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ON AUSLANDER-REITEN COMPONENTS AND PROJECTIVE LATTICES OF *p*-GROUPS

Dedicated to Professor Yukio Tsushima on his 60th birthday

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Introduction

Let G be a finite group, p a prime number which divides the order of G, and $(K, \mathcal{O}, k)$ a p-modular system, i.e., $\mathcal{O}$ is a complete discrete valuation ring of characteristic zero with maximal ideal (π), $k(:= \mathcal{O}/(\pi))$ is the residue field of $\mathcal{O}$ of characteristic $p > 0$, and K is the field of fractions of $\mathcal{O}$. R is used to denote either $\mathcal{O}$ or k. All the RG-modules considered here are R-free and finitely generated over R.

Let $\Gamma(RG)$ be the Auslander-Reiten quiver of RG. For a connected component Θ of $\Gamma(RG)$, we denote by Θ_s the stable part of Θ obtained from Θ by removing all projective RG-modules and arrows attached to them. In [16], P. J. Webb showed that the tree class of Θ_s is either a Euclidean diagram or one of the infinite trees A_∞, B_∞, C_∞, D_∞ and A_∞^∞ if the modules in Θ do not lie in a block of cyclic defect.

It was shown in [10] that if G is a p-group and $\mathcal{O}G$ is of infinite representation type, and furthermore if $(\pi) \supsetneqq (2)$ in the case where $p = 2$ and G is the Klein four group, then the stable part of the connected component of $\Gamma(\mathcal{O}G)$ containing the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$ has tree class A_∞. The purpose of this paper is to show the following.

Theorem. *Let G be a p-group and Δ the connected component of $\Gamma(\mathcal{O}G)$ containing the projective $\mathcal{O}G$-lattice $\mathcal{O}G$. Suppose that $\mathcal{O}G$ is of infinite representation type. Suppose further that $(\pi) \supsetneqq (2)$ in the case where $p = 2$ and G is the Klein four group. Then the tree class of the stable part Δ_s of Δ is A_∞.*

It is known that the group ring $\mathcal{O}G$ of a finite p-group G is of finite representation type if and only if one of the following cases arises: (i) $G = C_2$; (ii) $G = C_3$ and $(3) \supseteq (\pi^3)$; (iii) $G = C_p$ and $(p) \supseteq (\pi^2)$; (iv) $G = C_{p^2}$ and $(p) = (\pi)$, where C_{p^n} is the cyclic group of order p^n. See [4]. Also, it is known that if G is the Klein four group and $(\pi) = (2)$, then the tree class of the stable part of the connected component of $\Gamma(\mathcal{O}G)$ containing the projective $\mathcal{O}G$-lattice $\mathcal{O}G$ is $\tilde{D}_4$ (Proposition 3.4 of [5]).

In the rest of this paper G will always be a finite p-group. In Sections 1, we consider the Auslander-Reiten sequence where the projective $\mathcal{O}G$-lattice $\mathcal{O}G$ occurs. We treat the middle term of the Auslander-Reiten sequence terminating in the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$ in Section 2. In Section 3, the case where the projective-free part Δ_s of the connected component Δ of $\Gamma(\mathcal{O}G)$ containing $\mathcal{O}G$ has tree class A_∞^∞ is excluded. Also, we exclude the case where the tree class of Δ_s is B_∞ or C_∞ in Section 4. In Section 5, we show that the tree class of any connected component of $\Gamma(\mathcal{O}G)$ not containing $\mathcal{O}_G$ is not Euclidean. The proof of Theorem is completed in Section 6.

The notation is standard. For a non-projective indecomposable RG-module W, we write $\mathcal{A}(W)$ for the Auslander-Reiten sequence $0 \to \tau W \to M(W) \to W \to 0$, where τ is the Auslander-Reiten translation and we denote by $M(W)$ the middle term of $\mathcal{A}(W)$. It is known that $\tau = \Omega$ if $R = \mathcal{O}$, and $\tau = \Omega^2$ if $R = k$, where Ω is the Heller operator (see [13] and [1]). The trivial RG-module will be denoted by R_G. For an RG-module W, W^* means the dual RG-module $\mathrm{Hom}_R(W, R)$ of W. For $\mathcal{O}G$-lattices V and W, set $\underline{\mathrm{Hom}}_{\mathcal{O}G}(V, W) := \mathrm{Hom}_{\mathcal{O}G}(V, W)/\mathcal{P}\mathrm{Hom}_{\mathcal{O}G}(V, W)$, where $\mathcal{P}\mathrm{Hom}_{\mathcal{O}G}(V, W)$ is the subspace of $\mathrm{Hom}_{\mathcal{O}G}(V, W)$ of all projective maps from V to W. Also, the kG-module $W/\pi W$ is denoted by $\overline{W}$. Concerning some basic facts and terminologies used here, we refer to [12, 7, 2, 14].

1. Projective $\mathcal{O}G$-lattices and Auslander-Reiten sequences

Let G be a finite p-group and $J := J(\mathcal{O}G)$ the Jacobson radical of the group ring $\mathcal{O}G$. Then $J = \pi\mathcal{O}G + \sum_{g\in G} \mathcal{O}(g-1)$ is the unique maximal $\mathcal{O}G$-submodule of $\mathcal{O}G$. The following fact seems to be well-known, but we give an elementary proof here for convenience.

Lemma 1.1. *J is decomposable if and only if $(\pi) = (|G|)$, i.e., G is the cyclic group of order p and $(\pi) = (p)$.*

Proof. Suppose that J is decomposable. Considering a kG-decomposition $\overline{J} = (\mathcal{O}\cdot(\pi 1) + \pi J)/\pi J \oplus (\sum_{g\in G} \mathcal{O}(g-1) + \pi J)/\pi J \cong k_G \oplus \Omega k_G$, we have an $\mathcal{O}G$-decomposition $J = X \oplus Y$ such that $\overline{X} \cong k_G$ and $\overline{Y} \cong \Omega k_G$. Since $J \otimes_{\mathcal{O}} X^*$ is a maximal submodule of $\mathcal{O}G \otimes_{\mathcal{O}} X^*$ ($\cong \mathcal{O}G$), it follows that $J \cong J \otimes_{\mathcal{O}} X^* = \mathcal{O}_G \oplus (Y \otimes_{\mathcal{O}} X^*)$. Thus we may assume that $X \cong \mathcal{O}_G$. Then we see that $X \subseteq \mathcal{O}\hat{G}$, where $\hat{G} = \sum_{g\in G} g$, which implies that $Y \subseteq \sum_{g\in G} \mathcal{O}(g-1)$. As $\pi 1 \in J = X + Y$, we have $\pi 1 = r\hat{G} + \sum_{g\in G} r_g(g-1)$ for some r, $r_g \in \mathcal{O}$. This forces that $\pi = r|G|$ and $(\pi) = (|G|)$.

Conversely, if $(\pi) = (|G|)$, then we see that $J = \mathcal{O}\hat{G} \oplus \sum_{g\in G} \mathcal{O}(g-1)$. □

Next, let

$$I := \mathcal{O}G + \pi^{-1}(\hat{G} - |G|1)\mathcal{O}G,$$

where $\hat{G} = \sum_{g\in G} g$. Then I is the unique minimal $\mathcal{O}G$-submodule of $K \otimes_{\mathcal{O}} \mathcal{O}G$ containing $\mathcal{O}G$ properly, since $\pi^{-1}(\hat{G} - |G|1)$ generates the simple socle of $\pi^{-1}\mathcal{O}G/\mathcal{O}G$.

In this section we assume that $(\pi) \supsetneqq (|G|)$, so J is indecomposable. Then I is isomorphic to $\Omega^{-1}J$ (see, e.g., [11]), and the Auslander-Reiten sequence $\mathcal{A}(I)$ terminating in I has the form $0 \to J \to M(I)_s \oplus \mathcal{O}G \to I \to 0$, where $M(I)_s$ is the projective-free part of $M(I)$. Note that $\mathcal{A}(I)$ is the only Auslander-Reiten sequence where $\mathcal{O}G$ occurs.

Lemma 1.2. *Suppose that* $(\pi) \supsetneqq (|G|)$. *Then the short exact sequence* $\overline{\mathcal{A}(I)}$ *obtained from* $\mathcal{A}(I)$ *by reducing each term mod* (π) *is the direct sum of the standard Auslander-Reiten sequence* $0 \to \Omega k_G \to \mathrm{Rad}(kG)/\mathrm{Soc}(kG) \oplus kG \to \Omega^{-1}k_G \to 0$ *and a split sequence* $0 \to k_G \to k_G \oplus k_G \to k_G \to 0$.

Proof. See [11]. Note that the argument in the proof of Theorem 9 of [11] holds if J is indecomposable. □

Now let us define an $\mathcal{O}G$-submodule M of $K \otimes_{\mathcal{O}} \mathcal{O}G$ as follows:

$$M := \pi\mathcal{O}G + \sum_{g\in G}(g-1)\mathcal{O}G + \pi^{-1}(\hat{G} - |G|1)\mathcal{O}G.$$

We shall show that M is isomorphic to the projective-free part $M(I)_s$ of the middle term $M(I)$ of the Auslander-Reiten sequence $\mathcal{A}(I)$ except the case where $|G| = p$ and $(\pi) = (p)$.

Lemma 1.3. *Suppose that* $(\pi) \supsetneqq (|G|)$. *Then we have that* $\overline{M} \cong k_G \oplus k_G \oplus \mathrm{Rad}(kG)/\mathrm{Soc}(kG)$.

Proof. As $\hat{G} - |G|1 \in \pi M \cap \sum_{g\in G} \mathcal{O}\cdot(g-1)$, we have $\overline{M} = (\mathcal{O}\cdot(\pi 1) + \pi M)/\pi M \oplus (\sum_{g\in G} \mathcal{O}\cdot(g-1) + \pi M)/\pi M \oplus (\mathcal{O}\cdot\pi^{-1}(\hat{G} - |G|1) + \pi M)/\pi M$ as k-space. It is easily seen that $(\mathcal{O}\cdot(\pi 1) + \pi M)/\pi M \cong k_G$. Note that

$$\sum_{g\in G} \mathcal{O}\cdot(g-1) = \Omega\mathcal{O}_G$$

and

$$\begin{aligned}\left(\sum_{g\in G} \mathcal{O}\cdot(g-1) + \pi M\right)/\pi M &\cong \Omega\mathcal{O}_G/(\Omega\mathcal{O}_G \cap \pi M) \\ &= \Omega\mathcal{O}_G/(\pi\Omega\mathcal{O}_G + \mathcal{O}\cdot(\hat{G} - |G|1)).\end{aligned}$$

Since $\Omega\mathcal{O}_G/\pi\Omega\mathcal{O}_G \cong \mathrm{Rad}(kG)$ and

$$(\mathcal{O}\cdot(\hat{G}-|G|1)+\pi\Omega\mathcal{O}_G)/\pi\Omega\mathcal{O}_G = \mathrm{Soc}(\Omega\mathcal{O}_G/\pi\Omega\mathcal{O}_G),$$

we see that $(\sum_{g\in G}\mathcal{O}\cdot(g-1)+\pi M)/\pi M$ is isomorphic to $\mathrm{Rad}(kG)/\mathrm{Soc}(kG)$. To complete the proof, it suffices to show that $(\mathcal{O}\cdot\pi^{-1}(\hat{G}-|G|1)+\pi M)/\pi M$ is a kG-submodule of $\overline{M}$. Let x be any element of G. Then $\pi^{-1}(\hat{G}-|G|1)x = \pi^{-1}(\hat{G}-|G|x) = \pi^{-1}(\hat{G}-|G|1)+\pi^{-1}|G|(1-x)$. Since $\pi^{-1}|G|\in(\pi)$ by our assumption, it follows that $\pi^{-1}(\hat{G}-|G|1)x \in \mathcal{O}\cdot\pi^{-1}(\hat{G}-|G|1)+\pi M$. □

Lemma 1.4. *Let G be a finite p-group, and suppose that $(\pi)\supsetneqq(|G|)$. Suppose that M' is an $\mathcal{O}G$-submodule of I which contains J as a maximal $\mathcal{O}G$-submodule. Then $M'=M$ or $M'\cong\mathcal{O}G$ as $\mathcal{O}G$-lattices.*

Proof. Suppose that $M'\neq M$. Note that $I = J+\mathcal{O}\cdot 1+\mathcal{O}\cdot\pi^{-1}(\hat{G}-|G|1)$ as $\mathcal{O}$-modules. Since $M'\neq M$, M' contains an element $m := 1+\alpha\pi^{-1}(\hat{G}-|G|1)$ for some $\alpha\in\mathcal{O}$. Then $M' = m\mathcal{O}G+J = m\mathcal{O}G+\sum_{g\in G}\mathcal{O}\cdot(g-1)+\mathcal{O}\cdot(\pi 1)$ as $\mathcal{O}$-module. Let x be any element of G. Then $m(x-1) = (1-\alpha|G|\pi^{-1})(x-1)$ and $x-1\in m\mathcal{O}G$ since $|G|\pi^{-1}\in(\pi)$ by our assumption. Also, we see that $\pi 1 = \pi m-\alpha\sum_{g\in G}(g-1)\in m\mathcal{O}G$. Thus we have that $M' = m\mathcal{O}G$. As $\mathrm{rank}_{\mathcal{O}}M' = |G|$, it follows that $M'\cong\mathcal{O}G$. □

Proposition 1.5. *Suppose that $(\pi)\supsetneqq(|G|)$. Then M is isomorphic to the projective-free part $M(I)_s$ of the middle term $M(I)$ of the Auslander-Reiten sequence $\mathcal{A}(I)$. In particular, $\mathcal{A}(I)$ has the form $0\to J\to M\oplus\mathcal{O}G\to I\to 0$.*

Proof. Since $\mathrm{rank}_{\mathcal{O}}M(I)_s = |G| = \mathrm{rank}_{\mathcal{O}}I$, an irreducible map from $M(I)_s$ to I is a monomorphism. Hence we may regard that $J\subset\mathcal{O}G\subset I\subset K\otimes_{\mathcal{O}}\mathcal{O}G$ and $M(I)_s\subset I\subset K\otimes_{\mathcal{O}}\mathcal{O}G$. Note that $\mathcal{O}G$ and $M(I)_s$ are maximal $\mathcal{O}G$-submodules of I, and so $I/\mathcal{O}G\cong k_G\cong I/M(I)_s$. Here we claim that $M(I)_s\not\subseteq\mathcal{O}G$: Indeed, if $M(I)_s\subseteq\mathcal{O}G$, the maximality forces that $M(I)_s=\mathcal{O}G$. However, Lemma 1.2 implies that $\overline{M(I)_s}\cong k_G\oplus k_G\oplus\mathrm{Rad}(kG)/\mathrm{Soc}(kG)$, a contradiction.

Now since $\mathcal{O}G\subsetneqq\mathcal{O}G+M(I)_s\subseteq I$ and I is the unique minimal $\mathcal{O}G$-submodule of $K\otimes_{\mathcal{O}}\mathcal{O}G$ containing $\mathcal{O}G$, we have that $\mathcal{O}G+M(I)_s = I$. Thus it follows that $\mathcal{O}G/\mathcal{O}G\cap M(I)_s\cong(\mathcal{O}G+M(I)_s)/M(I)_s\cong I/M(I)_s\cong k_G$. Therefore $\mathcal{O}G\cap M(I)_s$ is a maximal $\mathcal{O}G$-submodule of $\mathcal{O}G$ and we get $\mathcal{O}G\cap M(I)_s = J$. Also, it follows that $M(I)_s/\mathcal{O}G\cap M(I)_s\cong(M(I)_s+\mathcal{O}G)/\mathcal{O}G\cong I/\mathcal{O}G\cong k_G$. Hence J is a maximal $\mathcal{O}G$-submodule of $M(I)_s$ and the result follows by Lemma 1.4. □

2. Trivial $\mathcal{O}G$-lattices and Auslander-Reiten sequences

Let G be a finite p-group and $\mathcal{O}_G$ the trivial $\mathcal{O}G$-lattice. Then $\underline{\mathrm{End}}_{\mathcal{O}G}(\mathcal{O}_G) \cong \mathcal{O}/(|G|)$ and $\pi^{-1}|G| \cdot \mathrm{id}_{\mathcal{O}_G}$ is a generator of $\mathrm{Soc}(\underline{\mathrm{End}}_{\mathcal{O}G}(\mathcal{O}_G))$. The Auslander-Reiten sequence $\mathcal{A}(\mathcal{O}_G)$ terminating in $\mathcal{O}_G$ is constructed as pullback of the projective cover of $\mathcal{O}_G$ along $\pi^{-1}|G| \cdot \mathrm{id}_{\mathcal{O}_G}$ (see [13, 15]):

$$\begin{array}{ccccccccccl} 0 & \longrightarrow & \Omega\mathcal{O}\mathcal{O}_G & \longrightarrow & M(\mathcal{O}_G) & \longrightarrow & \mathcal{O}_G & \longrightarrow & 0 & & : \mathcal{A}(\mathcal{O}_G) \\ & & \| & & \downarrow & \text{pull back} & \downarrow{\scriptstyle \pi^{-1}|G|\cdot \mathrm{id}_{\mathcal{O}_G}} & & & & \\ 0 & \longrightarrow & \Omega\mathcal{O}_G & \longrightarrow & \mathcal{O}G & \xrightarrow{\varepsilon} & \mathcal{O}_G & \longrightarrow & 0 & & : \text{projective cover,} \end{array}$$

where ε is the augmentation map. Here $M(\mathcal{O}_G) = \{(x, y) \mid x \in \mathcal{O}_G,\ y \in \mathcal{O}G,\ \pi^{-1}|G|x = \varepsilon(y)\} \subset \mathcal{O}_G \oplus \mathcal{O}G$. Hence we see that $M(\mathcal{O}_G) \cong \pi^{-1}|G|\mathcal{O}G + \sum_{g\in G}(g-1)\mathcal{O}G \subseteq \mathcal{O}G$.

Lemma 2.1 (Proposition 3.2 of [9]). *The middle term $M(\mathcal{O}_G)$ of $\mathcal{A}(\mathcal{O}_G)$ is indecomposable.*

In [3], J. F. Carlson and A. Jones defined the exponent $\exp(W)$ of an $\mathcal{O}G$-lattice W as the least power π^a of π such that $\pi^a \cdot \mathrm{id}_W$ is projective.

Lemma 2.2. *Let W be a non-projective indecomposable $\mathcal{O}G$-lattice. Suppose that the Auslander-Reiten sequence $\overline{\mathcal{A}(W)}$ modulo (π) does not split. Then* $\exp(W) = \pi$.

Proof. Let ρ be a generator of $\mathrm{Soc}(\underline{\mathrm{End}}_{\mathcal{O}G}(W))$. Then $\overline{\mathcal{A}(W)}$ is the pullback of the projective cover of $\overline{W}$ along the kG-endomorphism $\overline{\rho}$ of $\overline{W}$. By the assumption, $\overline{\rho}$ is not projective. In particular, $\rho \notin \pi\,\mathrm{End}_{\mathcal{O}G}(W)$. Thus it follows that $\pi\,\mathrm{End}_{\mathcal{O}G}(W) \subseteq \mathcal{P}\,\mathrm{End}_{\mathcal{O}G}(W)$ and $\pi \cdot \mathrm{id}_W$ is projective. □

Lemma 2.3. (1) $\exp(J) = \pi$.
(2) $\exp(M(\mathcal{O}_G)) = \pi^{n-1}$, *where* $(|G|) = (\pi^n)$.
(3) *J is isomorphic to $M(\mathcal{O}_G)$ if and only if* $(|G|) = (\pi^2)$.

Proof. (1) In the case where $(\pi) = (|G|)$, J is isomorphic to $\mathcal{O}_G \oplus \Omega\mathcal{O}_G$ and so $\exp(J) = \pi$. If $(\pi) \supsetneqq (|G|)$, J is indecomposable and non-projective by Lemma 1.1, and the Auslander-Reiten sequence $\overline{\mathcal{A}(J)}$ modulo (π) does not split by Lemma 1.2. Hence the result follows by Lemma 2.2.
(2) Since $\exp(\mathcal{O}_G) = \pi^n$, the assertion holds by Theorem 2.4 of [3].
(3) Suppose that $J \cong M(\mathcal{O}_G)$. Then since $\exp(J) = \exp(M(\mathcal{O}_G))$, we obtain $(\pi) = (\pi^{-1}|G|)$ by (1) and (2). The converse is clear by the definition. □

From Lemma 2.3 (3), we get the following immediately.

REMARK 2.4. J is isomorphic to the middle term $M(\mathcal{O}_G)$ of the Auslander-Reiten sequence $\mathcal{A}(\mathcal{O}_G)$ if and only if one of the following cases arises:
(1) $|G| = p^2$ and $(\pi) = (p)$;
(2) $|G| = p$ and $(\pi^2) = (p)$.
In these cases, $\mathcal{O}_G$ belongs to the connected component Δ of $\Gamma(\mathcal{O}G)$ containing $\mathcal{O}G$ by Proposition 1.5. Hence the tree class of Δ_s is not A_∞^∞ by Lemma 2.1.

3. Indecomposability of M

In this section, let G be a p-group and we assume that $(\pi) \supsetneqq (|G|)$. Then J and I are indecomposable by Lemma 1.1. We consider the indecomposability of the projective-free part $M(I)_s$ of the middle term of the Auslander-Reiten sequence $\mathcal{A}(I)$ terminating in I. We have seen in Proposition 1.5 that $M(I)_s = M := \pi\mathcal{O}G + \sum_{g\in G}(g-1)\mathcal{O}G + \pi^{-1}(\hat{G} - |G|1)\mathcal{O}G$. We begin with the following easy fact.

Lemma 3.1. *Let W be a kG-module. Suppose that there are two kG-decompositions: $W = X \oplus Y = X' \oplus Y'$ such that X, X' are semisimple and none of Y and Y' has a simple summand. Then we have*
(1) $\mathrm{Soc}(Y) = \mathrm{Soc}(Y')$.
(2) *The projection map $\pi_{X'} : W \to X'$ induces an isomorphism $\pi_{X'}|_X : X \xrightarrow{\sim} X'$.*

Proof. (1) Let $Y = \bigoplus_j Y_j$ be an indecomposable decomposition of Y, and let y be any element in $\mathrm{Soc}(Y_j)$. Note that $\mathrm{Soc}(Y_j) \subseteq \mathrm{Rad}(Y_j)$ as Y_j is indecomposable. Thus there are some elements $a_t \in Y_j$ and $z_t \in \mathrm{Rad}(kG)$ such that $\sum a_t z_t = y$. Since each $a_t \in X' \oplus Y'$, we see that $y \in \mathrm{Soc}(Y')$.
(2) It is enough to show that $\pi_{X'}|_X$ is monomorphism since $\dim_k X = \dim_k X'$. By (1) we see that $\mathrm{Ker}\,(\pi_{X'}|_X) = X \cap Y' \subseteq X \cap \mathrm{Soc}(Y') = X \cap \mathrm{Soc}(Y) = 0$. □

The following lemma will be used later.

Lemma 3.2. (1) *Let L be any $\mathcal{O}G$-lattice of $\mathcal{O}$-rank one. Then $M \otimes_{\mathcal{O}} L \cong M$. In particular, $L \mid M$ if and only if $\mathcal{O}_G \mid M$.*
(2) $M^* \cong M$.

Proof. Since $\mathcal{A}(I) \otimes_{\mathcal{O}} L : 0 \to J \otimes_{\mathcal{O}} L \to (M(I)_s \otimes_{\mathcal{O}} L) \oplus (\mathcal{O}G \otimes_{\mathcal{O}} L) \to I \otimes_{\mathcal{O}} L \to 0$ is an Auslander-Reiten sequence and $\mathcal{O}G \otimes_{\mathcal{O}} L \cong \mathcal{O}G$ occurs in its middle term, $\mathcal{A}(I) \otimes_{\mathcal{O}} L$ is isomorphic to $\mathcal{A}(I)$. Hence (1) holds. Also, $\mathcal{A}(I)^* : 0 \to I^* \to M(I)_s^* \oplus \mathcal{O}G^* \to J^* \to 0$ is an Auslander-Reiten sequence where $\mathcal{O}G$ occurs. Thus $\mathcal{A}(I)^*$ is isomorphic to $\mathcal{A}(I)$ and (2) holds. □

Lemma 3.3. *Suppose that G is neither the Klein four group nor a dihedral 2-group. If M is decomposable, then M has some direct summand of $\mathcal{O}$-rank one.*

Proof. By Lemma 1.3, $\overline{M} \cong k_G \oplus k_G \oplus \mathrm{Rad}(kG)/\mathrm{Soc}(kG)$. If $G = C_3$, the conclusion is clearly holds and thus we may assume that $G \neq C_3$, which implies that $\mathrm{Rad}(kG)/\mathrm{Soc}(kG)$ is indecomposable of dimension greater than one by our assumption and Theorem E of [16]. Assume to the contrary that M is decomposable but does not have any direct summand of $\mathcal{O}$-rank one. Then we have an indecomposable decomposition $M = X \oplus Y$ such that $\overline{X} \cong k_G \oplus k_G$ and $\overline{Y} \cong \mathrm{Rad}(kG)/\mathrm{Soc}(kG)$.

First we claim that X contains $\hat{G} = \sum_{g \in G} g$: From the proof of Lemma 1.3, we have two kG-decompositions $\overline{M} = (\mathcal{O} \cdot (\pi 1) + \pi M)/\pi M \oplus (\mathcal{O} \cdot \pi^{-1}(\hat{G} - |G|1) + \pi M)/\pi M \oplus (\sum_{g \in G} \mathcal{O} \cdot (g-1) + \pi M)/\pi M = \overline{X} \oplus \overline{Y}$. By Lemma 3.1, X contains an element of the form $\pi 1 + \alpha$ for some $\alpha \in \sum_{g \in G}(g-1)\mathcal{O}G + \pi M$. Hence we see that $X \ni (\pi 1 + \alpha)\hat{G} = \beta \hat{G}$ for some $\beta (\neq 0) \in \mathcal{O}$. Since X is a pure $\mathcal{O}$-submodule of M, X contains $\hat{G}$.

From the above claim, $K \otimes_{\mathcal{O}} X$ affords an ordinary character $\mathbf{1} + \eta$, where $\mathbf{1}$ is the trivial character of G and η is some linear character of G. Now $K \otimes_{\mathcal{O}} (X \oplus Y)$ affords the regular character of G. Since the multiplicity of $\mathbf{1}$ in the regular character is one, it follows that $\eta \neq \mathbf{1}$. Hence we have that $\eta(g) \neq 1$ for some $g \in G$. Since the order of g is a power of p, $\mathcal{O}$ contains primitive p-th roots of unity. Therefore $\mathcal{O}G$ has at least p non-isomorphic $\mathcal{O}G$-lattices of $\mathcal{O}$-rank one. Moreover, if G is not cyclic, $\mathcal{O}G$ has at least p^2 non-isomorphic $\mathcal{O}G$-lattices of $\mathcal{O}$-rank one.

Here, we claim that $\mathrm{rank}_{\mathcal{O}} X \geq p$, and moreover, $\mathrm{rank}_{\mathcal{O}} X \geq p^2$ unless G is cyclic: Let L be any $\mathcal{O}G$-lattice of $\mathcal{O}$-rank one and λ the ordinary linear character of G afforded by L. Then, by Lemma 3.2 (1), it follows that $X \otimes_{\mathcal{O}} L \cong X$ since $\overline{X \otimes_{\mathcal{O}} L} \cong k_G \oplus k_G$. This implies that λ is a constituent of the character afforded by X.

Now the above claim yields a contradiction if p is odd or G is not cyclic. Thus, in the rest of this proof, we assume that $G = \langle x \rangle$ is the cyclic 2-group of order 2^n with $n \geq 2$. Furthermore, we may assume that $\sqrt{-1} \notin \mathcal{O}$: Indeed, if $\sqrt{-1} \in \mathcal{O}$, then $\mathcal{O}G$ has at least four non-isomorphic $\mathcal{O}G$-lattices of $\mathcal{O}$-rank one and so $\mathrm{rank}_{\mathcal{O}} X \geq 4$, a contradiction.

Put $a := \sum_{i=0}^{2^{n-1}-1} x^{2i}$, $b := ax \in \mathcal{O}G$ and $U := \mathcal{O} \cdot a + \mathcal{O} \cdot b \subset \mathcal{O}G$. Then U is a pure $\mathcal{O}G$-submodule of $\mathcal{O}G$ and $0 \to U \xrightarrow{\imath} \mathcal{O}G$ is an injective hull of U, where $\imath$ is the inclusion map. Note that $U \cong {\mathcal{O}_{\langle x^2 \rangle}}^{\langle x \rangle}$.

Now we claim that $\Omega Y \cong U$: Indeed, X affords an ordinary character $\mathbf{1} + \eta$, where η is the linear character with $\eta(x) = -1$, as $\sqrt{-1} \notin \mathcal{O}$. Since both $Y \oplus \Omega Y$ and $Y \oplus X$ afford the regular character of G, ΩY affords the character $\mathbf{1} + \eta$. In particular $\langle x^2 \rangle$ acts on ΩY trivially. Since $\overline{Y} \cong \mathrm{Rad}(kG)/\mathrm{Soc}(kG)$ is uniserial of length $|G| - 2$, we see that $\overline{\Omega Y}$ is uniserial of length two. Thus $\overline{\Omega Y}$ is projective as $k(\langle x \rangle / \langle x^2 \rangle)$-module. This implies that $\Omega Y \cong {\mathcal{O}_{\langle x^2 \rangle}}^{\langle x \rangle} \cong U$.

Next, let us consider the Auslander-Reiten sequence $\mathcal{A}(U)$ terminating in $U \cong \Omega Y$. Since $\mathrm{rank}_{\mathcal{O}} Y + \mathrm{rank}_{\mathcal{O}} \Omega Y = |G| = \mathrm{rank}_{\mathcal{O}} I$ and $\Omega^2 Y \cong Y$, the middle term of $\mathcal{A}(U)$ is just I. Since $\overline{I} \cong k_G \oplus \Omega^{-1} k_G$ (See Lemma 1.2), the Auslander-Reiten sequence $\overline{\mathcal{A}(U)}$ modulo (π) does not split. So $\pi \cdot \mathrm{id}_U$ is projective by Lemma 2.2. Hence we have a factorization $\pi \cdot \mathrm{id}_U = f \circ \imath : U \xrightarrow{\imath} \mathcal{O}G \xrightarrow{f} U$ for some $\mathcal{O}G$-homomorphism f from $\mathcal{O}G$ to U. Put $f(1) = \alpha a + \beta b$ for some $\alpha, \beta \in \mathcal{O}$. Then $\pi a = \pi \cdot \mathrm{id}_U(a) = [f \circ \imath](a) = 2^{n-1}(\alpha a + \beta b)$ and it follows that $\pi = 2^{n-1}\alpha$. This forces that $n = 2$ and $(\pi) = (2)$ since $n \geq 2$. However, in this case, $\mathcal{O}_G$ is a direct summand of M by Remark 2.4, a contradiction. □

Lemma 3.4. *Suppose that G is the Klein four group and $(\pi) \supsetneq (2)$. Then the projective-free part M of the middle term of the Auslander-Reiten sequence $\mathcal{A}(I)$ terminating in I is indecomposable.*

Proof. Let Δ be the connected component of $\Gamma(\mathcal{O}G)$ containing the projective $\mathcal{O}G$-lattice $\mathcal{O}G$. Then from our assumption, Δ does not contain the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$ by the argument in the proof of Lemma 4.2 of [10].

Now, $\mathcal{O}G$ has three non-isomorphic non-trivial $\mathcal{O}G$-lattices of $\mathcal{O}$-rank one, say L_1, L_2, L_3. Let η_i $(1 \leq i \leq 3)$ be the linear character afforded by L_i. Note that M affords the regular character $\mathbf{1} + \eta_1 + \eta_2 + \eta_3$ of G. Thus some direct summand X of M affords a character χ having the trivial character $\mathbf{1}$ as a constituent. Since $\mathcal{O}_G$ is not contained in Δ, the character χ has η_i as a constituent for some i, $1 \leq i \leq 3$.

Next, consider the action of the automorphism group $\mathrm{Aut}(G)$ of G. $\mathrm{Aut}(G)$ acts on $\{\eta_1, \eta_2, \eta_3\}$ transitively. On the other hand, for any $\sigma \in \mathrm{Aut}(G)$, $\mathcal{A}(I)^\sigma : 0 \to J^\sigma \to M^\sigma \oplus \mathcal{O}G^\sigma \to I^\sigma \to 0$ is isomorphic to $\mathcal{A}(I)$. Since X^σ is a direct summand of $M^\sigma \cong M$ and $\mathbf{1}^\sigma = \mathbf{1}$, we see that $X^\sigma \cong X$. This forces $\chi = \mathbf{1} + \eta_1 + \eta_2 + \eta_3$, and hence $X = M$. □

Lemma 3.5. *Suppose that G is a dihedral 2-group of order $2^n \geq 8$. Then the projective-free part M of the middle term of the Auslander-Reiten sequence $\mathcal{A}(I)$ terminating in I is indecomposable.*

Proof. It is known that $\mathrm{Rad}(kG)/\mathrm{Soc}(kG)$ is a direct sum of two uniserial modules, say H_1 and H_2, which are non-isomorphic duals (see 3.1 Lemma of [6]).

Here, we claim that M does not have any direct summand of $\mathcal{O}$-rank one: Indeed, if M has a direct summand of $\mathcal{O}$-rank one, then $\mathcal{O}_G$ is a direct summand of M by Lemma 3.2 (1). Thus J is isomorphic to the middle term of the Auslander-Reiten sequence $\mathcal{A}(\mathcal{O}_G)$ terminating in $\mathcal{O}_G$ by Lemma 2.1. However, this contradicts Remark 2.4.

Now we assume to the contrary that M is decomposable. Since $\overline{M} \cong k_G \oplus k_G \oplus \mathrm{Rad}(kG)/\mathrm{Soc}(kG)$ by Lemma 1.3, one of the following two cases would occur:
Case (I): $M = X \oplus Y$, where X is indecomposable and $\overline{X} \cong k_G \oplus k_G$, and $\overline{Y} \cong \mathrm{Rad}(kG)/\mathrm{Soc}(kG)$, or
Case (II): $M = X \oplus Y$, where both X and Y are indecomposable, and $H_1 \mid \overline{X}$ and

$H_2 \mid \overline{Y}$.

First, we assume Case (I). Note that $\overline{Y}$ has no simple direct summand. Thus, using an argument similar to one in the proof of Lemma 3.3, we can derive a contradiction.

Next, assume Case (II). Since $M = X \oplus Y$ affords the regular character of G and the multiplicity of the trivial character $\mathbf{1}$ in it is one, we may assume that $\mathbf{1}$ is a constituent of the character afforded by X and $\mathbf{1}$ does not appear as a constituent in the character afforded by Y. Then, since $\mathbf{1}^* = \mathbf{1}$ and $M^* \cong M$ by Lemma 3.2 (2), it follows that $X^* \cong X$ and $Y^* \cong Y$. Thus we see that $(\overline{X})^* \cong \overline{X^*} \cong \overline{X}$. However, this implies that $H_2 \cong H_1^*$ is a direct summand of $(\overline{X})^* \cong \overline{X}$, a contradiction. □

Proposition 3.6. *Let G be a finite p-group. Then M is indecomposable except the following cases*:
(1) $|G| = p$ *and* $(\pi) = (p)$,
(2) $|G| = p$ *and* $(\pi^2) = (p)$,
(3) $|G| = p^2$ *and* $(\pi) = (p)$.

Proof. Assume that M is decomposable. Then $\mathcal{O}_G$ is a direct summand of M by Lemmas 3.2 (1), 3.3, 3.4 and 3.5. Hence, unless $|G| = p$ and $(\pi) = (p)$, J is just the middle term of the Auslander-Reiten sequence $\mathcal{A}(\mathcal{O}_G)$ terminating in $\mathcal{O}_G$, and the result follows by Remark 2.4. □

REMARK 3.7. Let G be a finite p-group and Δ the connected component of $\Gamma(\mathcal{O}G)$ containing $\mathcal{O}G$. Then we see that the tree class of Δ_s is not A_∞^∞ from Proposition 3.6 and Remark 2.4.

4. Endomorphism rings

In this section, we assume that G is a finite p-group as usual and consider the endomorphism rings of $J = J(\mathcal{O}G) = \pi\mathcal{O}G + \sum_{g\in G}(g-1)\mathcal{O}G$ and of $M = \pi\mathcal{O}G + \sum_{g\in G}(g-1)\mathcal{O}G + \pi^{-1}(\hat{G} - |G|1)\mathcal{O}G$.

Lemma 4.1. *Let W be an indecomposable $\mathcal{O}G$-lattice. Suppose that W has an $\mathcal{O}G$-submodule V satisfying the following two conditions*:
(i) $W/V \cong \mathcal{O}/(\pi)$; *and*
(ii) *For any* $f \in \mathrm{End}_{\mathcal{O}G}(W)$, $f(V) \subseteq V$.
Then we have that $\mathrm{End}_{\mathcal{O}G}(W)/\mathrm{Rad}(\mathrm{End}_{\mathcal{O}G}(W)) \cong \mathcal{O}/(\pi)$ *as ring.*

Proof. Choose and fix an element $e \in W \setminus V$. For an endomorphism f of W, put $f(e) = \alpha \cdot e + \beta$ for some $\alpha \in \mathcal{O}$ and some $\beta \in V$. Then it follows that $\mathrm{Im}(f - \alpha \cdot \mathrm{id}_W) \subseteq V$ and $f - \alpha \cdot \mathrm{id}_W \in \mathrm{Rad}(\mathrm{End}_{\mathcal{O}G}(W))$. □

Lemma 4.2. *Let G be a p-group, and suppose that J and M are indecomposable. Then both* $\mathrm{End}_{\mathcal{O}G}(J)/\mathrm{Rad}(\mathrm{End}_{\mathcal{O}G}(J))$ *and* $\mathrm{End}_{\mathcal{O}G}(M)/\mathrm{Rad}(\mathrm{End}_{\mathcal{O}G}(M))$ *are isomorphic to* $\mathcal{O}/(\pi)$ *as ring.*

Proof. Since $\{\pi 1\} \cup \{g-1\}_{g\in G}$ is an $\mathcal{O}$-basis of J, we have that $\sum_{g\in G}(g-1)\mathcal{O}G = \{x \in J \mid x\hat{G} = 0\}$, where $\hat{G} = \sum_{g\in G} g$. Thus, for any $f \in \mathrm{End}_{\mathcal{O}G}(J)$, we see that $f(\sum_{g\in G}(g-1)\mathcal{O}G) \subseteq \sum_{g\in G}(g-1)\mathcal{O}G$. Hence $\pi J + \sum_{g\in G}(g-1)\mathcal{O}G$ is a maximal $\mathcal{O}G$-submodule of J satisfying the two conditions in Lemma 4.1.

Also, $\sum_{g\in G}(g-1)\mathcal{O}G + \pi^{-1}(\hat{G} - |G|1)\mathcal{O}G = \{x \in M \mid x\hat{G} = 0\}$. Thus $\pi M + \sum_{g\in G}(g-1)\mathcal{O}G + \pi^{-1}(\hat{G} - |G|1)\mathcal{O}G$ is a maximal $\mathcal{O}G$-submodule of M satisfying the two conditions in Lemma 4.1. $\square$

REMARK 4.3. Let G be a p-group and suppose that $\mathcal{O}G$ is of infinite representation type. Let Δ be the connected component of $\Gamma(\mathcal{O}G)$ containing the projective $\mathcal{O}G$-lattice $\mathcal{O}G$.

(1) Suppose that M is indecomposable. Then J lies at the end of Δ. Also, the length of $\mathrm{rad}(\mathrm{Hon}_{\mathcal{O}G}(J, M))/\mathrm{rad}^2(\mathrm{Hom}_{\mathcal{O}G}(J, M))$ as $\mathrm{End}_{\mathcal{O}G}(J)$-module and that as $\mathrm{End}_{\mathcal{O}G}(M)$-module are the same by Lemma 4.2. Therefore, the tree class of Δ_s is neither B_∞ nor C_∞.

(2) Suppose that M is decomposable. Then $\mathcal{O}_G$ is isomorphic to a direct summand of M by Proposition 3.6 and Remark 2.4. Hence, unless G is the Klein four group and $(\pi) = (2)$, the tree class of Δ_s is A_∞ by Theorem of [10], and J lies at the second row from the end of Δ.

5. Euclidean diagrams

Let G be a p-group and Θ a connected component of $\Gamma(\mathcal{O}G)$. In this section, we shall show that if Θ does not contain the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$, then the tree class of the stable part Θ_s of Θ is not Euclidean. For this purpose, we recall some additive function due to T. Okuyama.

For any $\mathcal{O}G$-lattices X and W, $\underline{\mathrm{Hom}}_{\mathcal{O}G}(X, W) := \mathrm{Hom}_{\mathcal{O}G}(X, W)/\mathcal{P}\mathrm{Hom}_{\mathcal{O}G}(X, W)$ is an $\mathcal{O}$-torsion module. $d(X, W)$ denotes the composition length of $\underline{\mathrm{Hom}}_{\mathcal{O}G}(X, W)$ as $\mathcal{O}$-module. Put $d_X(W) := d(X, W) + d(\Omega^{-1}X, W)$.

Lemma 5.1 (Okuyama). *Let G be a p-group and Θ a connected component of $\Gamma(\mathcal{O}G)$.*

(1) *Let X be an indecomposable $\mathcal{O}G$-lattice not contained in Θ. Suppose that $X^* \otimes W$ is not projective for any $\mathcal{O}G$-lattice W in Θ. Then d_X is an additive function for Θ_s (not necessarily Ω-periodic).*

(2) *Let W be a non-projective indecomposable $\mathcal{O}G$-lattice and P_W the projective cover of W. Then we have that* $\mathrm{rank}_{\mathcal{O}} P_W \leq |G| d_{\mathcal{O}_G}(W)$.

Proof. See Corollary 2.4 of [10] for (1), and Lemma 1.3 of [10] for (2). □

Proposition 5.2. *Let G be a p-group, and let Θ be any connected component of $\Gamma(\mathcal{O}G)$ not containing the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$. Then the tree class of Θ_s is not Euclidean.*

Proof. Assume that the tree class of Θ_s is Euclidean. By Lemma 5.1 (1), $d_{\mathcal{O}_G}$ is an additive function for Θ_s and $d_{\mathcal{O}_G}$ takes bounded values by Corollary 2.4 of [16]. Hence $\{\mathrm{rank}_{\mathcal{O}} W\}_{W\in\Theta}$ is bounded by Lemma 5.1 (2). This implies that $\mathcal{O}G$ is of finite representation type by Theorem 2 of [17]. Thus, $\Theta = \Gamma(\mathcal{O}G)$ must contain $\mathcal{O}_G$, a contradiction. □

Lemma 5.3. *Suppose that G is a p-group and $\mathcal{O}G$ is of infinite representation type. Furthermore, in the case where $p = 2$ and G is the Klein four group, suppose that $(\pi) \supsetneqq (2)$. Let Δ be the connected component of $\Gamma(\mathcal{O}G)$ containing the projective $\mathcal{O}G$-lattice $\mathcal{O}G$. Then the tree class of Δ_s is not Euclidean.*

Proof. First, we assume that G is cyclic. Since $\mathcal{O}G$ is of infinite representation type and any $\mathcal{O}G$-lattice is Ω-periodic, Δ_s is an infinite tube by [8].

Next, assume that G is not cyclic and either of the following two conditions holds: (i) $|G| \gneqq p^2$, or (ii) $(\pi) \supsetneqq (p)$. Then, Δ does not contain the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$ (see the argument in the proof of Lemma 4.2 of [10]). Hence the result follows by Proposition 5.2.

Finally, assume that $G \cong C_p \times C_p$ and $(\pi) = (p)$ (p: odd). By Remark 2.4, Δ contains $\mathcal{O}_G$. Hence the tree class of Δ is A_∞ by Theorem of [10]. □

6. Proof of Theorem

Suppose that G is a p-group and $\mathcal{O}G$ is of infinite representation type. Let Δ be the connected component of $\Gamma(\mathcal{O}G)$ containing the projective $\mathcal{O}G$-lattice $\mathcal{O}G$. If G is cyclic, then Δ_s is an infinite tube by [8]. Hence, in the rest, we assume that G is not cyclic. Then, by a result of Webb (Theorem A of [16]), the tree class of Δ_s is either an infinite Dynkin diagram or a Euclidean diagram. Moreover, by Remarks 3.7, 4.3 and Lemma 5.3, the tree class of Δ_s is not A_∞^∞, B_∞, C_∞ or Euclidean. Thus, in order to show that the tree class of Δ_s is A_∞, we have to exclude only the case of D_∞.

Lemma 6.1. *The tree class of Δ_s is not D_∞.*

Proof. Assume that the tree class of Δ_s is D_∞. Then, by Remark 4.3 (2), M is indecomposable and J lies at the end of Δ_s.

Now a part of Δ is as follows for some indecomposable $\mathcal{O}G$-lattice Z:

$$
\begin{array}{ccccccccccccc}
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\Omega^2 Z & \to & \Omega M & \to & \Omega Z & \to & M & \to & Z & \to & \Omega^{-1} M & \to & \Omega^{-1} Z \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\Omega J & & & & J & \to & \mathcal{O}G & \to & I & & & & \Omega^{-1} I.
\end{array}
$$

Considering the dual lattices, we get the Auslander-Reiten sequence $0 \to Z^* \to M^* \to (\Omega Z)^* \to 0$. As $M^* \cong M$ by Lemma 3.2 (2), we see that $(\Omega Z)^* \cong Z$.

Since M affords the regular character of G, so does $Z \oplus \Omega Z \cong Z \oplus Z^*$. Note that the multiplicity of the trivial character $\mathbf{1}$ in the regular character is one. This implies that $\mathbf{1}$ appears as a constituent in the character afforded by Z or in the one afforded by Z^*, but not in the both, a contradiction. $\square$

We have now completed the proof of the Theorem.

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