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ON J_R-HOMOMORPHISMS

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1. Introduction

In [8] Snaith proved the Adams conjecture for suspension spaces. In this paper we shall prove an analogous result to Snaith's theorem ([8], Corollary 5.2) for the Real Adams operation ψ^3 and a Real J-map J_R (see §2). This is proved by using the results of Seymour [7]. And as an application we shall determine an undecided order in the theorem of [6].

Here we shall inherit the notations and terminologies in [2], §1 and [6].

2. Homomorphism J_R

In [6] we defined the homomorphisms $J_{R,n}$ and J_R for doubly indexed suspension spaces $\Sigma^{p,q}X$, $p\geqq 0$ and $q\geqq 1$. Clearly, these definitions are also valid for any finite pointed τ-complex. But the natural map obtained in this manner

$$J_R\colon \widetilde{KR}^{-1}(X)\to \pi_s^{0,0}(X)$$

is not a homomorphism in general. As in the usual case we see that this map satisfies the following formula:

$$J_R(\alpha+\beta)=J_R(\alpha)+J_R(\beta)+J_R(\alpha)J_R(\beta) \qquad \alpha,\beta\in\widetilde{KR}^{-1}(X)$$

where ab $(a, b\in\pi_s^{0,0}(X))$ denotes the product of a and b induced by the loop composition in $\Omega^{n,n}\Sigma^{n,n}$ (cf. [9],p. 314).

3. Adams operation ψ^3 in KR-theory

In this section we recall the construction of the Real Adams operation ψ_R^3 described in [7], §4.

Let S_3 be the symmetric group with two generators a, b satisfying

$$a^3=b^2=1,\ bab=a^2$$

and let Z_3 be the cyclic subgroup of S_3 generated by a. From the above relations we see that $\tau(a)=a^2$, $\tau(b)=b$ induces an automorphic involution τ on

S_3. Z_3 is closed under the involution τ. Therefore S_3 (resp. Z_3) is regarded as a Real group with the involution τ (resp. its restriction to Z_3) in the sense of Atiyah-Segal [4].

We know that all simple S_3- and Z_3-modules over $\boldsymbol{C}$ are as follows:

$$\begin{aligned} (3.1) \qquad & S_3\colon \tilde{1} = \{\boldsymbol{C} \mid a=1, b=1\}, \ \tilde{M} = \{\boldsymbol{C} \mid a=1, b=-1\}, \\ & \tilde{M}_1 = \{\boldsymbol{C}^2 \mid \ av = Av, \ bv = Bv, \ v \in \boldsymbol{C}^2\} \\ & Z_3\colon 1 = \{\boldsymbol{C} \mid a=1\}, \ M_1 = \{\boldsymbol{C} \mid av = \zeta v, \ v \in \boldsymbol{C}\}, \\ & M_2 = \{\boldsymbol{C} \mid av = \zeta^2 v, \ v \in \boldsymbol{C}\} \end{aligned}$$

where $A=\begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in GL(2, \boldsymbol{C})$ and $\zeta = \exp(2\pi i/3)$ (see, e.g., [5], §32).

Let G denote either S_3 or Z_3. Clearly each G-module listed above is a Real G-module with the conjugate linear involution induced by complex conjugation. This fact shows that the forgetful map $R_R(G) \to R(G)$, which is injective in general, is surjective where $R_R(G)$ is the Grothendieck group of Real G-modules and $R(G)$ is the complex representation ring of G.

Let X be a Real space with trivial G-action and $F \to X$ be a Real G-vector bundle in the sense of [4], §6. Then we see easily that the decomposition of F as a complex G-vector bundle ([4], §8)

$$(3.2) \qquad \bigoplus_{M} \mathrm{Hom}^G(\underline{M}, F) \otimes \underline{M} \xrightarrow{\cong} F$$

becomes an isomorphism of Real G-vector bundles. Here M runs through the simple G-modules over $\boldsymbol{C}$ and $\underline{M}$ denotes the product bundle $M \times X$ over X. And so we see that (3.2) induces a natural isomorphism $KR_G(X) \cong KR(X) \otimes R(G)$.

Let $E \to X$ be a Real vector bundle over X with the involution $\tau_E\colon E \to E$. We define a Real structure $\tilde{\tau}_E$ on $E^{\otimes 3}$ by $\tilde{\tau}_E = (1 \otimes t)\tau_E^{\otimes 3}$ where $t\colon E^{\otimes 2} \to E^{\otimes 2}$ is the switching map. Then $E^{\otimes 3}$ becomes a Real S_3-vector bundle with the S_3-action permuting the factors.

Applying (3.2) to $E^{\otimes 3}$ we have an isomorphism of Real S_3-vector bundles

$$\begin{aligned} (3.3) \qquad E^{\otimes 3} \cong & \mathrm{Hom}^{S_3}(\underline{\tilde{1}}, E^{\otimes 3}) \otimes \underline{\tilde{1}} \oplus \mathrm{Hom}^{S_3}(\underline{\tilde{M}}, E^{\otimes 3}) \otimes \underline{\tilde{M}} \\ & \oplus \mathrm{Hom}^{S_3}(\underline{\tilde{M}_1}, E^{\otimes 3}) \otimes \underline{\tilde{M}_1} \end{aligned}$$

with the notations of (3.1). And by (3.3), as a Real Z_3-vector bundle we obtain

$$\begin{aligned} (3.4) \qquad E^{\otimes 3} \cong & (\mathrm{Hom}^{S_3}(\underline{\tilde{1}}, E^{\otimes 3}) \oplus \mathrm{Hom}^{S_3}(\underline{\tilde{M}}, E^{\otimes 3})) \otimes \underline{1} \\ & \oplus \mathrm{Hom}^{S_3}(\underline{\tilde{M}_1}, E^{\otimes 3}) \otimes (\underline{M}_1 \oplus \underline{M}_2). \end{aligned}$$

Put

$$\begin{aligned} V_0 &= \mathrm{Hom}^{S_3}(\underline{\tilde{1}}, E^{\otimes 3}) \oplus \mathrm{Hom}^{S_3}(\underline{\tilde{M}}, E^{\otimes 3}), \\ V_1 &= \mathrm{Hom}^{S_3}(\underline{\tilde{M}_1}, E^{\otimes 3}) \end{aligned}$$

and

$$N = 1 \oplus M_1 \oplus M_2, \text{ the regular representation of } Z_3,$$

then by (3.4)

$$E^{\otimes 3} \cong V_0 \otimes \underline{1} \oplus V_1 \otimes (\underline{M}_1 \oplus \underline{M}_2) \tag{3.5}$$

as a Real Z_3-vector bundle and so

$$[E^{\otimes 3}] = ([V_0]-[V_1]) \otimes 1 + [V_1] \otimes N$$

in $KR_{Z_3}(X) = KR(X) \otimes R(Z_3)$ where $[A]$ denotes the isomorphism class of A.

Here we define ψ_R^3 by

$$\psi_R^3([E]) = [V_0]-[V_1]\,. \tag{3.6}$$

Then we can easily check that ψ_R^3 satisfies the properties of Adams operation. And moreover by [3], Proposition 2.5 we see that forgetting the Real structure, ψ_R^3 is reduced to the complex Adams operation ψ_U^3.

4. Real Adams conjecture for ψ_R^3

The purpose of this section is to prove the following theorem.

Theorem 4.1. *Let X be a finite pointed τ-complex. Then*

$$J_R(\psi_R^3(x)) = J_R(x) \quad \textit{for any} \quad x \in \widetilde{KR}^{-1}(X)$$

in $\pi_s^{0,0}(X)\left[\frac{1}{3}\right]$.

Let Y be a τ-space with trivial Z_3-action. As in §3 we assume here that $E^{\otimes 3}$ has the twisted Real structure for a Real vector bundle E over Y. We have the following lemmas as in [7], §1.

Lemma 4.2 (cf. [7], Proposition 1.2). *There is a natural isomorphism of Real Z_3-vector bundles*

$$(E \oplus F)^{\otimes 3} \cong E^{\otimes 3} \oplus F^{\otimes 3} \oplus (U'(E,F) \otimes \underline{N})$$

for Real vector bundles E and F over Y.

Lemma 4.3 (cf. [7], p.399). *For the trivial Real vector bundle $\underline{n}$ of dimension n over Y there is a canonical isomorphism of Real Z_3-vector bundles*

$$\theta_n \colon \underline{n}^{\otimes 3} \to \underline{n} \oplus (\underline{n}' \otimes \underline{N})$$

such that

$$\pi_n \theta_n((\Sigma_{i=1}^n z_i e_i)^{\otimes 3}, x) = (\Sigma_{i=1}^n z_i^3 e_i,\, x) \quad (z_i \in \boldsymbol{C},\, x \in X)$$

where let π_n denote the projection of $\underline{n} \oplus (\underline{n}' \otimes \underline{N})$ onto $\underline{n}$ and let $e_1, \cdots, e_n$ denote *the standard basis of* $\boldsymbol{C}^n$.

Let $f_k\colon \underline{n}^{\otimes 3}\to\underline{n}\oplus(\underline{n}'\otimes\underline{N})$ ($k=1,2$) be isomorphisms of Real Z_3-vector bundles. Consider the direct sum

$$f_1\oplus f_2\colon\ 2\underline{n}^{\otimes 3}\to 2\underline{n}\oplus(2\underline{n}'\otimes\underline{N})\,.$$

By Lemma 4.2, adding $U'(\underline{n},\underline{n})\otimes\underline{N}$ to the above isomorphism we have an isomorphism of Real Z_3-vector bundles

$$(\underline{2n})^{\otimes 3}\to\underline{2n}\oplus((\underline{2n})'\otimes\underline{N})$$

for which we write f_1+f_2.

By modifying the proof of [7], Proposition 2.5 we get the following

Lemma 4.4 (cf. [7], Proposition 2.5). *Given an isomorphism of Real* Z_3-*vector bundles* $f\colon\underline{n}^{\otimes 3}\to\underline{n}\oplus(\underline{n}'\otimes\underline{N})$, *there is an isomorphism of Real* Z_3-*vector bundles* $g\colon\underline{n}^{\otimes 3}\to\underline{n}\oplus(\underline{n}'\otimes\underline{N})$ *such that* $f+g$ *is homotopic to* θ_{2n} *through Real* Z_3-*isomorphism.*

Define a map $\delta\colon\boldsymbol{C}^n\to\boldsymbol{C}^n$ by $\delta(z_1,\cdots,z_n)=(z_1^3,\cdots,z_n^3)$ $(z_i\in\boldsymbol{C})$. Then δ induces a base-point-preserving τ-map of $\Sigma^{n,n}$ into itself which we denote by the same letter δ. Now, according to [2], Theorem 12.5

$$\pi_{n,n}(\Sigma^{n,n})=Z[\rho]/(1-\rho^2)$$

for $n\geqq 1$. We observe $[\delta]^\tau\in\pi_{n,n}(\Sigma^{n,n})$, the τ-homotopy class of δ.

Lemma 4.5. *With the above notations, we have*

$$[\delta]^\tau=\frac{1+3^n}{2}+\frac{1-3^n}{2}\rho\quad(n\geqq 1)$$

in $\pi_{n,n}(\Sigma^{n,n})$.

Proof. We have

$$\psi(1)=1,\ \phi(1)=1,\ \psi(\rho)=-1\ \text{ and }\ \phi(\rho)=1$$

where ψ and ϕ are the forgetful and fixed-point homomorphisms respectively. So putting $[\delta]^\tau=x+y\rho$ $(x,y\in Z)$ we have

$$x=\frac{1+3^n}{2}\ \text{ and }\ y=\frac{1-3^n}{2}$$

since $\psi([\delta]^\tau)=3^n$ and $\phi([\delta]^\tau)=1$ by the definition. q.e.d.

For a τ-map σ of $\Sigma^{l,l}$ into itself we define a τ-map $t_\sigma\colon\Omega^{m,m}\Sigma^{m,m}\to\Omega^{l+m,l+m}\Sigma^{l+m,l+m}$ by $t_\sigma(\eta)=\sigma\wedge\eta$ $(\eta\in\Omega^{m,m}\Sigma^{m,m})$ where '$\wedge$' denotes the smash product upon one point compactification. Let ε be a τ-map of $\Sigma^{n,n}$ into itself such that

$$[\varepsilon]^\tau=\frac{1+3^n}{2}-\frac{1-3^n}{2}\rho$$

in $\pi_{n,n}(\Sigma^{n,n})$. Then $[\varepsilon\delta]^\tau=3^n$ for δ as in Lemma 4.5. Hence we have

$$\varepsilon\wedge\delta\simeq_\tau\varepsilon\delta\wedge 1\simeq_\tau 3^n\colon \Sigma^{2n,2n}\to\Sigma^{2n,2n}$$

where 1 is the identity map of $\Sigma^{n,n}$.

For a τ-map $h\colon X\to GL(n, \boldsymbol{C})$ we define a τ-map $\tilde{h}\colon X\to\Omega_0^{3n,3n}\Sigma^{3n,3n}$ to be the composition

$$X\xrightarrow{h} GL(n, \boldsymbol{C})\overset{i}{\subset}\Omega^{n,n}\Sigma^{n,n}\xrightarrow{t_\delta}\Omega^{2n,2n}\Sigma^{2n,2n}$$
$$\xrightarrow{t_\varepsilon}\Omega^{3n,3n}\Sigma^{3n,3n}\xrightarrow{\tilde{t}}\Omega^{3n,3n}\Sigma^{3n,3n}.$$

Here i is the canonical inclusion map and $\tilde{t}$ is the map given by adding a fixed map of degree (-3^n) to the elements of $\Omega^{3n,3n}\Sigma^{3n,3n}$ with respect to the loop addition along fixed coordinates of $\Sigma^{3n,3n}$. By ad$h$ we denote the adjoint of $\tilde{h}$. Then, by the definition of $J_{R,3n}$ we have

Lemma 4.6. *With the above notations*

$$[\mathrm{ad}\, h]^\tau = 3^n J_{R,3n}([jh]^\tau)$$

where j is a canonical inclusion map of $GL(n, \boldsymbol{C})$ into $GL(3n, \boldsymbol{C})$.

As we note in [6] we have

$$\widetilde{KR}^{-1}(X)\cong\widetilde{KR}(\Sigma^{0,1}X)\cong[X, GL(\infty, \boldsymbol{C})]^\tau.$$

So we see that any Real vector bundle over $\Sigma^{0,1}X$ is obtained from the clutching of the trivial bundles $E_1=\boldsymbol{C}^m\times\Sigma_+^{0,1}X$ and $E_2=\boldsymbol{C}^m\times\Sigma_-^{0,1}X$ by a base-point-preserving τ-map from X to $GL(m, \boldsymbol{C})$. Here,

$$\Sigma_+^{0,1}X=\{t\wedge x\in SX\,|\,t\geqq 0\},\ \Sigma_-^{0,1}X=\{t\wedge x\in SX\,|\,t\leqq 0\},$$
$$X=\Sigma_+^{0,1}X\cap\Sigma_-^{0,1}X$$

and we consider that $\boldsymbol{C}^m$ has the natural Real structure, i.e., $\boldsymbol{C}^m=\boldsymbol{R}^{m,m}$.

Proof of Theorem 4.1. Denote by E_α the associated vector bundle with a base-point-preserving τ-map $\alpha\colon X\to GL(m, \boldsymbol{C})$. From (3.5) we have a decomposition

$$E_\alpha^{\otimes 3}\cong V_0\oplus V_1\otimes(\underline{M_1}\oplus\underline{M_2})$$

as a Real Z_3-vector bundle over $\Sigma^{0,1}X$ where '$\otimes\underline{1}$' is omitted for the simplicity. Also we have a vector bundle V_1^* over $\Sigma^{0,1}X$ such that $V_1\oplus V_1^*\cong\underline{2s}$ where let dim $V_1=s$. Adding $V_1\oplus V_1^*$ to the above isomorphism we obtain an isomorphism

$$E_{\alpha}^{\otimes 3}\oplus\underline{2s}\cong(V_0\oplus V_1^*)\oplus(V_1\otimes\underline{N})\,.$$

By Lemmas 4.2 and 4.3, adding $((\underline{2s})'\oplus U'(E_{\alpha}, \underline{2s}))\otimes\underline{N}$ we obtain an isomorphism

$$(4.1)\qquad \beta\colon (E_{\alpha}\oplus\underline{2s})^{\otimes 3}\xrightarrow{\cong}(V_0\oplus V_1^*)\oplus((\underline{2s})'\oplus U'(E_{\alpha}, \underline{2s})\oplus V_1)\otimes\underline{N}\,.$$

And by (3.6) we have

$$(4.2)\qquad [V_0\oplus V_1^*]\cong\psi_R^3([E_{\alpha}])+\underline{2s}\,.$$

Observe the restrictions of (4.1) over $\Sigma_+^{0,1}X$ and $\Sigma_-^{0,1}X$, then β yields an isomorphism of trivial bundles over each space since $\Sigma_{\pm}^{0,1}X$ are contractible. Therefore we have a homotopy commutative diagram

$$(4.3)\qquad \begin{array}{ccc} \underline{m}^{\otimes 3} & \xrightarrow{\beta_+} & \underline{m}\oplus(\underline{m}'\otimes\underline{N}) \\ \Big\downarrow f'^{\otimes 3} & & \Big\downarrow g'\oplus(g''\otimes 1) \\ \underline{m}^{\otimes 3} & \xrightarrow{\beta_-} & \underline{m}\oplus(\underline{m}'\otimes\underline{N})\,. \end{array}$$

Here the dotted arrows denote isomorphisms which are defined only over X and $\beta_{\pm}$ are given by $\beta_{\pm}(v, x)=(\beta(v, *), x)$ $(x\in \boldsymbol{C}^{m^3}, x\in\Sigma_{\pm}^{0,1}X)$ respectively where $*$ is the base-point of $\Sigma^{0,1}X$.

Applying Lemma 4.4 to the horizontal isomorphisms in (4.3) we obtain the following homotopy commutative diagram

$$(4.4)\qquad \begin{array}{ccc} (\underline{2m})^{\otimes 3} & \xrightarrow{\theta_{2m}} & \underline{2m}\oplus((\underline{2m})'\otimes\underline{N}) \\ \Big\downarrow \tilde{f}^{\otimes 3} & & \Big\downarrow \tilde{g}\oplus(\tilde{g}'\otimes 1) \\ (\underline{2m})^{\otimes 3} & \xrightarrow{\theta_{2m}} & \underline{2m}\otimes((\underline{2m})'\otimes\underline{N}) \end{array}$$

where $\tilde{f}$, $\tilde{g}$ and $\tilde{g}'$ are isomorphisms over X which are naturally induced from f', g' and g'' respectively.

Put $n=2m$ in (4.4). By Lemma 4.3 we see that the composition

$$\underline{n}\xrightarrow{\Delta_n}\underline{n}^{\otimes 3}\xrightarrow{\theta_n}\underline{n}\oplus(\underline{n}'\otimes\underline{N})\xrightarrow{\pi_n}\underline{n}$$

induces a constant τ-map

$$\gamma\colon X\to\Omega^{n,n}\Sigma^{n,n}$$

given by $\gamma(x)=\delta$ $(x\in X)$ where $\Delta_n(u)=u^{\otimes 3}$ and δ is as in Lemma 4.5. Besides we see that $\tilde{f}$ and $\tilde{g}$ induce τ-maps

$$f\colon X\to GL(n, \boldsymbol{C})\quad\text{and}\quad g\colon X\to GL(n, \boldsymbol{C})$$

in the natural way. By the commutativity of (4.4), we have

$$(ig)\circ\gamma\simeq_\tau\gamma\circ(if)\colon X\to\Omega^{n,n}\Sigma^{n,n}$$

where $i\colon GL(n,\boldsymbol{C})\subset\Omega^{n,n}\Sigma^{n,n}$ denotes the inclusion map and $f\circ h$ is given by $(f\circ h)(x)(z)=f(x)(h(x)(z))$ $(x\in X,\ z\in\Sigma^{n,n})$ for τ-maps $f,\ h\colon X\to\Omega^{n,n}\Sigma^{n,n}$. Therefore we obtain

$$\gamma\wedge ig\simeq_\tau\gamma\wedge if\colon X\to\Omega^{2n,2n}\Sigma^{2n,2n} \tag{4.5}$$

where $f\wedge h$ is given by $(f\wedge h)(x)(z_1\wedge z_2)=f(x)(z_1)\wedge h(x)(z_2)$ $(x\in X,\ z_1, z_2\in\Sigma^{n,n})$ for τ-maps $f, h\colon X\to\Omega^{n,n}\Sigma^{n,n}$. Therefore, by (4.5) and Lemma 4.6 we obtain

$$3^n J_{R,3n}([jf]^\tau)=3^n J_{R,3n}([jg]^\tau)\,.$$

This shows

$$J_{R,3n}([jf]^\tau)=J_{R,3n}([jg]^\tau)$$

in $[\Sigma^{3n,3n}X,\ \Sigma^{3n,3n}]^\tau\left[\frac{1}{3}\right]$. Consequently, passing the direct limit we have

$$J_R(\{E_\alpha\})=J_R(\psi_R^3(\{E_\alpha\}))$$

in $\pi_s^{0,0}(X)\left[\frac{1}{3}\right]$ where $\{A\}$ denotes the stable isomorphism class of A, because the vector bundles associated with f and α are stably equivariant and g respresents $\psi_R^3(\{E_\alpha\})$ stably by (4.2). This completes the proof.

5. $J_R(\pi_{m,n}(GL(\infty,C))$

In [6] we showed that if p is odd and k is even then the image $J_R(\pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C}))$ $(p>k\geqq 0)$ is a cyclic group and its order is either $m(2p)$ or $2m(2p)$. Here we prove the following

Therem 5.1. *The image $J_R(\pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C}))$ is a cyclic group of order $m(2p)$ for $p>k\geqq 0$, p odd and k even.*

Proof. Consider the following diagram

$$\begin{array}{ccc}
\pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C})) & \xrightarrow{\qquad c_1\qquad} & \pi_{4p-1}(GL(\infty,\boldsymbol{C})) \\
\uparrow\cong & & \uparrow\cong \\
\pi_{0,4k-1}(GL(\infty,\boldsymbol{C}))=\pi_{4k-1}(GL(\infty,\boldsymbol{R})) & \xrightarrow{c_2} & \pi_{4k-1}(GL(\infty,\boldsymbol{C}))
\end{array}$$

where the isomorphisms are the complex and Real Thom isomorphisms, and c_1 and c_2 are the natural complexification homomorphisms. Then we see easily that this diagram is commutative and c_2 is an isomorphism since k is even. Therefore c_1 becomes an isomorphism. Let g be a generator of $\pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C}))=Z$. Then we have

$$\psi_R^3(g) = 3^{2p}g$$

because $c_1\psi_R^3=\psi_U^3 c_1$ and $\psi_U^3(c_1(g))=3^{2p}c_1(g)$. Moreover we have $\nu_2(3^{2p}-1)=\nu_2(m(2p))$ by [1], Lemma 2.12 (ii). When we denote by G the quotient module of $\pi_{2p-2k,2p+2k-1}(GL(\infty, \boldsymbol{C}))$ by $(\psi_R^3-1)\,(\pi_{2p-2k,2p+2k-1}(GL(\infty, \boldsymbol{C})))$ we obtain

$$G_{(2)} \cong Z_{2^{\nu_2(m(2p))}}$$

by the above arguments where $G_{(2)}$ denotes the module obtained from G by localizing at the prime ideal (2). Now Theorem 4.1 yields the following 2-local factrization:

$$\begin{array}{ccc} \pi_{2p-2k,2p+2k-1}(GL(\infty, \boldsymbol{C}))_{(2)} & \xrightarrow{J_{R(2)}} & \pi^s_{2p-2k,2p+2k-1(2)} \\ & \searrow \quad \nearrow & \\ & G_{(2)} & \end{array}$$

This result and the theorem of [6] show that the order of $J_R(\pi_{2p-2k,2p+2k-1}(GL(\infty, \boldsymbol{C})))$ is equal to $m(2p)$.

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References

[1] J.F. Adams: *On the groups J(X) II*, Topology **3** (1965), 137–171.

[2] S. Araki and M. Murayama: *τ-Cohomology theories*, Japan. J. Math. **4** (1978), 363–416.

[3] M.F. Atiyah: *Power operation in K-theory*, Quart. J. Math., Oxford, **17** (1966), 165–193.

[4] M.F. Atiyah and G.B. Segal: *Equivariant K-theory and completion*, J. Differential Geometry **3** (1969), 1–18.

[5] C.W. Curtis and I. Reiner: Representation theory of finite groups and associative algebras, Pure and Applied Mathematics vol. XI, J. Wiley and Sons, Inc., 1962.

[6] H. Minami: *On Real J-homomorphisms*, Osaka J. Math. **16** (1979), 529–537.

[7] R.M. Seymour: *Vector bundles invariant under the Adams operations*, Quart. J. Math., Oxford, **25** (1974), 395–414.

[8] V. Snaith: *The complex J-homomorphism* I, Proc. London Math. Soc. **34** (1977), 269–302.

[9] R.M.W. Wood: *Framing the exceptional Lie group G_2*, Topology **15** (1976), 303–320.