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ON $J_R$-HOMOMORPHISMS

HARUO MINAMI

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1. Introduction

In [8] Snaith proved the Adams conjecture for suspension spaces. In this paper we shall prove an analogous result to Snaith's theorem ([8], Corollary 5.2) for the Real Adams operation $\psi^3$ and a Real $J$-map $J_R$ (see §2). This is proved by using the results of Seymour [7]. And as an application we shall determine an undecided order in the theorem of [6].

Here we shall inherit the notations and terminologies in [2], §1 and [6].

2. Homomorphism $J_R$

In [6] we defined the homomorphisms $J_{R_n}$ and $J_R$ for doubly indexed suspension spaces $\Sigma^p q X, p \geq 0$ and $q \geq 1$. Clearly, these definitions are also valid for any finite pointed $\tau$-complex. But the natural map obtained in this manner

$$J_R: \widetilde{K\Gamma}(X) \to \pi^{0,0}_*(X)$$

is not a homomorphism in general. As in the usual case we see that this map satisfies the following formula:

$$J_R(\alpha+\beta)=J_R(\alpha)+J_R(\beta)+J_R(\alpha)J_R(\beta) \quad \alpha, \beta \in \overline{K\Gamma}(X)$$

where $ab (a, b \in \pi^{0,0}_*(X))$ denotes the product of $a$ and $b$ induced by the loop composition in $\Omega^n \Sigma^n$ (cf. [9], p. 314).

3. Adams operation $\psi^3$ in $KR$-theory

In this section we recall the construction of the Real Adams operation $\psi^3_R$ described in [7], §4.

Let $S_3$ be the symmetric group with two generators $a$, $b$ satisfying

$$a^3 = b^2 = 1, \quad bab = a^2$$

and let $Z_3$ be the cyclic subgroup of $S_3$ generated by $a$. From the above relations we see that $\tau(a)=a^2$, $\tau(b)=b$ induces an automorphic involution $\tau$ on
$S_3$. $Z_3$ is closed under the involution $\tau$. Therefore $S_3$ (resp. $Z_3$) is regarded as a Real group with the involution $\tau$ (resp. its restriction to $Z_3$) in the sense of Atiyah-Segal [4].

We know that all simple $S_3$- and $Z_3$-modules over $C$ are as follows:

\begin{align}
S_3: & \begin{cases} 
\bar{1} = \{ C|a = 1, b = 1 \}, 
\bar{M} = \{ C|a = 1, b = -1 \}, 
\end{cases} \\
& M_1 = \{ C^2| av = Av, bv = Bv, v \in C^2 \} \\
& Z_3: \begin{cases} 
1 = \{ C|a = 1 \}, 
M_1 = \{ C| av = \zeta v, v \in C \}, 
M_2 = \{ C| av = \zeta^2 v, v \in C \} 
\end{cases}
\end{align}

where $A = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in GL(2, C)$ and $\zeta = \exp(2\pi i/3)$ (see, e.g., [5], §32).

Let $G$ denote either $S_3$ or $Z_3$. Clearly each $G$-module listed above is a Real $G$-module with the conjugate linear involution induced by complex conjugation. This fact shows that the forgetful map $R_R(G) \rightarrow R(G)$, which is injective in general, is surjective where $R_R(G)$ is the Grothendieck group of Real $G$-modules and $R(G)$ is the complex representation ring of $G$.

Let $X$ be a Real space with trivial $G$-action and $F \rightarrow X$ be a Real $G$-vector bundle in the sense of [4], §6. Then we see easily that the decomposition of $F$ as a complex $G$-vector bundle ([4], §8)

\begin{align}
\bigoplus_M \Hom^G(M, F) \otimes M & \sim \rightarrow F
\end{align}

becomes an isomorphism of Real $G$-vector bundles. Here $M$ runs through the simple $G$-modules over $C$ and $M$ denotes the product bundle $M \times X$ over $X$. And so we see that (3.2) induces a natural isomorphism $KR_R(X) \approx KR(X)$ $\otimes R(G)$.

Let $E \rightarrow X$ be a Real vector bundle over $X$ with the involution $\tau_E: E \rightarrow E$. We define a Real structure $\tau_E$ on $E^{S_3}$ by $\tau_E = (1 \otimes t) \tau^{S_3}$ where $t: E^{S_2} \rightarrow E^{S_2}$ is the switching map. Then $E^{S_3}$ becomes a Real $S_3$-vector bundle with the $S_3$-action permuting the factors.

Applying (3.2) to $E^{S_3}$ we have an isomorphism of Real $S_3$-vector bundles

\begin{align}
E^{S_3} \approx \Hom^S(\bar{1}, E^{S_3}) \otimes \bar{1} & \oplus \Hom^S(\bar{M}, E^{S_3}) \otimes \bar{M} \\
& \oplus \Hom^S(M_1, E^{S_3}) \otimes M_1,
\end{align}

with the notations of (3.1). And by (3.3), as a Real $Z_3$-vector bundle we obtain

\begin{align}
E^{S_3} \approx (\Hom^S(\bar{1}, E^{S_3}) & \oplus \Hom^S(\bar{M}, E^{S_3})) \otimes 1 \\
& \oplus \Hom^S(M_1, E^{S_3}) \otimes (M_1 \oplus M_2).
\end{align}

Put

\begin{align}
V_0 = \Hom^S(\bar{1}, E^{S_3}) & \oplus \Hom^S(\bar{M}, E^{S_3}), \\
V_1 = \Hom^S(M_1, E^{S_3})
\end{align}
and

\[ N = 1 \oplus M_1 \oplus M_2, \text{the regular representation of } Z_3, \]

then by (3.4)

\[ E^\Theta = V_0 \oplus 1 \oplus V_1 \otimes (M_1 \oplus M_2) \]

(3.5)

as a Real \( Z_3 \)-vector bundle and so

\[ [E^\Theta] = ([V_0] - [V_1]) \otimes 1 + [V_1] \otimes N \]

in \( KR_{Z_3}(X) = KR(X) \otimes R(Z_3) \) where \([A]\) denotes the isomorphism class of \( A \).

Here we define \( \psi^3_k \) by

\[ \psi^3_k([E]) = [V_0] - [V_1]. \]

Then we can easily check that \( \psi^3_k \) satisfies the properties of Adams operation. And moreover by [3], Proposition 2.5 we see that forgetting the Real structure, \( \psi^3_k \) is reduced to the complex Adams operation \( \psi_d^3 \).

4. Real Adams conjecture for \( \psi^3_k \)

The purpose of this section is to prove the following theorem.

**Theorem 4.1.** Let \( X \) be a finite pointed \( \tau \)-complex. Then

\[ J_{\theta}(\psi^3_k(x)) = J_{\theta}(x) \quad \text{for any } x \in \tilde{KR}^{-1}(X) \]

in \( \pi^0(X)[1/3] \).

Let \( Y \) be a \( \tau \)-space with trivial \( Z_3 \)-action. As in §3 we assume here that \( E^\Theta \) has the twisted Real structure for a Real vector bundle \( E \) over \( Y \). We have the following lemmas as in [7], §1.

**Lemma 4.2** (cf. [7], Proposition 1.2). There is a natural isomorphism of Real \( Z_3 \)-vector bundles

\[ (E \oplus F)^\Theta = E^\Theta \oplus F^\Theta \oplus (U'F \otimes N) \]

for Real vector bundles \( E \) and \( F \) over \( Y \).

**Lemma 4.3** (cf. [7], p.399). For the trivial Real vector bundle \( n \) of dimension \( n \) over \( Y \) there is a canonical isomorphism of Real \( Z_3 \)-vector bundles

\[ \theta_n : n^\Theta \to n \oplus (n' \otimes N) \]

such that

\[ \pi_n \theta_n((\Sigma_{i=1}^n z_i e_i)^\Theta, x) = (\Sigma_{i=1}^n z_i^3 e_i, x) \quad (z_i \in C, x \in X) \]

where let \( \pi_n \) denote the projection of \( n \oplus (n' \otimes N) \) onto \( n \) and let \( e_1, \ldots, e_n \) denote
the standard basis of \( C^n \).

Let \( f_k: n^0 \rightarrow n \oplus (n' \otimes N) \) \((k=1,2)\) be isomorphisms of Real \( Z_3 \)-vector bundles. Consider the direct sum

\[
f_1 \oplus f_2: 2n^0 \rightarrow 2n \oplus (2n' \otimes N).
\]

By Lemma 4.2, adding \( U'(n,n) \otimes N \) to the above isomorphism we have an isomorphism of Real \( Z_3 \)-vector bundles

\[
(2n)^0 \rightarrow 2n \oplus ((2n')^0 \otimes N)
\]

for which we write \( f_1 + f_2 \).

By modifying the proof of [7], Proposition 2.5 we get the following

**Lemma 4.4** (cf. [7], Proposition 2.5). *Given an isomorphism of Real \( Z_3 \)-vector bundles \( f: n^0 \rightarrow n \oplus (n' \otimes N) \), there is an isomorphism of Real \( Z_3 \)-vector bundles \( g: n^0 \rightarrow n \oplus (n' \otimes N) \) such that \( f + g \) is homotopic to \( \theta_{n'\otimes N} \) through Real \( Z_3 \)-isomorphism.*

Define a map \( \delta: C^n \rightarrow C^n \) by \( \delta(z_1, \ldots, z_n) = (z_1^2, \ldots, z_n^2) \) \((z_i \in C)\). Then \( \delta \) induces a base-point-preserving \( \tau \)-map of \( \Sigma^n \) into itself which we denote by the same letter \( \delta \). Now, according to [2], Theorem 12.5

\[
\pi_{n,\tau}(\Sigma^n) = Z[\rho]/(1 - \rho^2)
\]

for \( n \geq 1 \). We observe \([\delta] \in \pi_{n,\tau}(\Sigma^n)\), the \( \tau \)-homotopy class of \( \delta \).

**Lemma 4.5.** With the above notations, we have

\[
[\delta] = \frac{1 + 3^n}{2} - \frac{1 - 3^n}{2} \rho \quad (n \geq 1)
\]

in \( \pi_{n,\tau}(\Sigma^n) \).

**Proof.** We have

\[
\varphi(1) = 1, \phi(1) = 1, \varphi(\rho) = -1 \quad \text{and} \quad \phi(\rho) = 1
\]

where \( \varphi \) and \( \phi \) are the forgetful and fixed-point homomorphisms respectively. So putting \([\delta] = x + y\rho \) \((x, y \in Z)\) we have

\[
x = \frac{1 + 3^n}{2} \quad \text{and} \quad y = \frac{1 - 3^n}{2}
\]

since \( \varphi([\delta]) = 3^n \) and \( \phi([\delta]) = 1 \) by the definition. q.e.d.

For a \( \tau \)-map \( \sigma \) of \( \Sigma^l \) into itself we define a \( \tau \)-map \( t_\sigma: \Omega^m \Sigma^m,\tau \rightarrow \Omega^l \Sigma^l,\tau \) by \( t_\sigma(\eta) = \sigma \wedge \eta \) \((\eta \in \Omega^m \Sigma^m,\tau)\) where \( \wedge \) denotes the smash product upon one point compactification. Let \( \varepsilon \) be a \( \tau \)-map of \( \Sigma^n,\tau \) into itself such that

\[
[\varepsilon] = \frac{1 + 3^n}{2} - \frac{1 - 3^n}{2} \rho
\]
in $\pi_{n,\ast}(\Sigma^{n,\ast})$. Then $[\epsilon \delta]'' = 3^n$ for $\delta$ as in Lemma 4.5. Hence we have

$$\epsilon \wedge \delta = \epsilon \delta \wedge 1 = 3^n : \Sigma^{2n,2n} \to \Sigma^{2n,2n}$$

where 1 is the identity map of $\Sigma^{n,\ast}$.

For a $\tau$-map $h : X \to GL(n, \mathbb{C})$ we define a $\tau$-map $\tilde{h} : X \to \Omega^{3n,3n}_0 \Sigma^{3n,3n}$ to be the composition

$$X \to GL(n, \mathbb{C}) \subset \Omega^{n,\ast} \Sigma^{n,\ast} \to \Omega^{2n,2n} \Sigma^{2n,2n} \to \Omega^{3n,3n}_0 \Sigma^{3n,3n}.$$

Here $i$ is the canonical inclusion map and $\tilde{i}$ is the map given by adding a fixed map of degree $(-3^n)$ to the elements of $\Omega^{3n,3n}_0 \Sigma^{3n,3n}$ with respect to the loop addition along fixed coordinates of $\Sigma^{3n,3n}$. By $\text{ad}h$ we denote the adjoint of $\tilde{h}$. Then, by the definition of $J_{3n}$ we have

**Lemma 4.6.** With the above notations

$$[\text{ad} h]^* = 3^s J_{3n}([jh]^*)$$

where $j$ is a canonical inclusion map of $GL(n, \mathbb{C})$ into $GL(3n, \mathbb{C})$.

As we note in [6] we have

$$\widetilde{KR}^{-1}(X) \simeq \widetilde{KR}(\Sigma^{0,1}X) \simeq [X, GL(\infty, \mathbb{C})]^\tau.$$

So we see that any Real vector bundle over $\Sigma^{0,1}X$ is obtained from the clutching of the trivial bundles $E_1 = \mathbb{C}^m \times \Sigma^{0,1}X$ and $E_2 = \mathbb{C}^m \times \Sigma^{0,1}X$ by a base-point-preserving $\tau$-map from $X$ to $GL(m, \mathbb{C})$. Here,

$$\Sigma^{0,1}X = \{ t \wedge x \in SX | t \geq 0 \}, \quad \Sigma^{0,1}X = \{ t \wedge x \in SX | t \leq 0 \},$$

$$X = \Sigma^{0,1}X \cap \Sigma^{0,1}X$$

and we consider that $\mathbb{C}^m$ has the natural Real structure, i.e., $\mathbb{C}^m = \mathbb{R}^{m,\ast}$.

**Proof of Theorem 4.1.** Denote by $E_{\alpha}$ the associated vector bundle with a base-point-preserving $\tau$-map $\alpha : X \to GL(m, \mathbb{C})$. From (3.5) we have a decomposition

$$E_{\alpha}^{03} \cong V_0 \oplus V_1 \otimes (M_1 \oplus M_2)$$

as a Real $Z_3$-vector bundle over $\Sigma^{0,1}X$ where `$\otimes 1$' is omitted for the simplicity. Also we have a vector bundle $V_{\uparrow}$ over $\Sigma^{0,1}X$ such that $V_1 \oplus V_{\uparrow} \simeq 2s$ where let $\dim V_1 = s$. Adding $V_1 \oplus V_{\uparrow}$ to the above isomorphism we obtain an isomorphism
By Lemmas 4.2 and 4.3, adding \(((2s)' \oplus U'(E_a, 2s)) \otimes N\) we obtain an isomorphism

\[
(4.1) \quad \beta: (E_a \oplus 2s)^{03} \cong (V_0 \oplus V_1) \oplus (V_1 \otimes N).
\]

And by (3.6) we have

\[
(4.2) \quad [V_0 \oplus V_1] \cong \psi_2([E_a]) + 2s.
\]

Observe the restrictions of (4.1) over \(\Sigma^{0.1}_1\) and \(\Sigma_{-1, x}\), then \(\beta\) yields an isomorphism of trivial bundles over each space since \(\Sigma^{0.1}_\pm\) are contractible. Therefore we have a homotopy commutative diagram

\[
\begin{array}{ccc}
m^{03} & \xrightarrow{\beta^+} & m \oplus (m' \otimes N) \\
\downarrow f'^{03} & & \downarrow (g' \oplus (g'' \otimes 1)) \\
m^{03} & \xrightarrow{\beta^-} & m \oplus (m' \otimes N).
\end{array}
\]

Here the dotted arrows denote isomorphisms which are defined only over \(X\) and \(\beta_{\pm}\) are given by \(\beta_{\pm}(v, x) = (\beta(v, *), x) (x \in C^{m^3}, x \in \Sigma^{0.1}_\pm X)\) respectively where * is the base-point of \(\Sigma^{0.1}_0\).

Applying Lemma 4.4 to the horizontal isomorphisms in (4.3) we obtain the following homotopy commutative diagram

\[
\begin{array}{ccc}
(2m)^{03} & \xrightarrow{\theta_{2m}} & 2m \oplus ((2m)' \otimes N) \\
\downarrow f^{03} & & \downarrow (g \oplus (g' \otimes 1)) \\
(2m)^{03} & \xrightarrow{\theta_{2m}} & 2m \oplus ((2m)' \otimes N)
\end{array}
\]

where \(f, g\) and \(g'\) are isomorphisms over \(X\) which are naturally induced from \(f', g'\) and \(g''\) respectively.

Put \(n=2m\) in (4.4). By Lemma 4.3 we see that the composition

\[
\begin{array}{ccc}
n & \xrightarrow{\Delta_n} & n^{03} \\
\downarrow n & & \downarrow \theta_n \\
n & \xrightarrow{\pi_n} & n
\end{array}
\]

induces a constant \(\tau\)-map:

\[
\gamma: X \rightarrow \Omega^*_x \Sigma^{*, x}
\]

given by \(\gamma(x) = \delta (x \in X)\) where \(\Delta_n(u) = u^{03}\) and \(\delta\) is as in Lemma 4.5. Besides we see that \(f\) and \(g\) induce \(\tau\)-maps

\[
f: X \rightarrow GL(n, C) \quad \text{and} \quad g: X \rightarrow GL(n, C)
\]

in the natural way. By the commutativity of (4.4), we have
where \( i: GL(n, \mathbb{C}) \subset \Omega^n.\Sigma^n. \) denotes the inclusion map and \( f \circ h \) is given by \((f \circ h)(x)(z) = f(x)(h(x)(z)) (x \in X, z \in \Sigma^n.)\) for \( \tau \)-maps \( f, h: X \to \Omega^n.\Sigma^n. \). Therefore we obtain

\[
(\gamma \circ \gamma \circ \gamma \circ \gamma)(jg) = \gamma \circ \gamma \circ \gamma \circ \gamma(f) = f \circ h \circ f \circ h
\]

where \( f \circ h \) is given by \((f \circ h)(x)(z_1 \wedge z_2) = f(x)(z_1) \wedge h(x)(z_2) (x \in X, z_1, z_2 \in \Sigma^n.)\) for \( \tau \)-maps \( f, h: X \to \Omega^n.\Sigma^n. \). Therefore, by (4.5) and Lemma 4.6 we obtain

\[
3^n J_{R,3n}(\{jg\}) = 3^n J_{R,3n}(\{if\})
\]

This shows

\[
J_{R,3n}(\{if\}) = J_{R,3n}(\{jg\})
\]

in \([\Sigma^{3n,3n} X, \Sigma^{3n,3n}] [\frac{1}{3}]\). Consequently, passing the direct limit we have

\[
J_R(\{E_\alpha\}) = J_R(\phi_{\beta}^\alpha(\{E_\alpha\}))
\]

in \(\pi^0.\phi(X)[\frac{1}{3}]\) where \(\{A\}\) denotes the stable isomorphism class of \(A\), because the vector bundles associated with \(f\) and \(\alpha\) are stably equivariant and \(g\) represents \(\phi_{\beta}^\alpha(\{E_\alpha\})\) stably by (4.2). This completes the proof.

5. \(J_R(\pi_{m,n}(GL(\infty, \mathbb{C}))\))

In [6] we showed that if \(p\) is odd and \(k\) is even then the image \(J_R(\pi_{2p-2k,2p+2k-1}(GL(\infty, \mathbb{C}))) (p > k \geq 0)\) is a cyclic group and its order is either \(m(2p)\) or \(2m(2p)\). Here we prove the following

**Theorem 5.1.** The image \(J_R(\pi_{2p-2k,2p+2k-1}(GL(\infty, \mathbb{C}))\) is a cyclic group of order \(m(2p)\) for \(p > k \geq 0, p\) odd and \(k\) even.

**Proof.** Consider the following diagram

\[
\begin{CD}
\pi_{2p-2k,2p+2k-1}(GL(\infty, \mathbb{C})) @>c_1>> \pi_{4p-1}(GL(\infty, \mathbb{C}))
\end{CD}
\]

\[
\begin{CD}
\pi_{0,4k-1}(GL(\infty, \mathbb{C})) @>c_2>> \pi_{4k-1}(GL(\infty, \mathbb{R}))
\end{CD}
\]

where the isomorphisms are the complex and Real Thom isomorphisms, and \(c_1\) and \(c_2\) are the natural complexification homomorphisms. Then we see easily that this diagram is commutative and \(c_2\) is an isomorphism since \(k\) is even. Therefore \(c_1\) becomes an isomorphism. Let \(g\) be a generator of \(\pi_{2p-2k,2p+2k-1}(GL(\infty, \mathbb{C}))\) = \(Z\). Then we have
\[ \psi_k^3 (g) = 3^g g \]

because \( c_1 \psi_k^3 = \psi_k^3 c_1 \) and \( \psi_k^3 (c_1 (g)) = 3^g c_1 (g) \). Moreover we have \( \nu_2 (3^{2^p} - 1) = \nu_2 (m(2p)) \) by [1], Lemma 2.12 (ii). When we denote by \( G \) the quotient module of \( \pi_{2p-2k,2p+2k-1} (GL(\infty, C)) \) by \( (\psi_k^3 - 1) (\pi_{2p-2k,2p+2k-1} (GL(\infty, C))) \) we obtain

\[ G_{(2)} \cong \mathbb{Z}_{2^{m(2p)}} \]

by the above arguments where \( G_{(2)} \) denotes the module obtained from \( G \) by localizing at the prime ideal \( (2) \). Now Theorem 4.1 yields the following 2-local factorization:

\[ \pi_{2p-2k,2p+2k-1} (GL(\infty, C))_{(2)} \xrightarrow{J_{(2)}} \pi_{2p-2k,2p+2k-1} (GL(\infty, C))_{(2)} \]

\[ \downarrow \]

\[ \uparrow \]

\[ G_{(2)} \]

This result and the theorem of [6] show that the order of \( J_{(2)} (\pi_{2p-2k,2p+2k-1} (GL(\infty, C))) \) is equal to \( m(2p) \).

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**References**


