Title	Logarithmic Sobolev inequalities for differentiable paths
Author(s)	Léandre, Rémi
Citation	Osaka Journal of Mathematics. 2000, 37(1), p. 139-145
Version Type	VoR
URL	https://doi.org/10.18910/11551
rights	
Note	

Léandre, R.
Osaka J. Math.
37 (2000), 139 – 145

LOGARITHMIC SOBOLEV INEQUALITIES FOR DIFFERENTIABLE PATHS

RÉMI LÉANDRE

(Received March 20, 1998)

1. Introduction

Measures over differentiable paths can be interesting in two ways:

-)The spin representation of a loop groop is very well understood for C^1 loops ([25]).

-)There is a Morse theory theory for C^1 loops, which is involved with the energy functional (See [11] and the references therein).

For the first of these two reasons, there exist measures which are introduced for differentiable paths or differentiable loops of a Lie group. In the case of a loop group, they are introduced in [20] or in [21], in order to understand what is a string structure over the Brownian bridge. Let us recall that string structures over the Brownian bridge are very important in order to define the Dirac operator over the loop space (See [26], [29], [14]).

In the case of an analytical approach of the Morse theory related to the symplectic action, we can work with the measure defined over the loop space by the classical Brownian bridge (See [19], [10]). But if the function over the loop space is the energy functional (See [11]), we need to introduce a new measure, which is involved with differentiable paths. It is the subject of [22].

In [22], we introduced a measure over the path space. We give integration by parts formulas for differentiable paths, and we establish a Sobolev Calculus, which is in the spirit of [13] and [17], [18]. The integration by parts formulas are very badly written in terms of the tangent vector fields which are considered in [22]. In [17] or in [6], there is the remark that if we follow Bismut's indication to writte the shape of the tangent vector fields ([2]), the integration by parts formulas are better written.

In the first part, we consider the measure of [22] over the space of differentiable paths, and we establish a Bismut Calculus associated to it. The main result is the following: the transformation which gives a vector field in the manner of [13] or in the manner of [22] into a vector into a vector field of Bismut is path by path bounded, but not uniformly bounded. In order to control this transformation, we have to assume that a quantity of the form $exp[I(\gamma)]$ ($I(\gamma)$ is the square of the supremum of the modulus of the speed of the path) is in L^2. It is possible that this assumption is satisfied if we introduce a small parameter, which leads to a formalism analogous to small time asymptotics of heat kernels. (The reader who is interested by short time asymptotics expansions can see the surveys of Kusuoka ([15]), Léandre ([16]) or Watanabe ([27])). If this integrability condition is checked, or in other words, if the parameter of the equation which gives the measure is small enough, the cylindrical functionals belong to Bismut's Sobolev spaces with one derivative in L^2.

In the second part of this work, we show that there is a Clark-Ocone formula in this context, associated to Bismut's derivatives of a functional, when we have fixed the speed of the path in the starting time. The proof follows the ideas of [6].

This allows us, by following the method of [3] to deduce a Logarithmic Sobolev inequality over the differentiable path space, if we suppose given the speed of the path at the starting time. For that, we take the tangent space of [22]. But this gives a weighted Logarithmic Sobolev inequality, because the transformation which gives a tangent vector of [22] into a Bismut tangent vector is not uniformly bounded.

We thank S. Fang for helpfull discussions.

2. Bismut Calculus for Differentiable Paths

Let M be a compact Riemannian manifold. We endow it with the Levi-Civita connection. Let B_s be a Brownian motion over $T_x(M)$ starting from x. We consider the stochastic differential equation:

$$\begin{aligned} d\gamma_s &= \tau_s \epsilon (C + B_s) ds \\ \gamma_0 &= x \end{aligned} \tag{1.1}$$

ϵ is a small parameter. C follows a Gaussian law of covariance Id and average 0 over $T_x(M)$, independent of B_s. τ_s is the parrallel transport over the path γ_s, starting from Id.

[22] has introduced the following tangent space of a path $\gamma_.$: it is constituted of the path $\tau_s H_s$, where H_s has two derivatives with $H_0 = 0$. [22] considers the Hilbert structure $\|H_0'\|^2 + \int_0^1 \|d^2/ds^2 H\|^2 ds$. Let us recall one of the main theorem of [22]. Let F be a cylindrical functional over $P_x(M)$, the based path space of C^1 differentiables paths over M.

We get:

$$E[< dF, X >] = E[F div X] \tag{1.2}$$

if $X_t = \tau_t H_t$ where H_t is deterministic. The expression of $div X$ is given in [22], and is slightly complicated:

$$(1.3)\qquad \begin{aligned} divX &= -\int_0^1 < \tau_s^{-1}R(d/ds\gamma_s, X_s)\frac{d/ds\gamma_s}{\epsilon}, \delta B_s > \\ &+\int_0^1 \frac{< d^2/ds^2H_s, \delta B_s >}{\epsilon} + \frac{< d/dsH_0', C >}{\epsilon} \end{aligned}$$

R is the curvature tensor of the manifold. In order to simplify the expression of the divergence, we give the notion of Bismut's tangent vector field.

Let d/dsH_0 and d^2/ds^2K_s be deterministic given. A Bismut's vector field X_t^B is given by $X_t^B = \tau_t H_t^B$ where H_t^B is the solution of the differential equation:

$$(1.4)\qquad \begin{aligned} H_0^B &= 0 \\ d/dsH_0^B &= \epsilon H_0 \\ d^2/ds^2H_t^B &= \epsilon d^2/ds^2K_t + \tau_t^{-1}R(d/ds\gamma_t, X_t^B)d/ds\gamma_t \end{aligned}$$

By using the Gronwall lemma, we get that

$$(1.5)\qquad \begin{aligned} (\int_0^1 \|d^2/ds^2H_t^B\|^2 dt)^{\frac{1}{2}} &\leq C((\int_0^1 \|d^2/ds^2K\|^2 ds)^{\frac{1}{2}} \\ &+\|d/dsH_0\|^2 + 1)exp[C\sup \|d/ds\gamma_t\|^2] \end{aligned}$$

We can choose ϵ small enough such that for a vector field $d/dsH_0 + \int_0^s d^2/ds^2K_u du$ adapted in L^2, X_t^B is still adapted in L^2.

We define a tangent vector as $X_t^B(d/dsH_0, d^2/ds^2K)$ associated to the solution of (1.4).

We get:

Theorem I. 1. *Let ϵ be small enough. Let d/dsH_0 and let d^2/ds^2K be deterministic. We get for all cylindrical functionals*:

$$(1.6)\qquad \begin{aligned} E[< dF, X^B(d/dsH_0, d^2/ds^2K.) >] &= E[< d/dsH_0, \frac{d/ds\gamma_0}{\epsilon} > \\ &+\int_0^1 < \tau_s d^2/ds^2K, \frac{\delta\nabla d/ds\gamma_s}{\epsilon} >] \end{aligned}$$

$\delta\nabla d/ds\gamma_s$ is the formal acceleration of γ_s given by $\epsilon\tau_s\delta B_s$ and δ is the Itô integral.

This integration by parts formula leads to a first order Sobolev Calculus, called Bismut's Sobolev Calculus, which is not equivalent to the Sobolev Calculus developped in [22]. Namely the transformation $(d/dsH_0, d^2/ds^2K_.) \to X^B_.(d/dsH_0, d^2/ds^2K_.)$ is not bounded in γ.

For a cylindrical functional, we put:

$$(1.7)\qquad \begin{aligned} < dF, X^B(d/dsH_0, d^2/ds^2K_.) > &= < A, d/dsH_0 > \\ &+ \int_0^1 < C, d^2/ds^2K_s > ds \end{aligned}$$

in a unique way. We call $A = dF^{0,B}$ and $C_s = dF^B_s$ such that:

$$(1.8)\qquad \begin{aligned} < dF, X^B(d/dsH_0, d^2/ds^2K) > &= < dF^{0,B}, d/dsH_0 > \\ &+ \int_0^1 < dF^B_s, d^2/ds^2K_s > ds \end{aligned}$$

We get:

DEFINITION I. 2: The first order Bismut Sobolev norm $W_{1,2}$ are given by:

$$(1.9)\qquad ||F||_{1,2} = ||F||_{L^2} + ||\ ||dF^{0,B}||\ ||_{L^2} + ||(\int_0^1 ||dF^B_s||^2 ds)^{\frac{1}{2}}||_{L^2}$$

From the estimate (1.5), we get:

Theorem I. 3. *If ϵ is small enough, $||F||_{1,2}$ is finite for any cylindrical functional F.*

3. Clark-Ocone Formula

We fix C in the equation (1.2). We get a Sobolev Calculus as in the previous part with C fixed, or in other words, if $d/ds\gamma_0$ fixed. This implies that in the new Sobolev Calculus that $H'_0 = 0$. We suppose ϵ small enough in order that the exponential which appears in (1.5) is in L^2. We do in the sequel as if $\epsilon = 1$, in order to simplify the formulas.

Let F be a cylindrical functional. Since C is supposed fixed, we have for a suitable vector fields $Z_s(F)$ adapted to the filtration generated by the process $B_.$,

$$(2.1)\qquad F = E[F] + \int_0^1 < \tau_s Z_s(F), \delta\nabla d/ds\gamma_s >$$

Let Z_s be another adapted vector fields. We have:

$$(2.2)\quad \begin{aligned} E[\int_0^1 < dF_s^B, Z_s > ds] &= E[F\int_0^1 < \tau_s Z_s, \delta\nabla d/ds\gamma_s >] \\ &= E[\int_0^1 < \tau_s Z_s(F), \delta\nabla d/ds\gamma_s > \\ &\qquad \int_0^1 < \tau_s Z_s, \delta\nabla d/ds\gamma_s >] \\ &= E[\int_0^1 Z_s(F) Z_s ds] \end{aligned}$$

Moreover, we get the following Clark-Ocone formula (See [24], [6]):

Theorem II. 1.

$$(2.3)\qquad F = E[F] + \int_0^1 < \tau_s E^{G_s}[dF_s^B], \delta\nabla d/ds\gamma_s >$$

where G_s is the filtration spanned by B_s.

4. Logarithmic Sobolev Inequalities

We follow exactly the method of Capitaine-Hsu-Ledoux ([3]), with C fixed. We do as if ϵ was small enough.

We underline that there are the Bismut tangent space and the tangent space of [22]. This leads to an H-derivative in the sense of Bismut:

$$(3.1)\qquad < dF, X^B(d^2/ds^2K_.) >= \int_0^1 < dF_s^B, d^2/ds^2K_s > ds$$

and to an H-derivative in the sense of [22]:

$$(3.2)\qquad < dF, X(d^2/ds^2K_.) >= \int_0^1 < dF_s^{J.L}, d^2/ds^2K_s > ds$$

where $X(d^2/ds^2K_.) = \tau_t(\int_0^t\int_0^s d^2/ds^2K_u dudv)$.

Moreover by the estimates (1.5), we get:

$$(3.3)\qquad \int_0^1 ||dF_s^B||^2 ds \leq C\exp[\lambda \sup ||B_s||^2]\int_0^1 ||dF_s^{J.L}||^2 ds$$

for λ small enough, by the assumption which is done.

The method of Capitaine-Hsu-Ledoux shows that:

$$E[F^2 \mathrm{Log} F^2] \leq E[F^2]\mathrm{Log} E[F^2] + 2E[\int_0^1 ||dF_s^B||^2 ds] \tag{3.4}$$

Let be $I(\gamma) = \sup ||d/ds\gamma_t||^2$. Since $\exp[\lambda \sup ||B_s||^2] \leq C\exp[\lambda' I(\gamma)]$, we deduce the following Logarithmic Sobolev inequality:

Theorem III. 1.

$$E[F^2 \mathrm{Log} F^2] \leq E[F^2]\mathrm{Log} E[F^2] + CE[\exp[\lambda' I(\gamma)] \int_0^1 ||dF_s^{J.L}||^2 ds] \tag{3.5}$$

where the exponential is in L^2 for ϵ small enough.

References

[1] S. Aida: *Logarithmic Sobolev inequalities on loop spaces over Riemannian manifolds*, In "Stochastic Analysis and applications", Davies I.M. Elworthy K.D. Truman A. edit. World Scientific, Singapore. (1996), 1–20.

[2] J. M. Bismut: Large deviations and the Malliavin Calculus, Progress in Math. **45**, Birkhauser, Basel, 1984.

[3] M. Capitaine E. P. Hsu M. Ledoux: *Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces*, Preprint.

[4] B. K. Driver T. Lohrentz: *Logarithmic Sobolev inequalities for pinned loop groups*, J.F.A. **140** (1996), 381-448.

[5] S. Fang: *Girsanov theorem and quasi-invariance on the path space over loop groups*, Preprint.

[6] S. Fang P. Malliavin: *Stochastic analysis on the path space of a Riemannian manifold*, J.F.A. **118** (1993), 249-274.

[7] E. Getzler: *An extension of Gross's log Sobolev inequality for the loop space of a compact Lie group*, In "Probabi. model in Math. Phys." Morrow G. edit. World Scientific Singapore, (1991), 73-97.

[8] L. Gross: *Potential theory on Hilbert spaces*, J.F.A. **1** (1967), 123-181.

[9] L. Gross: *Logarithmic Sobolev inequality on loop group*, J.F.A. **102** (1991), 268-313.

[10] J. M. Guilarte: *The supersymmetric sigma model, topological mechanics and knot invariants*, Jour. Geometry. Physics, **7** (1990), 255-302.

[11] J. M. Guilarte: *Sphalerons and instantons in two dimensional field theory*, Preprint.

[12] E. P. Hsu: *Logarithmic Sobolev inequalities on path spaces over Riemannian manifolds*, C.R.A.S. Série I. **320** (1995), 1009-1014.

[13] J. Jones R. Léandre : *L^p Chen forms over loop spaces*, In "Stochastic Analysis", Barlow M. Bingham N. edit. Cambridge University Press, Cambridge, (1991), 104-162.

[14] J. Jones R. Léandre: *A stochastic approach to the Dirac operator over the free loop space*, In "Loop spaces and groups of diffeomorphisms", Proceedings of The Steklov Institute **217** (1997), 253-282.

[15] S. Kusuoka: *More recent theory of Malliavin Calculus*, Sugaku **5.2** (1992), 155-173.

[16] R. Léandre: *Applications quantitatives et qualitatives du Calcul de Malliavin*, "Colloque Franco-Japonais", Métivier M. Watanabe S. edit. L.N.M. **1322**. Heidelberg, (1988), 109-133. English translation: "Geometry of random motion" Durrett R Pinsky edit. Contemporary math. **73** (1988), 173-197.

[17] Léandre R.: *Integration by parts formulas and rotationally invariant Sobolev Calculus on the free loop space*, "XXVII Winter School of theoretical physic", Gielerak R. Borowiec A. edit. J. of Geometry and Physics, **11**, (1993), 517-528.

[18] R. Léandre: *Invariant Sobolev Calculus on the free loop space*, Acta Applicandae Mathematicae, **46** (1997), 267-350.

[19] R. Léandre: *Stochastic Wess-Zumino-Witten model over a symplectic manifold*, Journal of Geometry and Physics, **21** (1997), 307-336.

[20] R. Léandre: *Hilbert space of spinors fields over the free loop space*, Reviews in Mathematical physics, **9.2** (1997), 243-277.

[21] R. Léandre: *Stochastic gauge transform of the string bundle*, Journal of Geometry and Physics, **26** (1998), 1-25.

[22] R. Léandre: *Stochastic Wess-Zumino-Witten model for the measure of Kontsevitch*, To be published in "Stochastic Analysis", Dalang R. Dozzi M. Russo F. edit.

[23] P. Malliavin: Stochastic analysis, Grundleheren der math. Wissen. Springer, Heidelberg, 1997.

[24] D. Nualart: Malliavin Calculus and related topics, Springer, Heidelberg, 1995.

[25] A. Pressley G. Segal: Loop groups, Oxford University Press, Oxford, 1986.

[26] G. Segal: *Elliptic cohomology (after Landweber, Stong, Ochanine, Witten and others)*, Séminaire Bourbaki 1987-1988. n 695. Astérisque **161-162** (1988), 187-201.

[27] S. Watanabe: *Stochastic analysis and its applications*, Sugaku **5.1** (1992), 51-71.

[28] Ed. Witten: *Supersymmetry and Morse theory*, J. Dif. Geometry **17** (1982), 661-692.

[29] Ed. Witten: *The index of Dirac operators in loop space*, In " Elliptic curves and modular forms in Algebraic topology", Landweber P.S. edit. L.N.M. **1326.** Springer, Heidelberg, (1988), 161-181.

département de Mathématiques
Institut Elie Cartan
Faculté des Sciences
Université de Nancy I
54000. Vandoeuvre-les-Nancy
FRANCE