t)}) = \left(\quad Q_{t,t-1} \quad \right) = \left(\overbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}^{4s} \right),$$

$\cdots\cdots\cdots\cdots\cdots\cdots$

$$R(\bar{u}^{(1)}) = \left(\quad Q_{1,1} \quad \right) = \left(\overbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}^{4s} \right), \quad R(u^{(1)}) = \left(\quad Q_{1,\kappa} \quad \right) = \left(\overbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}^{4s} \right),$$

$$R(u^{(\beta)}) = \begin{pmatrix} \\ Q_{\beta,\kappa} \\ \\ \end{pmatrix} = \left(\begin{pmatrix} 1\ 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\ 0\ 1\ 0 \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots 0\ 0\ 1\ 0 \cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots 0\ 0 \cdots\cdots\cdots \\ \ddots \\ \ddots \\ \ddots \end{pmatrix}\right)$$

$$R(u^{(\alpha)}) = \begin{pmatrix} \\ Q_{\alpha,\kappa} \\ \\ \end{pmatrix} = \left(\begin{pmatrix} 1\ 1\ 0\ 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\ 0\ 1\ 1\ 0\ 0 \cdots\cdots\cdots\cdots\cdots\cdots \\ 0\ 0\ 0\ 0\ 1\ 1 \cdots\cdots\cdots\cdots\cdots\cdots \\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \cdots\cdots\cdots\cdots \\ \cdots\cdots 0\ 0 \cdots\cdots\cdots\cdots \ddots \cdots\cdots\cdots \\ \ddots \\ \ddots \\ \ddots \end{pmatrix}\right)$$

and any commutator B of $R(a)$ has the following form;

$$B = \begin{pmatrix} B_{\lambda_1} & & & & & & & & & \\ & B_{\lambda_2} & & & & & & & 0 & \\ & & B_{t-1} & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & B_\kappa & & & & & \\ & & & & & B'_\xi & & & & \\ & & & & & & B'_\eta & & & \\ & & * & & & & & B'_t & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & B'_\alpha \end{pmatrix}$$

where $Q_{\xi,\lambda_1}B_{\lambda_1} = B'_\xi Q_{\xi,\lambda_1}$, $Q_{t,\lambda_1}B_{\lambda_1} = B'_t Q_{t,\lambda_1}$, $Q_{\eta,\lambda_2}B_{\lambda_2} = B'_\eta Q_{\eta,\lambda_2}$, $Q_{t,\lambda_2}B_{\lambda_2} = B'_t Q_{t,\lambda_2} \cdots\cdots Q_{1,\kappa}B_\kappa = B'_1 Q_{1,\kappa}$, $Q_{\beta\kappa}B_\kappa = B'_\beta Q_{\beta\kappa}$, $Q_{\alpha,\kappa}B_\kappa = B'_\alpha Q_{\alpha,\kappa}$.

Then from $Q_{t,t-1}B_{t-1} = B'_t Q_{t,t-1}$, $Q_{t-1,t-1}B_{t-1} = B'_{t-1}Q_{t-1,t-1}$, $\cdots\cdots$ and $Q_{1,\kappa}B_\kappa = B'_1 Q_{1,\kappa}$, we have $B_{t-1} = B_{t-2} = \cdots\cdots = B_\kappa = B'_t = \cdots\cdots = B'_1$.

Now put $B_{t-1} = \bar{B} = [\bar{b}_{ij}]$ and from $Q_{\beta\kappa}B_\kappa = B'_\beta Q_{\beta\kappa}$ we have

$$\bar{B} = \begin{pmatrix} \bar{b}_{11} & 0 & \bar{b}_{13} & 0 \cdots\cdots \bar{b}_{1,4s-1} & 0 \\ \bar{b}_{21} & \bar{b}_{22} & \bar{b}_{23} & \cdots\cdots\cdots \bar{b}_{2,4s-1}, & b_{2,4s} \\ \bar{b}_{31} & 0 & \bar{b}_{33} & 0 \cdots\cdots \bar{b}_{3,4s-1}, & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \end{pmatrix} \quad \text{and} \quad B'_\beta = \begin{pmatrix} \bar{b}_{11} & \bar{b}_{13} \cdots\cdots \bar{b}_{1,4s-1} \\ \bar{b}_{31} & \bar{b}_{32} \cdots\cdots \bar{b}_{3,4s-1} \\ \cdots\cdots\cdots & \cdots\cdots \end{pmatrix}.$$

On the other hand from $Q_{t,\lambda_1}B_{\lambda_1}=B'_tQ_{t,\lambda_1}$, $Q_{t,\lambda_2}B_{\lambda_2}=B'_tQ_{t\ \lambda_2}$, $Q_{\eta,\lambda_2}B_{\lambda_2}=B'_\eta Q_{\eta,\lambda_2}$ and $Q_{\alpha\kappa}B_\kappa=B'_\alpha Q_{\alpha\kappa}$, we have $B_{\lambda_1}=B_{\lambda_2}=\begin{bmatrix} B_0 & \\ & B_0\end{bmatrix}$ where

$$B_0=\begin{pmatrix} \bar{b}_{22} & \bar{b}_{26}\cdots\cdots \\ \bar{b}_{62} & \bar{b}_{66}\cdots\cdots \\ \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \end{pmatrix}=B'_\eta \quad\text{and}\quad B'_\alpha=\begin{pmatrix} \bar{b}_{22} & \bar{b}_{24}\cdots\cdots \\ \bar{b}_{42} & \bar{b}_{44}\cdots\cdots \\ \cdots\cdots\cdots\cdots & \end{pmatrix}.$$

Next from $Q_{\xi,\lambda_1}B_{\lambda_1}=B'_\xi Q_{\xi\lambda_1}$ we have

$$B_0=\begin{pmatrix} b & & \\ & \ddots & \\ & & b\end{pmatrix} \quad\text{and}\quad B'_\xi=\begin{pmatrix} b & & \\ & \ddots & \\ & & b\end{pmatrix}. \quad \text{Hence} \quad B=\begin{pmatrix} b & & \\ & \ddots & \\ * & & b\end{pmatrix}$$

and $R(a)$ is directly indecomposable and $\mathfrak{M}$ is directly indecomposable.

5. **Lemma 5.** *Suppose that* $Ne_1=Au_1^{(\lambda)}\oplus Au_1^{(2)}$, $Ne_2=Au_2^{(\lambda)}\oplus Au_2^{(3)}$, $Ne_3=Au_3^{(\lambda)}\oplus Au_3^{(4)}$ *where* $Au_1^{(\lambda)}\cong Au_2^{(\lambda)}\cong Au_3^{(\lambda)}$. *Then there is a directly indecomposable A-left module of arbitrary high degrees.*

Proof. We construct an A-left module $\mathfrak{M}$ as follows;

$$\mathfrak{M}=Ae_1m_{1,1}+Ae_2m_{2,1}+Ae_2m_{2,2}+Ae_3m_{3,1}+Ae_3m_{3,2}+Ae_1m_{1,2}+Ae_1m_{1,3}+\cdots\cdots$$
$$+Ae_3m_{3,2s}+Ae_1m_{1,2s},$$

where

$$u_1^{(2)}m_{1,1}\neq 0,\ u_1^{(\lambda)}m_{1,1}=u_2^{(\lambda)}m_{2,1},\ u_2^{(3)}m_{2,1}=u_2^{(3)}m_{2,2},\ \cdots\cdots$$
$$u_3^{(\lambda)}m_{3,2s}=u_1^{(\lambda)}m_{1,2s},\ u_1^{(2)}m_{1,2s}\neq 0.$$

Then the representation $R(a)$ by $\mathfrak{M}$ has the following form;

$$R(a)=\begin{pmatrix} I_{2s}\times y_1 & & & & & & \\ 0 & I_{2s}\times y_2 & & & & & \\ 0 & 0 & I_{2s}\times y_3 & & & & \\ Q_{\lambda 1}\times z_{\lambda 1} & Q_{\lambda 2}\times z_{\lambda 2} & Q_{\lambda 3}\times z_{\lambda 3} & I_{3s}\times x_\lambda & & & \\ Q_{21}\times z_{21} & 0 & 0 & 0 & I_{s+1}\times x_2 & & \\ & Q_{32}\times z_{32} & 0\cdots & \cdots 0 & & I_s\times x_3 & \\ & & Q_{43}\times z_{43} & 0\cdots & \cdots 0 & & I_s\times x_4 \end{pmatrix}$$

where

$$a=y_1e_1+y_2e_2+y_3e_3+x_\lambda e_\lambda+x_2e'_2+x_3e'_3+x_4e'_4+z_{\lambda 1}u_1^{(\lambda)}+\cdots\cdots+z_{43}u_3^{(4)}+\cdots\cdots.$$

and

$$R(u_1^{(\lambda)}) = \begin{pmatrix} & & \\ & Q_{\lambda_1} & \\ & & \end{pmatrix} = \left(\begin{array}{c@{}l} \begin{array}{cc} \overbrace{\hphantom{0\cdots0}}^{s} & \overbrace{\hphantom{0\cdots0}}^{s} \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} & \begin{array}{l} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} s \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} s \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} s \end{array} \end{array} \right),$$

$$R(u_2^{(\lambda)}) = \begin{pmatrix} & & \\ & Q_{\lambda_2} & \\ & & \end{pmatrix} = \left(\begin{array}{c@{}l} \begin{array}{cc} \overbrace{\hphantom{0\cdots0}}^{s} & \overbrace{\hphantom{0\cdots0}}^{s} \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} & \begin{array}{l} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} s \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} s \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} s \end{array} \end{array} \right),$$

$$R(u_3^{(\lambda)}) = \begin{pmatrix} & & \\ & Q_{\lambda_3} & \\ & & \end{pmatrix} = \left(\begin{array}{c@{}l} \begin{array}{cc} \overbrace{\hphantom{0\cdots0}}^{s} & \overbrace{\hphantom{0\cdots0}}^{s} \end{array} & \\ \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} & \begin{array}{l} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} s \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} s \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} s \end{array} \end{array} \right),$$

$$R(u_1^{(2)}) = \begin{pmatrix} & & \\ & Q_{21} & \\ & & \end{pmatrix} = \left(\begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{1\cdots\cdots0}}^{s} & \overbrace{\hphantom{0\cdots\cdots0}}^{s} \end{array} \\ \begin{pmatrix} 1 & \cdots\cdots & 0 & 0 & \cdots\cdots & 0 \\ \vdots & \ddots & \vdots & 1 & \cdots\cdots & 0 \\ \vdots & & \ddots\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots\cdots & 1 & \vdots & & \ddots\ \vdots \\ 0 & \cdots\cdots & 0 & 0 & \cdots\cdots & 1 \end{pmatrix} \end{array} \right),$$

$$R(u_2^{(3)}) = \begin{pmatrix} & & \\ & Q_{32} & \\ & & \end{pmatrix} = \left(\begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{1\cdots\cdots0}}^{s} & \overbrace{\hphantom{1\cdots\cdots0}}^{s} \end{array} \\ \begin{pmatrix} 1 & \cdots\cdots & 0 & 1 & \cdots\cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & & \ddots\ \vdots & \vdots & & \ddots\ \vdots \\ 0 & \cdots\cdots & 1 & 0 & \cdots\cdots & 1 \end{pmatrix} \end{array} \right),$$

$$R(u_3^{(4)}) = \begin{pmatrix} & & \\ & Q_{43} & \\ & & \end{pmatrix} = \left(\begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{1\cdots\cdots0}}^{s} & \overbrace{\hphantom{1\cdots\cdots0}}^{s} \end{array} \\ \begin{pmatrix} 1 & \cdots\cdots & 0 & 1 & \cdots\cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & & \ddots\ \vdots & \vdots & & \ddots\ \vdots \\ 0 & \cdots\cdots & 1 & 0 & \cdots\cdots & 1 \end{pmatrix} \end{array} \right),$$

and any commutator B of $R(a)$ is as follows;

$$B = \begin{pmatrix} B_1 & & & & & & \\ & B_2 & & & & & \\ & & B_3 & & & & \\ & & & B'_\lambda & & & \\ & & * & & B'_2 & & \\ & & & & & B'_3 & \\ & & & & & & B'_4 \end{pmatrix}$$

where $Q_{\lambda 1}B_1 = B'_\lambda Q_{\lambda 1}$, $Q_{\lambda 2}B_2 = B'_\lambda Q_{\lambda 2}$, $Q_{\lambda 3}B_3 = B'_\lambda Q_{\lambda 3}$, $Q_{21}B_1 = B'_2 Q_{21}$, $Q_{32}B_2 = B'_3 Q_{32}$, $Q_{43}B_3 = B'_4 Q_{43}$. Hence by the same computation as above lemmas, from $Q_{\lambda 1}B_1 = B'_\lambda Q_{\lambda 1}$, $Q_{\lambda 2}B_2 = B'_\lambda Q_{\lambda 2}$ and $Q_{\lambda 3}B_3 = B'_\lambda Q_{\lambda 3}$ we have

$$B'_\lambda = \begin{pmatrix} \overline{B}_1 & 0 & 0 \\ 0 & \overline{B}_2 & 0 \\ 0 & 0 & \overline{B}_3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \overline{B}_2 & \\ & \overline{B}_3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \overline{B}_2 & \\ & \overline{B}_1 \end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix} \overline{B}_1 & \\ & \overline{B}_3 \end{pmatrix}.$$

Next from $Q_{32}B_2 = B'_3 Q_{32}$ and $Q_{43}B_3 = B'_4 Q_{43}$ we have $\overline{B}_1 = \overline{B}_2 = \overline{B}_3 = B'_3 = B'_4$. Moreover from $Q_{21}B_1 = B'_2 Q_{21}$ we have

$$\overline{B}_1 = \begin{pmatrix} b & & \\ & \ddots & \\ & & b \end{pmatrix} = \overline{B}_2 = \overline{B}_3, \quad B'_2 = \begin{pmatrix} b & & \\ & \ddots & \\ & & b \end{pmatrix}. \quad \text{Hence } B = \begin{pmatrix} b & & \\ & \ddots & \\ * & & b \end{pmatrix}.$$

Therefore $R(a)$ is directly indecomposable and $\mathfrak{M}$ is directly indecomposable.

6. Lemma 6. *Suppose that* $Ne_1 = Au_1^{(\kappa_1)} \oplus Au_1^{(\kappa_2)}$, $Ne_2 = Au_2^{(\kappa_2)} \oplus Au_2^{(\kappa_3)}$, $Ne_3 = Au_3^{(\kappa_3)} \oplus A_3^{(0)} \oplus Au_3^{(\kappa_4)}$, $Ne_4 = Au_4^{(\kappa_4)}$, *where* $Au_i^{(\kappa_{i+1})} \cong Au_{i+1}^{(\kappa_{i+1})}$. *Then there exists a directly indecomposable A-left module of arbitrary high degrees.*

Proof. Now we construct $\mathfrak{M}$ as follows;

$$\begin{aligned} \mathfrak{M} = {} & Ae_4 m_{4,1} + Ae_4 m_{4,2} + \cdots\cdots\cdots + Ae_4 m_{4,2s} \\ & + Ae_3 m_{3,1} + Ae_3 m_{3,2} + \cdots\cdots\cdots + Ae_3 m_{3,6s+1} \\ & + Ae_2 m_{2,1} + Ae_2 m_{2,2} + \cdots\cdots\cdots + Ae_2 m_{2,4s+1} \\ & + Ae_1 m_{1,1} + Ae_1 m_{1,2} + \cdots\cdots\cdots + Ae_1 m_{1,2s+1} \end{aligned}$$

where $u_4^{(\kappa_4)} m_{4,1} = u_3^{(\kappa_4)} m_{3,1} + u_3^{(\kappa_4)} m_{3,3} + u_3^{(\kappa_4)} m_{3,4}$, $u_4^{(\kappa_4)} m_{4,2} = u_3^{(\kappa_4)} m_{3,4} + u_3^{(\kappa_4)} m_{3,6} + u_3^{(\kappa_4)} m_{3,7}$, $\cdots\cdots$, $u_4^{(\kappa_4)} m_{4,2s} = u_3^{(\kappa_4)} m_{3,6s-2} + u_3^{(\kappa_4)} m_{3,6s} + u_3^{(\kappa_4)} m_{3,6s+1}$, $u_3^{(0)} m_{3,1} = 0$, $u_3^{(\kappa_4)} m_{3,2} = 0$, $u_3^{(0)} m_{3,3} = 0$, $u_3^{(\kappa_3)} m_{3,4} = 0$, $u_3^{(\kappa_4)} m_{3,5} = 0$, $u_3^{(0)} m_{3,6} = 0$, $\cdots\cdots$, $u_3^{(\kappa_4)} m_{3,6s-4} = 0$, $u_3^{(0)} m_{3,6s-3} = 0$, $u_3^{(\kappa_3)} m_{3,6s-2} = 0$, $u_3^{(\kappa_4)} m_{3,6s-1} = 0$, $u_3^{(0)} m_{3,6s} = 0$, $u_3^{(0)} m_{3,6s+1} = 0$, $u_3^{(\kappa_3)} m_{3,1} = u_2^{(\kappa_3)} m_{2,1}$, $u_3^{(\kappa_3)} m_{3,2} = u_2^{(\kappa_3)} m_{2,2}$ $u_3^{(\kappa_3)} m_{3,3} = u_2^{(\kappa_3)} m_{2,3}$, $u_2^{(\kappa_3)} m_{2,4} = u_3^{(\kappa_3)} m_{3,5} + u_3^{(\kappa_3)} m_{3,6}$, $\cdots\cdots$, $u_3^{(\kappa_3)} m_{3,\lambda} = u_2^{(\kappa_3)} m_{2,\lambda-2p}$, $u_2^{(\kappa_3)} m_{2,4+4q} = u_3^{(\kappa_3)} m_{3,5+6q} + u_3^{(\kappa_3)} m_{3,6+6q}$, $u_2^{(\kappa_2)} m_{2,1} = u_1^{(\kappa_2)} m_{1,1}$, $u_1^{(\kappa_2)} m_{1,2} = u_2^{(\kappa_2)} m_{2,2} + u_2^{(\kappa_2)} m_{2,3}$, $u_2^{(\kappa_2)} m_{2,4} = 0$, $\cdots\cdots$, $u_2^{(\kappa_2)} m_{2,\lambda} = u_1^{(\kappa_2)} m_{1,\lambda-2p}$, $u_1^{(\kappa_2)} m_{1,2p} = u_2^{(\kappa_2)} m_{2,2+4(p-1)} + u_2^{(\kappa_2)} m_{2,3+4(p-1)}$,

$u_2^{(\kappa_2)}m_{2,4\gamma}=0$, $u_1^{(\kappa_1)}m_{1,1}\neq 0$, $u_1^{(\kappa_1)}m_{1,2}=0$, $\cdots\cdots$, $u_1^{(\kappa_1)}m_{1,2s}=0$, $\cdots\cdots$, $u_1^{(\kappa_1)}m_{1,2s+1}$ $\neq 0$.

Then the representation $R(a)$ by $\mathfrak{M}$ has the following form;

$$R(a)=\left(\begin{array}{ccccccccc}
I_{2s}\times y_4 &&&&&&&& \\
0 & I_{6s+1}\times y_3 &&&&&&& \\
0 & 0 & I_{4s+1}\times y_2 &&&&&& \\
0 & 0 & 0 & I_{2s+1}\times y_1 &&&&& \\
Q_{44}\times z_{44} & Q_{43}\times z_{43} & 0 & 0 & I_{4s+1}\times x_{\kappa_4} &&&& \\
 & Q_{03}\times z_{03} & 0 & 0 & 0 & I_{3s}\times x_{\kappa_0} &&& \\
 & Q_{33}\times z_{33} & Q_{32}\times z_{32} & 0 & 0 & 0 & I_{5s+1}\times x_{\kappa_3} && \\
 & & Q_{22}\times z_{22} & Q_{21}\times z_{21} & 0 & 0 & 0 & I_{3s+1}\times x_{\kappa_2} & \\
 & & & Q_{11}\times z_{11} & 0 & 0 & 0 & 0 & I_{s+1}\times x_{\kappa_1}
\end{array}\right),$$

where $a=y_4e_4+\cdots+y_1e_1+x_{\kappa_4}e_{\kappa_4}+\cdots+x_{\kappa_1}e_{\kappa_1}+z_{44}u_4^{(\kappa_4)}+\cdots+z_{11}u_1^{(\kappa_1)}+\cdots$

and

$$R(u_4^{(\kappa_4)})=\left(\begin{array}{c} \\ Q_{44} \\ \\ \end{array}\right)=\left(\left(\begin{array}{cccccc}
1 & 0\cdots\cdots\cdots\cdots\cdots\cdots &&&& \\
1 & 0\cdots\cdots\cdots\cdots\cdots\cdots &&&& \\
1 & 1 & 0\cdots\cdots\cdots\cdots\cdots &&& \\
0 & 1 & 0\cdots\cdots\cdots\cdots\cdots &&& \\
0 & 1 & 1 & 0\cdots\cdots\cdots\cdots && \\
 & 0 & 1 & 0\cdots\cdots\cdots\cdots && \\
 & 0 & 1 & 1 & 0\cdots\cdots\cdots & \\
 &&& \ddots && \\
 &&&& \ddots & \\
 &&&& \ddots & 0 \\
 &&&& \cdots\cdots\cdots\cdots 1 & 1 \\
 &&&& \cdots\cdots\cdots\cdots 0 & 1
\end{array}\right) \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right.$$

$$\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \cdots\cdots\cdots\cdots 0 \;\; 1 \end{array}$$

$$R(u_3^{(\kappa_4)})=\left(\begin{array}{c} \\ Q_{43} \\ \\ \end{array}\right)=\left(\begin{array}{c}
\begin{array}{cc} \overbrace{\hspace{12em}}^{3s+1} & \overbrace{\hspace{12em}}^{3s} \end{array} \\
\left(\begin{array}{ccccccc:ccccccc}
1 & 0 & 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 &&&&& 0 & 0 & 0\cdots\cdots\cdots\cdots\cdots\cdots &&&& \\
0 & 1 & 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &&&&& \cdots 0 & 0 & 0\cdots\cdots\cdots\cdots\cdots\cdots &&&& \\
0 & 0 & 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &&&&& \cdots 0 & 1 & 0\cdots\cdots\cdots\cdots\cdots\cdots &&&& \\
0 & 0 & 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &&&&& \cdots 0 & 0 & 1\cdots\cdots\cdots\cdots\cdots\cdots &&&& \\
\cdots\cdots\cdots 1 & 0 & 0\cdots\cdots\cdots\cdots\cdots &&&&& \cdots\cdots\cdots\cdots 0 & 0 & 0\cdots\cdots\cdots\cdots &&&& \\
\cdots\cdots\cdots 0 & 1 & 0\cdots\cdots\cdots\cdots\cdots &&&&& \cdots\cdots\cdots\cdots 0 & 0 & 0\cdots\cdots\cdots\cdots &&&& \\
\cdots\cdots\cdots 0 & 0 & 0\cdots\cdots\cdots\cdots\cdots &&&&& \cdots\cdots\cdots\cdots 0 & 1 & 0\cdots\cdots\cdots\cdots &&&& \\
\cdots\cdots\cdots 0 & 0 & 0\cdots\cdots\cdots\cdots\cdots &&&&& \cdots\cdots\cdots\cdots 0 & 0 & 1\cdots\cdots\cdots\cdots &&&& \\
 & \ddots &&&&&& & \ddots &&&&& \\
 && \ddots &&&&& && \ddots &&&& \\
 &&& \cdots\cdots\cdots\cdots 1 & 0 & 0\cdots && \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 & 0 & 0 &&&& \\
 &&& \cdots\cdots\cdots\cdots 0 & 1 & 0\cdots && \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 & 0 & 0 &&&& \\
 &&& \cdots\cdots\cdots\cdots 0 & 0 & 0\cdots && \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 & 1 & 0 &&&& \\
 &&& \cdots\cdots\cdots\cdots 0 & 0 & 0 & 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 & 0 & 1 &&&& \\
 &&& \cdots\cdots\cdots\cdots 0 & 0 & 0 & 1 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 & 0 & 0 &&&&
\end{array}\right)
\end{array}\right)$$

$$R(u_3^{(0)}) = \left(\begin{array}{c} \\ Q_{03} \\ \\ \end{array}\right) = \left(\begin{array}{cc} \overbrace{\hphantom{0\;0\;0\cdots\cdots\cdots\cdots\cdots\cdots\cdots0}}^{3s+1} & \overbrace{\hphantom{1\;0\;0\cdots\cdots\cdots\cdots\cdots\cdots}}^{3s} \\ \multicolumn{2}{c}{\left(\begin{array}{lcl}
0\;0\;0\cdots\cdots\cdots\cdots\cdots\cdots & 0 & 1\;0\;0\cdots\cdots\cdots\cdots\cdots \\
0\;0\;0\cdots\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots0\;1\;0\cdots\cdots\cdots\cdots\cdots \\
0\;0\;1\cdots\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots0\;0\;0\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots0\;0\;0\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots1\;0\;0\cdots\cdots\cdots \\
\cdots\cdots0\;0\;0\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots0\;1\;0\cdots\cdots\cdots \\
\cdots\cdots0\;0\;1\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots0\;0\;0\cdots\cdots\cdots \\
\qquad\qquad\ddots & \vdots & \qquad\qquad\qquad\ddots \\
\qquad\cdots\cdots0\;0\;0 & 0 & \cdots\cdots\cdots\cdots\cdots\cdots1\;0\;0 \\
\qquad\cdots\cdots0\;0\;0 & 0 & \cdots\cdots\cdots\cdots\cdots\cdots0\;1\;0 \\
\qquad\cdots\cdots0\;0\;1 & 0 & \cdots\cdots\cdots\cdots\cdots\cdots0\;0\;0
\end{array}\right)} \end{array}\right)$$

$$R(u_3^{(\kappa_3)}) = \left(\begin{array}{c} \\ Q_{33} \\ \\ \end{array}\right) = \left(\begin{array}{cc} \overbrace{\hphantom{1\;0\;0\cdots\cdots\cdots\cdots\cdots\cdots0}}^{3s+1} & \overbrace{\hphantom{0\;0\;0\cdots\cdots\cdots\cdots\cdots\cdots}}^{3s} \\ \multicolumn{2}{c}{\left(\begin{array}{lcl}
1\;0\;0\cdots\cdots\cdots\cdots\cdots & 0 & 0\;0\;0\cdots\cdots\cdots\cdots\cdots \\
0\;0\;0\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots1\;0\;0\cdots\cdots\cdots\cdots\cdots \\
0\;1\;0\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots0\;0\;0\cdots\cdots\cdots\cdots\cdots \\
0\;0\;1\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots0\;0\;0\cdots\cdots\cdots\cdots\cdots \\
0\;0\;0\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots0\;0\;1\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots1\;0\;0 & \vdots & \qquad\ddots \\
\cdots\cdots0\;0\;0 & \vdots & \\
\cdots\cdots0\;1\;0 & \vdots & \qquad\qquad\ddots \\
\cdots\cdots0\;0\;1 & \vdots & \\
\cdots\cdots0\;0\;0 & \vdots & \qquad\qquad\qquad\ddots \\
\qquad\qquad\ddots & \vdots & \\
\qquad\cdots\cdots1\;0\;0\cdots & \vdots & \cdots\cdots\cdots\cdots\cdots\cdots0\;0\;0 \\
\qquad\cdots\cdots0\;0\;0\cdots & \vdots & \cdots\cdots\cdots\cdots\cdots\cdots1\;0\;0 \\
\qquad\cdots\cdots0\;1\;0\cdots & \vdots & \cdots\cdots\cdots\cdots\cdots\cdots0\;0\;0 \\
\qquad\cdots\cdots0\;0\;1\cdots & \vdots & \cdots\cdots\cdots\cdots\cdots\cdots0\;0\;0 \\
\qquad\cdots\cdots0\;0\;0 & 0 & \cdots\cdots\cdots\cdots\cdots\cdots0\;0\;1 \\
\qquad\cdots\cdots0\;0\;0 & 1 & \cdots\cdots\cdots\cdots\cdots\cdots0\;0\;0
\end{array}\right)} \end{array}\right)$$

$$R(u_2^{(\kappa_3)}) = \left(\begin{array}{c} \\ Q_{32} \\ \\ \end{array}\right) = \left(\begin{array}{cc} \overbrace{\hphantom{1\;0\cdots\cdots\cdots\cdots\cdots0}}^{2s+1} & \overbrace{\hphantom{0\;0\cdots\cdots\cdots\cdots\cdots}}^{2s} \\ \multicolumn{2}{c}{\left(\begin{array}{lcl}
1\;0\cdots\cdots\cdots\cdots\cdots & 0 & 0\;0\cdots\cdots\cdots\cdots\cdots \\
0\;0\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots1\;0\cdots\cdots\cdots\cdots\cdots \\
0\;1\;0\cdots\cdots\cdots\cdots & \vdots & \cdots0\;0\cdots\cdots\cdots\cdots\cdots \\
0\;0\;0\cdots\cdots\cdots\cdots & \vdots & \cdots0\;1\;0\cdots\cdots\cdots\cdots \\
0\;0\;0\cdots\cdots\cdots\cdots & \vdots & \cdots0\;1\;0\cdots\cdots\cdots\cdots \\
\quad1\;0\cdots\cdots\cdots & \vdots & \cdots\cdots0\;0\cdots\cdots\cdots \\
\quad0\;0\cdots\cdots\cdots & \vdots & \cdots\cdots1\;0\cdots\cdots\cdots \\
\quad0\;1\;0\cdots\cdots & \vdots & \cdots\cdots0\;0\cdots\cdots\cdots \\
\quad0\;0\;0\cdots\cdots & \vdots & \cdots\cdots0\;1\;0\cdots\cdots \\
\quad0\;0\;0\cdots\cdots & \vdots & \cdots\cdots0\;1\;0\cdots\cdots \\
\qquad\qquad\ddots & \vdots & \qquad\qquad\ddots \\
\qquad\qquad1\;0\cdots & \vdots & \cdots\cdots\cdots\cdots\cdots0\;0 \\
\qquad\qquad0\;0 & 0 & \cdots\cdots\cdots\cdots\cdots1\;0 \\
\qquad\qquad0\;1 & 0 & \cdots\cdots\cdots\cdots\cdots0\;0 \\
\qquad\qquad0\;0 & 0 & \cdots\cdots\cdots\cdots\cdots0\;1 \\
\qquad\qquad0\;0 & 0 & \cdots\cdots\cdots\cdots\cdots0\;1 \\
\qquad\qquad0\;0 & 1 & \cdots\cdots\cdots\cdots\cdots0\;0
\end{array}\right)} \end{array}\right)$$

$$R(u_2^{(\kappa_2)})=\begin{pmatrix} & & \\ & Q_{22} & \\ & & \end{pmatrix}=\left(\begin{array}{c} \overbrace{\qquad\qquad\qquad\qquad}^{2s+1}\quad\overbrace{\qquad\qquad\qquad}^{2s} \\ \begin{pmatrix} 1\ 0\cdots\cdots\cdots\cdots\cdots\cdots0 & 0\ 0\cdots\cdots\cdots\cdots\cdots \\ 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\vdots & \cdots1\ 0\cdots\cdots\cdots\cdots\cdots \\ 0\ 1\cdots\cdots\cdots\cdots\cdots\cdots\vdots & \cdots0\ 0\cdots\cdots\cdots\cdots\cdots \\ \quad 1\ 0\cdots\cdots\cdots\cdots\cdots\vdots & \cdots\cdots0\ 0\cdots\cdots\cdots \\ \quad 0\ 0\cdots\cdots\cdots\cdots\cdots\vdots & \cdots\cdots1\ 0\cdots\cdots\cdots \\ \quad 0\ 1\cdots\cdots\cdots\cdots\cdots\vdots & \cdots\cdots0\ 0\cdots\cdots\cdots \\ \qquad\ddots\qquad\qquad\qquad\vdots & \qquad\qquad\ddots \\ \qquad\qquad\ddots\qquad\qquad\vdots & \qquad\qquad\qquad\ddots \\ \qquad\qquad\qquad\ddots 1\ 0\cdots\vdots & \cdots\cdots\cdots\cdots\cdots\cdots0\ 0 \\ \qquad\qquad\qquad\quad 0\ 0\cdots\vdots & \cdots\cdots\cdots\cdots\cdots\cdots1\ 0 \\ \qquad\qquad\qquad\quad 0\ 1\ 0 & \cdots\cdots\cdots\cdots\cdots\cdots0\ 0 \\ \qquad\qquad\qquad\quad 0\ 0\ 1 & \cdots\cdots\cdots\cdots\cdots\cdots0\ 0 \end{pmatrix}\end{array}\right)$$

$$R(u_1^{(\kappa_2)})=\begin{pmatrix} & & \\ & Q_{21} & \\ & & \end{pmatrix}=\left(\begin{array}{c} \overbrace{\qquad\qquad\quad}^{s+1}\quad\overbrace{\qquad\qquad}^{s} \\ \begin{pmatrix} 1\ 0\cdots\cdots\cdots0 & 0\cdots\cdots\cdots \\ 0\cdots\cdots\cdots\cdots\vdots & \cdots1\ 0\cdots\cdots \\ 0\ 0\cdots\cdots\cdots\vdots & \cdots1\ 0\cdots\cdots \\ \quad 1\ 0\cdots\cdots\vdots & \cdots\cdots0\cdots\cdots \\ \quad 0\cdots\cdots\cdots\vdots & \cdots\cdots1\cdots\cdots \\ \quad 0\cdots\cdots\cdots\vdots & \cdots\cdots1\cdots\cdots \\ \qquad\ddots\qquad\quad\vdots & \qquad\ddots \\ \qquad\quad\ddots\qquad\vdots & \qquad\quad\ddots \\ \qquad\qquad\ddots\quad\vdots & \qquad\qquad\ddots \\ \qquad\qquad\quad 1\ 0 & \cdots\cdots\cdots\cdots0 \\ \qquad\qquad\quad 0\cdots & \cdots\cdots\cdots\cdots1 \\ \qquad\qquad\quad 0\ 0 & \cdots\cdots\cdots\cdots1 \\ \qquad\qquad\quad 0\ 1 & \cdots\cdots\cdots\cdots0 \end{pmatrix}\end{array}\right)$$

$$R(u_1^{(\kappa_1)})=\begin{pmatrix} & & \\ & Q_{11} & \\ & & \end{pmatrix}=\left(\begin{array}{cc} \overbrace{\qquad\quad}^{s+1} & \overbrace{\qquad\quad}^{s} \\ \multicolumn{2}{c}{\begin{pmatrix} 1\cdots\cdots0 & 0\cdots\cdots0 \\ \vdots\ \ddots\quad\ \vdots & \vdots\qquad\quad\vdots \\ \vdots\qquad\ddots\ \vdots & \vdots\qquad\quad\vdots \\ 0\cdots\cdots1 & 0\cdots\cdots0 \end{pmatrix}}\end{array}\right)$$

and any commutator B of $R(a)$ has the following form;

$$B=\begin{pmatrix} B_4 & & & & & & & & \\ & B_3 & & & & & & & \\ & & B_2 & & & & & & \\ & & & B_1 & & & & & \\ & & & & B'_{\kappa_4} & & & & \\ & & * & & & B_0' & & & \\ & & & & & & B'_{\kappa_3} & & \\ & & & & & & & B'_{\kappa_2} & \\ & & & & & & & & B'_{\kappa_1} \end{pmatrix}$$

where $Q_{44}B_4=B'_{\kappa_4}Q_{44}$, $Q_{43}B_3=B'_{\kappa_4}Q_{43}$, $Q_{03}B_3=B'_0Q_{03}$, $Q_{33}B_3=B'_{\kappa_3}Q_{33}$,
$Q_{32}B_2=B'_{\kappa_3}Q_{32}$, $Q_{22}B_2=B'_{\kappa_2}Q_{22}$, $Q_{21}B_1=B'_{\kappa_2}Q_{21}$, $Q_{11}B_1=B'_{\kappa_1}Q_{11}$.

Then from $Q_{11}B_1 = B'_{\kappa_1}Q_{11}$, we have $B_1 = \begin{bmatrix} B'_{\kappa_1} & 0 \\ * & * \end{bmatrix}$. Next from $Q_{43}B_3 = B'_{\kappa_4}Q_{43}$, $\cdots\cdots$ and, $Q_{21}B_1 = B'_{\kappa_2}Q_{21}$, we have

$$B'_{\kappa_4} = \begin{pmatrix} b_{11} & 0\cdots\cdots 0 & b_{15} & 0\cdots\cdots\cdots & & \\ 0 & b_{22} & 0\cdots\cdots 0 & b_{26} & & \\ \vdots & 0 & b_{33} & 0\cdots\vdots & \ddots & \\ \vdots & \vdots & 0 & b_{44} \quad \vdots & & \ddots \\ b_{51} & 0 & \vdots & b_{55} & & \ddots \\ & b_{62} & \vdots & & \ddots & \\ & & \ddots & & & \ddots \\ & & & \ddots & & \ddots \end{pmatrix}, \quad B'_0 = \begin{pmatrix} b_{22} & & b_{26} & & & \\ & b_{33} & & b_{37} & & \\ & & b_{44} & & \ddots & \\ b_{62} & & b_{66} & & & \ddots \\ & \ddots & & \ddots & & \ddots \\ & & \ddots & & \ddots & \\ & & & \ddots & & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

and from $Q_{44}B_4 = B'_{\kappa_4}Q_{44}$ we have

$$B'_{\kappa_4} = \begin{pmatrix} b_{11} & & \\ & \ddots & \\ & & b_{11} \end{pmatrix}, \quad B_4 = \begin{pmatrix} b_{11} & & \\ & \ddots & \\ & & b_{11} \end{pmatrix},$$

$$B'_0 = \begin{pmatrix} b_{11} & & \\ & \ddots & \\ & & b_{11} \end{pmatrix}, \quad \text{and} \quad B_i = \begin{pmatrix} b_{11} & & \\ & \ddots & \\ & & b_{11} \end{pmatrix}, \quad B'_j = \begin{pmatrix} b_{11} & & \\ & \ddots & \\ & & b_{11} \end{pmatrix}.$$

Hence B has just one eigenvalue and $R(a)$ is directly indecomposable. Therefore $\mathfrak{M}$ is directly indecomposable.

Moreover by the same way as this lemma we have the following

Lemma 7. *Suppose that* $Ne_1 = Au_1^{(\kappa_1)}$, $Ne_2 = Au_2^{(\kappa_1)} \oplus Au_2^{(\kappa_?)}$, $Ne_3 = Au_3^{(\kappa_2)} + Au_3^{(\kappa_0)} + Au_3^{(\kappa_3)}$ *and* $Ne_4 = Au_4^{(\kappa_3)} + Au_4^{(\kappa_4)}$ *where* $Au_i^{(\kappa_i)} \cong Au_{i+1}^{(\kappa_i)}$. *Then there exists a directly indecomposable A–left module of arbitrary high degrees.*

Proof. Now we construct an A–left module $\mathfrak{M}$ as follows:

$$\begin{aligned} \mathfrak{M} = & Ae_1m_{1,1} + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + Ae_1m_{1,2s} \\ & + Ae_2m_{2,1} + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + Ae_2m_{2,6s+1} \\ & + Ae_3m_{3,1} + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + Ae_3m_{3,11s+1} \\ & + Ae_4m_{4,1} + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + Ae_4m_{4,5s+1} \end{aligned}$$

where $u_1^{(\kappa_1)}m_{1,1} = u_2^{(\kappa_1)}m_{2,1} + u_2^{(\kappa_1)}m_{2,2} + u_2^{(\kappa_1)}m_{2,3}$, $\cdots\cdots$, $u_1^{(\kappa_1)}m_{1,2s} = u_2^{(\kappa_1)}m_{2,6s-3} + u_2^{(\kappa_1)}m_{2,6s-1} + u_2^{(\kappa_1)}m_{2,6s+1}$, $u_2^{(\kappa_1)}m_{2,4} = 0$, $u_2^{(\kappa_1)}m_{2,6} = 0$, $u_2^{(\kappa_1)}m_{2,10} = 0$, $u_2^{(\kappa_1)}m_{2,12} = 0$, $\cdots\cdots$, $u_2^{(\kappa_1)}m_{2,6s-2} = 0$, $u_2^{(\kappa_1)}m_{2,6s} = 0$, $u_2^{(\kappa_2)}m_{2,1} = u_3^{(\kappa_2)}m_{3,1}$, $u_2^{(\kappa_2)}m_{2,2} = u_3^{(\kappa_2)}m_{3,2} + u_3^{(\kappa_2)}m_{3,3}$, $u_2^{(\kappa_2)}m_{2,3} = u_3^{(\kappa_2)}m_{3,4}$, $u_2^{(\kappa_2)}m_{2,4} = u_3^{(\kappa_2)}m_{3,6} + u_3^{(\kappa_2)}m_{3,7}$, $u_2^{(\kappa_2)}m_{2,5} = u_3^{(\kappa_2)}m_{3,8}$, $u_2^{(\kappa_2)}m_{2,6} = u_3^{(\kappa_2)}m_{3,10} + u_3^{(\kappa_2)}m_{3,11}$, $\cdots\cdots$, $u_2^{(\kappa_2)}m_{2,6s-5} = u_3^{(\kappa_2)}m_{3,11s-10}$, $u_2^{(\kappa_2)}m_{2,6s-4} = u_3^{(\kappa_2)}m_{3,11s-9} + u_3^{(\kappa_2)}m_{3,11s-8}$, $u_2^{(\kappa_2)}m_{2,6s-3} = u_3^{(\kappa_2)}m_{3,11s-7}$, $u_2^{(\kappa_2)}m_{2,6s-2} = u_3^{(\kappa_2)}m_{3,11s-4} + u_3^{(\kappa_2)}m_{3,11s-3}$, $u_2^{(\kappa_2)}m_{2,6s-1} = u_3^{(\kappa_2)}m_{3,11s-2}$, $u_2^{(\kappa_2)}m_{2,6s} = u_3^{(\kappa_2)}m_{3,11s-1} + u_3^{(\kappa_2)}m_{3,11s}$, $u_2^{(\kappa_2)}m_{2,6s+1} = u_3^{(\kappa_2)}m_{3,11s+1}$, $u_3^{(\kappa_0)}m_{3,1} = 0$, $u_3^{(\kappa_3)}m_{3,2} = 0$, $u_3^{(\kappa_0)}m_{3,3} = 0$, $u_3^{(\kappa_0)}m_{3,4} = 0$,

$u_3^{(\kappa_2)}m_{3,5}=0,\ u_3^{(\kappa_3)}m_{3,6}=0,\ u_3^{(\kappa_0)}m_{3,7}=0,\ u_3^{(\kappa_0)}m_{3,8}=0,\ u_3^{(\kappa_2)}m_{3,9}=0,\ u_3^{(\kappa_3)}m_{3,10}=0,$
$u_3^{(\kappa_0)}m_{3,11}=0,\ \cdots\cdots,\ u_3^{(\kappa_0)}m_{3,11s-10}=0,\ u_3^{(\kappa_3)}m_{3,11s-9}=0,\ u_3^{(\kappa_0)}m_{3,11s-8}=0,$
$u_3^{(\kappa_0)}m_{3,11s-7}=0,\ u_3^{(\kappa_2)}m_{3,11s-6}=0,\ u_3^{(\kappa_3)}m_{3,11s-5}=0,\ u_3^{(\kappa_0)}m_{3,11s-4}=0,\ u_3^{(\kappa_0)}m_{3,11s-3}=0,$
$u_3^{(\kappa_2)}m_{3,11s-2}=0,\ u_3^{(\kappa_3)}m_{3,11s-1}=0,\ u_3^{(\kappa_0)}m_{3,11s}=0,\ u_3^{(\kappa_0)}{}_{3,11s+1}=0,\ u_4^{(\kappa_3)}m_{4,1}=u_3^{(\kappa_3)}m_{3,1},$
$u_4^{(\kappa_3)}m_{4,2}=u_3^{(\kappa_3)}m_{3,3},\ u_4^{(\kappa_3)}m_{4,3}=u_3^{(\kappa_3)}m_{3,4}+u_3^{(\kappa_3)}m_{3,5},\ u_4^{(\kappa_3)}m_{4,4}=u_3^{(\kappa_3)}m_{3,7}+u_3^{(\kappa_3)}m_{3,8},$
$u_4^{(\kappa_3)}m_{4,5}=u_3^{(\kappa_3)}m_{3,9}+u_3^{(\kappa_3)}m_{3,11},\ \cdots\cdots,\ u_4^{(\kappa_3)}m_{4,5s-4}=u_3^{(\kappa_3)}m_{3,11s-10},\ u_4^{(\kappa_3)}m_{4,5s-3}=$
$u_3^{(\kappa_3)}m_{3,11s-8},\ u_4^{(\kappa_3)}m_{4,5s-2}=u_3^{(\kappa_3)}m_{3,11s-7}+u_3^{(\kappa_3)}m_{3,11s-6},\ u_4^{(\kappa_3)}m_{4,5s-1}=u_3^{(\kappa_3)}m_{3,11s-4}+$
$u_3^{(\kappa_3)}m_{3,11s-3}=0,\ u_4^{(\kappa_3)}m_{4,5s}=u_3^{(\kappa_3)}m_{3,11s-2}+u_3^{(\kappa_3)}m_{3,11s},\ u_4^{(\kappa_3)}m_{4,5s+1}=u_3^{(\kappa_3)}m_{3,11s+1},$
$u_4^{(\kappa_4)}m_{4,1}\neq 0,\ u_4^{(\kappa_4)}m_{4,2}=0,\ u_4^{(\kappa_4)}m_{4,3}=0,\ u_4^{(\kappa_4)}m_{4,4}=u_4^{(\kappa_4)}m_{4,5},\ \cdots\cdots,\ u_4^{(\kappa_4)}m_{4,5s-4}\neq 0,$
$u_4^{(\kappa_4)}m_{4,5s-3}=0,\ u_4^{(\kappa_4)}m_{4,5s-2}=0,\ u_4^{(\kappa_4)}m_{4,5s-1}=u_4^{(\kappa_4)}m_{4,5s},\ u_4^{(\kappa_4)}m_{4,5s}\neq 0.$

Then it is shown by the same method as above that $\mathfrak{M}$ is directly indecomposable.

7. In this chapter we shall prove the main theorem. First we shall prove the following

Theorem 1. *Suppose that* $Ne=Au_1\oplus Au_2\oplus Au_3$. *Then an arbitrary* Ae*-left module* $\mathfrak{M}$ *is the direct sum of* Aem_i *which are homomorphic to* Ae *or* Aem_j+Aem_{j+1} *such that* $u_1m_j\neq 0$, $u_2m_j=0$, $u_3m_j=u_3m_{j+1}$, $u_1m_{j+1}=0$, $u_2m_{j+1}\neq 0$.

Proof. We may assume that A is the basic algebra.

(i) Suppose that $\mathfrak{M}=Aem_1+Aem_2$. Then the representation $R(a)$ by $\mathfrak{M}$ has the following form;

$$R(a)=\begin{pmatrix} I_2\times y & & & \\ Q_{11}\times z_{11} & I_{s_1}\times x_1 & & \\ Q_{21}\times z_{21} & & I_{s_2}\times x_2 & \\ Q_{31}\times z_{31} & & & I_{s_3}\times x_3 \end{pmatrix}$$

where $s_i\leq 2$ and $a=ye+x_1e_1+x_2e_2+x_3e_3+z_{11}u_1+z_{21}u_2+z_{31}u_3+\cdots\cdots$.

Now we remark that if we put

$$T=\begin{pmatrix} M_1 & & & \\ & N_1^{-1} & & \\ & & N_2^{-1} & \\ & & & N_3^{-1} \end{pmatrix},$$

we have the transformation

$$T^{-1}R(a)T=\begin{pmatrix} I_2\times y & & & \\ N_1Q_{11}M_1 & I_{s_1}\times x_1 & & \\ N_2Q_{21}M_1 & & I_{s_2}\times x_2 & \\ N_3Q_{31}M_1 & & & I_{s_3}\times x_3 \end{pmatrix}. \tag{II}$$

First if $s_1=s_2=s_3=2$, $R(a)=\begin{pmatrix} I_2\times y &&&\\ I_2\times z_{11} & I_2\times x_1 &&\\ I_2\times z_{21} && I_2\times x_2 &\\ I_2\times z_{31} &&& I_2\times x_3\end{pmatrix}$

and therefore $R(a)$ is directly decomposable and

$$R(a)=\begin{pmatrix} y &&&&&&&\\ z_{11} & x_1 &&&&&&\\ z_{21} & 0 & x_2 &&&&&\\ z_{31} & 0 & 0 & x_3 &&&&\\ &&&& y &&&\\ &&&& z_{11} & x_1 &&\\ &&&& z_{21} & 0 & x_2 &\\ &&&& z_{31} & 0 & 0 & x_3 \end{pmatrix}.$$

If $s_1=s_2=2$ $s_3=1$, $R(a)=\begin{pmatrix} I_2\times y &&&\\ I_2\times z_{11} & I_2\times x_1 &&\\ I_2\times z_{21} && I_2\times x_2 &\\ [1, u]\times z_{31} &&& x_3\end{pmatrix}.$

Then by (II) $R(a)$ is similar to

$$R_1(a)=\begin{pmatrix} I_2\times y &&&\\ I_2\times z_{11} & I_2\times x_1 &&\\ I_2\times z_{21} && I_2\times x_2 &\\ [1\ 0]\times z_{31} &&& x_3\end{pmatrix}.$$

Hence $R(a)$ is similar to

$$R_2(a)=\begin{pmatrix} y &&&&&&\\ z_{11} & x_1 &&&&&\\ z_{21} & 0 & x_2 &&&&\\ z_{31} & 0 & 0 & x_3 &&&\\ &&&& y &&\\ &&&& z_{11} & x_1 &\\ &&&& z_{21} & 0 & x_2 \end{pmatrix}$$

and $R(a)$ is directly decomposable.

If $s_1=2 \quad s_2=s_3=1$, $R(a) = \begin{pmatrix} I_2\times y & & & \\ I_2\times z_{11} & I_2\times x_1 & & \\ [1\ u]\times z_{21} & & x_2 & \\ [1\ v]\times z_{31} & & & x_3 \end{pmatrix}$.

Then by (II) $R(a)$ is similar to

$$R_1(a) = \begin{pmatrix} I_2\times y & & & \\ I_2\times z_{11} & I_2\times x_1 & & \\ [1\ 0]\times z_{21} & & x_2 & \\ [0\ 1]\times z'_{31} & & & x_3 \end{pmatrix},$$

and by the same way as above arguments $R(a)$ is directly decomposable.

If $s_1=s_2=s_3=1$, $R(a) = \begin{pmatrix} I_2\times y & & & \\ [1\ u]\times z_{11} & x_1 & & \\ [1\ v]\times z_{21} & & x_2 & \\ [1\ w]\times z_{31} & & & x_3 \end{pmatrix}$

is similar to

$$R_1(a) = \begin{pmatrix} I_2\times y & & & \\ [1\ u']\times z_{11} & x_1 & & \\ [0\ 1]\times z_{21} & & x_2 & \\ [1\ 0]\times z_{31} & & & x_3 \end{pmatrix}.$$

Now it is shown by the simple computation of eigenvalues of any commutator of $R_1(a)$ that $R_1(a)$ is directly indecomposable.

Hence $\mathfrak{M}=Aem_1+Aem_2$, where $u_1m_1=u_1m_2$, $u_2m_1\neq 0$, $u_2m_2=0$, $u_3m_1=0$, $u_3m_2\neq 0$, is directly indecomposable. From now we shall say that such a module has the type (*).

(ii) Suppose that $\mathfrak{M}=Aem_1+Aem_2+Aem_3$. Now we consider the two cases.

(a) Suppose that Aem_1+Aem_2 is directly decomposable and $Aem_1\frown Aem_2=0$. Then the representation $R(a)$ by $\mathfrak{M}$ has the following form;

$$R(a) = \begin{pmatrix} I_3\times y & & & \\ Q_{11}\times z_{11} & I_{s_1}\times x_1 & & \\ Q_{21}\times z_{21} & & I_{s_2}\times x_2 & \\ Q_{31}\times z_{31} & & & I_{s_3}\times x_3 \end{pmatrix} \quad \text{where } s_i\leq 3.$$

If $s_1 = s_2 = s_3 = 3$, $R(a) = \begin{pmatrix} I_3 \times y & & & \\ I_3 \times z_{11} & I_3 \times x_1 & & \\ I_3 \times z_{21} & & I_3 \times x_2 & \\ I_3 \times z_{31} & & & I_3 \times x_3 \end{pmatrix}$

and it is clear by the same way as above that $R(a)$ is directly decomposable.

If $s_1 = s_2 = 3$ $s_3 = 2$, $R(a) = \begin{pmatrix} I_3 \times y & & & \\ I_3 \times z_{11} & I_3 \times x_1 & & \\ I_3 \times z_{21} & & I_3 \times x_2 & \\ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & w' \end{bmatrix} \times z_{31} & & & I_2 \times x_3 \end{pmatrix}$

and is similar to

$$R(a) = \begin{pmatrix} I_3 \times y & & & \\ I_3 \times z_{11} & I_3 \times x_1 & & \\ I_3 \times z_{21} & & I_3 \times x_2 & \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times z_{31} & & & I_2 \times x_3 \end{pmatrix}$$

and $R(a)$ is directly decomposable.

If $s_1 = s_2 = 3$ $s_3 = 1$, $R(a) \sim R_1(a) = \begin{pmatrix} I_3 \times y & & & \\ I_3 \times z_{11} & I_3 \times x_1 & & \\ I_3 \times z_{21} & & I_3 \times x_2 & \\ [1,0,0] \times z_{31} & & & x_3 \end{pmatrix}$ [13]

and $R(a)$ is directly decomposable.

If $s_1 = 3$ $s_2 = s_3 = 2$, $R(a) \sim R_1(a) = \begin{pmatrix} I_3 \times y & & & \\ I_3 \times z_{11} & I_3 \times x_1 & & \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times z_{21} & & I_2 \times x_2 & \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times z_{31} & & & I_2 \times x_3 \end{pmatrix}$

and $R(a)$ is directly decomposable.

If $s_1 = 3$ $s_2 = 2$ $s_3 = 1$, $R(a) \sim R_1(a) = \begin{pmatrix} I_3 \times y & & & \\ I_3 \times z_{11} & I_3 \times x_1 & & \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times z_{21} & & I_2 \times x_2 & \\ [0\ 0\ 1] \times z_{31} & & & x_3 \end{pmatrix}$

13) $\sim$ denotes the similarity.

and $R(a)$ is directly decomposable.

$$\text{If } s_1=3 \quad s_2=s_3=1, \quad R(a)\sim R_1(a)=\left(\begin{array}{llll} I_3\times y & & & \\ I_3\times z_{11} & I_3\times x_1 & & \\ [1\,0\,0]\times z_{21} & & x_2 & \\ [0\,0\,1]\times z_{31} & & & x_3 \end{array}\right)$$

and $R(a)$ is directly decomposable.

$$\text{If } s_1=2 \quad s_2=2 \quad s_3=2, \quad R(a)\sim R_1(a)=\left(\begin{array}{llll} I_3\times y & & & \\ \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\times z_{11} & I_2\times x_1 & & \\ \begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\times z_{21} & & I_2\times x_2 & \\ \begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}\times z_{31} & & & I_2\times x_3 \end{array}\right)$$

and $R(a)$ is directly decomposable.

$$\text{If } s_1=s_2=2 \quad s_3=1, \quad R(a)\sim R_1(a)=\left(\begin{array}{llll} I_3\times y & & & \\ \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\times z_{11} & I_2\times x_1 & & \\ \begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\times z_{21} & & x_2 & \\ [0\,1\,w]\times z_{31} & & & x_3 \end{array}\right)$$

and $R(a)$ is directly decomposable.

$$\text{If } s_1=2 \quad s_2=s_3=1, \quad R(a)\sim R_1(a)=\left(\begin{array}{llll} I_3\times y & & & \\ \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\times z_{11} & I_2\times x_1 & & \\ [0\,0\,1]\times z_{21} & & x_2 & \\ [0\,1\,w]\times z_{31} & & & x_3 \end{array}\right)$$

and $R(a)$ is directly decomposable.

$$\text{If } s_1=s_2=s_3=1, \quad R(a)\sim R_1(a)=\left(\begin{array}{llll} I_3\times y & & & \\ [1\,0\,0]\times z_{11} & x_1 & & \\ [0\,1\,0]\times z_{21} & & x_2 & \\ [0\,0\,1]\times z_{31} & & & x_3 \end{array}\right)$$

and $R(a)$ is directly decomposable.

(b) Suppose that Aem_1+Aem_2 is directly indecomposable. Then the representation $R(a)$ by $\mathfrak{M}$ has the followimg form;

$$R(a) = \begin{pmatrix} I_3 \times y & & & \\ Q_{11} \times z_{11} & I_{s_1} \times x_1 & & \\ Q_{21} \times z_{21} & & I_{s_2} \times x_2 & \\ Q_{31} \times z_{31} & & & I_{s_3} \times x_3 \end{pmatrix} \quad \text{where } s_i \leq 2.$$

If $s_1 = s_2 = s_3 = 2$, $R(a)$ is directly decomposable.

If $s_1 = s_2 = 2 \quad s_3 = 1$, $R(a) \sim R_1(a) = \begin{pmatrix} I_3 \times y & & & \\ \begin{bmatrix} 1 & u & 0 \\ 0 & 0 & 1 \end{bmatrix} \times z_{11} & I_2 \times x_1 & & \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times z_{21} & & I_2 \times x_2 & \\ [0\ 1\ 0] \times z_{31} & & & x_3 \end{pmatrix}$

and $R(a)$ is directly decomposable.

If $s_1 = 2 \quad s_2 = s_3 = 1$, $R(a) \sim R_1(a) = \begin{pmatrix} I_3 \times y & & & \\ \begin{bmatrix} 1 & u & 0 \\ 0 & 0 & 1 \end{bmatrix} \times z_{11} & I_2 \times x_1 & & \\ [1\ 0\ 0] \times z_{21} & & x_2 & \\ [0\ 1\ 0] \times z_{31} & & & x_3 \end{pmatrix}$

and $R(a)$ is directly decomposable.

If $s_1 = s_2 = s_3 = 1$,

$$R(a) \sim R_1(a) = \begin{pmatrix} I_3 \times y & & & \\ [1\ u\ u'] \times z_{11} & x_1 & & \\ [1\ 0\ 0] \times z_{21} & & x_2 & \\ [0\ 1\ 0] \times z_{31} & & & x_3 \end{pmatrix} \sim R_2(a) = \begin{pmatrix} I_3 \times y & & & \\ [0\ 0\ 1] \times z_{11} & x_1 & & \\ [1\ 0\ 0] \times z_{21} & & x_2 & \\ [0\ 1\ 0] \times z_{31} & & & x_3 \end{pmatrix}$$

and $R(a)$ is directly decomposable.

(iii) Now we shall prove this theorem by induction on the number of generators of $\mathfrak{M}$. Hence we assume that $\mathfrak{M}' = \sum_{i=1}^{s-1} Aem_i$ is the direct sum of Aem_i which is homomophic to Ae or $Aem_j + Aem_{j+1}$ which have the type (*). Then the representation $R(a)$ by $\mathfrak{M} = \sum_{i=1}^{s} Aem_i$ is as follows:

$$R(a) = \begin{pmatrix} I_s \times y & & & \\ Q_{11} \times z_{11} & I_{s_1} \times x_1 & & \\ Q_{21} \times z_{21} & & I_{s_2} \times x_2 & \\ Q_{31} \times z_{31} & & & I_{s_3} \times x_3 \end{pmatrix}$$

where $s_i \leq s$ and

$$Q_{11}=\left(\begin{array}{l@{}r}
1\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & x_1^{(1)} \\
\quad\ddots & \vdots \\
\qquad 1\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & x_1^{(t_1)} \\
\qquad\quad 1\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & x_2^{(1)} \\
\qquad\qquad\ddots & \vdots \\
\qquad\qquad\quad 1\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & x_2^{(t_2)} \\
\qquad\qquad\qquad 1\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & x_3^{(1)} \\
\qquad\qquad\qquad\quad\ddots & \vdots \\
\qquad\qquad\qquad\qquad 1\ 0\cdots 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & x_3^{(t_3)} \\
\qquad\qquad\quad\cdots\cdots\underbrace{0\quad 0}_{t_4}\ 1\ u\ 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & x_5^{(1)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 1\ u\ 0\ 0 & \vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ddots\quad\ \vdots & \vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1\ u\ 0\ 0\cdots\cdots\cdots\cdots\cdots & x_5^{(t_5)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\cdots\cdots 0\ 1\ 0\cdots\cdots\cdots\cdots\cdots & x_6^{(1)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ddots & \vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1\ 0\cdots\cdots\cdots\cdots & x_6^{(t_6)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\cdots 0\ \underbrace{0\cdots 0}_{t_7}\ \underbrace{0\cdots\cdots 0}_{t_8} & x
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}t_1 \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}t_2 \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}t_3 \\
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\}t_5 \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}t_6 \\
\\
\end{array}$$

$$Q_{21}=\left(\begin{array}{l@{}r}
1\cdots 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & y_1^{(1)} \\
\vdots\ \ddots\ \vdots\ \ \vdots & \vdots \\
0\cdots 1\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & y_1^{(t_1)} \\
\ \cdots 0\ 1\cdots 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & y_2^{(1)} \\
\quad\ \vdots\ \ \vdots\ \ddots\ \vdots\ \ \vdots & \vdots \\
\qquad 0\cdots 1\ 0\cdots 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & y_2^{(t_2)} \\
\quad\cdots\cdots\cdots 0\cdots 0\ 1\cdots 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & y_4^{(1)} \\
\qquad\qquad\underbrace{\vdots\quad\ \vdots}_{t_3}\ \ \vdots\ \ddots\ \vdots\ \ \vdots & \vdots \\
\qquad\qquad\qquad\quad 0\cdots 1\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & y_4^{(t_4)} \\
\qquad\qquad\qquad\quad\cdots\cdots 0\ 1\ 0\ 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & y_5^{(\ )} \\
\qquad\qquad\qquad\qquad\qquad\qquad 1\ 0\ 0\ 0 & \vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 1\ 0\cdots\cdots & \vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ddots & \vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1\ 0\ 0\ 0 & \vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\ 1\ \overbrace{0\ 0\cdots\cdots}^{t_6}\cdots\cdots\overbrace{\cdots\cdots}^{t_8} & y_5^{(t_5)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ 0\ 0\cdots 0\ 1\cdots 0\ 0\cdots 0 & y_7^{(1)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \vdots\qquad\ \vdots\ \vdots\ \ddots\ \vdots\ \vdots\quad\ \vdots & \vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \vdots\qquad\ \vdots\ 0\cdots 1\ \vdots\quad\ \vdots & y_7^{(t_7)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ 0\cdots\cdots 0\ 0\cdots 0\ 0\cdots 0 & y
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}t_1 \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}t_2 \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}t_4 \\
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\}t_5 \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}t_7 \\
\\
\end{array}$$

$$
Q_{31}=\left(\begin{array}{l}
1\cdots 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots z_1^{(1)}\\
\vdots\ \ddots\ \vdots\ \vdots\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots\\
0\cdots 1\ 0\cdots 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots z_1^{(t_1)}\\
\quad\cdots 0\ 0\cdots 0\ 1\cdots 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots z_3^{(1)}\\
\qquad\qquad\ \vdots\quad \ddots\ \vdots\ \vdots\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots\\
\qquad\qquad\ \vdots\qquad\ 1\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots z_3^{(t_3)}\\
\qquad\ \underbrace{\quad\ }_{t_2}\cdots\cdots 0\ 1\cdots 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots z_4^{(1)}\\
\qquad\qquad\qquad\quad \vdots\ \ \vdots\ \ddots\ \vdots\ \vdots\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
\qquad\qquad\qquad\quad \vdots\qquad\ 1\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots z_4^{(t_4)}\\
\qquad\qquad\qquad\quad \cdots\cdots 0\ 0\ 1\ 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots z_5^{(1)}\\
\qquad\qquad\qquad\qquad\qquad\qquad\ 0\ 1\ 0\ 0\qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \ddots\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 0\ 1\ 0\ 0\quad \overbrace{\qquad}^{t_6}\quad \overbrace{\qquad}^{t_7}\qquad\quad \vdots\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ 0\ 1\ 0\ 0\cdots\cdots\cdots\cdots\cdots\cdots z_5^{(t_5)}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\ 0\cdots 0\ 0\cdots 0\ 1\cdots 0\ z_8^{(1)}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \vdots\quad \vdots\quad \vdots\quad \vdots\ \ \vdots\ \ddots\ \vdots\ \ \vdots\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \vdots\quad \vdots\quad \vdots\quad \vdots\ \ 0\quad 1\ \ z_8^{(t_8)}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ 0\quad 0\ 0\quad 0\ 0\cdots 0\ z
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l}\\ \\ \\ \end{array}\right\} t_1\\
\left.\begin{array}{l}\\ \\ \\ \end{array}\right\} t_3\\
\left.\begin{array}{l}\\ \\ \\ \end{array}\right\} t_4\\
\left.\begin{array}{l}\\ \\ \\ \\ \\ \end{array}\right\} t_5\\
\left.\begin{array}{l}\\ \\ \\ \end{array}\right\} t_8\\
\\
\end{array}
$$

and $x \neq 0$ means $x_i^{(j)}=0$, $y \neq 0$ means $y_i^{(j)}=0$ and $z \neq 0$ means $z_i^{(j)}=0$.

First if $x \neq 0$ $y \neq 0$ $z \neq 0$, $R(a)$ is directly decomposable. Next if $x \neq 0$ $y \neq 0$ $z=0$, $R(a)$ is similar to $R_1(a)$ such that all $z_i^{(j)}=0$ and $R(a)$ is directly decomposable. If $x \neq 0$ $y=z=0$, we may assume that all $y_i^{(j)}=0$. Then Q_{31} may to replaced by Q'_{31} such that $z_3^{(\kappa)}=z_5^{(\lambda)}=z_8^{(\mu)}=0$ and if t_3, t_5 or t_8 is not zero, $R(a)$ is decomposable into direct components of desired types by the assumption of induction. If t_3, t_5 and t_8 is zero, $z_1^{(\xi)}=0$ for $\xi \neq 1$ and $z_4^{(\eta)}=0$ for $\eta \neq 1$. Then we may replace Q_{31} by

$$
Q'_{31}=\left(\begin{array}{l}
1\ 0\cdots\cdots 0\cdots\cdots\cdots\cdots z_1^{(1)}\\
0\ 1\cdots 0\ \vdots\qquad\qquad\quad 0\\
\vdots\ \vdots\ \ddots\ \vdots\ \vdots\qquad\qquad\quad \vdots\\
0\ 0\cdots 1\ 0\cdots 0\cdots\cdots 0\\
\ \ \cdots\cdots 0\cdots 0\ 1\cdots 0\ 0\\
\ \ \underbrace{\quad}_{t_1-1}\qquad\ \vdots\ \ddots\ \vdots\ \vdots\\
\qquad\qquad\quad \underbrace{0\cdots 1}_{t_4}\ 0
\end{array}\right)
\quad\text{or}\quad
Q''_{31}=\left(\begin{array}{l}
1\cdots 0\cdots\cdots\cdots\cdots\cdots 0\\
\vdots\ \ddots\ \vdots\quad \vdots\ \vdots\qquad\ \vdots\\
0\cdots 1\ 0\cdots 0\ 0\cdots\cdots 0\\
\cdots\cdots\cdots 0\ 1\ 0\cdots\cdots z_4^{(1)}\\
\underbrace{\quad}_{t_1}\qquad \vdots\ 0\ 1\cdots 0\ 0\\
\qquad\quad\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots\ \vdots\\
\qquad\quad\ 0\ \underbrace{0\cdots 1}_{t_4-1}\ 0
\end{array}\right)
$$

and $R(a)$ is directly decomposable. Hence we may assume that $x=y=z=0$.

First we may assume that $x_i^{(j)}=0$. Moreover we may assume that there exists just one $y_\alpha^{(\beta)}$ such that $y_\alpha^{(\beta)} \neq 0$ and all other $y_\lambda^{(\mu)}=0$. Of course all $y_i^{(j)}$ may be zero. Now if $y_1^{(1)} \neq 0$, Q_{21} may be replaced by

$$Q'_{21}=\left(\begin{array}{llllllll}
0\ 0\cdots 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & & y_1^{(1)} \\
0\ 1 & & & & & & & 0 \\
\quad \ddots & & & & & & & \vdots \\
\quad\quad 1 & & & & & & & \vdots \\
\underbrace{\qquad}_{t_1-1}\ 1 & & & & & & & \vdots \\
\quad\quad \ddots & & & & & & & \vdots \\
 & 1\ 0\cdots 0\ 0 & & & & & & \vdots \\
 & \underbrace{\quad}_{t_2}\ 0\ \ 0\ 1\cdots 0 & & & & & & \vdots \\
 & \quad \underbrace{\quad}_{t_3}\ \vdots\ \ddots\ \vdots & & & & & & \vdots \\
 & \quad\quad 0\cdots 1 & 0\cdots\cdots\cdots & & & & & \vdots \\
 & \quad\quad \underbrace{\quad}_{t_4} & 1\ 0\ 0\ 0 & & & & & \vdots \\
 & & \quad 1\ 0\ 0\ 0 & & & & & \vdots \\
 & & & \ddots & & & & \vdots \\
 & & & 1\ 0\ 0\ 0 & & & & \vdots \\
 & & & & 1\ 0\ 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & 0 \\
 & & & & \qquad\quad \vdots & 1\cdots 0 & \vdots & \vdots \\
 & & & & \qquad\quad \vdots & \vdots\ \ddots\ \vdots & \vdots & \vdots \\
 & & & \cdots\cdots\cdots & \cdots\cdots 0 & 0\cdots 1 & 0\cdots 0 & 0
\end{array}\right)$$

and Q_{31} is replaced by

$$Q'_{31}=\left(\begin{array}{llllllll}
1\ 0\cdots 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & & 0 \\
0\ 1\cdots 0 & & & & & & & z_1^{(2)} \\
\vdots\ \vdots\ \ddots\ \vdots & & & & & & & \vdots \\
0\ 0\cdots 1 & 0\cdots 0\ 0\cdots\cdots\cdots & & & & & & \vdots \\
\cdots\cdots\cdots\cdots & \cdots 0\ 1\cdots 0 & & & & & & \vdots \\
\underbrace{\qquad}_{t_1-1} & \underbrace{\quad}_{t_2}\ \ \vdots\ \ddots\ \vdots & & & & & & \vdots \\
 & \qquad 0\cdots 1 & 0\cdots\cdots\cdots & & & & & \vdots \\
 & \cdots\cdots 0 & 1\cdots 0 & & & & & \vdots \\
 & \underbrace{\quad}_{t_3}\ \vdots & \vdots\ \ddots\ \vdots & & & & & \vdots \\
 & \qquad\ \vdots & 0\cdots 1 & 0\cdots\cdots\cdots & & & & \vdots \\
 & & \cdots\cdots 0 & 0\ 1\ 0\ 0 & & & & \vdots \\
 & & \underbrace{\quad}_{t_4} & \quad 0\ 1\ 0\ 0 & & & & \vdots \\
 & & & \qquad \ddots & & & & \vdots \\
 & & & & 0\ 1\ 0\cdots 0\ 0\cdots & \overbrace{0\cdots 0}^{t_8} & & \vdots \\
 & & & & & 1\cdots 0 & & \vdots \\
 & & & & & \vdots\ \ddots\ \vdots & & \vdots \\
 & & & & & 0\cdots 1 & & z_8^{(t_8)}
\end{array}\right)$$

and from the assumption of induction, $\mathfrak{M}$ is decomposed into direct components of desired types.

Next if $y_2^{(1)} \neq 0$, Q_{31} is replaced by Q'_{31} such that $z_1^{(1)} \neq 0$, $z_3^{(1)} \neq 0$ and $z_4^{(1)} \neq 0$. If $z_1^{(1)} \neq 0$, $z_3^{(1)} = z_4^{(1)} = 0$. Hence in this case the theorem is trivial. If $z_1^{(1)} = 0$, Q_{31} may be replaced by

$$
Q'_{31} = \left(\begin{array}{llllllll}
1 & 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 \\
\ddots & \vdots & & & & & & \vdots \\
\quad 1 & 0\cdots 0 & 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 \\
\underbrace{\cdots 0}_{t_1} & \underbrace{0\cdots 0}_{t_2} & 1 \;\; 0 \cdots\cdots w' \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 \\
 & & 0 \;\; 1 \qquad \vdots & & & & & \vdots \\
 & & \vdots \quad \ddots \;\; \vdots & & & & & \vdots \\
 & & \vdots \qquad 1 \;\; 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 \\
 & & 0 \cdots\cdots 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots z_4^{(1)} \\
 & & \underbrace{\qquad}_{t_3-1} \;\vdots & 1\cdots 0 & 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 \\
 & & \quad\;\; \vdots & \vdots \ddots \vdots & \vdots & & & \vdots \\
 & & \quad\;\; \vdots & 0\cdots 1 & 0 \cdots\cdots\cdots & & & \vdots \\
 & & \quad\;\; \vdots & \underbrace{0\cdots 0}_{t_4-1} & 0\;1\;0\;0 & & & \vdots \\
 & & & & \quad 0\;1\;0\;0 & & & \vdots \\
 & & & & \qquad\quad \ddots & & & \vdots \\
 & & & & \qquad 0\;1\;0 & 0\cdots 0 \;\; 0\cdots 0 & 0\cdots 0 & \vdots \\
 & & & & & & 1\cdots 0 & 0 \\
 & & & & & & \vdots \;\iddots\; \vdots & \vdots \\
 & & & & & & \underbrace{0\cdots 1}_{t_8} & 0
\end{array}\right)
$$

and $\mathfrak{M}$ is decomposed into direct components of desired types. In other cases by the same way as above we can prove this theorem.

Now we shall prove the main theorem.

Theorem2. *Suppose that $N^2=0$ and the ground field K is algebraically closed. Then A is of bounded representation type if and only if the following conditions are satisfied;*

(1) *$Ne_\lambda(e_\lambda N)$ $(\lambda=1, \cdots\cdots, n)$ do not contain at least two simple components isomorphic to each other.*

(2) *$Ne_\lambda(e_\lambda N)$ $(\lambda=1, \cdots\cdots, n)$ are the direct sums of at most three simple componedts.*

(3) *There is no chain such that $\{Ne_{\kappa_1}=Ne_\kappa, \cdots\cdots, Ne_{\kappa_{m-1}}, Ne_{\kappa_m}=Ne_\kappa, \cdots\cdots\}$.*

(4) *If $Ne_\kappa(e_\kappa N)$ is the direct sum of three simple components and $Ne_\lambda(e_\lambda N)$ is the direct sum of three simple component or divided, there is no chain which connects Ne_κ and Ne_λ ($e_\kappa N$ and $e_\lambda N$).*

(5) *If Ne_1, Ne_2 and Ne_3 (e_1N, e_2N and e_3N) are the direct sums of two simple components, $Ne_1(e_1N)$ is not divieed into Ne_2 and Ne_3 (e_2N and e_3N).*

(6) *Suppose that $\{Ne_1, \cdots\cdots, Ne_r\}$ ($\{e_1N, \cdots\cdots, e_rN\}$) is a chain. Ne_1 or $Ne_r(e_1N$ or $e_rN)$ is the direct sum of three simple components or, if $Ne_j(e_jN)$ $(j\neq 1, r)$ is the direct sum of three simple components, the chain is $\{Ne_1, Ne_2, Ne_3\}$ or $\{Ne_4, Ne_2, Ne_5, Ne_6\}$ where Ne_2 is the direct sum of three simple components and Ne_4 and Ne_6 are simple.*

Proof. The "only if" part is clear from above lemmas. Hence we shall prove the "if" part. This proof is quite long and we shall show this proof in outline.

Now we consider the following six cases. Namely

(1) $\{Ne_1, \cdots\cdots, Ne_r\}$ is such a chain that $Ne_1, \cdots\cdots, Ne_{r-1}$ are the direct sums of two simple components and Ne_r is the direct sum of three simple components.

(2) $\{Ne_1, \cdots\cdots, Ne_{r-1}, Ne_{r_1}, Ne_{r_2}\}$ is such a chain that $Ne_1, \cdots\cdots, Ne_{r-1}$ are the direct sums of two simple component and Ne_{r_1} and Ne_{r_2} are simple and isomorphic to a simple component of Ne_{r-1}.

(3) $\{Ne_1, Ne_2, Ne_3\}$ is such a chain that Ne_2 is the direct sum of three components and other Ne_i are the direct sums of two simple components.

(4) $\{Ne_1, Ne_2, Ne_3, Ne_4, Ne_5\}$ is such a chain that $Ne_2 = Au_2^{(\eta_1)} \oplus Au_2^{(\eta_2)}$, $Ne_4 = Au_4^{(\eta_2)} \oplus Au_4^{(\eta_3)}$ and $Ne_1 \simeq Au_2^{(\eta_1)}$, $Ne_3 \simeq Au_2^{(\eta_2)} \simeq Au_4^{(\eta_2)}$, $Ne_5 \simeq Au_4^{(\eta_3)}$.

(5) $\{Ne_1, Ne_2, Ne_3, Ne_4\}$ is such a chain that Ne_2 is the direct sum of three simple components and Ne_1 and Ne_4 are simple.

(6) $\{Ne_1, Ne_2, Ne_3, Ne_4\}$ is such a chain that Ne_3 is simple and Ne_2 is divided into Ne_3 and Ne_4.

Moreover we may assume that A is a basic algebra and A has just one chain.

[The case 1]: Suppose that $Ne_i = Au_i^{(\eta_i)} \oplus Au_i^{(\eta_{i+1})}$ $(i=1, \cdots\cdots, r-1)$ and $Ne_r = Au_r^{(\eta_r)} \oplus Au_r^{(0)} \oplus Au_r^{(\eta_{r+1})}$ where $Au_{i-1}^{(\eta_i)} \simeq Au_i^{(\eta_i)}$. Now let $\mathfrak{M} = \sum_i \sum_{\kappa_i} Ae_i m_{i,\kappa_i}$ be an arbitrary directly indecomposable A-left module. Then the representation $R(a)$ by $\mathfrak{M}$ is as follows;

$$R(a) = \begin{pmatrix}
I_{\kappa_1} \times y_1 & & & & & & & & & \\
0 & \ddots & & & & & & & & \\
\vdots & & \ddots & & & & & & & \\
0 & & & I_{\kappa_r} \times y_r & & & & & & \\
Q_{11} \times z_{11} & 0 & \cdots & 0 & I_s \times x_{\eta_1} & & & & & \\
Q_{21} \times z_{21} & Q_{22} \times z_{22} & & \vdots & & \ddots & & & & \\
 & Q_{32} \times z_{32} & & \vdots & & & \ddots & & & \\
 & & \ddots & 0 & & & & \ddots & & \\
 & & & Q_{rr} \times z_{rr} & 0 & \cdots & 0 & I_{s_r} \times x_{\eta_r} & & \\
 & & & Q_{0r} \times z_{0r} & 0 & \cdots & & 0 & I_{s_0} \times x_0 & \\
 & & & Q_{r+1,r} \times z_{r+1,r} & 0 & \cdots & & & 0 & I_{s_{r+1}} \times x_{\eta_{r+1}}
\end{pmatrix}$$

where $R(u^{(n_i)}_{i-1})=[Q_{i,i-1}]$. Now we may assume that Q_{rr}, Q_{0r} and $Q_{r+1,r}$ have the following form;[14]

$$
Q_{rr}=\left(\begin{array}{ccccccccccccc}
I_{t_1} & & & & & & & & & & & & \\
 & I_{t_2^{(2r-2)}} & & & & & & & & & & & \\
 & & I_{t_2^{(2r-3)}} & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & \\
 & & & & I_{t_2^{(1)}} & & & & & & & & \\
 & & & & & I_{t_3^{(1)}} & & & & & & & \\
 & & & & & & \ddots & & & & & & \\
 & & & & & & & I_{t_3^{(2r-2)}} & & & & & \\
 & & & & & & & & 1\ u\ 0\ 0\cdots & & & & \\
 & & & & & & & & 1\ u\ 0\ 0 & & & & \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & 1\ u & \left.\right\}t_4 & 0\cdots 0 & & \\
 & & & & & & & & 0 & & & & \\
 & & & & & & & & \vdots & & I_{t_5} & & \\
 & & & & & & & & 0 & & & &
\end{array}\right)
$$

$$
Q_{0r}=\left(\begin{array}{ccccccccc}
I_{t_1} & & & & & & & & \\
 & I_{t_2^{(2r-2)}} & & & & & & & \\
 & & I_{t_2^{(2r-3)}} & \cdots & I_{t_2^{(1)}} & 0\cdots\cdots 0 & 0\ 0 & & \\
 & & & & & & 1\ 0\ 0\ 0 & & \\
 & & & & & & 1\ 0\ 0\ 0 & & \\
 & & & & & & \ddots & & \\
 & & & & & & 1\ 0\ 0\ 0 & & \\
 & & & & & & 1\ 0 & \overbrace{0\cdots 0}^{t_5} &
\end{array}\right)
$$

$$
Q_{r+1,r}=\left(\begin{array}{ccccccc}
I_{t_1} & 0\cdots\cdots 0 & & & & & \\
 & & I_3^{(1)} & & & & \\
 & & & I_3^{(2)} & & & \\
 & & & & \ddots & & \\
 & & & & & I_3^{(2r-2)} & \\
 & & & & & & 0\ 0\ 0\ 0 \\
 & & & & & & 0\ 1\ 0\ 0 \\
 & & & & & & 0\ 1\ 0\ 0 \\
 & & & & & & 0\ 1\ 0\ 0 \\
 & & & & & & \ddots \\
 & & & & & & 0\ 1\ 0\ 0 \\
 & & & & & & 0\ 1\ 0\cdots 0
\end{array}\right)
$$

14) $I_{t_\kappa^{(\lambda)}}$ and $I_{t_i^{(j)}}$ are on the same row or column if $t_\kappa^{(\lambda)}=t_i^{(j)}$.

where $t_1+t_2^{(2r-2)}+t_2^{(2r-3)}+\cdots\cdots+t_2^{(1)}+t_3^{(1)}+\cdots\cdots+t_3^{(2r-2)}+2t_4+t_5=\kappa_r$. Next we break up $Q_{ij}(Q_{ij}\neq Q_{rr},\ Q_{0r},\ Q_{r+1,r})$ into submatrices corresponding to the divisions of Q_{rr}. Then

$$Q_{ij}=\begin{pmatrix} Z^{(i,j)}_{t_1,t_1} & Z^{(i,j)}_{t_1,t_2^{(2r-2)}} & \cdots\cdots & Z^{(i,j)}_{t_1,t_5} \\ Z^{(i,j)}_{t_2^{(2r-2)},t_1} & Z^{(i,j)}_{t_2^{(2r-2)},t_2^{(2r-2)}} & \cdots\cdots & Z^{(i,j)}_{t_2^{(2r-2)},t_5} \\ \vdots & & & \vdots \\ Z^{(i,j)}_{t_5,t_1} & \cdots\cdots & \cdots\cdots & Z^{(i,j)}_{t_5,t_5} \end{pmatrix}.$$

First by (II) we may replace $Z^{(i,j)}_{t_1\kappa}$ and $Z^{(i,j)}_{\lambda t_1}$ $(\kappa\neq t_1,\ \lambda\neq t_1)$ by 0 and $Z^{(i,j)}_{t_1t_1}$ by the following matrices;

$$Z^{(j,j-1)}_{t_1,t_1}=\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & 0 \\ & & \vdots \\ & & 0 \end{pmatrix} \quad\text{or}\quad Z^{(j,j)}_{t_1,t_1}=\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Then from the indecomposability of $R(a)$, $t_1=0$ and by the same way as this $t_5=0$. Similarly we may replace $Z^{(r,r-1)}_{t_2^{(1)},t_2^{(1)}}$ by $I_{t_2^{(1)}}$, $Z^{(r,r-1)}_{t_3^{(1)},t_2^{(1)}}$ by $I_{t_3^{(1)}}\times u^{(1)}$ and other $Z^{(r,r-1)}_{t_2^{(1)},\kappa}$, $Z^{(r,r-1)}_{t_3^{(1)},t_3^{(1)}}$ and $Z^{(r,r-1)}_{t_3^{(1)},\lambda}$ by 0. Moreover we may assume to replace $Z^{(i,j)}_{t_2^{(2r-2)},t_2^{(2r-2)}}$, $Z^{(i,j)}_{t_2^{(2r-3)},t_2^{(2r-3)}}$, $\cdots\cdots$, $Z^{(i,j)}_{t_2^{(2)},t_2^{(2)}}$, $Z^{(i,j)}_{t_3^{(2)},t_3^{(2)}}$, $\cdots\cdots$, $Z^{(i,j)}_{t_4,t_4}$ by $I_{t_p^{(q)}}$ and other $Z^{(i,j)}_{\kappa\lambda}$ by 0. Next $Z^{(r-1,r-1)}_{t_2^{(2)},t_2^{(2)}}$ may be replaced by $I_{t_2^{(2)}}$, $Z^{(r-1,r-1)}_{t_2^{(2)},t_3^{(2)}}$ by $I_{t_3^{(2)}}\times u^{(2)}$, and $Z^{(i,j)}_{t_2^{(2r-2)},t_2^{(2r-2)}}$, $\cdots\cdots$, $Z^{(i,j)}_{t_2^{(3)},t_2^{(3)}}$, $Z^{(i,j)}_{t_2^{(1)},t_2^{(1)}}$, $Z^{(i,j)}_{t_3^{(3)},t_3^{(3)}}$, $\cdots\cdots$, $Z^{(i,j)}_{t_4,t_4}$ by $I_{t_i^{(j)}}$ and other $Z^{(i,j)}_{\kappa\lambda}$ by 0.

In this way we may replace $Q_{i,i-1}$ by

$$
Q'_{i,i-1} = \left(\begin{array}{ccccccccccccc}
\left.\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\right\}t_2^{(2r-2)} &&&&&&&&&&&& \\
& \ddots &&&&&&&&&&& \\
&& \left.\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\right\}t_2^{(2(r-i)+2)} &&&&&&&&&& \\
&&& \left.\begin{matrix}1&&\\ \vdots&\ddots&\\ \vdots&&1\end{matrix}\right\}t_2^{(2(r-i)+1)} &&&&&&&&& \\
&&& \vdots \quad \vdots & \left.\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\right\}t_2^{(2(r-i))} &&&&&&&& \\
&&& \vdots \quad \vdots && \ddots &&&&&&& \\
&&& \vdots \quad \vdots &&& \left.\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\right\}t_2^{(1)} &&&&&& \\
&&& \vdots \quad \vdots &&&& \begin{matrix}0&&\\&\ddots&\\&&0\end{matrix} &&&&& \\
&&& \begin{matrix}u^{(2(r-i)+1)}&\\ \ddots&\\ &u^{(2(r-i)+1)}\end{matrix} & \cdots & \cdots & \cdots & \cdots & \left.\begin{matrix}0&&\\&\ddots&\\&&0\end{matrix}\right\}t_2^{(2(r-i)+1)} &&&& \\
&&&&&&&&& \left.\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\right\}t_3^{(2(r-i)+1)} &&& \\
&&&&&&&&&& \ddots && \\
&&&&&&&&&&& \left.\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\right\}t_3^{(2(r-1))} & \\
&&&&&&&&&&&& \left.\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\right\}t_4
\end{array}\right)
$$

and Q_{ii} by

$$
Q'_{ii} = \left(\begin{array}{ccccccccccccc}
\left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\} t_2^{(2r-2)} & & & & & & & & & & & & \\
& \ddots & & & & & & & & & & & \\
& & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\} t_2^{(2(r-i)+3)} & & & & & & & & & & \\
& & & t_2^{(2(r-i)+2)}\left\{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right. & \cdots & \cdots & \cdots & \cdots & \begin{matrix}u^{(2(r-i)+2)} & & \\ \vdots & \ddots & \\ \vdots & & u^{(2(r-i)+2)}\end{matrix} & & & & \\
& & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\} t_2^{(2(r-i)+1)} & & & & & & & & \\
& & & & & \ddots & & & & & & & \\
& & & & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\} t_2^{(1)} & & & & & & \\
& & & & & & & \begin{matrix}0 & & \\ & \ddots & \\ & & 0\end{matrix} & & & & & \\
& & & & & & & & t_2^{(2(r-i)+2)}\left\{\begin{matrix}0 & & \\ & \ddots & \\ & & 0\end{matrix}\right. & & & & \\
& & & & & & & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\} t_3^{(2(r-i)+3)} & & & \\
& & & & & & & & & & \ddots & & \\
& & & & & & & & & & & t_3^{(2r-2)}\left\{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right. & \\
& & & & & & & & & & & & t_4\left\{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right.
\end{array}\right)
$$

where we may assume that $t_2^{(2(r-i)+1)} = t_3^{(2(r-i)+1)}$ and $t_2^{(2(r-i)+2)} = t_3^{(2(r-i)+2)}$. Then from the indecomposability of $R(a)$ $Q_{i,i-1}$ for $i \neq r+1$ are 1 or $\begin{bmatrix}1 \\ u^{(2(r-i)+1)}\end{bmatrix}$ and Q_{ii} are 1 or $[1, u^{(2(r-i))}]$.

Thus an arbitrary indecomposable representation has the following form;

$$
\text{(i)} \qquad R(a) = \left(\begin{array}{cccccccc}
y_1 & & & & & & & \\
0 & y_2 & & & & & & \\
\vdots & 0 & \ddots & & & & & \\
\vdots & \vdots & \ddots & & & & & \\
0 & \vdots & & y_r & & & & \\
z_{11} & 0 & \cdots & 0 & x_{\eta_1} & & & \\
z_{21} & z_{22} & 0 \cdots \vdots \cdots & 0 & & x_{\eta_2} & & \\
 & z_{32} & \vdots & & & & \ddots & \\
 & \ddots & \vdots & & & & \ddots & \\
 & & z_{rr} & 0 & \cdots & \cdots & x_{\eta_r} & \\
 & & z_{0r} & 0 & \cdots & \cdots & 0 \;\; x_0 & \\
 & & z_{r+1r} & 0 & \cdots & \cdots & 0 & x_{\eta_{r+1}}
\end{array}\right)
$$

where z_{ij} may be zero,

$$
\text{(ii)}\ R(a)=\begin{pmatrix}
y_1 &&&&&&& \\
\vdots & \ddots &&&&&& \\
\vdots && y_{i-1} &&&&& \\
\vdots && \vdots & y_i &&&& \\
\vdots && \vdots & \vdots & y_i &&& \\
\vdots && \vdots & \vdots & \vdots & \ddots && \\
\vdots && \vdots & \vdots & \vdots & y_r && \\
\vdots && \vdots & \vdots & \vdots & \vdots & y_r & \\
z_{11} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_1} \\
z_{21} & z_{22} & 0 & \vdots & \vdots & \vdots & \vdots & \ddots \\
& \ddots & z_{i-1,i-1} & \cdots & \cdots & \cdots & \cdots & x_{\eta_{i-1}} \\
&& z_{i,i-1} & z_{ii} & 0 & \cdots & \cdots & x_{\eta_i} \\
&& z'_{i,i-1} & 0 & z'_{ii} & \cdots & \cdots & x_{\eta_i} \\
&&& z_{i+1,i} & 0 & \vdots & \vdots & \ddots \\
&&&& \ddots & \vdots & \vdots & \ddots \\
&&&&& z_{rr} & 0\ \cdots & x_{\eta_r} \\
&&&&& 0 & z'_{rr}\ \cdots & x_{\eta_r} \\
&&&&& z_{0r} & 0\ \cdots & x_0 \\
&&&&& 0 & z_{r+1,r}\ \cdots & x_{\eta_{r+1}}
\end{pmatrix}
$$

$$
\text{(iii)}\ R(a)=\begin{pmatrix}
y_1 &&&&&& \\
\vdots & y_2 &&&&& \\
\vdots & \vdots & \ddots &&&& \\
\vdots & \vdots & y_{i-1} &&&& \\
\vdots & \vdots & \vdots & y_{i-1} &&& \\
\vdots & \vdots & \vdots & \vdots & \ddots && \\
\vdots & \vdots & \vdots & \vdots & y_r && \\
\vdots & \vdots & \vdots & \vdots & \vdots & y_r & \\
z_{11} & 0 & \cdots & \cdots & \cdots & \cdots & x_{\eta_1} \\
z_{21} & z_{22} & \vdots & \vdots & \vdots & \vdots & \ddots \\
& \ddots & z_{i-1,i-1} & z'_{i-1,i-1} & \cdots & \cdots & x_{\eta_{i-1}} \\
&& z_{i,i-1} & 0 & \cdots & \cdots & x_{\eta_i} \\
&& 0 & z'_{i,i-1} & \cdots & \cdots & x_{\eta_i} \\
&&& \ddots & \vdots & \vdots & \ddots \\
&&&& z_{rr} & 0\ \cdots & x_{\eta_r} \\
&&&& 0 & z'_{rr}\ \cdots & x_{\eta_r} \\
&&&& z_{0r} & 0\ \cdots & x_0 \\
&&&& 0 & z'_{r+1,r}\ \cdots & x_{\eta_{r+1}}
\end{pmatrix}
$$

and the degree of $R(a)$ is bounded and is less than $4r+1$.

[The case 2]; Suppose that $Ne_i = Au_i^{(\eta_i)} \oplus Au_i^{(\eta_{i+1})}$ ($i=1, \cdots\cdots, r-1$) and $Ne_{r_1} = Au_{r_1}^{(\eta_r)}$, $Ne_{r_2} = Au_{r_2}^{(\eta_r)}$ where $Au_{r-1}^{(\eta_r)} \cong Au_{r_1}^{(\eta_r)} \cong Au_{r_2}^{(\eta_r)}$. Then by the same way as [the case 1] an arbitrary directly indecomposable representation has one of the following forms;

$$\text{(i)} \qquad R(a) = \left(\begin{array}{ccccccccc}
y_1 &&&&&&&& \\
\vdots & y_2 &&&&&&& \\
\vdots & \vdots & \ddots &&&&&& \\
\vdots & \vdots && y_{r-1} &&&&& \\
\vdots & \vdots && \vdots & y_{r_1} &&&& \\
\vdots & \vdots && \vdots & \vdots & y_{r_1} &&& \\
z_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_1} && \\
z_{21} & z_{22} && \vdots & \vdots & \vdots && \ddots & \\
& \ddots & \ddots & \vdots & \vdots & \vdots &&& \ddots \\
&&& z_{rr-1} & z_{rr_1} & z_{rr_2} &&& x_{\eta_r}
\end{array}\right)$$

$$\text{(ii)} \quad R(a) = \left(\begin{array}{cccccccccccccccccc}
y_1 &&&&&&&&&&&&&&&&& \\
\vdots & \ddots &&&&&&&&&&&&&&&& \\
\vdots && y_{i-1} &&&&&&&&&&&&&&& \\
\vdots && \vdots & y_i &&&&&&&&&&&&&& \\
\vdots && \vdots & \vdots & y_i &&&&&&&&&&&&& \\
\vdots && \vdots & \vdots & \vdots & \ddots &&&&&&&&&&&& \\
\vdots && \vdots & \vdots & \vdots && y_{r-1} &&&&&&&&&&& \\
\vdots && \vdots & \vdots & \vdots && \vdots & y_{r-1} &&&&&&&&&& \\
\vdots && \vdots & \vdots & \vdots && \vdots & \vdots & y_{r_1} &&&&&&&&& \\
\vdots && \vdots & \vdots & \vdots && \vdots & \vdots & \vdots & y_{r_2} &&&&&&&& \\
z_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_1} &&&&&&& \\
& \ddots & \vdots & \vdots & \vdots && \vdots & \vdots & \vdots & \vdots && \ddots &&&&&& \\
&& z_{i-1,i-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_{i-1}} &&&&& \\
&& z_{ii-1} & z_{ii} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_i} &&&& \\
&& z'_{ii-1} & 0 & z'_{ii} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_i} &&& \\
&&&& & \ddots & \vdots & \vdots & \vdots & \vdots &&&&&& \ddots && \\
&&&&&& z_{rr-1} & 0 & z_{rr_1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_r} & \\
&&&&&& 0 & z'_{rr-1} & 0 & z_{rr_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_r}
\end{array}\right)$$

$$
\text{(iii) } R(a)=\left(\begin{array}{cccccccccccccccc}
y_1 \\
\vdots & y_2 \\
\vdots & \vdots & \ddots \\
\vdots & \vdots & y_{i-1} \\
\vdots & \vdots & \vdots & y_{i-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots & \vdots & y_{r-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & y_{r-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & y_{r_1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & y_{r_2} \\
z_{11} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_1} \\
z_{21} & z_{22} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots \\
 & & z_{i-1i-1} & z'_{i-1i-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_{i-1}} \\
 & & z_{ii-1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_i} \\
 & & 0 & z'_{i-1i-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_i} \\
 & & & & \vdots & \vdots & \vdots & \vdots & & & & & & \ddots \\
 & & & & z_{rr-1} & 0 & z_{rr_1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_r} \\
 & & & & 0 & z'_{rr-1} & 0 & z_{rr_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{\eta_r}
\end{array}\right)
$$

and the degree of $R(a)$ is less than $4r-1$.

[The case 3]; Suppose that $Ne_1=Au_1^{(\kappa_1)}\oplus Au_1^{(\kappa_2)}$, $Ne_2=Au_2^{(\kappa_2)}\oplus Au_2^{(0)}\oplus Au_2^{(\kappa_3)}$ and $Ne_3=Au_3^{(\kappa_3)}\oplus Au_3^{(\kappa_4)}$. Then let $\mathfrak{M}=\sum_i\sum_{\lambda_i}Ae_im_{i\lambda_i}$ be an arbitrary left module and the representation $R(a)$ by $\mathfrak{M}$ has the following form;

$$
R(a)=\left(\begin{array}{cccccccc}
I_{\lambda_1}\times y_1 \\
0 & I_{\lambda_2}\times y_2 \\
0 & 0 & I_{\lambda_3}\times y_3 \\
Q_{11}\times z_{11} & 0 & 0 & I_{s_1}\times x_{\kappa_1} \\
Q_{21}\times z_{21} & Q_{22}\times z_{22} & 0 & 0 & I_{s_2}\times x_{\kappa_2} \\
0 & Q_{02}\times z_{02} & 0 & 0 & 0 & I_{s_0}\times x_{\kappa_0} \\
0 & Q_{32}\times z_{32} & Q_{33}\times z_{33} & 0 & 0 & 0 & I_{s_3}\times x_{\kappa_3} \\
0 & 0 & Q_{43}\times z_{43} & 0 & 0 & 0 & 0 & I_{s_4}\times x_{\kappa_4}
\end{array}\right)
$$

First by [the case 1] Q_{22}, Q_{02}, Q_{32}, Q_{11}, Q_{21} may be replaced by

$$Q'_{22} = \begin{pmatrix}
\underbrace{0\cdots0}_{t_{16}} \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_1 & & & & & & & & & & \\
& \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_2 & & & & & & & & & \\
& & \ddots & & & & & & & & \\
& & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{10} & & & & & & & \\
& & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t'_8 & & & & & & \\
& & & & & \ddots & & & & & \\
& & & & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t'_9 & & & & \\
& & & & & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{11} & & & \\
& & & & & & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t'_7 & & \\
& & & & & & & & & t_{12}\left\{\begin{matrix}1 & & & & \\ & \ddots & & & \\ & & 1 & 0 & 0\end{matrix}\right. & \\
& & & & & & & & & & t_{13}\left\{\begin{matrix}0 & 1 & u & 0 & 0\cdots\cdots\cdots \\ & & 1 & u & 0\ 0 \\ & & & \ddots & \\ & & & & 1\ u\ 0\ 0\end{matrix}\right. \\
& & & & & & & & & & t_{14}\left\{\begin{matrix}1 & u & 0 & 0 \\ & \ddots & & \\ & & 1\ u & 0\ 0\end{matrix}\right. \\
& & & & & & & & & & t_{15}\left\{\begin{matrix}1 & u & 0 & 0 \\ & & \ddots & \\ & & & 1\ u\end{matrix}\right.
\end{pmatrix}$$

$$
Q'_{02}=\left(\begin{array}{ccccccccccc}
\left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{16} & & & & & & & & & & \\
 & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{1} & & & & & & & & & \\
 & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{2} & & & & & & & & \\
 & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{3} & & & & & & & \\
 & & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{4} & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & t_{9}\left\{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\;0\cdots\cdots\cdots\cdots 0\;0\end{matrix}\right. & & & & \\
 & & & & & & 0\cdots\cdots\cdots\cdots & t_{13}\left\{\begin{matrix}0\,1\,0\,0 & & \\ & 0\,1\,0\,0 & \\ & \ddots & \\ & & 0\,1\,0\,0\end{matrix}\right. & & & \\
 & & & & & & & & t_{14}\left\{\begin{matrix}0\,1\,0\,0 & \\ \ddots & \\ & 0\,1\,0\,0\end{matrix}\right. & & \\
 & & & & & & & & & t_{15}\left\{\begin{matrix}0\,1\,0\,0 & \\ \ddots & \\ & 0\,1\end{matrix}\right. &
\end{array}\right)
$$

$$
Q'_{32}=\left(\begin{array}{ccccccc}
\left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{16} & & & & & & \\
 & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{1} & & & & & \\
 & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{2} & & & & \\
 & & & t_{3}\left\{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\;0\cdots 0\;0\end{matrix}\right. & & & \\
 & & & 0\;0\cdots 0\;1 & \left.\begin{matrix} & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_{10} & & \\
 & & & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t'_{8} & \\
 & & & & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t'_{9}
\end{array}\right.
$$

$$
\left(\begin{array}{llllll}
\left.\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\right\}t_{11} &&&&&\\
& \left.\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\right\}t'_{7} &&&&\\
&& t_{12}\left\{\begin{matrix}1&&\\&\ddots&\\&&1\;0\;0\cdots\cdots\end{matrix}\right. &&&\\
&&& t_{13}\left\{\begin{matrix}1\,0\,0\,0&&\\&1\,0\,0\,0&\\&\ddots&\\&&1\,0\,0\,0\end{matrix}\right. &&\\
&&&& t_{14}\left\{\begin{matrix}1\,0\,0\,0&\\ &\ddots\\&1\,0\,0\,0\end{matrix}\right. &\\
&&&&& t_{15}\left\{\begin{matrix}1\,0\,0\,0&&\\&\ddots&\\&&1\,0\,0\,0\\&&\quad 1\,0\end{matrix}\right.
\end{array}\right)
$$

$$
Q'_{11} = \left(\begin{array}{l}
\left.\begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\right\}t_{1} \\
\qquad\underbrace{0\cdots0}_{t_2}\;0 \\
\qquad\qquad\quad 1 \\
\qquad\qquad\qquad \ddots \\
\qquad\qquad\underbrace{\qquad}_{t_4}\;1\;\underbrace{0\cdots0}_{t_5}\;0 \\
\qquad\qquad\qquad\qquad\qquad \left.\begin{matrix}1\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots u & \\ \ddots \qquad\qquad\qquad\qquad\qquad\quad \vdots & \\ \quad 1\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots u\end{matrix}\right\}t_7 \\
\qquad\qquad\qquad\qquad\qquad\; t_8\left\{\begin{matrix}1&&\\&\ddots&\\&&1\;\underbrace{0\cdots0}_{t_9}\;0\end{matrix}\right. \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \ddots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\underbrace{\qquad}_{t_{10}}\;1\;\underbrace{0\cdots0}_{t_{11}}\;\underbrace{0\cdots0}_{t_7}\;0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\underbrace{\qquad}_{t_{13}}\;1\;\underbrace{0\cdots0}_{t_{14}}
\end{array}\right)
$$

$$Q'_{21} = \begin{pmatrix}
\left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_1 & & & & & & & & & & & & & & & \\
& \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_2 & & & & & & & & & & & & & & \\
& & t_3\left\{\begin{matrix}0 \\ \vdots \\ 0\end{matrix}\right. & & & & & & & & & & & & & \\
& & & \left.\begin{matrix}0 & 1 & & \\ & & \ddots & \\ & & & 1\end{matrix}\right\}t_4 & & & & & & & & & & & & \\
& & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t_5 & & & & & & & & & & & \\
& & & & & t_6\left\{\begin{matrix}0 \\ \vdots \\ 0\end{matrix}\right. & & & & & & & & & & \\
& & & & & & \underbrace{\begin{matrix}0 & 1 & & \\ & & \ddots & \\ & & & 1\end{matrix}}_{t_7} & & & & & & & & & \\
& & & & & & & \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{t_8} & & & & & & & & \\
& & & & & & & \vdots & \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{t_9} & & & & & & & \\
& & & & & & & \vdots & \vdots & \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{t_{10}} & & & & & & \\
& & & & & & & \begin{matrix} v_1 & \cdots & & \cdots & & \cdots & 0 \\ & \ddots & & & & & \vdots \\ & & v_1 & \cdots & & \cdots & 0 \\ & & v_2 & \cdots & & \cdots & 0 \\ & & & \ddots & & \cdots & \vdots \\ & & & & v_2 & \cdots & 0 \end{matrix} & & & \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix}\right.\begin{matrix} \}t_8 \\ \\ \}t_9 \end{matrix} & & & & & \\
& & & & & & & & & & \left.\begin{matrix}0 & 1 & & \\ & & \ddots & \\ & & & 1\end{matrix}\right\}t_{11} & & & & & \\
& & & & & & & & & & & \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\}t'_7 & & & & \\
& & & & & & & & & & & & \left.\begin{matrix}0 \\ \vdots \\ 0\end{matrix}\right\}t_{12} & & & \\
& & & & & & & & & & & & & \left.\begin{matrix}0 & 1 & & \\ & & \ddots & \\ & & & 1\end{matrix}\right\}t_{13} & & \\
& & & & & & & & & & & & & & t_{14}\left\{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right. & \\
& & & & & & & & & & & & & & & t_{15}\left\{\begin{matrix}0 \\ \vdots \\ 0\end{matrix}\right.
\end{pmatrix}$$

Now Q_{43} may be replaced by

$$Q'_{43}=\begin{pmatrix} 1 & 0\cdots\cdots 0 & 0\cdots\cdots 0\\ & \ddots & \vdots \quad\vdots & \vdots\\ 0 & \ddots & \vdots \quad \vdots & \vdots\\ & & 1 \quad 0\cdots\cdots 0 \end{pmatrix}$$

and Q_{33} is broken up into submatrices to correspond to divisions of Q'_{32} and Q'_{43} as follows;

$$Q_{33}=\begin{pmatrix} C_{16,16} & C_{16,1}\cdots\cdots & C_{16,15} & D_{16,16} & D_{16,1}\cdots\cdots & D_{16,15}\\ C_{1,16} & C_{11} & \vdots & D_{1,16}\cdots\cdots & & \vdots\\ \vdots & & \vdots & \vdots & & \vdots\\ C_{15,16} & \cdots\cdots & C_{15,15} & D_{15,16} & \cdots\cdots & D_{15,15} \end{pmatrix}.$$

First $D_{16,16}$ and $D_{7,16}$ may be replaced by

$$D'_{16,16}=\begin{pmatrix} I_{t_{16}^{(1)}} & 0\cdots\cdots\cdots 0\\ 0 & \\ \vdots & *\\ 0 & \end{pmatrix} \quad\text{and}\quad D'_{7,16}=\begin{pmatrix} I_{t_{16}^{(1)}}\times u^{(1)} & 0\cdots\cdots\cdots 0\\ 0 & \\ \vdots & \\ \vdots & *\\ 0 & \end{pmatrix},$$

and other $D_{16,\kappa}$ and $D_{\lambda,16}$ are replaced by

$$D'_{16,\kappa}=\begin{pmatrix} 0\cdots\cdots\cdots 0\\ \\ *\\ \\ \end{pmatrix}\}t_{16}^{(1)} \quad\text{and}\quad D'_{\lambda,16}=\begin{pmatrix} \overbrace{0}^{t_{16}^{(1)}} & \\ \vdots & *\\ 0 & \end{pmatrix}$$

and $D_{7,\kappa}$ is replaced by $D'_{7,\kappa}=\begin{pmatrix} 0\cdots\cdots 0\\ *\end{pmatrix}$ where $\kappa\neq 7$. Next $D_{16,16}$ and $D_{8,16}$ may be replaced by

$$D''_{7,16}=\begin{pmatrix} I_{t_{16}^{(1)}} & 0\cdots\cdots\cdots\cdots 0 & \\ 0 & I_{t_{16}^{(2)}} & 0\cdots\cdots\cdots 0\\ \vdots & 0 & \\ \vdots & \vdots & *\\ 0 & 0 & \end{pmatrix} \quad\text{and}\quad D''_{8,16}=\begin{pmatrix} 0 & I_{t_{16}^{(2)}}\times g^{(2)} & 0\cdots\cdots 0\\ \vdots & \vdots & \\ \vdots & \vdots & *\\ \vdots & \vdots & \\ 0 & 0 & \end{pmatrix},$$

$$D'_{7,16} \text{ by } D''_{7,16} = \begin{pmatrix} I_{t^{(1)}_{16}}\times g^{(1)} & 0\cdots\cdots 0 & \\ 0 & 0 & \\ \vdots & \vdots & * \\ 0 & 0 & \end{pmatrix}, \quad D_{\kappa',16} \text{ by } D'_{\kappa,16} = \begin{pmatrix} 0 & 0 & \\ \vdots & \vdots & * \\ 0 & 0 & \end{pmatrix},$$

$$D_{16,\lambda} \text{ by } D''_{16,\lambda} = \begin{pmatrix} 0\cdots\cdots 0 \\ 0\cdots\cdots 0 \\ * \end{pmatrix} \text{ and } D_{8,\mu} \text{ by } D'_{8,\mu} = \begin{pmatrix} 0\cdots\cdots 0 \\ * \end{pmatrix}.$$

In this way $D_{16,16}$, $D_{7,16}$, $D_{8,16}$, $D_{9,16}$, $D_{10,16}$, $D_{11,16}$ and $D_{12,16}$ are replaced by

$$D'_{16,16} = \begin{pmatrix} I_{t^{(1)}_{16}} & & 0 \\ & \ddots & \\ & & I_{t^{(8)}_{16}} \\ & & 0 \\ 0 & & \vdots \\ & & 0 \end{pmatrix}, \quad D'_{7,16} = \begin{pmatrix} I_{t^{(1)}_{16}}\times g^{(1)} & 0\cdots\cdots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix}, \quad D'_{8,16} = \begin{pmatrix} 0 & I_{t^{(2)}_{16}}\times g^{(2)} & 0\cdots \\ \vdots & 0 & \\ \vdots & \vdots & 0 \\ 0 & 0 & \end{pmatrix},$$

$$D'_{9,16} = \begin{pmatrix} 0\,0 & I_{t^{(3)}_{16}}\times g^{(3)} & 0 & 0 & 000 \\ \vdots\,\vdots & 0 & \vdots & I_{t^{(5)}_{16}}\times g^{(5)} & \vdots\,\vdots\,\vdots \\ \vdots\,\vdots & \vdots & \vdots & 0 & \vdots\,\vdots\,\vdots \\ \vdots\,\vdots & \vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots \\ 0\,0 & 0 & 0 & 0 & 000 \end{pmatrix}, \quad D'_{10,16} = \begin{pmatrix} 000 & I_{t^{(4)}_{16}}\times g^{(4)} & 0 & 000 \\ \vdots\,\vdots\,\vdots & 0 & I_{t^{(5)}_{16}}\times \bar{g}^{(5)} & \vdots\,\vdots\,\vdots \\ \vdots\,\vdots\,\vdots & \vdots & 0 & \vdots\,\vdots\,\vdots \\ \vdots\,\vdots\,\vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots \\ 000 & 0 & 0 & 000 \end{pmatrix},$$

$$D'_{11\ 16} = \begin{pmatrix} 0\,0\,0\,0\,0\, I_{t^{(6)}_{16}}\times g^{(6)}\, 0\,0 \\ 0 \end{pmatrix} \text{ and } D'_{12,16} = \begin{pmatrix} 0\,0\,0\,0\,0\,0\, I_{t^{(7)}_{16}}\times g^{(7)}\, 0 \\ 0 \end{pmatrix}.$$

Moreover D_{11} is replaced by $D'_{11} = \begin{pmatrix} I_{t^{(1)}_1} & \\ & I_{t^{(2)}_1} \\ 0 & \end{pmatrix}$, $D_{12,1}$ by $D'_{12,1} = \begin{pmatrix} I_{t^{(1)}_1}\times h & 0 \\ 0 & 0 \end{pmatrix}$,

$$D_{22} \text{ by } D'_{22} = \begin{pmatrix} I_{t^{(1)}_2} & & & \\ & I_{t^{(2)}_2} & & \\ & & I_{t^{(3)}_2} & \\ & & 0 & I_{t^{(2)}_4} \\ & & \vdots & \vdots \\ & & 0 & 0 \end{pmatrix}, \quad D_{10,2} \text{ by } D'_{10,2} = \begin{pmatrix} 0 & 0 & 0\,0 \\ 0 & 0 & 0\,0 \\ 0 & 0 & 0\,0 \\ 0 & I_{t^{(2)}_2}\times j^{(2)} & 0\,0 \\ 0 & 0 & 0\,0 \\ 0 & 0 & 0\,0 \\ 0 & 0 & 0\,0 \end{pmatrix}.$$

$$D_{12,2} \text{ by } D'_{12,2} = \begin{pmatrix} 0 & 0\,0\,0 \\ 0 & 0\,0\,0 \\ I_{t^{(1)}_2}\times j^{(1)} & 0\,0\,0 \\ 0 & 0\,0\,0 \end{pmatrix}, \quad D_{8,2} \text{ by } D_{8,2}' = \begin{pmatrix} 0\,0 & 0 & 0 \\ 0\,0 & I_{t^{(3)}_2}\times j^{(3)} & 0 \\ 0\,0 & 0 & 0 \end{pmatrix},$$

$$,\quad D_{33} \text{ by } D'_{33} = \begin{pmatrix} I_{t_3} \\ 0 \end{pmatrix},$$

$$D_{10,10} \text{ by } D'_{10,10} = \begin{pmatrix} 0 & & \\ 0 & & \\ 0 & & 0 \\ I_{t_{10}^{(1)}} & & \\ & I_{t_{10}^{(2)}} & \\ & & I_{t_{10}^{(4)}} \\ & 0 & 0 \\ & & \vdots \\ & & 0 \end{pmatrix}, \quad D_{9,10} \text{ by } D'_{9,10} = \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots \\ I_{t_{10}^{(1)}} \times y^{(1)} & \vdots & \vdots \\ 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix},$$

$$D_{15,10} \text{ by } D'_{15,10} = \begin{pmatrix} 0 & I_{t_{10}^{(2)}} \times y^{(2)} & 0 \\ 0 & 0 & \vdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}, \quad D_{88} \text{ by } D'_{88} = \begin{pmatrix} 0 \cdots\cdots 0 \\ I_{t_8^{(1)}} \\ \\ 0 \end{pmatrix},$$

$$D_{99} \text{ by } D'_{99} = \begin{pmatrix} I_{t_9^{(1)}} \\ \\ 0 \end{pmatrix}, \quad D_{11,11} \text{ by } D'_{11,11} = \begin{pmatrix} I_{t_{11}^{(1)}} & & 0 \\ & I_{t_{11}^{(2)}} & \\ 0 & & I_{t_{11}^{(3)}} \end{pmatrix},$$

$$D_{15,11} \text{ by } D'_{15,11} = \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots \\ I_{t_{11}^{(1)}} \times v^{(1)} & \vdots & \vdots \\ 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}, \quad D_{13,11} \text{ by } D'_{13,11} = \begin{pmatrix} 0 & I_{t_{11}^{(2)}} \times v^{(2)} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D_{77} \text{ by } D'_{77} = \begin{pmatrix} & 0 \\ I_{t_7^{(1)}} & \\ & \\ & I_{t_7^{(2)}} \\ 0 & \end{pmatrix}, \quad D_{15,7} \text{ by } D'_{15,7} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & \vdots \\ I_{t_7^{(1)}} \times x & \vdots \\ 0 & 0 \end{pmatrix},$$

$$D_{12,12} \text{ by } D_{12,12} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_{t_{12}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad D_{13,13} \text{ by } D'_{13,13} = \begin{pmatrix} 0 \\ \\ I_{t_{13}} \\ \\ 0 \end{pmatrix},$$

$$D_{14,14} \text{ by } D'_{14,14} = \begin{pmatrix} I_{t_{14}} \\ \\ 0 \end{pmatrix} \quad \text{and} \quad D_{15,15} \text{ by } D'_{15,15} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_{t_{15}} \end{pmatrix}$$

and other D_{ij} are replaced by 0.

Next we may replace $C_{\kappa\lambda}$ by the scalar forms by the same way as above.

Thus an arbitrary directly indecomposable representation has one of the following forms;

$$R_1(a)=\left(\begin{array}{cccccccc}
y_1 &&&&&&& \\
0 & y_2 &&&&& \multicolumn{2}{c}{} \\
0 & 0 & y_3 &&& 0 && \\
z_{11} & 0 & 0 & x_1 &&&& \\
z_{21} & z_{22} & 0 & 0 & x_2 &&& \\
0 & z_{32} & 0 & 0 & 0 & x_3 && \\
0 & z_{42} & z_{43} & 0 & 0 & 0 & x_4 & \\
0 & 0 & z_{53} & 0 & 0 & 0 & 0 & x_5
\end{array}\right)$$

$$R_2(a)=\left(\begin{array}{cccccccccccc}
y_1 &&&&&&&&&&& \\
0 & y_2 &&&&&&&&&& \\
\vdots & 0 & y_2 &&&&&&&&& \\
\vdots & \vdots & 0 & y_3 &&&&&&&& \\
0 & \vdots & \vdots & 0 & y_3 &&&& 0 &&& \\
z_{11} & 0 & \cdots & \cdots & 0 & x_1 &&&&&& \\
z_{21} & z_{22} & 0 & \cdots & \cdots & 0 & x_2 &&&&& \\
0 & 0 & z'_{22} & \cdots & \cdots & \cdots & 0 & x_2 &&&& \\
0 & z_{32} & 0 & 0 & \cdots & \cdots & \cdots & 0 & x_3 &&& \\
0 & z_{42} & 0 & z_{43} & 0 & \cdots & \cdots & \cdots & 0 & x_4 && \\
0 & 0 & z'_{42} & 0 & z'_{43} & 0 & \cdots & \cdots & \cdots & 0 & x_4 & \\
0 & 0 & 0 & z_{53} & z'_{53} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{array}\right)$$

$$R_3(a)=\left(\begin{array}{ccccccccccccc}
y_1 &&&&&&&&&&&& \\
0 & y &&&&&&&&&&& \\
\vdots & 0 & y_2 &&&&&&&&&& \\
\vdots & \vdots & 0 & y_2 &&&&&&&&& \\
\vdots & \vdots & \vdots & 0 & y_2 &&&&& 0 &&& \\
0 & \vdots & \vdots & \vdots & 0 & y_3 &&&&&&& \\
z_{11} & 0 & 0 & \cdots & \cdots & 0 & x_1 &&&&&& \\
z_{21} & 0 & z_{22} & 0 & \cdots & \cdots & 0 & x_2 &&&&& \\
0 & z'_{21} & 0 & z'_{22} & 0 & \cdots & \cdots & 0 & x_2 &&&& \\
 & & 0 & z'_{32} & 0 & \cdots & \cdots & \cdots & 0 & x_3 &&& \\
 & & z_{42} & 0 & z_{43} & 0 & \cdots & \cdots & \cdots & 0 & x_4 && \\
 & & 0 & z'_{42} & 0 & z'_{43} & 0 & \cdots & \cdots & \cdots & 0 & x_4 & \\
 & & & & z_{53} & z'_{53} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{array}\right)$$

$$R_4(a)=\left(\begin{array}{ccccccccccccccccc}
y_1 \\
0 & y_1 \\
\vdots & 0 & y_2 \\
\vdots & \vdots & 0 & y_2 \\
\vdots & \vdots & \vdots & 0 & y_2 & & & & & & & & 0 \\
\vdots & \vdots & \vdots & \vdots & 0 & y_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_3 \\
0 & 0 & \vdots & \vdots & \vdots & \vdots & 0 & y_3 \\
z_{11} & z'_{11} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_1 \\
z_{21} & 0 & z_{22} & 0 & 0 & \cdots & \cdots & \cdots & 0 & x_2 \\
0 & z'_{21} & 0 & z'_{22} & 0 & 0 & \cdots & \cdots & \cdots & 0 & x_2 \\
0 & 0 & 0 & 0 & z_{22} & z_{22}^{(2)} & 0 & \cdots & \cdots & \cdots & 0 & x_2 \\
\vdots & \vdots & 0 & z'_{32} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_3 \\
\vdots & \vdots & 0 & 0 & 0 & z_{32}^{(2)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 \\
\vdots & \vdots & z_{42} & 0 & 0 & 0 & z_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 \\
\vdots & \vdots & 0 & 0 & z_{42}^{(1)} & 0 & 0 & z_{43}^{(1)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 \\
0 & 0 & 0 & 0 & 0 & 0 & z_{53} & z_{53}^{(1)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{array}\right)$$

$$R_5(a)=\left(\begin{array}{cccccccccccccccc}
y_1 \\
0 & y_1 \\
\vdots & 0 & y_2 \\
\vdots & \vdots & 0 & y_2 \\
\vdots & \vdots & \vdots & 0 & y_2 & & & & & & & 0 \\
\vdots & \vdots & \vdots & \vdots & 0 & y_3 \\
0 & \vdots & \vdots & \vdots & \vdots & 0 & y_3 \\
z_{11} & \vdots & 0 & \cdots & \cdots & \cdots & 0 & x_1 \\
z_{21} & \vdots & z_{22} & 0 & \cdots & \cdots & \cdots & 0 & x_2 \\
z'_{21} & 0 & 0 & z'_{22} & 0 & \cdots & \cdots & \cdots & 0 & x_2 \\
0 & z_{21}^{(1)} & 0 & 0 & z_{22}^{(1)} & 0 & \cdots & \cdots & \cdots & 0 & x_2 \\
\vdots & 0 & z_{32} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 \\
\vdots & 0 & 0 & 0 & z_{32}^{(1)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 \\
\vdots & 0 & 0 & z'_{42} & 0 & z'_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 \\
\vdots & 0 & 0 & 0 & z_{42}^{(1)} & 0 & z_{43}^{(1)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 \\
0 & 0 & 0 & 0 & 0 & z'_{53} & z_{53}^{(1)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{array}\right)$$

$$R_6(a)=\left(\begin{array}{ccccccccccc}
y_1 \\
0 & y_2 \\
\vdots & 0 & y_2 \\
\vdots & \vdots & 0 & y_3 & & & & 0 \\
0 & \cdots & \cdots & 0 & y_3 \\
z_{11} & 0 & 0 & \cdots & 0 & x_1 \\
z_{21} & z_{22} & 0 & \cdots & \cdots & 0 & x_2 \\
0 & 0 & z_{32} & 0 & 0 & \cdots & 0 & x_3 \\
\vdots & z_{42} & 0 & z_{43} & 0 & \cdots & \cdots & 0 & x_4 \\
\vdots & 0 & z'_{42} & 0 & z'_{43} & 0 & \cdots & \cdots & 0 & x_4 \\
0 & 0 & 0 & z_{53} & z'_{53} & 0 & \cdots & \cdots & \cdots & 0 & x_5
\end{array}\right)$$

$$R_7(a)=\begin{pmatrix}
y_1 &&&&&&&&&&&&& \\
0 & y_2 &&&&&&&&&&&& \\
\vdots & 0 & y_2 &&&&&&&&&&& \\
\vdots & \vdots & 0 & y_2 &&&&&& 0 &&&& \\
\vdots & \vdots & \vdots & 0 & y_3 &&&&&&&&& \\
0 & \cdots & \cdots & \cdots & 0 & y_3 &&&&&&&& \\
z_{11} & 0 & \cdots & \cdots & \cdots & 0 & x_1 &&&&&&& \\
z_{21} & z_{22} & 0 & \cdots & \cdots & \cdots & 0 & x_2 &&&&&& \\
z'_{21} & 0 & z'_{22} & 0 & \cdots & \cdots & \cdots & 0 & x_2 &&&&& \\
0 & 0 & 0 & z''_{32} & 0 & \cdots & \cdots & \cdots & 0 & x_3 &&&& \\
0 & z_{32} & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & x_3 &&& \\
0 & 0 & z_{42}' & 0 & z'_{43} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_4 && \\
\vdots & \vdots & 0 & z''_{42} & 0 & z''_{43} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_4 & \\
0 & 0 & 0 & 0 & z_{53} & z''_{53} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{pmatrix}$$

$$R_8(a)=\begin{pmatrix}
y_1 &&&&&&&&&&&&&& \\
0 & y_1 &&&&&&&&&&&&& \\
\vdots & 0 & y_2 &&&&&&&&&&&& \\
\vdots & \vdots & 0 & y_2 &&&&&&&&&&& \\
\vdots & \vdots & \vdots & 0 & y_2 &&&&&& 0 &&&& \\
\vdots & \vdots & \vdots & \vdots & 0 & y_3 &&&&&&&&& \\
0 & 0 & \cdots & \cdots & \cdots & 0 & y_3 &&&&&&&& \\
z_{11} & z'_{11} & 0 & \cdots & \cdots & \cdots & 0 & x_1 &&&&&&& \\
z_{21} & 0 & z_{22} & 0 & \cdots & \cdots & \cdots & 0 & x_2 &&&&&& \\
0 & z'_{21} & 0 & z'_{22} & 0 & \cdots & \cdots & \cdots & 0 & x_2 &&&&& \\
0 & 0 & 0 & 0 & z''_{32} & 0 & \cdots & \cdots & \cdots & 0 & x_3 &&&& \\
0 & 0 & z_{32} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_3 &&& \\
0 & 0 & 0 & 0 & z''_{42} & z''_{43} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_4 && \\
0 & 0 & 0 & z_{42} & 0 & 0 & z_{43} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_4 & \\
0 & 0 & 0 & 0 & 0 & z''_{53} & z_{53} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{pmatrix}$$

$$R_9(a)=\begin{pmatrix}
y_1 &&&&&&&&&&&&&&&&&& \\
0 & y_1 &&&&&&&&&&&&&&&&& \\
\vdots & 0 & y_2 &&&&&&&&&&&&&&&& \\
\vdots & \vdots & 0 & y_2 &&&&&&&&&&&&&&& \\
\vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&&&&&&& \\
\vdots & \vdots & \vdots & \vdots & \vdots & y_2 &&&&&&&& 0 &&&&& \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_3 &&&&&&&&&&&& \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_3 &&&&&&&&&&& \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & y_3 &&&&&&&&&& \\
z_{11} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_1 &&&&&&&&& \\
z_{21} & 0 & z_{22} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&&&& \\
0 & z'_{21} & 0 & z'_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&&& \\
0 & z''_{21} & 0 & 0 & z''_{22} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&& \\
\vdots & \vdots & 0 & 0 & z''_{32} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 &&&&& \\
\vdots & \vdots & 0 & 0 & 0 & z^{(2)}_{32} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 &&&& \\
\vdots & \vdots & z_{42} & 0 & 0 & 0 & z_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 &&& \\
\vdots & \vdots & 0 & z'_{42} & 0 & 0 & 0 & z'_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 && \\
0 & 0 & 0 & 0 & 0 & z''_{42} & 0 & 0 & z''_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 & \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & z_{53} & z'_{53} & z''_{53} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{pmatrix}$$

$$R_{10}(a)=\begin{pmatrix}
y_1 &&&&&&&&&&&&& \\
0 & y_1 &&&&&&&&&&&& \\
\vdots & 0 & y_2 &&&&&&&&&&& \\
\vdots & \vdots & 0 & y_2 &&&&&&&&&& \\
\vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&& \\
\vdots & \vdots & \vdots & \vdots & 0 & y_3 &&&&&& 0 && \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & y_3 &&&&&&& \\
z_{11} & 0 & 0 & 0 & \cdots & \cdots & 0 & x_1 &&&&&& \\
z_{21} & 0 & z_{22}^{(1)} & z_{22}^{(2)} & 0 & \cdots & \cdots & 0 & x_2 &&&&& \\
0 & z'_{21} & 0 & 0 & z'_{22} & 0 & \cdots & \cdots & 0 & x_2 &&&& \\
0 & 0 & 0 & z_{32}^{(2)} & 0 & 0 & \cdots & \cdots & \cdots & 0 & x_3 &&& \\
0 & 0 & z_{42}^{(1)} & 0 & 0 & z_{43}^{(1)} & 0 & \cdots & \cdots & \cdots & 0 & x_4 && \\
0 & 0 & 0 & 0 & z'_{42} & 0 & z'_{43} & 0 & \cdots & \cdots & \cdots & 0 & x_4 & \\
0 & 0 & 0 & 0 & 0 & z_{53}^{(1)} & z'_{53} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{pmatrix}$$

$$R_{11}(a)=\begin{pmatrix}
y_1 &&&&&&&&&&&&&& \\
0 & y_1 &&&&&&&&&&&&& \\
\vdots & 0 & y_2 &&&&&&&&&&&& \\
\vdots & \vdots & 0 & y_2 &&&&&&&&&&& \\
\vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&&& \\
\vdots & \vdots & \vdots & \vdots & 0 & y_3 &&&&&& 0 &&& \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & y_3 &&&&&&&& \\
z_{11} & 0 & 0 & \cdots & \cdots & \cdots & 0 & x_1 &&&&&&& \\
z_{21} & 0 & z_{22} & 0 & \cdots & \cdots & \cdots & 0 & x_2 &&&&&& \\
0 & z'_{21} & 0 & z'_{22} & 0 & \cdots & \cdots & \cdots & 0 & x_2 &&&&& \\
0 & z''_{21} & 0 & 0 & z''_{22} & 0 & \cdots & \cdots & \cdots & 0 & x_2 &&&& \\
0 & 0 & 0 & 0 & z''_{32} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_3 &&& \\
0 & 0 & z_{42} & 0 & 0 & z_{43} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_4 && \\
0 & 0 & 0 & z'_{42} & 0 & 0 & z'_{43} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_4 & \\
0 & 0 & 0 & 0 & 0 & z_{53} & z'_{53} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{pmatrix}$$

$$R_{12}(a)=\begin{pmatrix}
y_1 &&&&&&&&&&&& \\
0 & y_2 &&&&&&&&&&& \\
\vdots & 0 & y_2 &&&&&&&&&& \\
\vdots & \vdots & 0 & y_2 &&&&&&&&& \\
\vdots & \vdots & \vdots & 0 & y_3 &&&&& 0 &&& \\
0 & \cdots & \cdots & \cdots & 0 & y_3 &&&&&&& \\
z_{11} & 0 & 0 & \cdots & \cdots & 0 & x_1 &&&&&& \\
z_{21} & z_{22} & 0 & 0 & \cdots & \cdots & 0 & x_2 &&&&& \\
0 & 0 & z_{22}^{(1)} & z_{22}^{(2)} & 0 & \cdots & \cdots & 0 & x_2 &&&& \\
\vdots & 0 & 0 & z_{32}^{(2)} & 0 & \cdots & \cdots & \cdots & 0 & x_3 &&& \\
\vdots & 0 & z_{42}^{(1)} & 0 & z_{43}^{(1)} & 0 & \cdots & \cdots & \cdots & 0 & x_4 && \\
\vdots & z_{42} & 0 & 0 & 0 & z_{43} & 0 & \cdots & \cdots & \cdots & 0 & x_4 & \\
0 & 0 & 0 & 0 & z_{53}^{(1)} & z_{53} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{pmatrix}$$

$$
R_{13}(a)=\left(\begin{array}{cccccccccccccccccccccccc}
y_1 &&&&&&&&&&&&&&&&&&&&&&& \\
0 & y_1 &&&&&&&&&&&&&&&&&&&&&& \\
\vdots & 0 & y_1 &&&&&&&&&&&&&&&&&&&&& \\
\vdots & \vdots & 0 & y_2 &&&&&&&&&&&&&&&&&&&& \\
\vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&&&&&&&&&&&& \\
\vdots & \vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&&&&&&&&&&& \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&&&& 0 &&&&&& \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&&&&&&&&& \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_3 &&&&&&&&&&&&&&& \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & y_3 &&&&&&&&&&&&&& \\
z_{11} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_1 &&&&&&&&&&&&& \\
0 & 0 & z''_{11} & \vdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_1 &&&&&&&&&&&& \\
0 & z'_{21} & 0 & 0 & z'_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&&&&&&& \\
z_{21} & 0 & 0 & z_{22} & 0 & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&&&&&& \\
0 & 0 & z''_{21} & 0 & 0 & z''_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&&&&& \\
\vdots & \vdots & \bar{z}''_{11} & 0 & 0 & 0 & \bar{z}''_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&&&& \\
\vdots & \vdots & 0 & 0 & z'_{32} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 &&&&&&& \\
\vdots & \vdots & 0 & 0 & 0 & 0 & \bar{z}''_{32} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 &&&&&& \\
\vdots & \vdots & 0 & 0 & 0 & 0 & 0 & z^{(2)}_{32} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 &&&&& \\
\vdots & \vdots & 0 & z_{42} & 0 & 0 & 0 & 0 & z_{43} & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 &&&& \\
\vdots & \cdots & \cdots & 0 & z'_{42} & 0 & 0 & 0 & z'_{43} & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 &&& \\
\vdots & \cdots & \cdots & \cdots & 0 & z''_{42} & 0 & 0 & 0 & \bar{z}_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 && \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & z^{(1)}_{42} & 0 & \bar{z}'_{43} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 & \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & z_{53} & \bar{z}_{53} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{array}\right)
$$

$$R_{14}(a)=\left(\begin{array}{cccccccccccccccccccccccccc}
y_1 \\
0 & y_1 \\
\vdots & 0 & y_1 \\
\vdots & \vdots & 0 & y_2 \\
\vdots & \vdots & \vdots & 0 & y_2 \\
\vdots & \vdots & \vdots & \vdots & 0 & y_2 & & & & & & & & & & & & & & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_3 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & y_3 \\
z_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_1 \\
0 & z''_{11} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_1 \\
z_{21} & 0 & 0 & z_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 \\
z'_{21} & 0 & 0 & 0 & z'_{22} & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 \\
0 & z''_{21} & 0 & 0 & 0 & 0 & z''_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 \\
0 & 0 & \bar{z}_{21} & 0 & 0 & 0 & 0 & \bar{z}_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 \\
0 & 0 & \bar{z}'_{21} & 0 & 0 & \vdots & 0 & 0 & \bar{z}'_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 \\
0 & 0 & 0 & 0 & z'_{32} & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 \\
\vdots & \vdots & \vdots & \vdots & 0 & z^{(2)}_{32} & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 \\
\vdots & \vdots & \vdots & 0 & 0 & \vdots & \vdots & 0 & \bar{z}'_{32} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 \\
\vdots & \vdots & 0 & z_{42} & 0 & 0 & \vdots & 0 & 0 & z_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 \\
\vdots & \vdots & \vdots & 0 & 0 & z^{(1)}_{42} & \vdots & 0 & 0 & z^{(1)}_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & \bar{z}_{42} & 0 & 0 & \bar{z}_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & z''_{42} & 0 & 0 & 0 & z''_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & z_{53} & z'_{53} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{array}\right)$$

$$R_{15}(a)=\left(\begin{array}{ccccccccccccccccccccccc}
y_1 &&&&&&&&&&&&&&&&&&&&&& \\
0 & y_1 &&&&&&&&&&&&&&&&&&&&& \\
\vdots & 0 & y_2 &&&&&&&&&&&&&&&&&&&& \\
\vdots & \vdots & 0 & y_2 &&&&&&&&&&&&&&&&&&& \\
\vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&&&&&&&&&&& \\
\vdots & \vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&&&&& 0 &&&&& \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&&&&&&&&& \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_2 &&&&&&&&&&&&&&& \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & y_3 &&&&&&&&&&&&&& \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & y_3 &&&&&&&&&&&&& \\
z_{11} & 0 & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_1 &&&&&&&&&&&& \\
0 & z_{21} & 0 & z_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&&&&&&& \\
0 & z'_{21} & 0 & 0 & z'_{22} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&&&&&& \\
z''_{21} & 0 & 0 & 0 & 0 & z''_{22} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{z}^{(1)}_{22} & \bar{z}^{(2)}_{22} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_2 &&&&&&&& \\
\vdots & \vdots & z^{(2)}_{32} & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 &&&&&&& \\
\vdots & \vdots & \vdots & 0 & z'_{32} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 &&&&&& \\
\vdots & \vdots & 0 & \vdots & \vdots & \vdots & 0 & \bar{z}^{(2)}_{32} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_3 &&&&& \\
\vdots & \vdots & z^{(1)}_{42} & 0 & \vdots & \vdots & \vdots & 0 & z^{(1)}_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 &&&& \\
\vdots & \vdots & 0 & z_{42} & 0 & 0 & 0 & 0 & z_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 &&& \\
\vdots & \vdots & \vdots & 0 & 0 & z''_{42} & 0 & \vdots & 0 & z''_{43} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 && \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & \bar{z}^{(1)}_{42} & 0 & 0 & \bar{z}^{(1)}_{52} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_4 & \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & z_{53} & z'_{53} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_5
\end{array}\right)$$

$$R_{16}(a)=\left(\begin{array}{ccccccccccccccccccccccccccccccccc}
y_1 \\
0 & y_1 \\
\vdots & 0 & y_1 \\
& & 0 & y_1 \\
& & & 0 & y_2 \\
& & & & 0 & y_2 \\
& & & & & 0 & y_2 \\
& & & & & & 0 & y_2 \\
& & & & & & & 0 & y_2 & & & & & & & & & & & & & & & & 0 \\
& & & & & & & & 0 & y_2 \\
& & & & & & & & & 0 & y_2 \\
& & & & & & & & & & 0 & y_3 \\
& & & & & & & & & & & 0 & y_3 \\
0 & 0 & 0 & 0 & \cdots & & & & & & & & 0 & y_3 \\
z_{11} & 0 & 0 & 0 & 0 & \cdots & & & & & & & & 0 & x_1 \\
0 & 0 & z_{11}^{(1)} & 0 & 0 & \cdots & & & & & & & & & 0 & x_1 \\
z_{21} & 0 & 0 & 0 & z_{22} & 0 & \cdots & & & & & & & & & 0 & x_2 \\
0 & z'_{21} & 0 & 0 & 0 & z'_{22} & 0 & \cdots & & & & & & & & & 0 & x_2 \\
\vdots & z''_{21} & 0 & 0 & 0 & 0 & z''_{22} & 0 & \cdots & & & & & & & & & 0 & x_2 \\
\vdots & 0 & z_{21}^{(1)} & 0 & 0 & 0 & 0 & z_{22}^{(1)} & 0 & \cdots & & & & & & & & & 0 & x_2 \\
\vdots & \vdots & z_{21}^{(2)} & 0 & 0 & 0 & 0 & 0 & z_{22}^{(2)} & 0 & 0 & \cdots & & & & & & & & 0 & x_2 \\
\vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & & & & 0 & x_2 \\
\vdots & \vdots & 0 & z_{21}^{(4)} & 0 & 0 & 0 & 0 & 0 & 0 & z_{22}^{(4)} & 0 & \cdots & & & & & & & & & 0 & x_2 \\
\vdots & \vdots & \vdots & 0 & \vdots & \vdots & z''_{32} & 0 & 0 & 0 & 0 & \cdots & & & & & & & & & & & 0 & x_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & z_{32}^{(2)} & 0 & 0 & \cdots & & & & & & & & & & & & 0 & x_3 \\
\vdots & \vdots & \vdots & \vdots & 0 & \vdots & 0 & 0 & 0 & z_{32}^{(3)} & 0 & \cdots & & & & & & & & & & & & & 0 & x_3 \\
\vdots & \vdots & \vdots & \vdots & 0 & \vdots & 0 & 0 & 0 & 0 & z_{32}^{(4)} & 0 & \cdots & & & & & & & & & & & & & 0 & x_3 \\
\vdots & \vdots & \vdots & \vdots & z_{42} & 0 & \cdots & \cdots & 0 & 0 & 0 & z_{43} & \cdots & & & & & & & & & & & & & & 0 & x_4 \\
\vdots & \vdots & \vdots & \vdots & 0 & z'_{42} & 0 & 0 & 0 & 0 & 0 & z'_{43} & 0 & \cdots & & & & & & & & & & & & & & 0 & x_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & z_{42}^{(1)} & 0 & 0 & 0 & 0 & z_{43}^{(1)} & 0 & \cdots & & & & & & & & & & & & & & 0 & x_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & z_{43}^{(3)} & 0 & z_{43}^{(3)} & 0 & \cdots & & & & & & & & & & & & & & & & 0 & x_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & 0 & z_{42}^{(4)} & 0 & 0 & z_{43}^{(4)} & 0 & \cdots & & & & & & & & & & & & & & & 0 & x_4 \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & z_{53} & z'_{53} & z''_{53} & 0 & \cdots & & & & & & & & & & & & & & & & 0 & x_5
\end{array}\right)$$

$$
R_{17}(a)=\left(\begin{array}{cccccccccccccccccccccccccccccccccc}
y_1 \\
0 & y_1 \\
 & 0 & y_1 \\
 & & 0 & y_1 \\
 & & & 0 & y_2 \\
 & & & & 0 & y_2 \\
 & & & & & 0 & y_2 \\
 & & & & & & 0 & y_2 \\
 & & & & & & & 0 & y_2 \\
 & & & & & & & & 0 & y_2 \\
 & & & & & & & & & 0 & y_2 & & & & & & & & & & & & & & 0 \\
 & & & & & & & & & & 0 & y_2 \\
 & & & & & & & & & & & 0 & y_3 \\
 & & & & & & & & & & & & 0 & y_3 \\
0 & & & & & & & & & & & & & 0 & y_3 \\
z_{11} & 0 & 0 & & & & & & & & & & & & 0 & x_1 \\
0 & 0 & z''_{11} & 0 & 0 & & & & & & & & & & & 0 & x_1 \\
z_{21} & 0 & 0 & 0 & z_{22} & 0 & & & & & & & & & & & 0 & x_2 \\
0 & z'_{21} & 0 & 0 & 0 & z'_{22} & 0 & & & & & & & & & & & 0 & x_2 \\
 & z''_{21} & 0 & 0 & & 0 & z''_{22} & 0 & & & & & & & & & & & 0 & x_2 \\
 & 0 & z'''_{21} & 0 & & & 0 & z'''_{22} & 0 & & & & & & & & & & & 0 & x_2 \\
 & & & \bar{z}_{21} & 0 & & & & & 0 & \bar{z}_{22} & 0 & & & & & & & & & 0 & x_2 \\
 & & & \bar{z}'_{21} & 0 & 0 & 0 & 0 & & & 0 & \bar{z}'_{22} & 0 & & & & & & & & & 0 & x_2 \\
 & & & 0 & & 0 & z''_{32} & 0 & & & & 0 & & & & & & & & & & & 0 & x_3 \\
 & & & & & & 0 & & z^{(1)}_{32} & & & & & & & & & & & & & & & 0 & x_3 \\
 & & & & & & & & & z^{(4)}_{32} & & 0 & & & & & & & & & & & & & 0 & x_3 \\
 & & & & 0 & & & & & & & \bar{z}'_{32} & 0 & & & & & & & & & & & & & 0 & x_3 \\
 & & & & z_{42} & 0 & & & & & & 0 & z_{43} & 0 & & & & & & & & & & & & & 0 & x_4 \\
 & & & & 0 & z'_{42} & & 0 & & & & & z'_{43} & 0 & & & & & & & & & & & & & & 0 & x_4 \\
 & & & & & 0 & & z'''_{42} & 0 & & & & 0 & z'''_{43} & 0 & & & & & & & & & & & & & & 0 & x_4 \\
 & & & & & & & 0 & z^{(1)}_{42} & 0 & & & & z^{(1)}_{43} & 0 & & & & & & & & & & & & & & & 0 & x_4 \\
 & & & & & & & & 0 & z^{(3)}_{42} & & & & 0 & z^{(3)}_{43} & 0 & & & & & & & & & & & & & & & 0 & x_4 \\
 & & & & & & & & & 0 & \bar{z}_{42} & 0 & 0 & 0 & \bar{z}_{43} & 0 & & & & & & & & & & & & & & & & 0 & x_4 \\
0 & & 0 & & 0 & & & & & 0 & 0 & 0 & z_{53} & z'_{53} & z''_{53} & 0 & & & & & & & & & & & & & & & & & 0 & x_5
\end{array}\right)
$$

Hence the degree of an arbitrary directly indecomposable A-left module is bounded and less than 34.

[The case 4, 5, 6] In these cases we can prove that the degree of an arbitrary directly indecomposable A-left module is bounded but this proof is quite same as [the case 3].

Thus the proof of this theorem is completed.

8. In [5] G. Köthe propounded the problem to determine the general type of algebras whose directly indecomposable left modules are cyclic and not necessarily homomorphic to Ae_κ. Now we call such an algebra the *Köthe* algebra. Then from the above theorem we can answer to this problem in a special case where $N^2=0$ and K is algebraically closed. Namely it is clear that the Köthe algebra is of bounded representation type but

(1) When each Ne_κ is the direct sum of at most two simple components, an algebra of bounded representation type is the Köthe algebra.

(2) When $\{Ne_1, \cdots\cdots, Ne_s\}$ is a chain such that Ne_s is the direct sum of three simple components, if $f(\kappa)\geq 2$ for all κ it is the Köthe algebra but if there exists μ such that $f(\mu)=1$ it is not necessarily the Köthe algebra.

(3) When $\{Ne_1, Ne_2, Ne_3\}$ is a chain such that Ne_2 is the direct sum of three simple components, if $f(\kappa)\geq 8$ for all κ it is the Köthe algebra but if there exists μ such that $f(\mu)\leq 7$ it is not necessarily the Köthe algebra.

(5) When $\{Ne_1, Ne_2, Ne_3, Ne_4\}$ is a chain such that Ne_2 is the direct sum of three simple components and Ne_1 and Ne_4 are simple, if $f(\kappa)\geq 5$ for all κ, it is the Köthe algebra but if there exists μ such that $f(\mu)\leq 4$, it is not nesessarily the Köthe algebra.

This proof is clear from the fact that $Ae_{\kappa 1}m=Ae_{\kappa 2}\pi_{21}m$ where π_{21} is the isomorphism such that $Ae_{\kappa,2}\pi_{21}=Ae_{\kappa,1}$.

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