Osaka Mathematical Journal
Vol. 6, No. 1, December, 1954.

On Multiple Distributions

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In the theory of quantumn wave fields, there appears a distribution called "invariant Δ-function" which gives the commutation relation between fields quantities. This Δ-function is not a function but a distribution and is considered to be defined by the wave equation $(\square-\kappa)\cdot\Delta=0$ with initial conditions $\Delta(x, 0)=0$, $\partial\Delta/\partial t(x, 0)=-\delta_0(x)$ (c.f. J. Schwinger ([9]), W. Pauli ([10])). Concerning this sorts of equations, we consider generally here about an equation of evolution in the sense of distribution.

L. Schwartz treats this problem ([3]). He considers distributions $U_x(t)\in\mathfrak{D}'(x)$ on the spacial variables $(x_1, \dots, x_n)$ where the time variable t is a parameter. For the simplicity we call hereafter this sort of distribution a *parametric distribution* and call a distribution on the space $(x_1, \dots, x_n, t)$ a *proper distribution.* He discusses mainly parametric distribution and parametric equation of evolution. Concerning the proper one L. Schwartz refers (§16) that a parmeteric distribution can be considered to be a proper distribution and also refers to a proper distributional equation. But the relation between parametric and proper distribution and the relation between parametric and proper distributional equation is not treated in detail. In this paper we start from proper distribution conversely and researches in what case it can be considered as parametric one and researches in what case a proper equation can correspond to a parametric equation.

To give a clarification of these relations we introduce the notation of multiple distributions defined in §3, and research (§3, §4) several properties of multiple distributions.

A parametric distribution ($\in\mathfrak{D}'(x)$) is a multiple distribution of a distribution ($\in\mathfrak{D}'(x, t)$) and the special distribution ($\in\mathfrak{D}'(t)$). In §5 we consider this special case and study relations between proper distribution and parametric continuous or parametric continuously differentiable distribution. As an example of applications we discuss in §6 relations between two sorts of equations.

The invariant Δ-function mentioned at the top will be clarified in the sense of the one in the proper distributional equation, and since its corresponding parametric equation can be solved, we obtaine the proper distributional solution with consideration of § 6. (Direct calculation as a proper one is also possible).

Concerning the topological terminologies used in the paper refer to N. Bourbaki ([4], [5]), C. Chevalley ([8]).

§ 1. Topologies defined by bounded sets.

In this section we modify a few the B. H. Arnold's results ([1]). Let $S=\{\theta, x, y, \cdots\}$ be a vector space over the real number field with zero vector θ, and let $\mathfrak{B}$ be any collection of subsets of S satisfying

(B1) For any $x \in S$, $\{x\} \in \mathfrak{B}$,

(B2) The union of any two sets of $\mathfrak{B}$ is a set of $\mathfrak{B}$,

(B3) Any subset of a set of $\mathfrak{B}$ is a set of $\mathfrak{B}$,

(B4) Any scaler multiple of a set of $\mathfrak{B}$ is a set of $\mathfrak{B}$,

(B5) The convex hull of a set of $\mathfrak{B}$ is a set of $\mathfrak{B}$.

We call the elements of $\mathfrak{B}$ *bounded sets* of the vector space S.

The following algebraic properties of $\mathfrak{B}$ hold in our cases too.

Lemma 1. *The linear sum of any tow bounded sets is bounded.*

DEFINITION 1. For any $X \subset S$ the *symmetric starlike hull* X^* *of* X is

$$X^* = \{U \lambda X \mid |\lambda| \leq 1\} .$$

Lemma 2.

(1) *For* $B \in \mathfrak{B}$, *we have* $B^* \in \mathfrak{B}$.

(2) *If* $|\lambda| \leq |\mu|$, *then* $\lambda X^* \subset \mu X^*$ *for* $X \subset S$.

THE TOPOLOGY IN S.

DEFINITION 2. A subset G of S is *open* if and only if whenever $g \in G$ there exists a convex set N such that for any $B \in \mathfrak{B}$ there exists a $\lambda > 0$ which satisfies $g + \lambda B \subset N \subset G$. ($N$ depends on g, but is independent from B).

Lemma 3. *Definition* 2 *makes* S *a topological space.*

Proof. It is evident that the empty set, the whole space and any union of open sets are open. If G and H are open, and $g \in G \cap H$, there exist sets N_1 and N_2 such that for any $B \in \mathfrak{B}$ there exist $\mu > 0$ and $\nu > 0$ which satisfy $g + \mu B^* \subset N_1 \subset G$ and $g + \nu B^* \subset N_2 \subset H$. Setting $\lambda = \min\{\nu, \mu\}$, we have $\lambda B \subset \lambda B^* \subset \mu B^* \cap \nu B^* \subset N_1 \cap N_2 \subset G \cap H$, so that $G \frown H$ is open and S is seen to be a topological space.

Lemma 4. *There is a fundamental system of convex balanced neighborhood of θ.*

Proof. If G is an open neighborhood of θ, there exists a convex set N such that $G \supset N \supset \lambda B$. For any point $x \in \bigcup_{0 \leq \alpha < 1} \alpha N = N_1$, there exist $0 < \alpha_0 < \alpha_1 < 1$ such that $\alpha_1 x/\alpha_0 \in \alpha_1 N$. Since $\alpha_1 N$ is convex and can swallow any $B \in \mathfrak{B}$ for some positive multiple μ, we have $x + (1-\alpha_0/\alpha_1)\mu B = (\alpha_0/\alpha_1)(\alpha_1 x/\alpha_0) + (1-\alpha_0/\alpha_1)\mu B \ni \alpha_1 N$. So N_1 is also an open convex set. Since $(-N_1)$ is also an open convex set we have a convex balanced open neighborhood of θ, N_0; $N_0 = N_1 \frown (-N_1) \subset G$.

Lemma 5. *This topology is compatible with the vector operation of S.*

Proof. First the mapping $(x, y) \rightarrow x+y$ is continuous jointly.

For any open set G_{x+y} which contains $x+y$, there exists a convex set N_{x+y} such that $G_{x+y} \supset N_{x+y} \ni x+y$. Now $N_{x+y} - (x+y)$ is a convex set which can swallow any $B \in \mathfrak{B}$ for some positive multiple, so it is a neighborhood of θ as can be seen in the proof of Lemma 4. By Definition 2, for any open set G and for any $x \ni S$ the subset $x+G$ is open. So $x + \{N_{x+y} - (x+y)\}/2$ is a convex neighborhood of x and $y + \{N_{x+y} - (x+y)\}/2$ is a convex neighborhood of y and we see

$$[\{x + \{N_{x+y} - (x+y)\}/2] + [y + \{N_{x+y} - (x+y)\}/2] = N_{x+y}.$$

Next the continuity of the mapping $(\lambda, x) \rightarrow \lambda x$ is seen as follows. A mapping $x \rightarrow \lambda_0 x$ is continuous in the neighforhood of $x = \theta$ for any fixed λ_0. If $\lambda_0 = 0$, this assertion is evident. If $\lambda_0 \neq 0$, $\lambda_0(N_0/\lambda_0) \subset N_0 \subset N \subset G$ for any neighborhood G of θ where N_0 is a convex balanced neighborhood of θ. But N/λ_0 is a neighborhood of θ so the mapping $x \rightarrow \lambda_0 x$ is continuous.

The mapping $\lambda \rightarrow \lambda x_0$ is continuous in the neighborhood of $\lambda = 0$ for any fixed x_0. For $\{x_0\}$, $\{x_0\}^* \in B$ and for any neighborhood G of θ, there exists $\mu > 0$ such that $\mu\{x_0\}^* \subset N_0 \subset N \subset G$. Then for any $|\lambda| \leq \mu$ we have $\lambda x \in N_0 \subset G$.

The mapping $(\lambda, x) \rightarrow \lambda x$ is continuous in the neigeborhood of $x = \theta$, $\lambda = 0$, since for any neighborhood G of θ we have $\lambda N_0 \subset N_0 \subset N \subset G$ for $|\lambda| \leqq 1$.

Therefore we see the .continuity of the mapping $(\lambda, x) \rightarrow \lambda x$ and Lemma 5 is proved.

TOPOLOGICAL BOUNDEDNESS.

We define a new concept of boundness in the usual way by

DEFINITION 3. A set $T\subset S$ is *topologically bounded* if and only if for each neighborhood U of θ there exist a λ with $T\subset\lambda U$.

We denote by $\mathfrak{T}$ the collection of all subsets of S which are topologically bounded.

Lemma 6. $\mathfrak{T}\supset\mathfrak{B}$, *and the collection* $\mathfrak{T}$ *satisfies the axioms B1)—B5).*

Proof. $\mathfrak{T}\supset\mathfrak{B}$ is the direct consequence of Definitions 2 and 3. So T evidently satisfies B1), B3), B4). B2) follows from the existence of a fundamental balanced neighborhood system of θ. B5) follows from the existence of a fundamental convex neighborhood system of θ.

Lemma 7. *The topologies defined in S by the collection* $\mathfrak{T}(\tau_{\mathfrak{T}})$ *and by the collection* $\mathfrak{B}$ $(\tau_{\mathfrak{B}})$ *are identical.*

Proof. $\tau_{\mathfrak{B}}$ is stronger than $\tau_{\mathfrak{T}}$ since $\mathfrak{B}\subset\mathfrak{T}$, and $\tau_{\mathfrak{T}}$ is stronger than $\tau_{\mathfrak{B}}$ by virtue of the definition of $\mathfrak{T}$.

TOPOLOGIES DEFINED BY BOUNDED SETS.

Theorem 1. *Definition* 2 *makes S a bornographic* ([7]) *locally convex topological vector space.*

Proof. The proof of Lemma 9 assures the bornography of this space.

Remark. If a locally convex topological vector space V is given and if we take the totality $\mathfrak{B}$ of bounded sets (in the natural tolplogy of V), $\mathfrak{B}$ satisfies B1)—B5) and the topology $\tau_{\mathfrak{B}}$ is stronger than the natural topology of V. But if V is a bornographic space the topology $\tau_{\mathfrak{B}}$ is identical with the old topology of V.

§ 2. Bounded sets in the product space.

NOTATIONS.

For any $0\leq\pi\leq\infty$ we consider the vector space of all real valued π-times differentiable functions having compact carriers. We denote the space $\mathfrak{D}^{\pi}$ ([2]) defined on the n-dimensional Euclidean space $R^{n}(x)$ by $\mathfrak{D}^{\pi}(x)$, similarly the one on $R^{m}(t)$ by $\mathfrak{D}^{\pi}(t)$ and the one on $R^{m+n}(x, t)$ by $\mathfrak{D}^{\pi}(x, t)$ where $m>0$ and $n\geq 0$, and denote the totality of bounded sets in their natural topology by $\mathfrak{B}_\pi(t)$ etc. Further we denote their strong dual spaces by $\mathfrak{D}^{(\pi)'}(x)$, $\mathfrak{D}^{(\pi)'}(t)$ etc, denote the convergence in the topology of $\mathfrak{D}^{(\pi)'}$ by the symbol $\xrightarrow{(\pi)'}$, and denote the bounded sets in $\mathfrak{D}^{(\pi)'}$ by $B_{(\pi)'}$.

ARNOLD'S FAMILY IN $\mathfrak{D}^\pi(t)$.

Now we take a sequence of functions $\{\phi_j(t) \mid \phi_j(t) \in \mathfrak{D}^\pi(t), \xrightarrow{(\nu)'} Q\}$ where Q is a definite distribution of $\mathfrak{D}^{\mu'}(t)$ for $\mu < \nu$, and often call it a (ν, Q)-sequence. We consider also occasionally a (ν, Q)-sequence each of them having a carrier contained in a fixed compact set K of R and call it a (ν, K, Q)*-sequence.*

We take the totality of the above (ν, Q)-sequences and denote it by $\mathfrak{B}'(t)$, and consider the minimum collection of subsets of $\mathfrak{D}^\pi(t)$ which satisfies axioms from B1) to B5) including both $\mathfrak{B}'(t)$ and $\mathfrak{B}_\pi(t)$. For the sake of simplicity we call such a collection an *Arnold's family.* In this case such an Arnold family $\mathfrak{B}^\circ(t)$ really exists and is uniquely determined and is given by a collection of sets of the following form

$$\mathfrak{B}^\circ(t) = \left\{ B^\circ(t) = ((B(t) \cup \bigcup_{i=1}^k \lambda_i B_i'(t))) \,\middle|\, \begin{matrix} k = 1, 2, \dots \\ B \in B_\pi(t), \ B' \in \mathfrak{B}'(t) \end{matrix} \right\},$$

where the symbol $((A))$ means the convex hull of a set A. In fact, Arnold's family must at least include this collection, and this collection satisfies B1)—B5), so this is indeed our Arnold's family. We denote this family by $\mathfrak{B}^\circ(t)$ and each set of it by $B^\circ(t)$. We denote by $\mathfrak{N}(t)$ a fundamental neighborhood system of θ which is induced by $\mathfrak{B}^\circ(t)$ obeying the method § 1, its element by $N(t)$, and denote the space $\mathfrak{D}^\pi(t)$ having this topology by $\mathfrak{D}_Q(t)$.

ARNOLD'S FAMILY IN THE SPACE $\mathfrak{D}^\pi(x) \otimes \mathfrak{D}^\pi(t)$.

We consider the tensor product space ([6]) $\mathfrak{D}^\pi(x) \otimes \mathfrak{D}^\pi(t)$ i.e.

$$\mathfrak{D}^\pi(x) \otimes \mathfrak{D}^\pi(t) = \{\textstyle\sum_i \varphi_i(x)\phi_i(t) \mid \varphi_i \in \mathfrak{D}^\pi(x), \ \phi_i \in \mathfrak{D}^\pi(t)\},$$

where $\sum_i$ means finite sum. We consider in this space the Arnold's family $\mathfrak{B}^\circ(x, t)$ which includes a family of subsets

$$\{B(x) \otimes B^\circ(t) \mid B \in \mathfrak{B}_\pi(x), \quad B^\circ \in \mathfrak{B}^\circ(t)\}$$

where

$$B(x) \otimes B^\circ(t) = \{\varphi(x)\phi(t) \mid \varphi \in B_\pi(x), \quad \phi \in B^\circ(t)\}.$$

Then $\mathfrak{B}^\circ(x, t)$ is also uniquely determined and is given by the collection of subsets $((B(x) \otimes B^\circ(t)))$ with their arbitrary subsets, where $B \in \mathfrak{B}_\pi(x)$ and $B^\circ \in \mathfrak{B}^\circ(t)$, since the operation contained in the axioms

B1)—B5) are closed either in $\mathfrak{B}_\pi(x)$ or in $\mathfrak{B}^\circ(t)$, and the Arnold's family must contain at least this family. We denote by $\mathfrak{N}(x, t)=\{N(x, t)\}$ a fundamental neighborhood system of θ which is induced by this family.

New Topology in the Space $\mathfrak{D}^\pi(x, t)$.

Similarly in the space $\mathfrak{D}^\pi(x, t)$ we find the Arnold's family which contains $\mathfrak{B}^\circ(x, t)$ and $\mathfrak{B}_\pi(x, t)$, i. e.

$$\{((B(x, t) \bigcup B^\circ(x, t))) \mid B \in \mathfrak{B}_\pi(x, t), \quad B^\circ \in \mathfrak{B}^\circ(x, t)\} .$$

A fundamental neighborhood system of θ is given by

$$\{((V(x, t) \bigcup N(x, t))) \mid V \in \mathfrak{B}(x, t), \quad N \in \mathfrak{N}(x, t)\} ,$$

where $\mathfrak{B}(x, t)$ means a fundamental neighbornood system of θ in the natural topology of $\mathfrak{D}^\pi(x, t)$. We denote the space $\mathfrak{D}^\pi(x, t)$ having this topology by $\mathfrak{D}_Q(x, t)$ or simply by $\mathfrak{D}_Q$.

The Space $\mathfrak{D}Q_\Lambda$, $\tilde{\mathfrak{D}}Q_\Lambda$.

Thus the space $\mathfrak{D}_Q(x, t)$ is introduced by a single distribution Q, but a similar process is possible for a fixed family of distributions $\{Q_\lambda \mid \lambda \in \Lambda\}$. That is to say $B^\circ(t)$ is expressed by

$$B^\circ(t) = ((B(t) \cup \bigcup_{k=1}^{s} \bigcup_{i=1}^{r} \bigcup_{j=1}^{\infty} \mu_{ik} \phi_{ij\lambda_k}(t))) ,$$

where $\phi_{ij\lambda_k} \xrightarrow{(\nu)'} Q_{\lambda_k}$, and of course ν is larger than the orders of the distributions Q_{λ_k}. The forms of Arnold's family in the other spaces, say, $\mathfrak{D}^\pi(x) \otimes \mathfrak{D}^\pi(t)$ and $\mathfrak{D}^\pi(x, t)$, are quite similar. We denote the space $\mathfrak{D}^\pi(t)$ or $\mathfrak{D}^\pi(x, t)$ having this topology by $\mathfrak{D}Q_\Lambda(t)$ or $\mathfrak{D}Q_\Lambda(x, t)$. The orders μ of the distributions Q_λ and the orders of the convergences $\nu=\nu(\lambda)$ can be various, but we have interest only in the case when both μ and ν are constants, and we consider only this case.

From the same family of distributions we can also construct another $B^\circ(t)$ as follows.

We take a family of sequences $\{\{\phi_{\lambda j} \mid j\} \mid \lambda \in \Lambda\}$ which satisfies the condition that for any neighborhood of θ, V, of $\mathfrak{D}^{(\nu)'}(t)$ there exists j_0 such that (1) for any $j \geq j_0$, for any $\lambda \in \Lambda$, $\phi_{\lambda j}-Q_\lambda \in V$, (2) $\bigcup_{j<j_0, \lambda \in \Lambda} \phi_{\lambda j} \in B_\pi(t)$. We call it a $(\nu, \tilde{Q})$-famaily and write its element by $B_i'(t)$. Now we consider $B^\circ(t)=((B(t) \cup \bigcup_{i=1}^{s} \rho_i B_i'(t)))$ or its arbitrary subset. The other forms of Armold's family are quite the same. We denote the space $\mathfrak{D}^\pi(t)$ or $\mathfrak{D}^\pi(x, t)$ having this topology by $\tilde{\mathfrak{D}}_Q(t)$ or by $\tilde{\mathfrak{D}}_Q(x, t)$. We often consider properties common to each of the spaces $\mathfrak{D}_Q(x, t)$, $\mathfrak{D}Q_\Lambda(x, t)$,

$\tilde{\mathfrak{D}}_{Q_\Lambda}(x, t)$. In such a case we denote them collectively by $\mathfrak{D}_P$, similarly denote the space $\mathfrak{D}_Q(t)$, $\mathfrak{D}_{Q_\Lambda}(t)$ and $\tilde{\mathfrak{D}}_{Q_\Lambda}(t)$ by $\mathfrak{D}_P(t)$.

PROPERTIES OF THE SPACE $\mathfrak{D}_P$.

Lemma 8. *For any neighborhood of θ, $N(x, t)$, in $\mathfrak{D}^\pi(x) \otimes \mathfrak{D}^\pi(t)$ contained in the space $\mathfrak{D}_P$ and for any bounded set $B_\pi(x)$, there exists a neighborhood $N(t)$ of θ in $\mathfrak{D}_P(t)$ such that $N(x, t) \supset B(x) \otimes N(t)$.*

Similarly for any bounded set $B^\circ(t)$ there exists a neighborhood of $V(x)$, θ in $D^\pi(x)$ such that $N(x, t) \supset V(x) \otimes B^\circ(t)$.

Proof. For any covex neighborhood $N(x, t)$, of θ, and for any bounded set $B_\pi(x)$, we consider the $N_{B(x)}$ such that $N_{B(x)} = \{g(t) \,|\, f(x)g(t) \in N(x, t)$ for all $f(x) \in B(x)\}$. Now any bounded set $B^\pi(x) \otimes B^\circ(t)$ in $\mathfrak{D}^\pi(x) \otimes \mathfrak{D}^\pi(t)$ is swallowed by $N(x, t)$ for some positive multiple λ, so for any bounded set $B^\circ(t)$ in $\mathfrak{D}^\pi(t)$, $N_{B(x)}$ must swallow $B^\circ(t)$ for the same positive multiple λ. While $N_{B(x)}$ is a convex set for a convex set $N(x, t)$, it must contain some neighborhood $N(t)$ of θ in $\mathfrak{D}^\pi(t)$. So the former part of Lemma holds, and the latter holds also quite similarly.

Corollary. *$T \in \mathfrak{D}'_P(x, t)$ is separately continuous for $\mathfrak{D}^\pi(x)$ and $\mathfrak{D}_P(t)$.*
The following property is evident.

(i) $$\tau_{D^\pi} > \tau_{D_Q} > \tau_{D_{Q_\Lambda}} > \tau_{\tilde{D}_Q} > \tau_{D^{(\nu)'}},$$

(ii) $$\tau_{D^{\pi_1}_{Q(\nu_1)}} > \tau_{D^{\pi_2}_{Q(\nu_2)}} \quad \text{for} \quad \pi_1 \leq \pi_2 \quad \text{and} \quad \nu_1 \geq \nu_2,$$

where $\tau_{\nu_1} > \tau_{\nu_2}$ means that the topology of the space indexed with ν_1 is finer than the topology of the one indexd with ν_2, and $\mathfrak{D}^{\pi_1}_{Q(\nu_1)}$, means the space with the topology induced by (ν_1, Q)-sequences.

§ 3. Properties of the space $'\mathfrak{D}_P$. (I)

We consider the strong dual space $\mathfrak{D}'_P$ of $\mathfrak{D}_P$ and the closure of the space $\mathfrak{D}^\nu$ in the topology of $\mathfrak{D}'_P$, and denote this closure by $'\mathfrak{D}_P$. Then $'\mathfrak{D}_P$ is a topological vector space and $\mathfrak{D}^\nu \subset {}'\mathfrak{D}_P \subset \mathfrak{D}'_P \subset \mathfrak{D}^{(\pi)'}$ with topologies $\tau_{\mathfrak{D}^\nu} > \tau_{'\mathfrak{D}_P} > \tau_{\mathfrak{D}^{(\pi)'}}$, $\tau_{\mathfrak{D}^\nu} > \tau_{'\tilde{\mathfrak{D}}_Q} > \tau_{'\mathfrak{D}_{Q_\Lambda}} > \tau_{'\mathfrak{D}_Q}$ and $\tau_{'\mathfrak{D}^{\pi_1}_{Q(\nu_1)}} > \tau_{'\mathfrak{D}^{\pi_2}_{Q(\nu_2)}}$ for $\pi_1 \leq \pi_2$, and $\nu_1 \geq \nu_2$.

If $T \in {}'\mathfrak{D}_P(x, t)$, we have a filter $\mathfrak{F}$ on $\mathfrak{D}^\nu(x, t)$ such that $\mathfrak{F} \xrightarrow{\mathfrak{D}'_Q} T$. Now in the inequality

$$\begin{aligned} &|\langle f, \varphi Q\rangle - \langle \bar{f}, \varphi Q\rangle| \\ &\leq |\langle f, \varphi Q\rangle - \langle f, \varphi\phi_i\rangle| + |\langle f, \varphi\phi_i\rangle - \langle T, \varphi\phi_i\rangle| \\ &+ |\langle T, \varphi\phi_i\rangle - \langle \bar{f}, \varphi\phi_i\rangle| + |\langle \bar{f}, \varphi\phi_i\rangle - \langle \bar{f}, \varphi Q\rangle|, \end{aligned}$$

where $\langle A, B\rangle$ means scaler product of A and B, for any $\varepsilon>0$ we can take an element F of the filtre $\mathfrak{F}$ such that for any two elements f and $\bar{f}$ of F, the 3rd and the 4th terms are smaller than $\varepsilon/4$ uniformly for $\varphi\in B_{\pi}(x)$, $\phi_j\in B^{\circ}(t)$, and we can choose j_0 such that for any $j>j_0$ the 2nd and the 5th terms are smaller than $\varepsilon/4$ since $\{\langle f(x, t), \varphi(x)\rangle_x | \varphi\in B_{\pi}\}\in B_{\nu}(t)$. So we see $\lim_{\mathfrak{F}\underset{\rightarrow}{D'_Q} T}\langle f, \varphi Q\rangle$ exists. Further if two filters $\mathfrak{F}$ and $\bar{\mathfrak{F}}$ converge to T in the topology of $\mathfrak{D}'_Q$ this limit must conincide. In fact we can take a set $F\in\mathfrak{F}$ and a set $\bar{F}\in\bar{\mathfrak{F}}$ and j_0 such that the same evaluation of this inequality for $f\in F$ and $\bar{f}\in\bar{F}$ can be done. This uniquely determined weak limit in $\mathfrak{D}^{(\pi)\prime}(x)$ is at the same time a strong limit since this convergence is uniform for $\varphi\in\mathfrak{B}_{\pi}(x)$.

Definition 4. *Multiple distribution* of a distribution $T\in{}'\mathfrak{D}_Q(x, t)$ and a distribution $Q(t)\in\mathfrak{D}^{(\pi)\prime}(t)$ is a distribution $T_Q\in\mathfrak{D}^{(\pi)\prime}(x)$ such that for $\varphi\in\mathfrak{D}^{\pi}(x)$, $\langle T_Q, \varphi\rangle=\lim_{\mathfrak{F}\underset{\rightarrow}{\mathfrak{D}'_Q} T}\langle f, \varphi Q\rangle$. Of course in the space ${}'\mathfrak{D}_{Q_\Lambda}$ or ${}'\tilde{\mathfrak{D}}_Q$, the same definition is poisible for any $\lambda\in\Lambda$.

A Characterization of Space ${}'\mathfrak{D}_P$.

Theorem 2. *If $T\in{}'\mathfrak{D}_Q$, then T is continuous with respect to any $(\nu, \rho Q)$-sequence uniformly for $\varphi\in B_{\pi}(x)$, where ρ is an arbitary constant, and the limit determined by this sequence coincides with ρT_Q.*

Proof. The former part of the theorem is seen to be true from the following inequality by the similar evalution as above:

$$\begin{aligned}&|\langle T, \varphi\phi_i\rangle-\langle T, \varphi\phi_k\rangle|\\&\leq|\langle T, \varphi\phi_j\rangle-\langle f, \varphi\phi_j\rangle|+|\langle f, \varphi\phi_j\rangle-\langle f, \varphi\phi_k\rangle|\\&+|\langle f, \varphi\phi_k\rangle-\langle T, \varphi\phi_k\rangle|.\end{aligned}$$

Denoting the limit of this Cauchy filter by $\langle T, \varphi Q\rangle$, we obtain the latter part of the theorem similarly by the following inequality.

$$\begin{aligned}&|\langle f, \rho\varphi Q\rangle-\langle T, \rho\varphi Q\rangle|\\&\leq|\langle f, \rho\varphi Q\rangle-\langle f, \varphi\phi_j\rangle|+|\langle f, \varphi\phi_j\rangle-\langle T, \varphi\phi_j\rangle|\\&+|\langle T, \varphi\phi_j\rangle-\langle T, \rho\varphi Q\rangle|<\varepsilon.\end{aligned}$$

Corollary. *If $T\in{}'\tilde{\mathfrak{D}}_Q$ and $\{\{\phi_{\lambda j}\}|\lambda\in\Lambda\}$ is a $(\nu, \rho\tilde{Q})$-family, then T is continuous with respect to the sequence $\{\phi_{\lambda j}|j=1, 2, \dots\}$ uniformly for $\lambda\in\Lambda$ and uniformly for $\varphi\in B_{\pi}(x)$.*

Proof. This is evident if we examine the proof of Theorem 2.

Remark. The topology $\tau_{D'_Q}$ is dependent on two constants π, ν.

However when T_Q can be defined, it is uniquely determined by T and Q and does not depend on π, ν.

The following lemma is occasionally used.

Lemma 9. *If* $\{\phi\} \in B_{(\nu)'}$ *and* $\{\beta_k(t) | k=1, 2, \dots\}$ *is a* $(0, K, 0)$*-sequence then* $\{\phi * \beta_k | k=1, 2, \dots\}$ *is a* $(\nu, 0)$*-sequencnce whose convergence is uniform for* $\phi \in B_{(\nu)'}$.

Proof. For any $u \in B_{(\nu)'}$, we have $\langle \phi * \beta, u \rangle = \langle \beta, \check{\phi} * u \rangle$ where $\vee$ means the reflection. Now $\{\check{\phi} * u | \phi \in B_{(\nu)'}, u \in B_{(\nu)}\}$ is a bounded set in the space C° (i. e., the space of continuous functions having topology of compact convergence), so $\langle \phi * \beta_k, u \rangle \to 0$ uniformly for $u \in B_{(\nu)}$ and for $\phi \in B_{(\nu)'}$.

Corollary. *If* $\{\phi\} = B_\nu \in \mathfrak{B}_\nu$, *and* $\{\beta_k | k=1, 2, \dots\}$ *is a* $(0, K, 0)$*-sequence, then* $(\phi * \beta_k) \overset{(\nu)}{\to} 0$ *uniformly for* $\phi \in B_\nu$.

Proof. For any $u \in B_{(\nu)'}$, $\langle \phi * \beta_k, u \rangle = \langle \beta_k, \check{\phi} * u \rangle \to 0$ uniformly for $u \in B_{(\nu)'}$ and $\phi \in B_{(\nu)}$, by Lemma 9. Since $\mathfrak{D}^\nu$ is bornographic, $\tau_{\mathfrak{D}^{(\nu)''}} = \tau_{\mathfrak{D}^{(\nu)}}$. This proves our corollary.

Theorem 3. (The Converse of Theorem 3)

If $T \in \mathfrak{D}^{(\pi)'}(x, t)$ *and* $\langle T, \varphi\phi_i \rangle$ *makes a Cauchy sequence uniformly for* $\varphi \in B_\pi(x)$ *with respect to any* $(\nu, \rho Q)$*-sequence* $\{\phi_i\}$ *for* $\rho = 0, 1$, *then* $T \in {}'\mathfrak{D}_Q$.

Proof. we take a sequence $\alpha_k(x)\alpha_k(t) \xrightarrow{(0)'} \delta(x)\delta(t)$ in $\mathfrak{D}(x, t)$ such that $\{\alpha_k(x)\}$ is a $(0, K, \delta(x))$-sequence and $\{\alpha_k(t)\}$ is a $(0, K, \delta(t))$-sequence where $\delta(x)$, $\delta(t)$ means Dirac's δ at the origin of $R^n(x)$ and $R_m(t)$ respectively. (Hereafter we call such a sequence *α-sequence*)-

For any $\varphi \in B_\pi(x)$ and $\phi_j \in B'(t)$ we have

$$\begin{aligned} &|\langle T*(\alpha_k(x)\alpha_k(t)),\ \varphi(x)\phi_j(t) \rangle - \langle T,\ \varphi\phi_j \rangle| \\ &\leq |\langle T,\ \{\varphi*(\delta(x) - \check{\alpha}_k(x))\} \times \phi_j(t) \rangle| \\ &+ |\langle T,\ \{\varphi * \check{\alpha}_k x)\} \times \{\phi_j(t)*(\delta(t) - \check{\alpha}_k(t))\} \rangle| . \end{aligned}$$

In the 2nd term, $(\delta - \check{\alpha}_k(x))$ is a $(0, K, 0)$ sequence, so $\varphi * (\delta - \check{\alpha}_k) \overset{(\pi)}{\to} 0$ by the above corollary. Since $\langle T, \varphi\phi \rangle$ is bounded for $\varphi\phi \in B_\pi(x) \otimes B^\circ(t)$ by assumption, $T \in \mathfrak{D}'_Q$ and Lemma 8 can be used. So for any $\varepsilon > 0$ we can take k_0, such that for any $k > k_0$,

$$|\langle T,\ \{\varphi*(\delta - \check{\alpha}_k)\} \times \phi_j(t) \rangle| < \varepsilon/2$$

uniformly for $\phi_j \in B'(t)$ and $\varphi \in B_\pi(t)$. In the 3rd term,

$$\{\varphi * \check{\alpha}_k(x) \mid \varphi \in B_\pi(x), \quad k=1, \quad 2 \quad \dots\} \in B_\pi(x)$$

and the term $\{\phi_j(t) * (\delta(t) - \check{\alpha}_k(t)) \mid k=1, 2, \dots\}$ is a $(\nu, 0)$–sequence whose convergence is uniform for ϕ_j. So by the assumption it follows that the 3rd term is $<\varepsilon/2$ uniformly for $\phi_j \in B'(t)$ and $\varphi \in B_\pi(t)$.

Corollary. *If* $T \in \mathfrak{D}^{(\pi)'}(x, t)$ *and* $\{\langle T, \varphi\phi_{\lambda j}\rangle \mid j=1, 2, \dots\}$ *makes a Cauchy sequence with respect to any* $\{\phi_{\lambda j}\}$ *of a* $(\nu, \rho\tilde{Q})$*–family* $\{\{\phi_{\lambda j}\} \mid \lambda \in \Lambda\}$ *for* $\rho=0, 1$, *uniformly for* $\varphi \in B_\pi(x)$ *and uniformly for* $\lambda \in \Lambda$, *then* $T \in {}'\tilde{\mathfrak{D}}_Q$.

The proof is quite similar to the proof of Theorem 3.

§ 4. The Properties of the Space $'\mathfrak{D}_P$ (II).

CONTINUITY OF MULTIPLE OPERATION.

Theorem 4. *The mapping* $T \to T_Q$ *is a continuous linear mapping from* $\mathfrak{D}_P(x, t)$ *to* $\mathfrak{D}^{(\pi)'}(x)$.

Proof. Linearlity is evident. Now for a neighborhood U of θ in the $\mathfrak{D}^{(\pi)'}(x)$ such that $U=\{T \mid \mathrm{Sup}_{\varphi \in B_\pi(x)} |\langle T, \varphi\rangle| \leq \varepsilon\}$, we can take a neighborhood N of θ in $'\mathfrak{D}_P$ such that

$$N=\{T \mid \mathrm{Sup}_{\varphi\phi \in B_\pi(x) \otimes B^\circ(t)} |\langle T, \varphi\phi\rangle| < \varepsilon\}$$

for the same $B_\pi(x)$. Then we see for any $T \in N$, $T_{Q_\lambda} \in U$, q. e. d.

MULTIPLE DISTRIBUTION BY DERIVATIVES OF Q.

We denote a differential operator in $R^{m}(t)$, such as $\sum_{|s| \leq \sigma} a_{s_1 \cdots s_m} \partial^{|s|}/\partial t_1^{s_1} \cdots \partial t_m^{s_m}$, where $|s|=s_1+\cdots+s_m$ and $a_{s_1 \cdots s_m}$ is a constant, by D^σ and its conjugate operator by $D^{\sigma *}$, i. e., $D^{\sigma *}=\sum_{|s| \leq \sigma}(-)^{|s|} a_{s_1 \cdots s_m} \partial^{|s|}/\partial t_1^{s_1} \cdots \partial t_m^{s_m}$.

Theorem 5. *If* $T \in {}'\mathfrak{D}_{D^\sigma(P)}^{(\pi)'}$ (π, ν, μ) *then* $D^{\sigma_1 *} T \in {}'\mathfrak{D}_P^{(\pi+\sigma)'}$ $(\pi+\sigma, \nu-\sigma, \mu-\sigma)$, *and* $T_{D_t Q_\lambda}=(D^* T)_{Q_\lambda}$. *Especially if* D^σ *is a product such that* $D^\sigma=D^{\sigma_1} D^{\sigma_2}$ *then from* $T \in {}'\mathfrak{D}_{D^\sigma P}$, *it follows that* $D^{\sigma_1 *} T \in {}'\mathfrak{D}_{D^{\sigma_2}(P)}$, *and* $T_{D^\sigma Q_\lambda}=(D^{\sigma_1 *} T)_{D^{\sigma_2} Q_\lambda}$.

REMARK. If $T \in \mathfrak{D}^{(\pi)'}$, then $D^{\sigma *} T \in \mathfrak{D}^{(\pi+\sigma)'}$ and a map of $(\pi', \nu', Q_\lambda(\mu))$–sequence by D^σ is a $(\pi'-\sigma, \nu'+\sigma, D^\sigma Q_\lambda(\mu+\sigma))$–sequence, where π' means $\phi_j \in D^{(\pi')}$, ν' means $\xrightarrow{(\nu')'}$, $Q_\lambda(\mu)$ means $Q_\lambda \in D^{(\mu)'}$. The notations (π, ν, μ) and $(\pi+\sigma, \nu-\sigma, \mu-\sigma)$ in Theorem 5 are used in similar meanings.

The theorem may be stated more generally. Consider a mapping L_t from $\mathfrak{D}^{\pi'}(t)$ into $\mathfrak{D}^{\pi}(t)$ which satisfies the following conditions. (i) L_t maps any $(\pi', \nu', \rho' Q_\lambda)$-sequence to a $(\pi, \nu, \rho L_t(Q_\lambda))$-sequence or maps any $(\pi', \nu', \rho'\tilde{Q})$-family to a $(\pi, \nu, \rho \widetilde{L_t(Q_\lambda)})$-family for $\rho' = 0,1$ and the some constants ρ, where $\nu \geq \mu$, $\nu' \geq \mu'$ (μ': order of $L_t\ Q$). (ii) $L^*(\mathfrak{D}^{\nu}) \subset \mathfrak{D}^{\nu'}$ where L^* is a conjugate operator of $\mathfrak{D}^{(\pi)'}(x, t)$ into $\mathfrak{D}^{(\pi')'}(x, t)$ defined by $\langle L^*T, \varphi\phi\rangle = \langle T, \varphi L_t(\phi)\rangle$ for $\varphi \in \mathfrak{D}^{\pi'}(x)$, $\phi \in \mathfrak{D}^{\pi'}(t)$ for $\pi' \geq \pi$.

Concerning this mapping L_t, the following Lemma holds.

Lemma 10. *If* $T \in {}'\mathfrak{D}_{L_tP}(\pi, \nu, \mu)$, *then* $L^*T \in {}'\mathfrak{D}_P(\pi', \nu', \mu')$ *and* $T_{\rho L_t Q_\lambda} = (L^*T)_{Q_\lambda}$ *where* ρ *is determined by the equality* $L_t(\nu'\ Q)$*-sequence (or formily)* $= (\nu, \rho L_t(Q))$*- sequence (or family).*

Proof. Take a filter $\mathfrak{F}$ on $\mathfrak{D}^{\nu}$ such that $\mathfrak{F} \xrightarrow{\mathfrak{D}'_{L_tP}} T$.

Then the filter $L^*(\mathfrak{F})$ converges to L^*T in the sense of ${}'\mathfrak{D}_P\ (\pi', \nu', \mu')$ as follows: For any $\varepsilon > 0$ there exists some $F \in \mathfrak{F}$ such that for any $f \in \mathfrak{F}$, for any $\varphi \in D^{\pi'}$ we have the following inequality for any ϕ_j of any $(\nu', \rho' Q)$-sequence (or $\phi_{\lambda j}$ of $(\nu', \rho'\tilde{Q})$-family) for $\rho' = 0, 1$,

$$\begin{aligned} &|\langle L^*f, \varphi\phi\rangle - \langle L^*T, \varphi\phi\rangle| \\ &= |\langle f, \varphi L_t\ (\phi)\rangle - \langle T, \varphi L_t(\phi)\rangle| < \varepsilon. \end{aligned}$$

Next for a (ν', Q_λ)-sequence we have

$$\begin{aligned} &\langle (L^*T)_{Q_\lambda}, \varphi\rangle = \lim_{j\to\infty} \langle L^*T, \varphi\phi_{\lambda j}\rangle = \lim_{j\to\infty} \langle T, \varphi L_t(\phi_{\lambda j})\rangle \\ &= \langle T_{\rho L_t Q_\lambda}, \varphi\rangle \text{ by the condition (1),} \qquad \text{q. e. d.} \end{aligned}$$

Proof of Theorem 5.

We can take D^σ as L_t in Lemma 10, since condition (ii) is evident for $\nu' = \nu - \sigma$ and condition (i) is satisfied for $\nu' = \nu - \sigma$, $\rho' = \rho$, $\pi' = \pi + \sigma$. The last part of the theorem follows from

$$\begin{aligned} &\langle (\mathfrak{D}^{\sigma *}T)_{Q_\lambda}, \varphi\rangle = \lim_{j\to\infty} \langle D^{\sigma *}T, \varphi\phi_{\lambda j}\rangle \\ &= \lim_{j\to\infty} \langle D^{\sigma_1 *}T, \varphi D_t^{\sigma_2}\phi_{\lambda j}\rangle = \langle (D^{\sigma_1 *}T)_{D^{\sigma_2}Q}, \varphi\rangle, \qquad \text{q. e. d.} \end{aligned}$$

Theorem 6. *If the topology of* ${}'\mathfrak{D}^{\pi}_{D^\sigma P}(\nu+\sigma)$ *is introduced by bounded sets such that every* $(\nu+\sigma, \rho' D^\sigma Q_\lambda)$*-sequence (or* $(\nu+\sigma, \rho'\tilde{D}^\sigma Q)$*-family) for* $\rho' = 0,1$, *is a map* D^σ *of a* $(\nu, \rho Q_\lambda)$*-sequence (or* $(\nu, \rho\tilde{Q})$*-family) and* *if* $D^*T\in{}'\mathfrak{D}_P$, *then we have* $T\in{}'\mathfrak{D}_{D^\sigma P}$.

Proof. If the topology of $\mathfrak{D}_{L_tP}$ is given by bounded sets such that for $\rho'=0,1$ each $(\nu',\ \rho'L_tQ_\lambda)$–sequence (or $(\nu',\ \rho'\widetilde{L_tQ})$–family) is a map of a $(\nu,\ \rho Q_\lambda)$–sequence (or $(\nu,\ \rho\tilde{Q})$–family) by the above L_t such that $L_t(\phi_1(t)*\phi_2(t))=L_t(\phi_1(t))*\phi_2(t))$, then we have for an α–sequence and for a $(\nu,\ Q)$–sequence (or family) $\{\phi\}$,

$$\begin{aligned}&|\langle T,\ \varphi L_t\rho\phi\rangle-\langle T*\alpha_K,\ \varphi L_t\rho\phi\rangle|\\&\quad\leq|\langle L^*T,\ \rho\{\varphi*(\delta-\check{\alpha}_K(x))\}\times\phi_j(t)\rangle|\\&\quad+|\langle L^*T,\ \rho(\varphi*\check{\alpha}_K(x))\times\{\phi_j*(\delta-\check{\alpha}_K(t))\}\rangle|\leq\varepsilon,\end{aligned}\qquad\text{q. e. d.}$$

CONTINUITY OF MULTIPLE OPERATIONS $\lambda\to T_{Q_\lambda}$.

Theorem 7. *If* $T\in{}'\tilde{\mathfrak{D}}_Q$ *and* $\Lambda=\{\lambda\}$ *is a topological space and the mapping* $\lambda\to Q_\lambda$ *is continuous as the mapping from* Λ *into* $\mathfrak{D}^{(\mu)'}(t)$, *then the mapping* $\lambda\to T_{Q_\lambda}$ *is a continuous mapping from* Λ *to* $\mathfrak{D}^{(\pi)'}(x)$.

Proof. For any $\varphi\in B_\pi(x)$, we take a $(\nu,\ \tilde{Q})$–family $\{\phi_{\lambda j}\,|\,\lambda\}$. We have

$$\begin{aligned}&|\langle T_{Q_\lambda},\ \varphi\rangle-\langle T_{Q_{\lambda'}},\ \varphi\rangle|\\&\quad\leq|\langle T_{Q_\lambda},\ \varphi\rangle-\langle T,\ \varphi\phi_{\lambda j}\rangle|+|\langle T,\ \varphi_{\lambda j}\rangle-\langle f,\ \varphi\phi_{\lambda j}\rangle|\\&\quad+|\langle f,\ \varphi\phi_{\lambda j}\rangle-\langle f,\ \varphi Q_\lambda\rangle|+|\langle f,\ \varphi Q_\lambda\rangle-\langle f,\ \varphi Q_{\lambda'}\rangle|\\&\quad+\{\text{corresponding terms of the 2nd, 3rd, 4th terms}\}.\end{aligned}$$

We take a filter $\mathfrak{F}$ on $\mathfrak{D}^\nu(x,\ t)$ such that $\mathfrak{F}\xrightarrow{\tilde{\mathfrak{D}}'_Q}T$. Now for any $\varepsilon>0$ there exists $F\in\mathfrak{F}$ such that for any $f\in F$ the 3rd and its corresponding terms are $<\varepsilon/7$ uniformly for $\lambda\in\Lambda$ and $j=1,\ 2,\ \cdots$ and $\varphi\in B_\pi(x)$. Regarding such an $f(x,\ t)$ we consider the 5th term. Since the mapping $\lambda\to Q_\lambda$ is continuous, we can take V_λ such that for any $\lambda'\in V_\lambda$ the 5th term is $<\varepsilon/7$ uniformly for $\varphi\in B_\pi(x)$, since $\{\langle f(x,\ t),\ \varphi(x)\rangle_x\,|\,\varphi\in B_\pi(x)\}\in B_\pi(t)$. Regarding such a λ' and an $f(x,\ t)$, the 2nd and the 4th and their corresponding terms can be made smaller than $\varepsilon/7$ uniformly for $\varphi\in B_\pi(x)$ by taking some j, q. e. d.

CONVOLUTION AND MULTIPLICATION OF A MULTIPLE DISTRIBUTION.

The following two lemmas may be used in the application.

Lemma 11. *If* $T\in{}'\mathfrak{D}_P^{(\pi)}$, $S\in\mathfrak{E}'(x)\cap\mathfrak{D}^{(\sigma)'}(x)$ *then* $(\delta(t)\times S)*T\in{}'\mathfrak{D}_P^{(\pi+\sigma)}$ *and* $\{(S\times\delta(t))*T\}_{Q_\lambda}=S\underset{(x)}{*}T_{Q_\lambda}$.

Proof. Take an α–sequence. Then for any $\varphi\phi_{\lambda j}\in B_{\pi+\sigma}(x)\otimes B'(t)$ we have

$$\begin{aligned}&\langle (\delta(t)\times S)*T,\ \varphi\phi_{\lambda j}\rangle - \langle (\delta\times S)*T*\alpha_k,\ \varphi\phi_{\lambda j}\rangle\\&= \langle T,\ (\delta(t)\times \check{S})*(\varphi\phi_{\lambda j})*(\delta-\check{\alpha}_K)\rangle\\&= \langle T,\ \{\check{S}*\varphi*(\delta(x)-\check{\alpha}_k(x))\}\times\phi_{\lambda j}\rangle(t)\\&\quad+\langle T,\ \{\check{S}*\varphi*\check{\alpha}_k(x)\}\times\{\phi_{\lambda j}*(\delta(t)-\check{\alpha}_K(t))\}\rangle.\end{aligned}$$

Now $\{\check{S}*\varphi\,|\,\varphi\in B_{\pi+\sigma}(x)\}\in\mathfrak{B}_\pi(x)$, so $\check{S}*\varphi*(\delta(x)-\check{\alpha}_K(x)\xrightarrow{(\pi)}0$,

$$\{\varphi*\check{\alpha}_K*S\,|\,\varphi\in B_{\pi+\sigma}(x),\ k=1,\ 2,\ \cdots\}\in\mathfrak{B}_\pi(x)$$

and $\{\phi_{\lambda j}*|\,\delta(t)-\check{\alpha}_k(t))\,|\,k=1,\ 2,\ \cdots\}$ is a $(\nu,\ 0)$–sequence or $\{\{\phi_{\lambda j}*(\delta(t)-\check{\alpha}_k(t))\,|\,k=1,\ 2,\ \cdots\}\,|\,\lambda\in\Lambda\}$ is a $(\nu,\ \tilde{0})$–family. So $T*(\delta(t)\times S)\in{}'\mathfrak{D}_P^{(\pi+\rho)}$, and we have

$$\begin{aligned}&\langle \{T*(\delta(t)\times S)\}_{Q_\lambda},\ \varphi\rangle = \lim_{j\to\infty}\langle (\delta\times S)*T,\ \varphi\phi_{\lambda j}\rangle\\&= \lim_{j\to\infty}\langle T,\ (\check{S}\underset{(x)}{*}\varphi)\times\phi_{\lambda j}\rangle = \langle T_{Q_\lambda},\ \check{S}\underset{(x)}{*}\varphi\rangle\\&= \langle S\underset{(x)}{*}T_{Q_\lambda},\ \varphi\rangle.\end{aligned}$$

Here $\underset{(x)}{*}$ means the convolution in the space $\mathfrak{D}^{\pi+\sigma}(x)$.

Corollary. *If* $T\in{}'\mathfrak{D}_P^\pi(x,\ t)$ *then* $D_x^\rho T\in{}'\mathfrak{D}_P^{\rho+\pi}$ *and* $(D_x^\rho T)_{Q_\lambda}=D_x^\rho(T_{Q_\lambda})$.

Proof. Take $D_x^\rho\,\delta(x)$ as $S(x)$ in Lemma 11, then we obtain

$$D_x^\rho T=(\delta(t)\times D_x^\rho\delta(x))*T\in{}'\mathfrak{D}_P.$$

and

$$(D_x^\rho T)_{Q_\lambda}=D_x^\rho\,\delta(x)\underset{(x)}{*}T_{Q_\lambda}=D_x^\rho(T_{Q_\lambda}).$$

Lemma 12. *If* $T\in{}'\mathfrak{D}_P(x,\ t),\ f(t)\in D^\kappa(t)$ *for* $\kappa\geq\nu,\ \kappa\geq\pi,\ g(x)\in\mathfrak{E}(x)$, *then* $(f(t)\,g(x)T)\in{}'\mathfrak{D}_P$ *and* $(f(t)\,g(x)T)_{Q_\lambda}=g(x).\ T_{f\cdot Q_\lambda}$.

Proof. Take an α–sequence. Then

$$\begin{aligned}&\langle (fgT)*(\delta-\alpha_k),\ \varphi\phi_{\lambda j}\rangle=\langle T,\ \{g(x)\ (\varphi*(\delta-\check{\alpha}_k(x))\}\times\phi_{\lambda j}f\rangle\\&\quad+\langle T,\ g(\varphi*\check{\alpha}_k)\times\{\phi_{\lambda j}*(\delta-\check{\alpha}_k(t))\}f\rangle.\end{aligned}$$

In the 2nd term $\{f(t)\,\phi_{\lambda j}(t)\,|\,\phi_{\lambda j}(t)\in B^0\}\in\mathfrak{B}^0$,
and $(g\cdot(\varphi*((\delta-\alpha_k)))\xrightarrow{(\pi)}0$. In the 3rd term we see

$\{g(\varphi*\check{\alpha}_k)\,|\,\varphi\in B_\pi(x),\ k=1,2,\ldots\}\in\mathfrak{B}_\pi(x)$, and $\{f(\phi_{\lambda j}*(\delta-\check{\alpha}_k(t))\,|\,k\}$ is a $(\nu,\ O)-$sequence or $\{\{f(\phi_{\lambda i}*(\delta-\check{\alpha}_k(t)))\,|\,k\}\,|\,\lambda\}$ is a $(\nu,\ \tilde{O})$–family. So we obtain the former part of the lemma. Now

$\lim_{j\to\infty}\langle fgT,\ \varphi\phi_j\rangle=\lim_{j\to\infty}\langle T,\ g\varphi f\phi_j\rangle$ and $\{f\phi_j\,|\,\phi_j$ runs through *a* $(\nu,\ Q)$-sequence$\}$

is a $(\nu,\ fQ)$–sequence, similarly $\{\{f\phi_{\lambda j}\}\,|\,\lambda\}$ is a $(\nu,\ f\tilde{Q})$–family, so it follows that $(fgT)_{Q_\lambda}=g\cdot T_{fQ_\lambda}$.

§ 5. Spaces of parametric destributions.

Hereafter we confine ourselves to some special cases. We take Dirac's δ and its μ–th derivative $\delta^{(\mu)}$ as Q, and t itself as λ and D_t as L_t. We treat only the case where m is 1, though quite similar results can be obtained in the case $m \neq 1$ too. We take an interval $\mathfrak{B}$; $a \leq t \leq b$, as Λ. Further we write $'\mathfrak{D}_{t_0^{(\mu)}}$ in place of $'\mathfrak{D}_{\delta_{t_0}^{(\mu)}}$ and $'\mathfrak{D}_{\mathfrak{B}^{(\mu)}}$ in place of $'\mathfrak{D}_{\delta_\Lambda^{(\mu)}}$, similarly $\tilde{\mathfrak{D}}_{\mathfrak{B}^{(\mu)}}$, and $'\mathfrak{D}_{t_0}$ for $'\mathfrak{D}_{\delta_{t_0}^{(0)}}$ and $'\mathfrak{D}_{\mathfrak{B}}$ for $'\mathfrak{D}_{\mathfrak{B}^{(0)}}$, $'\tilde{\mathfrak{D}}_{\mathfrak{B}}$ for $'\tilde{\mathfrak{D}}_{\mathfrak{B}^{(0)}}$. We use also notations $\partial^\mu T/\partial t_0^\mu$ in place of $T_{\delta_{t_0}^{(\mu)}}$ and T_t for T_{δ_t}. These designations are not so unreasonable, since, for example, if $T = f(t)\, S(x)$ where $f(t) \in \mathfrak{D}^{(\sigma)}(t)$ and $S(x) \in \mathfrak{D}'(x)$ then $T \in {'\mathfrak{D}}_{t^{(\sigma)}}$ and $T_{t_0^{(\sigma)}} = \partial^\sigma f/\partial t_0^\sigma \cdot S(x)$. Using these notations the theorems in § 4 are written in the following way.

Theorem 4'. *The mappings $T \to T_{t_0}$ and $T \to \partial^\mu T/\partial t_0^\mu$ are continuous.*

Theorem 5'. *If* $T \in {'\mathfrak{D}}_{t^{(\mu)}}^{(\pi)}$ then $\partial^\lambda T/\partial t^\lambda \in {'\mathfrak{D}}_{t^{(\mu-\lambda)}}^{(\pi+\lambda)}$ and $\partial^\mu T/\partial t_0^\mu = \partial^{(\mu-\lambda)}(\partial^\lambda T/\partial t^\lambda)/\partial t_0^{(\mu-\lambda)}$ for any $0 \leq \lambda \leq \mu$.

Theorem 7'. *If $T \in {'\tilde{\mathfrak{D}}}_{t^{(\mu)}}$, then the mappings $t_0 \to \partial^\mu T/\partial t_0^\mu$ is continuous.*

Theorem 8. *If for any $t \in \mathfrak{B}$, there corresponds $T_t \in \mathfrak{D}^{(\pi)\prime}(x)$ such that mapping $t \to T_t$ is continuous, we can define $(n+1)$–dimensional distribution $\tilde{T}$ on the interior of $\mathfrak{B}$ by $\langle \tilde{T}, \varphi(x, t) \rangle = \int_{\mathfrak{B}} \langle T_t, \varphi(x, t) \rangle_x \, dt$ where $\langle\ \rangle_x$ means the scaler product between $\mathfrak{D}^\pi(x)$ and $\mathfrak{D}^{(\pi)\prime}(x)$. Then $\tilde{T} \in {'\mathfrak{D}}_{\mathfrak{B}}(\nu)$ for any $\pi \geq \nu \geq 0$, and $\tilde{T}_t = T_t$.*

REMARK. It is evident that if $\nu_1 > \nu_2$, then $\tau_{'\mathfrak{D}_Q}(\nu_2)$ is finer than $\tau_{'\mathfrak{D}_Q}(\nu_1)$ and $'\mathfrak{D}_{Q_\lambda}(\nu_2) \subset {'\mathfrak{D}}_{Q_\lambda}(\nu_1)$. So if we can prove $\tilde{T} \in {'\tilde{\mathfrak{D}}}_{\mathfrak{B}}(\nu = 0)$, it follows $\tilde{T} \in {'\tilde{\mathfrak{D}}}_{\mathfrak{B}}(\nu > 0)$.

Proof. Manifestly $\tilde{T}$ is an additive operator, so we show its continuity on $\mathfrak{D}_{\mathfrak{B}}^{(\pi)}(x, t)$. Now a family $\{T_t \mid t \in \mathfrak{B}\}$ is a bounded set in $\mathfrak{D}^{(\pi)\prime}(x)$, and a family of functions $\{\varphi_t(x) \mid \varphi \in B_\pi(x, t),\ t \in \mathfrak{B}\} \in \mathfrak{B}_\pi(x)$. So there exists a number M such that for any $\varphi \in B_\pi(x, t)$, $|\langle T_t, \varphi_t(x) \rangle| \leq M$, i.e. $|\langle \tilde{T}, \varphi(x, t) \rangle| \leq M(b-a)$, which means continuity. We prove the second and third proposition generally about μ–times continuously differentiable distribution for $\mu \leq \pi$ using the α–sequence. For any $\varphi \in B_\pi(x)$ and for any element ϕ_{tj} of a $(\mu, \tilde{\delta}_t^{(\mu)})$–family we evaluate

$$|\langle \tilde{T}, \varphi\phi \rangle - \langle \tilde{T} * \alpha_k, \varphi\phi \rangle|$$
$$\leq |\langle\langle T_t, \varphi * (\delta - \check{\alpha}_k(x)) \rangle, \phi_{tj} \rangle|$$
$$+ |\langle\langle T_t, \varphi * \check{\alpha}_k \rangle, \phi_{tj} * (\delta(t) - \check{\alpha}_k(t)) \rangle| .$$

In the 2nd term $\varphi * (\delta - \check{\alpha}_k) \overset{(\pi)}{\longrightarrow} 0$ and $f_k(t) = \langle T_t, \varphi * (\delta - \check{\alpha}_k(x)) \rangle$ is a μ-times continuously differentiable function and $\text{Sup}_{t \in \mathfrak{V}} |\partial^\lambda f_k(t))/\partial t^\lambda| \xrightarrow{k \to \infty} 0$ for $0 \leq \lambda \leq \mu$. While we can take a $(\mu, \tilde{\delta}^{(\mu)})$-family $\{\phi_{tj} | t \in \mathfrak{V}, j = 1, 2, \cdots\}$ each of whose carrier is contained in a compact set $\mathfrak{V}$. So the 2nd term is smaller than $\varepsilon/2$ uniformly for j and t. In the 3rd term

$$\text{Sup}_{t \ni \mathfrak{V},\, K=1,2,\cdots} |\partial^\lambda \langle T_t, \varphi\check{\alpha} *_k \rangle / \partial t^\lambda| < M_\lambda, \quad 0 \leq \lambda \leq \mu .$$

On the other hand $\{\phi_{tj} * (\delta - \check{\alpha}_k(t)) | k = 1, 2, \cdots\}$ is a sequence which converges in the topology of $\mathfrak{D}^{(\mu)\prime}$ and $\mathfrak{E}'$. So the 3rd term is $< \varepsilon/2$ uniformly for j and t_0, and $\tilde{T} \in \bigcup_{\lambda=0}^{\mu} {}'\tilde{\mathfrak{D}}_{\mathfrak{V}(\lambda)}$ (where $\nu = \mu_1 = 0, 1, \cdots \mu$; μ_1 means μ in § 2). The last evaluation is done by taking a sequence $\phi_j \overset{(0)'}{\longrightarrow} \delta_{t_0}$, $\phi_j \in \mathfrak{D}(t)$

$$\lim_{j \to \infty} \langle \tilde{T}, (-1)^\mu \varphi \phi_j^{(\mu)} \rangle = \lim_{j \to \infty} \langle\langle T_t, \varphi \rangle, (-1)^\mu \phi_j^{(\mu)} \rangle$$
$$= \lim_{j \to \infty} \langle \partial^\mu \langle T_t, \varphi \rangle / \partial t^\mu, \phi_j \rangle = \langle T_{t_0}^{(\mu)}, \varphi \rangle ,$$

where $T_{t_0}^{(\mu)}$ means μ-th parametric derivative of T.

From this proof we see also that the following theorem holds.

Theorem 9. *If a parametric distribution T_t is μ-times continuously differentiable with respect t on $\mathfrak{V}$, then $\tilde{T}$ which is defined in Theorem 8 belongs to the space $\bigcap_{\rho=0}^{\mu} {}'\tilde{\mathfrak{D}}_{\mathfrak{V}(\rho)}$ and its μ-th parametric derivative $T_t^{(\mu)}$ is equal to $\partial^\mu T/\partial t_0^\mu$ or $(\partial^\mu \tilde{T}/\partial t^\mu)_{t_0}$ on $\mathfrak{V}$.*

Theorem 10. *If $T \in {}'\tilde{\mathfrak{D}}_{\mathfrak{V}}$ and $\tilde{T}$ is constructed from T_t on $\mathfrak{V}$ by Theorem 8, then $T = \tilde{T}$ on $\mathfrak{V}$.*

Proof. Since $\mathfrak{D}^\pi(x) \otimes \mathfrak{D}^\pi(t)$ is dense in the topology of $\mathfrak{D}_{\mathfrak{V}}$ in $\mathfrak{D}^\pi(x, t)$, we have only to prove $\langle T, u(x)v(t) \rangle = \langle \tilde{T}, u(x)v(t) \rangle$ for $u(x)v(t)$.

Now

$$|\langle T, uv \rangle - \langle \tilde{T}, uv \rangle| = |\langle T, uv \rangle - \langle\langle T, u\delta_{t_0} \rangle, v(t_0) \rangle|$$
$$\leq |\langle T, uv \rangle - \langle\langle T, u\phi_{tj} \rangle, v(t) \rangle|$$
$$+ |\langle\langle T, u\phi_{tj} \rangle, v(t) \rangle - \langle\langle T, u\delta_{t_0} \rangle, v(t_0) \rangle|$$

If we take $\phi_j \overset{(\nu)'}{\longrightarrow} \delta_0$ then $\{\tau_t\phi_j | j = 1, 2, \cdots, t \in \mathfrak{V}\}$ is a $(\nu, \widetilde{\tau_t \delta})$-family. So there exists j_0 such that for any $j > j_0$, $|\langle T, u\phi_{tj} \rangle - \langle T, u\delta_t \rangle| < \varepsilon/2M$ by Corollary of Theorem 2. If we take M such that $\text{Max} |v(t)| \leq M/(b-a)$

then the 3rd term is $<\varepsilon/2$. The 2nd term is smaller than

$$|\langle T, uv\rangle - \langle f, uv\rangle| + |\langle f, uv\rangle - \langle\langle f, u\phi_{tj}\rangle, v(t)\rangle| \\ + |\langle\langle f, u\phi_{tj}\rangle, v(t)\rangle - \langle\langle T, u\phi_{tj}\rangle, v(t)\rangle|.$$

If we take $\mathfrak{F} \xrightarrow{'\tilde{\mathfrak{D}}_{\mathfrak{B}}} T$ then the 1st term is $<\varepsilon/6$ and $|\langle f, u\phi_{tj}\rangle - \langle T, u\phi_{tj}\rangle| < \varepsilon/6M$ uniformly for t, j. For such an f we can take j such that the 2nd term $|\langle f, u\rangle - \langle f, u\phi_{tj}\rangle| < \varepsilon/6M$ uniformly for t. So we have $|\langle \tilde{T}, uv\rangle - \langle T, uv\rangle| < \varepsilon$, q. e. d.

Theorem 11. (THE CONVERSE OF THEOREM 10)

If $T \in {}'\tilde{\mathfrak{D}}_{\mathfrak{B}} \cap \bigcap_{\rho=1}^{\mu} {}'\tilde{\mathfrak{D}}_{\mathfrak{B}^{(\rho)}(\nu=\rho+1)}$ then the mapping $t \to T_t$ is μ-times continuously differentiable from $\mathfrak{B}$ to $\mathfrak{D}^{(\pi)\prime}(x)$, and its μ-th parametric derivative $T^{(\mu)}$ equals $\partial^{\mu}T/\partial t_0^{\mu}$.

Proof. We take a sequence $\{\phi_j\}$ such that $\phi_j \xrightarrow{(\mu-1)'} \delta^{(\mu-1)}$. Then $\{\tau_t\phi_j | j, t\}$ is a $(\mu-1, \tau_t\delta^{(\mu-1)})$-family where τ means a shift. So we have $|\langle T, u\tau_{-\Delta t}\phi_j\rangle - \langle T, u\tau_{-\Delta t}\delta_{t_0}^{(\mu-1)}\rangle| < \varepsilon$ uniformly for Δt where $\{\Delta t | t_0 + \Delta t \in \mathfrak{B}\}$. So for $\xi \neq 0$, there exists a $j_1(\xi)$ such that for any $j > j_1(\xi)$ and for any Δt with $|\Delta t| \leq |\xi|$,

$$|\langle T, u\tau_{-\Delta t}\phi_j\rangle - \langle T, u\tau_{-\Delta t}\delta_{t_0}^{(\mu-1)}\rangle| < \varepsilon|\xi|.$$

In the next place we can say $\lim_{\Delta t\to 0,\, j\to\infty}(\tau_{-\Delta t}\phi_j - \phi_j)/\Delta t \xrightarrow{(\mu+1)'} \delta^{(\mu)}$ as follows. For any $\varphi \in B_{(\mu+1)}$, we evaluate

$$|\langle\{\tau_{-\Delta t}\phi_j - \phi_j\}/\Delta t, \varphi\rangle - \langle\delta^{(\mu)}, \varphi\rangle| \\ \leq |\langle\phi_j, \{\varphi(t-\Delta t) - \varphi(t)\}/\Delta t\rangle - \langle\delta^{(\mu-1)}, \{\varphi(t-\Delta t) - \varphi(t)\}/\Delta t\rangle\} \\ + |\langle\delta^{(\mu-1)}, \{\varphi(t-\Delta t) - \varphi(t)\}/\Delta t\rangle - \langle\delta^{(\mu-1)}, -\varphi'\rangle|.$$

In the 2nd term a set B;

$$B = \{\{\varphi(t-\Delta t) - \varphi(t)\}/\Delta t = \psi_{\Delta t} \text{ and } -\varphi'(t) | \varphi \in B_{\mathfrak{D}^{(\mu+1)}}\},$$

is a bounded set in $B_{(\mu-1)}$. So if $\phi_j \xrightarrow{(\mu-1)'} \delta^{(\mu-1)}$, then there exists j_2 such that for any $j > j_2$,

$$|\langle\phi_j, \psi_{\Delta t}\rangle - \langle\delta^{(\mu-1)}, \psi_{\Delta t}\rangle| < \varepsilon/2 \quad \text{uniformly for } B_{(\mu+1)}, \Delta t.$$

The 3rd term is equal to

$$|\varphi^{(\mu)}(t_0 - \theta\Delta t) - \varphi^{(\mu)}(t_0)| \leq |\varphi^{(\mu+1)}(t_0 - \theta'\Delta t)|\,|\theta\Delta t|$$

where $0 < \theta, \theta', < 1$.

Since $\varphi \in B_{(\mu+1)}$, we have $\xi_0 > 0$ such that for any $|\Delta t| \leq \xi_0$ the 3rd term is $<\varepsilon/2$ uniformly for $\varphi \in B_{(\mu+1)}$. So there exists ξ_0 and j_2 such that for any $j > j_2$ and $|\Delta t| \leq \xi_0$ we have

$$|\langle \{\tau_{-\Delta t}\phi_j - \phi_j\}/\Delta t, \varphi\rangle - \langle \delta^{\mu}, \varphi\rangle| < \varepsilon,$$

and we obtain (1).

Now putting Max $(j_2, j_1(\xi)) = j_0(\xi)$ for $|\xi| \leq \xi_0$, we evaluate

$$\begin{aligned}&|\langle T, u\delta^{(\mu)}\rangle - \langle \{T^{(\mu-1)}_{t+\Delta t} - T^{(\mu-1)}_{t}\}/\Delta t, u\rangle| \\ &\leq |\langle T, u\delta^{(\mu)}\rangle - \langle T, u\{\tau_{-\Delta t}\phi_j - \phi_j\}/\Delta t\rangle| \\ &+ |\langle T, u\{\tau_{-\Delta t}\phi_j - \tau_{-\Delta t}\delta^{(\mu-1)}\}/\Delta t\rangle - \langle T, u\{\phi_j - \delta^{(\mu-1)}\}/\Delta t\rangle|.\end{aligned}$$

For any $\varepsilon > 0$ there exists ξ_0 such that for any $j > j_0(\xi_0)$ and $|\Delta t| < |\xi_0|$ the second term is $< \varepsilon/2$. Now for any Δt with $|\Delta t| < |\xi_0|$, if we take a ϕ_j with $j > j_0(\Delta t)$, we can make the 3rd term smaller than $\varepsilon/2$.

REMARK. We have assumed $\nu = \rho + 1$ in the space $'\tilde{\mathfrak{D}}_{\mathfrak{V}(\rho)}$ in this theorem. As the proof shows this condition can be weakened. That is to say, Theorem is also true for $T \in '\tilde{\mathfrak{D}}_{\mathfrak{V}} \cap '\mathfrak{D}^{\pi}_{L}$ where $'\mathfrak{D}^{\pi}_{L}$ is the dual space of the $\mathfrak{D}^{\pi}_{L}$ whose topology is induced by the bounded set defined by the boundedness of the difference quotient of ρ-th differential coefficient in place of by the bounded set defined by $(\rho+1, \tilde{\delta}')$—family. However it will not be sufficient to assume $\nu = \rho$, since $\{\tau_{-h}\delta - \delta\}/h \xrightarrow{(2)'} \delta'$ but not $\xrightarrow{(1)'} \delta'$.

§ 6. Application to the distributional differential equation of evolution.

L. Schwartz ([3]) treated the parametric equation of evolution of the following type.

$$(1) \qquad \partial U(x, t)/\partial t + \textstyle\sum_{|\rho| \leq \sigma} A_{\rho}(t) D^{\rho}_{x} U(x, t) = B(x, t),$$

where $A_{\rho}(t)$ is a function of $\mathfrak{D}^{(\pi+\rho)}$ and $B(x, t)$ is a continuous parametric distribution. $L^{\rho}(x)$ means a differential operator from the space $\mathfrak{D}^{(\pi)'}(x)$ to $\mathfrak{D}^{)\pi+\rho)'}(x)$ and B, A, U are all matrices.

We consider the corresponding proper distributional (in $\mathfrak{D}^{(\pi)'}(x, t)$) equation of this type and its proper distributional solution. (Initial condition on $t = t_0$ is given in the space $'\mathfrak{D}^{\pi}_{t_0}$).

Theorem 12. *If as a mapping* $t \to \mathfrak{D}^{(\pi)'}(x)$ *for* $\pi \geq 1$ *a parametric continuously differentiable distribution* $U(x, t)$ *satisfes parametric equation* (1) *under the above mentioned condition, then* $\tilde{U}$ *satisfies the corresponding proper distributional equation,* i.e.

$$\partial \tilde{U}(x, t)/\partial t + \textstyle\sum_{|\rho| \leq \sigma} A_{\rho}(t) D^{\rho}_{x} \tilde{U}(x, t) = \tilde{B}(x, t).$$

Proof. By Theorems 8 and 9, $\tilde{U}\in{}'\tilde{\mathfrak{D}}^{\pi}_{\mathfrak{V}}\frown{}'\tilde{\mathfrak{D}}^{\pi}_{\mathfrak{V}^{(1)}}$, and $\tilde{U}^{(1)}_{t_0}(x)=(\partial\tilde{U}/\partial t)_{t_0}$ where subscript (1) of $\tilde{U}$ means parametric derivative. By Theorem 8, $\tilde{B}(x,t)\in{}'\tilde{\mathfrak{D}}_{\mathfrak{V}}$ and $\tilde{B}_{t_0}(X)=B_{t_0}(X)$. By the Corollary of Lemma 11 and Lemma 12, $\sum A_\rho(x,t)D^\rho_x\tilde{U}(x,t)\in{}'\tilde{\mathfrak{D}}^{(\sigma+\pi)}_{\mathfrak{V}}$, and $\sum A_\rho(t)D^\rho_x U_t(x)=(\sum A_\rho(t)D^\rho_x\tilde{U}(x,t))_t$, where D^ρ_x in the left hand side of equality means differential operator from $\mathfrak{D}^{(\pi)\prime}(x)$ to $\mathfrak{D}^{(\pi+\sigma)\prime}(x)$ and D^ρ_x in the right hand side means differential operator of the same form from $\mathfrak{D}^{(\pi)\prime}(x,t)$ to $\mathfrak{D}^{(\pi+\sigma)\prime}(x,t)$.

Now we can rewrite parametric equation (1) as a proper equation of multiple distributions by $\delta(t)$, i.e.,

$$(\partial\tilde{U}(x,t)/\partial t+\sum A_\rho D^\rho_x\tilde{U}(x,t))_t=(\tilde{B}(x,t))_t$$

for any $t\in\mathfrak{V}$. So if we take $\sim$ on both side we obtain a proper equation in $\mathfrak{D}^{(\pi+\sigma)\prime}(x,t)$, $\partial\tilde{U}/\partial t+\sum A_\rho D^\rho_x\tilde{U}=\tilde{B}$, by Theorem 11.

Conversely the following theorem holds.

Theorem 13. *If a proper equation* (1) *is given, and the proper solution* $U(x,t)$ *belongs to* ${}'\tilde{\mathfrak{D}}_{\mathfrak{V}}\frown{}'\tilde{\mathfrak{D}}_{\mathfrak{V}'(\nu=2)}$ *for* $\pi\geq 2$, *then* $U_t(x)$ *satisfies the corresponding parametric equation.*

Proof. Manifestly ${}'\tilde{\mathfrak{D}}_{\mathfrak{V}'(\nu=2)}\subset{}'\tilde{\mathfrak{D}}^{\pi}_{\mathfrak{V}'(\nu=1)}$. So $\partial U/\partial t\in{}'\tilde{\mathfrak{D}}^{\pi}_{\mathfrak{V}}$ by Theorem 7′. It holds also that $\sum A_\rho(t)D^\rho_x U(x,t)\in\tilde{\mathfrak{D}}^{\pi+\sigma}_{\mathfrak{V}}$ by Corollary of Lemma 11 and Lemma 12. Therefore we can take the multiple distribution by δ_t of the distribution of both hand sides of the equation

$$(\partial U/\partial t)_t+(\sum A_\rho(t)D^\rho_x U(x,t))_t=(B(x,t))_t\,.$$

Since $U\in{}'\tilde{\mathfrak{D}}_{\mathfrak{V}'(\nu=2)}\frown{}'\tilde{\mathfrak{D}}_{\mathfrak{V}}$, $(\partial U/\partial t)_t$ equals parametric derivative by Theorem 11, and the second term equals $\sum A_\rho(t)D^\rho_x U_t(x)$ where D^ρ_x means an operator from $\mathfrak{D}^{(\pi)\prime}(x)$ to $\mathfrak{D}^{(\pi+\sigma)\prime}(x)$ and the third term equals $B_t(x)$. So this is itself a parametric equation whose solution is $U_t(x)$, q.e.d.

(Received September 1, 1954)

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